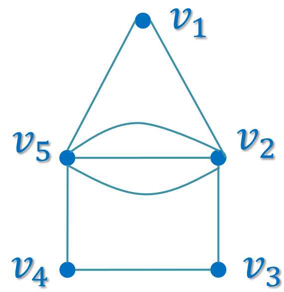
Ma/CS 6b

Class 12: Graphs and Matrices

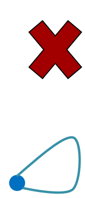
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 \end{pmatrix}$$

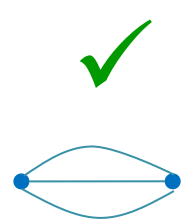


By Adam Sheffer

Non-simple Graphs

- In this class we allow graphs to be non-simple.
- We allow parallel edges, but not loops.

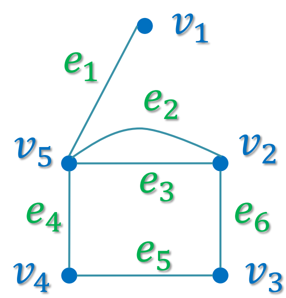




Incidence Matrix

- Consider a graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, \dots, v_n\}$ and the edges as $E = \{e_1, e_2, \dots, e_m\}$
  - The *incidence matrix* of G is an $n \times m$ matrix M. The cell M_{ij} contains 1 if v_i is an endpoint of e_j, and 0 otherwise.

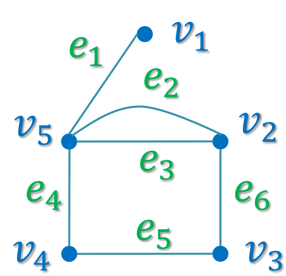
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$



Playing with an Incidence Matrix

- Let M be the incidence matrix of a graph $G = (V, E)$, and let $B = MM^T$.
  - What is the value of B_{ii}? The degree of v_i in G.
  - What is the value of B_{ij} for $i \neq j$? The number of edges between v_i and v_j.

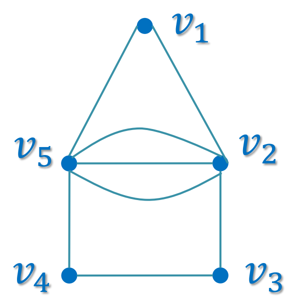
$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$



Adjacency Matrix

- Consider a graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, \ldots, v_n\}$.
  - The *adjacency matrix* of G is a symmetric $n \times n$ matrix A. The cell A_{ij} contains the number of edges between v_i and v_j.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 \end{pmatrix}$$



A Connection

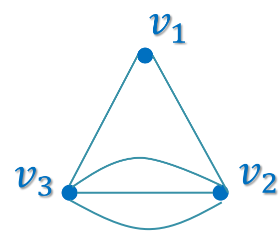
- **Problem.** Let $G = (V, E)$ be a graph with incidence matrix M and adjacency matrix A. Express MM^T using A.
- **Answer.** This is a $|V| \times |V|$ matrix.
  - $(MM^T)_{ii}$ is the degree of v_i.
  - $(MM^T)_{ij}$ for $i \neq j$ is number of edges between v_i and v_j.
  - Let D be a diagonal matrix with D_{ii} being the degree of v_i. We have

$$MM^T = A + D.$$

Example

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$



$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$MM^T = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} = A + D$$

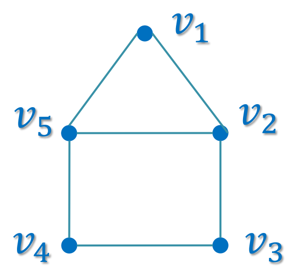
Multiplying by 1's

- Let 1_n denote the $n \times n$ matrix with 1 in each of its cells.
- **Problem.** Let $G = (V, E)$ be a graph with adjacency matrix A. Describe the values in the cells of $B = A1_{|V|}$.
- **Answer.** It is a $|V| \times |V|$ matrix.
  - The column vectors of B are identical.
  - The i'th element of each column is the degree of v_i.

Playing with an Adjacency Matrix

- Let A be the adjacency matrix of a simple graph $G = (V, E)$, and let $M = A^2$.
  - What is the value of M_{ii}? The degree of v_i in G.
  - What is the value of M_{ij} for $i \neq j$? The number of vertices that are adjacent to both v_i and v_j.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$



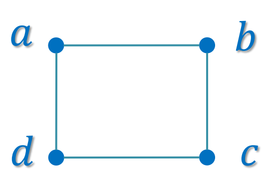
What about M^3?

