Signal Processing Primer:
Fourier Analysis, Sampling & Interpolation

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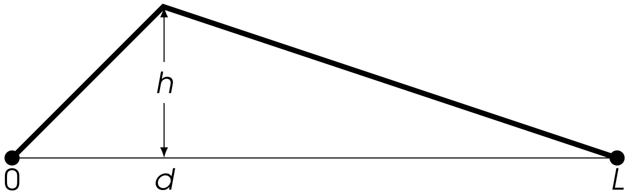
Subject:EE103 23B3933

October 10, 2023

Vibration of a String



Tied String of length L plucked to height h.

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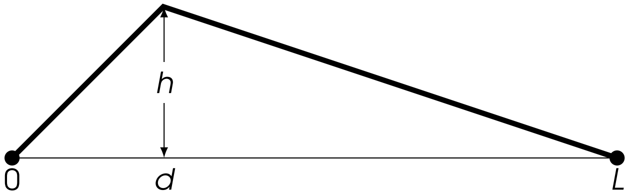
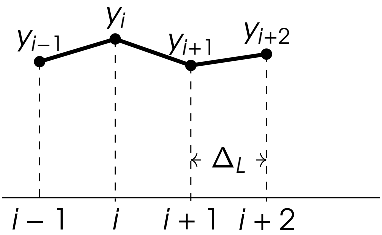


Tied String of length L plucked to height h.

What happens if you let it go?

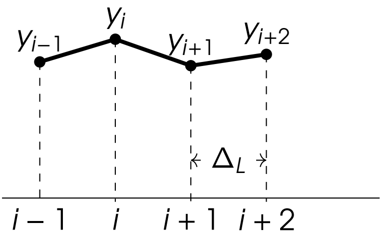
- ☐ String goes straight back to its original shape
- ☐ String snaps to two or more pieces
- ☐ String vibrates randomly, producing audible noise
- ☐ String continues to vibrate, producing sound waves

Particle Model



Particle neighbours in the string

Particle Model



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Force on the point i is $F_i = \frac{\kappa}{\Delta_L}(y_{i+1} - y_i) + \frac{\kappa}{\Delta_L}(y_{i-1} - y_i)$.

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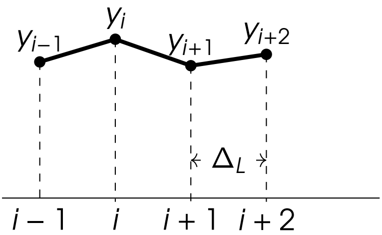

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$$\frac{\kappa}{\Delta_L}(y_{i+1} - y_i) + \frac{\kappa}{\Delta_L}(y_{i-1} - y_i) = m(\Delta_L)\frac{\partial^2 y}{\partial t^2}$$:- Newton's Law

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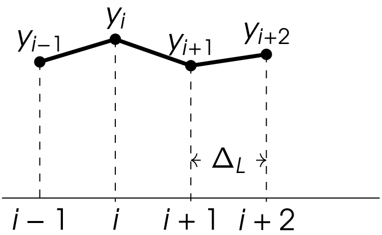

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$$= \rho\Delta_L\frac{\partial^2 y}{\partial t^2}.$$

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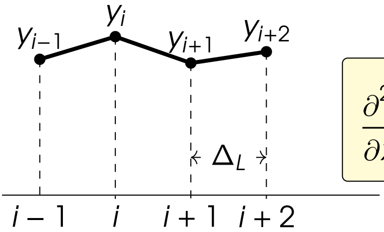

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}}$$

?

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Fourier's Solutions

$$y(x,t) = \sum_{m\in\mathbb{Z}} 2j\, c_m e^{-j\frac{\pi}{L}mt} \sin(\frac{\pi}{L}mx)$$

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$$= \sum_{m\geq 1} \Big(\tilde{a}_m \cos(\frac{\pi}{L}mt) + \tilde{b}_m \sin(\frac{\pi}{L}mt) \Big) \sin(\frac{\pi}{L}mx).$$

$$\frac{dy}{dt} = \sum_{m\geq 1} \Big(-\tilde{a}_m \sin(\frac{\pi}{L}mt)\frac{\pi m}{L} + \tilde{b}_m \cos(\frac{\pi}{L}mt)\frac{\pi m}{L} \Big) \sin(\frac{\pi}{L}mx)$$

$t=0$

$$\Rightarrow \sum_{m\geq 1} 0 + \tilde{b}_m (\;) = 0$$

$$\Rightarrow \boxed{\tilde{b}_m = 0}$$

Fourier's Solutions

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Opposite Camp

$$y(x,0) = \sum_{m\geq 1} \tilde{a}_m \sin(\frac{\pi}{L}mx)\,,\ 0 \leq x \leq L$$

"Initial Position"

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$\Rightarrow$ Any initial position can be expressed as sum of sinusoids??