- Let A be the adjacency matrix of a simple graph $G = (V, E)$. For $i \neq j$:
  - A_{ij} tells us if there is an edge $(v_i, v_j) \in E$.
  - $(A^2)_{ij}$ tells us how many vertices are adjacent to both v_i and v_j.
  - What is $(A^3)_{ij}$? It is the number of paths of length three between v_i and v_j.
  - In fact, $(A^2)_{ij}$ is the number of paths of length two between v_i and v_j.

The Meaning of M^k

- **Theorem.** Let A be the adjacency matrix of a (not necessarily simple) graph $G = (V, E)$. Then $\left(A^k\right)_{ij}$ is the number of (not necessarily simple) paths between v_i and v_j.

An Example



$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

$$A^2 = AA = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$A^3 = A^2A = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 \\ 4 & 0 & 4 & 0 \end{pmatrix}$$

The Meaning of M^k

- **Theorem.** Let M be the incidence matrix of a (not necessarily simple) graph $G = (V, E)$. Then $\left(M^k\right)_{ij}$ is the number of (not necessarily simple) paths between v_i and v_j.
- **Proof.** By induction on k.
  - Induction basis. Easy to see for $k = 1$ and $k = 2$.

The Induction Step

- We have $A^k = A^{k-1}A$. That is

$$\left(A^k\right)_{ij} = \sum_{m=1}^{|V|} \left(A^{k-1}\right)_{im} A_{mj}$$

  - By the induction hypothesis, $\left(A^{k-1}\right)_{im}$ is the number of paths of length $k - 1$ between v_i and v_m.
  - A_{mj} is the number of edges between v_m, v_j.

The Induction Step (cont.)

$$\left(A^k\right)_{ij} = \sum_{m=1}^{|V|} \left(A^{k-1}\right)_{im} A_{mj}$$

- For a fixed m, $\left(A^{k-1}\right)_{im} A_{mj}$ is the number of paths from v_i to v_j of length k with v_m as their penultimate vertex.
  - Thus, summing over every $1 \leq m \leq |V|$ results in the number of paths from v_i to v_j of length k.

Which Classic English Rock Band is More Sciency?

Computing the Number of Paths of Length k

- **Problem.** Consider a graph $G = (V, E)$, two vertices $s, t \in V$, and an integer $k > 0$. Describe an algorithm for finding the number of paths of length k between s and t.
  - Let A be the adjacency matrix of G.
  - We need to compute A^k, which involves $k - 1$ matrix multiplication.

Matrix Multiplication: A Brief History

- We wish to multiply two $n \times n$ matrices.
  - Computing one cell requires about n multiplications and additions. So computing an entire matrix takes cn^3 (for some constant c).
  - In 1969 Strassen found an improved algorithm with a running time of $cn^{2.807}$.
  - In 1987, Coppersmith and Winograd obtained an improved $cn^{2.376}$.
  - After over 20 years, in 2010, Stothers obtained $cn^{2.374}$.
  - Williams immediately improved to $cn^{2.3728642}$.
  - In 2014, Le Gall improved to $cn^{2.3728639}$.

A More Efficient Algorithm

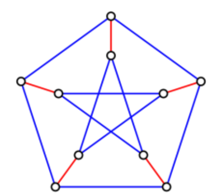
- To compute A^k, we do not need $k-1$ matrix multiplications.
- If k is a power of 2:
  - $A^2 = AA, A^4 = A^2A^2, \ldots, A^k = A^{k/2}A^{k/2}$.
  - Only $\log_2 k$ multiplications!
- If k is not a power of 2:
  - We again compute $A, A^2, A^4, \ldots, A^{\lfloor k \rfloor}$.
  - We can then obtain A^k by multiplying a subset of those.
  - For example, $A^{57} = A^{32}A^{16}A^8A$.
  - At most $2\log k - 1$ multiplications!

Connectivity and Matrices

- **Problem.** Let $G = (V, E)$ be a graph with an adjacency matrix A. Use $M = I + A + A^2 + A^3 + \cdots + A^{|V|-1}$ to tell whether G is connected.
- **Answer.**
  - G is connected iff every cell of A is positive.
  - The main diagonal of A is positive due to I.
  - A cell A_{ij} for $i \neq j$ contains the number of paths between v_i and v_j of length at most $k-1$. If the graph is connected, such paths exist between every two vertices.

And Now with Colors

- **Problem.** Consider a graph $G = (V, E)$, two vertices $s, t \in V$, and an integer $k > 0$. Moreover, every edge is colored either red or blue. Describe an algorithm for finding the number of paths of length k between s and t that have an even number of blue edges.



Solution

- We define two sets of matrices:
  - Cell ij of the matrix $E^{(m)}$ contains the number of paths of length m between v_i an v_j using an even number of blue edges.
  - Cell ij of the matrix $O^{(m)}$ contains the number of paths of length m between v_i an v_j using an odd number of blue edges.
- What are $E^{(1)}$ and $O^{(1)}$?
  - $E^{(1)}$ is the adjacency matrix of G after removing the blue edges. Similarly for $O^{(1)}$ after removing the red edges.