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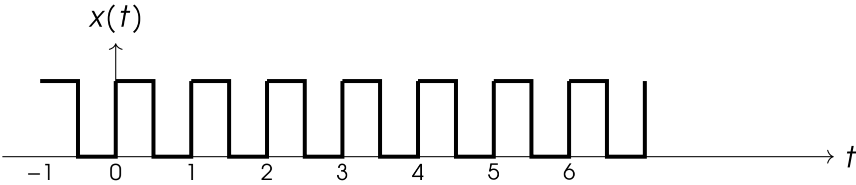
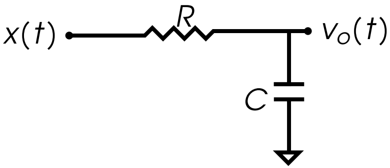
NOTE: RHS is an odd function, periodic with period $T = 2L$.

A Circuit Question





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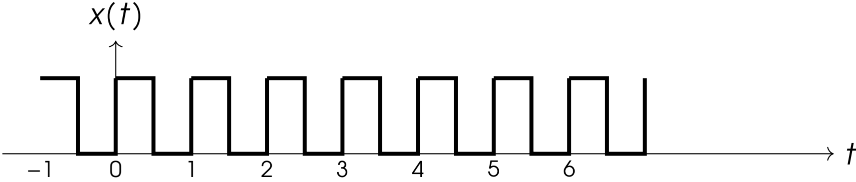
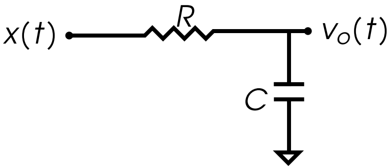


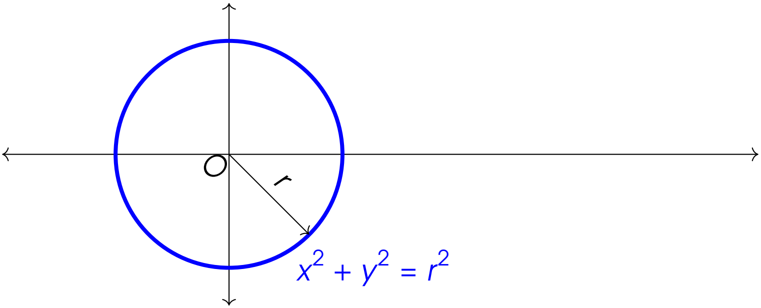


$$x(t) = \sum_m \alpha_m \Phi_m(t)$$

"Superposition"

$$v_o(t) = \sum_m \frac{1}{1 + j2\pi f_m RC} \alpha_m \Phi_m(t)$$

Drawing Signals



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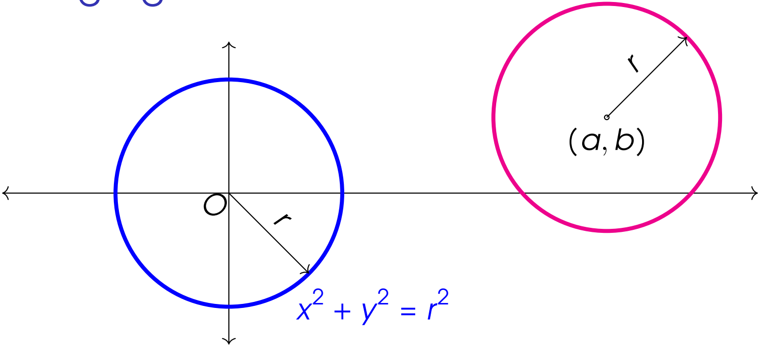

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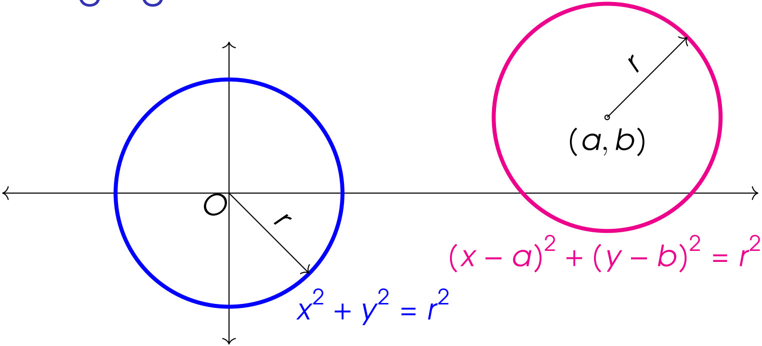

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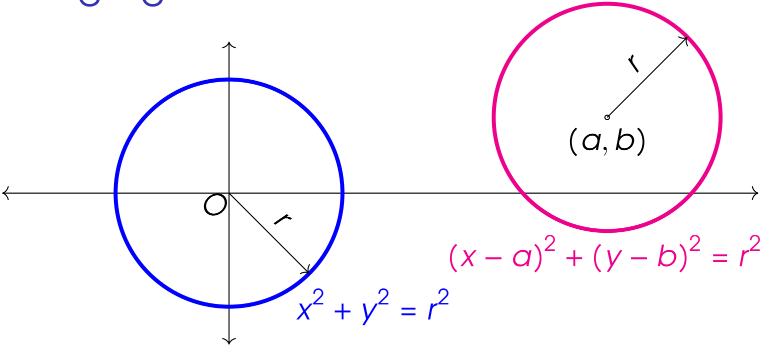

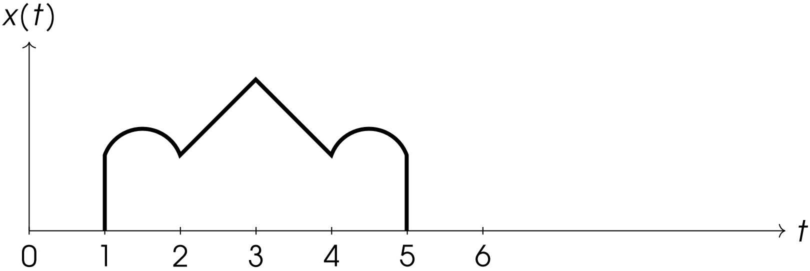

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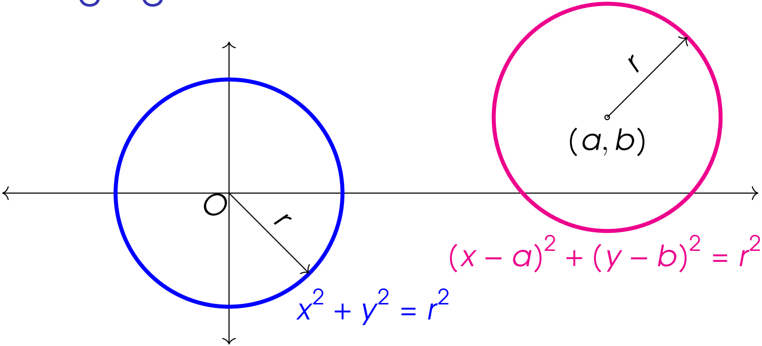