Solution (cont.)

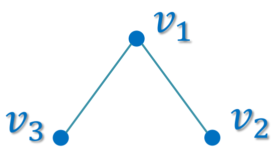
- We wish to compute $E^{(k)}$.
  - We know how to find $E^{(1)}$ and $O^{(1)}$.
  - How can we compute $E^{(m)}$ and $O^{(m)}$ using $E^{(m-1)}$ and $O^{(m-1)}$?
    $$E^{(m)} = E^{(m-1)}E^{(1)} + O^{(m-1)}O^{(1)}.$$
  - $\left(E^{(m-1)}E^{(1)}\right)_{ij}$ is the number of paths of length m between v_i and v_j with an even number of blue edges and whose last edge is red.
  - Similarly for $\left(O^{(m-1)}O^{(1)}\right)_{ij}$ except that the last edges of the paths is blue.

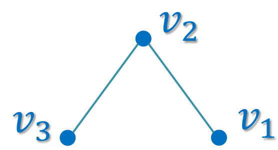
Solution (cont.)

- How can we similarly compute $O^{(m)}$ using $E^{(m-1)}$ and $O^{(m-1)}$?
  $$O^{(m)} = E^{(m-1)}O^{(1)} + O^{(m-1)}E^{(1)}.$$
- Concluding the solution.
  - We compute about $2k$ matrices.
  - Each computation involves two matrix multiplications and one addition.

Identical Graphs?

- Are the following two graphs identical?

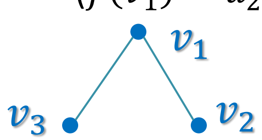


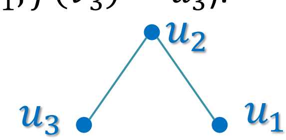


- Possible answers:
  - No, since in one v_1 has degree 2 and in the other degree 1.
  - Yes, because they have exactly the same structure.

Graph Isomorphism

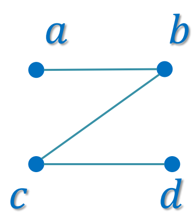
- An *isomorphism* from a graph $G = (V, E)$ to a graph $G' = (V', E')$ is a bijection $f: V \to V'$ such that for every $u, v \in V$, E has the same number of edges between u and v as E' has between $f(u)$ and $f(v)$.
  - We say that G and G' are *isomorphic* if there is an isomorphism from G to G'.
  - In our example, the graphs are isomorphic ($f(v_1) = u_2, f(v_2) = u_1, f(v_3) = u_3$).

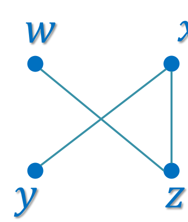




Uniqueness?

- **Question.** If two graphs are isomorphic, can there be more than one isomorphism from one to the other?
  - Yes!
  - $a \to w,\ b \to z, c \to x, d \to y.$
  - $a \to y, b \to z, c \to x, d \to w.$

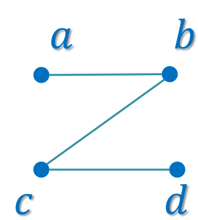




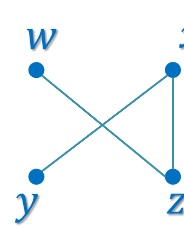
Isomorphisms and Adjacency Matrices

- What can we say about adjacency matrices of isomorphic graphs?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$



Answer

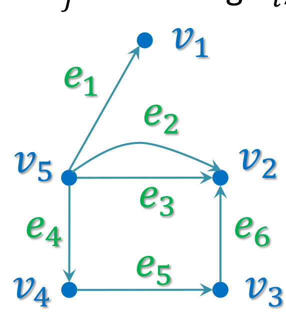
- Two graphs G, G' are isomorphic if and only if the adjacency matrix of G is obtained by permuting the rows and columns of the adjacency matrix of G'.
  - (The same permutation should apply both to the rows and to the column).
  - $1 \to 1, 2 \to 4, 4 \to 3, 3 \to 2$:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Incidence Matrix of a Directed Graph

- Consider a directed graph $G = (V, E)$.
  - We order the vertices as $V = \{v_1, v_2, \dots, v_n\}$ and the edges as $E = \{e_1, e_2, \dots, e_m\}$
  - The *incidence matrix* of G is an $n \times m$ matrix M. The cell M_{ij} contains -1 if e_j is entering v_i, and 1 if e_j is leaving v_i.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$



The End: Queen

Ph.D. in astrophysics

Degree in Biology

Electronics engineer

Freddie Mercury