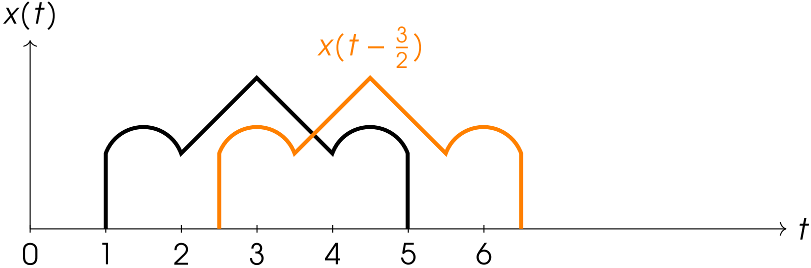

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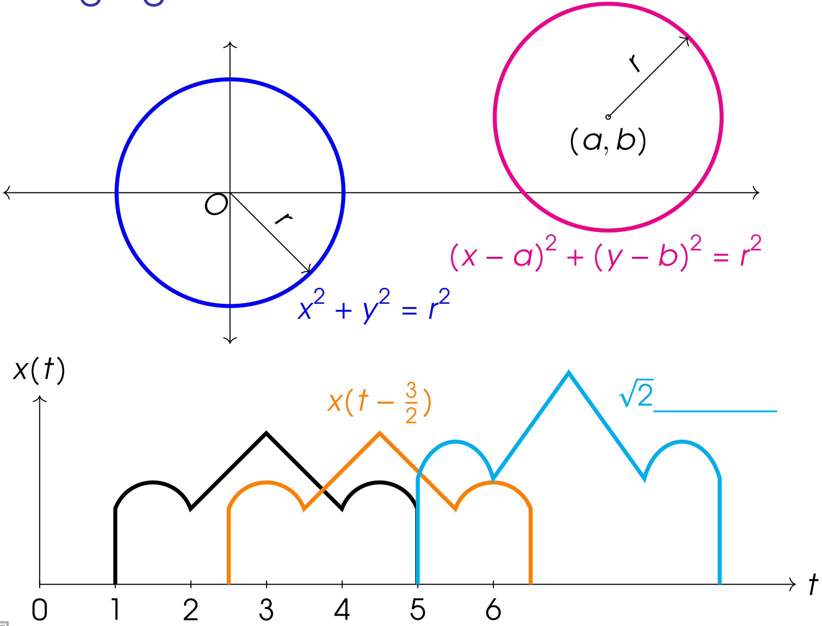

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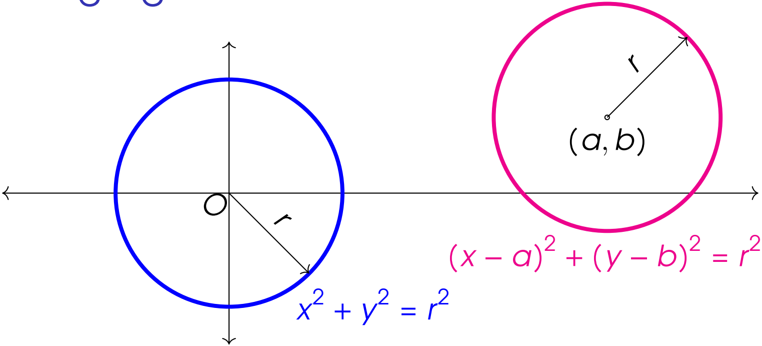

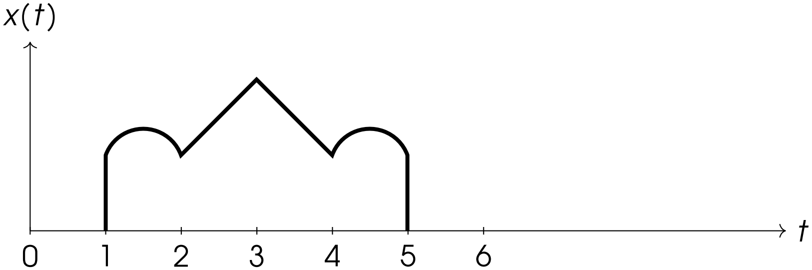

Dot-Product and Projections

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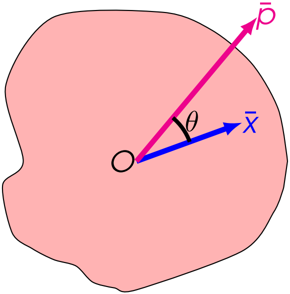
The dot product $\langle \bar{x}, \bar{p} \rangle$ of vectors $\bar{x}, \bar{p}$ measures their overlap.



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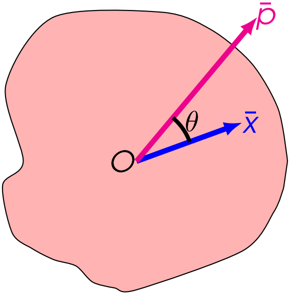

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$$\langle x(t), p(t) \rangle = 0 \Rightarrow \text{orthogonal}$$

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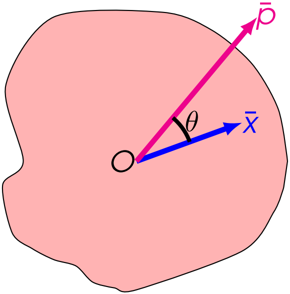

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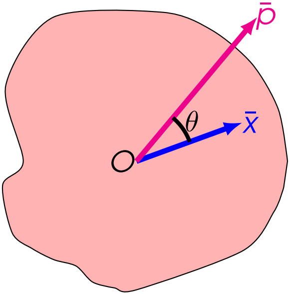

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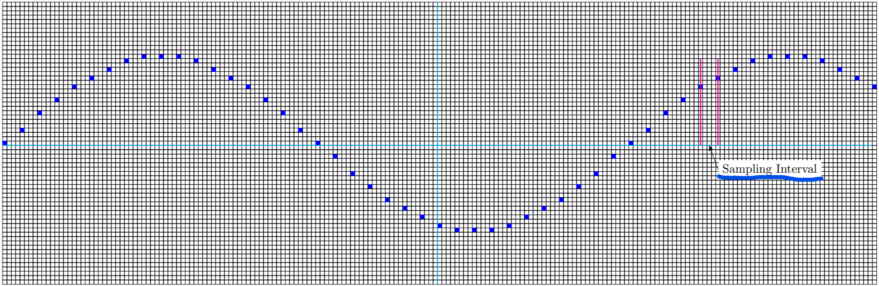
Bandlimited Signal

$$\langle x(t), \cos(2\pi f t) \rangle = \langle x(t), \sin(2\pi f t) \rangle = 0, \forall f \geq f_0$$

Digital Oscilloscope

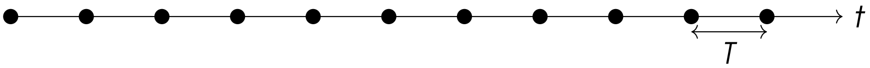


Digital-to-Analog :- "Sufficiently many samples interpolated"

Analog-to-Digital :- "Sample enough to preserve signal identity"

Modern Systems strive to stay digital till the last *whisker*.

Sampling Idea: All Zero Signal



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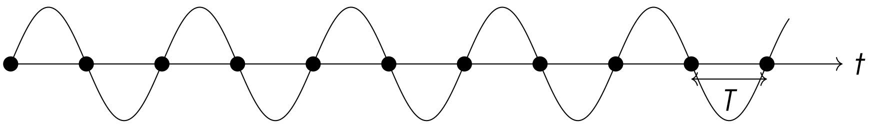

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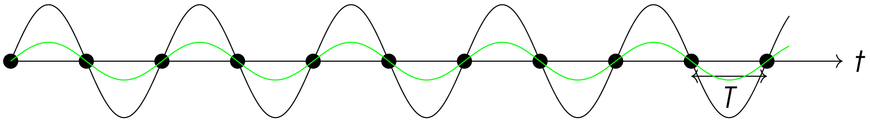

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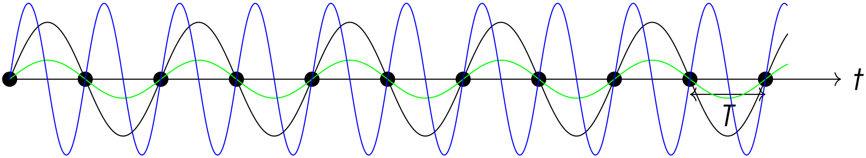

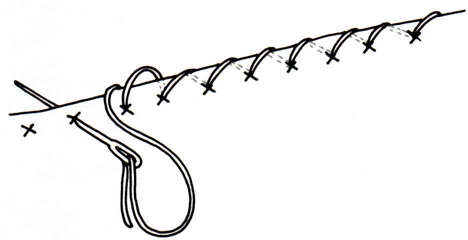

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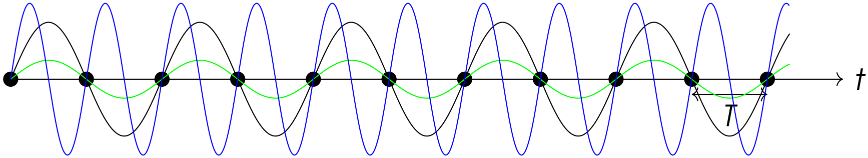
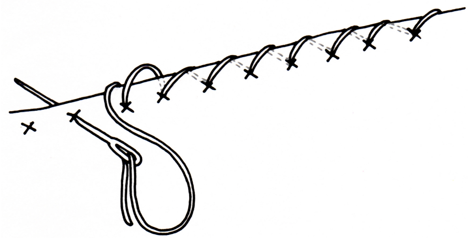


No non-zero continuous interpolator having only frequencies $< \frac{1}{2T}$.



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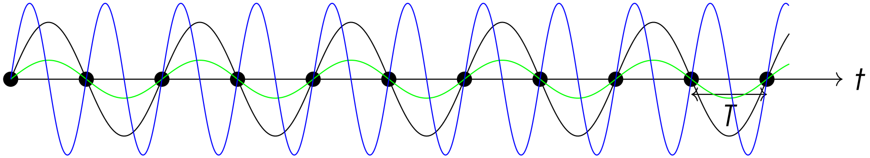

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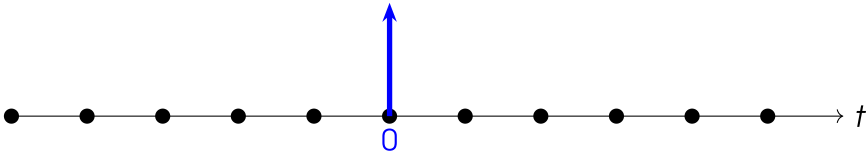
Shannon Interpolator: $\operatorname{sinc}\left(\frac{t}{T}\right)$



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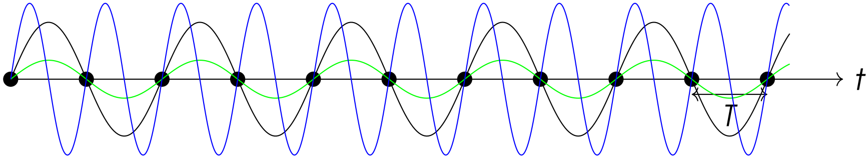

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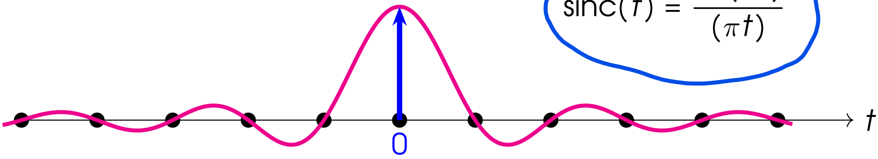
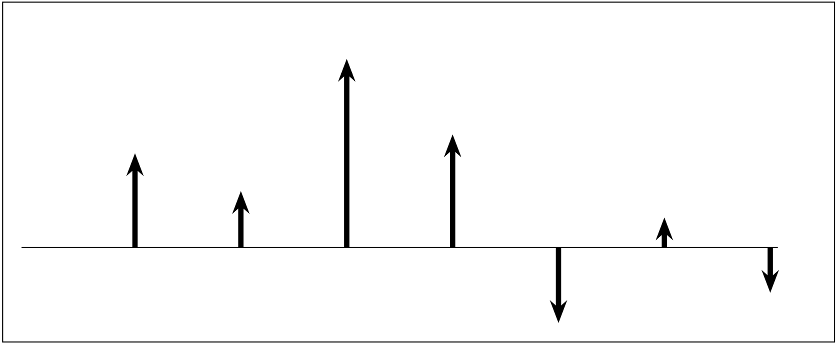
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{(\pi t)}$$



Digital-to-Analog



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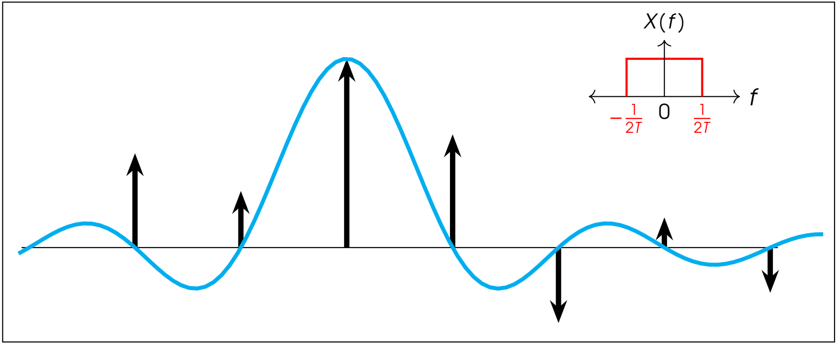

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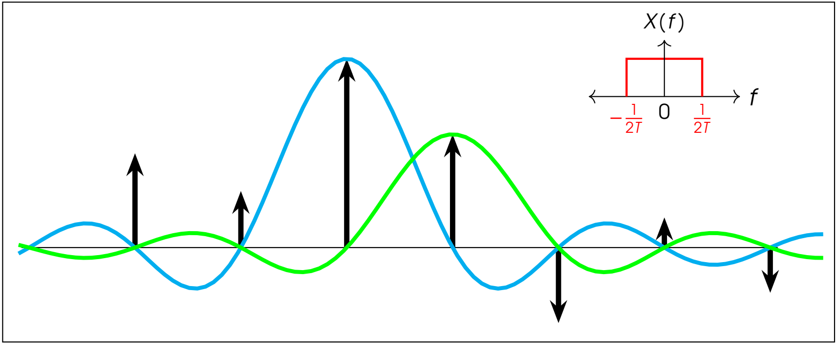

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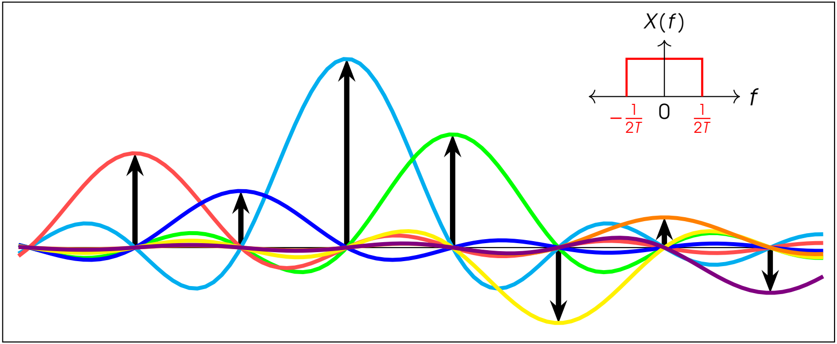

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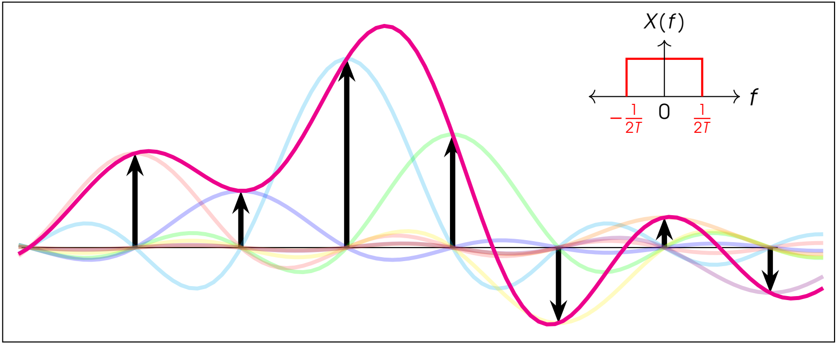

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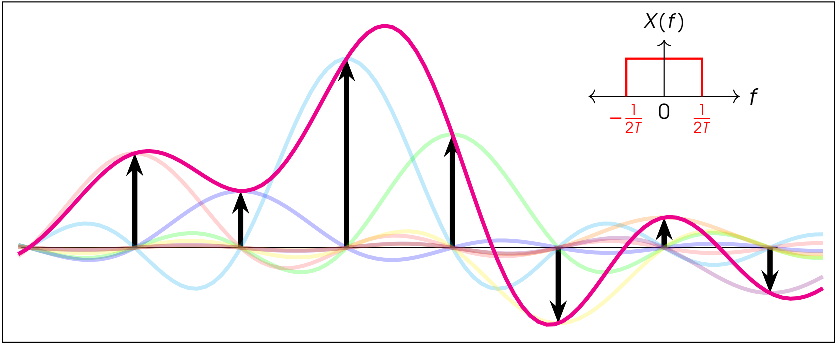
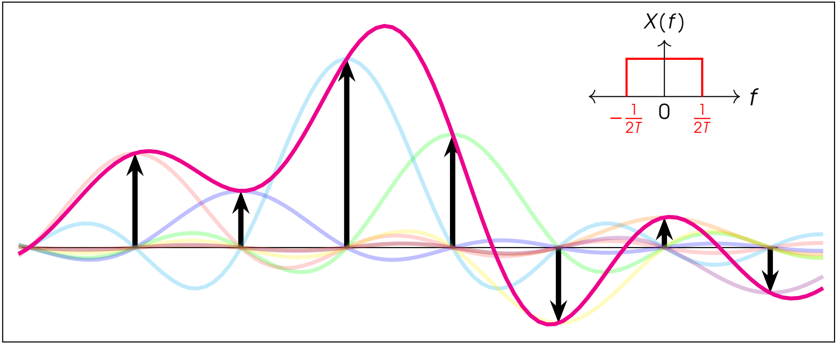


Using superposition: (Shannon Interpolation Formula)

$$x(t) = \sum_{n \in \mathbb{Z}} x[n] \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

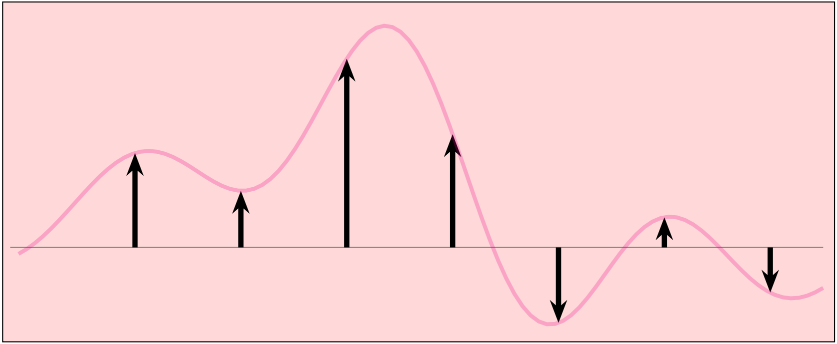
Digital-to-Analog



Using superposition: (Shannon Interpolation Formula)

$$x(t) = \sum_{n\in\mathbb{Z}} x[n]\,\mathrm{sinc}\left(\frac{t}{T} - n\right)$$ "Convolution"

Piece-wise Polynomials



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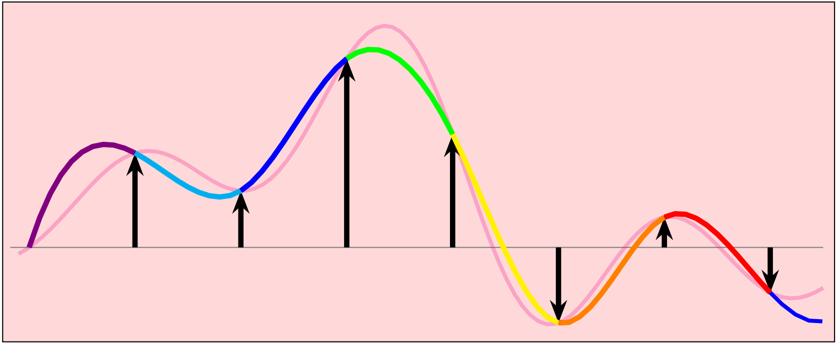

2D Demo: Interpolating Images; 1D : Audio (MP3 or WAV)

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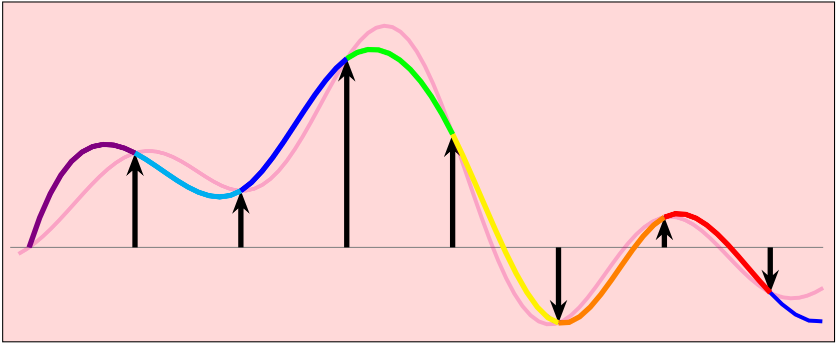

2D Demo: Interpolating Images; 1D : Audio (MP3 or WAV)

How to deal with Colors (multi-dimensional)