EE 113 - Control Systems Module

Madhu Belur

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Control and Computing Group
Department of Electrical Engineering
Indian Institute of Technology Bombay

Key elements of control

Control Theory

- Regulation & Tracking (often needs ‘feedback’): we saw Black’s amplifier

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  (fix sliding or rolling....):
  'trajectory is best amongst various paths'

Today

$s = j\omega$ $f(t)$ $\hat{f}(j\omega)$

- Convolution
- Differential equations
- Laplace transform (of signals) (like Fourier transform of signals):
  'Frequency domain' $F(j\omega)$ representation of
  a time-domain signal $f(t)$

  $f(t)$

  $\hat{f}(j\omega)$ $\cancel{f}(j\omega)$
- $f(t) \leftrightarrow F(s)$, or $F(j\omega)$, of $\hat{f}(s)$, $\hat{f}(j\omega)$,

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- Recall again: Laplace transform (and Fourier transform): we take for signals
- Systems have inputs and outputs as signals
- Ratio of Laplace transforms (output/input):

$$\frac{\text{output}}{\text{input}} \qquad \cancel{\frac{y(t)}{u(t)}} \qquad \frac{Y(s)}{U(s)}$$

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- Ratio of Laplace transforms (output/input):
  gain: 'transfer function'
- Poles, zeros

Convolution

$y(t) := \int \cdot$ ——

- $y(t) := \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau,$

Convolution

$u(t)$ $h(t)$ $\times$ $*$

- $y(t) = \int_{-\infty}^{\infty} u(\tau) h(t - \tau) d\tau$, denoted by $y = u * h$
  $y(t) = (u * h)(t)$

Convolution

$$\int_0^t u(\tau)\, h(t-\tau)\, d\tau .$$

$$u \longrightarrow \boxed{h} \longrightarrow y$$

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Convolution

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- Systems have 'memory' and a 'signature'. System convolves input u with its signature h and releases output y.

  $y = u * h = h * u$
- LTI systems: linear, constant-coefficient, ordinary differential equations: (linear-time-invariant)

  $u \rightarrow \boxed{\quad} \rightarrow y$
- Examples

$6\dot{u} + 5tu - 3y + \frac{d^6}{dt^6}y = 0$

$\frac{d}{dt}u = \dot{u}$

$u, y, \dot{u}, \dot{y}$

$F = Ma$

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F

$q(t) = R_T \, i(t)$

$u(t), y(t), \dot{u}, \dot{y}, \ddot{u}, \frac{dy}{du}$

$u \to \boxed{\text{LTI}} \to y$

$u_1 \to y_1$

$u_2 \to y_2$

$\alpha_1 u_1 + \alpha_2 u_2 \to \boxed{\text{Linear}} \to \alpha_1 y_1 + \alpha_2 y_2$

$\alpha_1, \alpha_2 \in \mathbb{R}$

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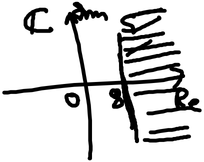
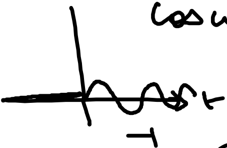
$$\sin \omega t \longrightarrow \frac{\omega}{s^2+\omega^2}$$



$$\cos \omega t \longrightarrow \frac{s}{s^2+\omega^2}$$



$$\int_0^{\infty} u(t)\, e^{-st}\, dt$$

$$\mathcal{L}^{-1}\left(\frac{10}{s-8}\right) \longrightarrow 10e^{+8t}$$

$$\frac{1}{(s+3)(s+2)} = \frac{a}{s+3} + \frac{b}{s+2}$$

$$\frac{a}{s+3} \longrightarrow ae^{-3t} \qquad \frac{b}{s+2} \longrightarrow be^{-2t}$$

$u(t) \longrightarrow U(s)$

$s = 3$

$U(0) = \int_0^\infty u(t)\,dt$

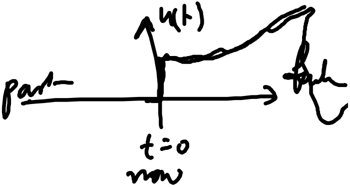
$\mathcal{L}(u)$

$s=0$

$u(t) \longrightarrow U(s)$

- Given $u(t)$, $U(s) := (L(u))(s) := \int_0^\infty u(t) e^{-st} dt$
- For suitable class of functions, Laplace transform is 'well-defined'.
- Look-up table, linearity, 'linear combination'
- Laplace transform of $f(t)$: exponentials, sinusoids, etc : 'strictly proper' $F(s)$.

$U(s)$



$U(s) = \int_0^\infty u(t)\, e^{-st}\, dt$

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$$10e^{-3t}u(t) \xrightarrow{\mathcal{L}} \frac{10}{s+3}$$

$<$

strictly proper

numer

deg n $\leq$ deg d

proper

denom

$$-\pi e^{-8t} \xrightarrow{\mathcal{L}} \frac{-\pi}{s+8}$$

e^{-8t}

$$e^{63t} \longrightarrow \frac{1}{s-63}$$

e^{6t}

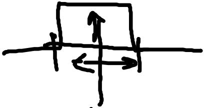
$\sin 8t$

$$\left(e^{-3t} + 7e^{63t}\right) \xrightarrow{\mathcal{L}} \frac{1}{s+3} + \frac{7}{s-63}$$

Impulse

distribution

δ function



- Impulse: δ: is nonzero for very small time around $t = 0$,
- Still manages area $= 1$ (becomes unbounded) $\quad \frac{1}{} = \frac{1}{1}$
- Has Fourier and Laplace transform as 1.
- When $u(t) = \delta$, then output $y(t) = L^{-1}G(s)$
- Examples

$$\mathcal{L}\left(\frac{d}{dt}y + 6y\right) = \mathcal{L}\left(3u - 63\frac{du}{dt}\right)$$

zero initial conditi

$$\mathcal{L}\left(\frac{d}{dt}y\right) = s\mathcal{L}(y)$$

$\mathcal{L}$ s under zero init

Laplace transform of (both sides in) a differential equation

$$sY(s) + 6Y(s) = 3U(s) - 63sU(s)$$

$$\mathcal{L}(f) = F(s)$$

$$(s+6)Y(s) = (3-63s)U(s)$$

$$\mathcal{L}(\dot{f}) = sF(s)$$

- $\frac{d}{dt}y + y = 2\frac{d}{dt}u - y$

$$f \xrightarrow{\mathcal{L}} F$$

$$\dot{f} \longrightarrow sF - f(0)$$

$$sY(s) + Y(s) = 2sU(s) - Y(s)$$

$$u \longrightarrow \boxed{G} \longrightarrow y$$

$$\frac{Y(s)}{U(s)} = G(s) = \frac{2s}{s+2}$$

$$\frac{3-63s}{s+6} = \frac{Y(s)}{U(s)}$$

zeros = 0

poles = −2

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$$\frac{d^2}{dt^2} y + 3\dot{y} + 2y = \dot{u} - 2u .$$

$$\frac{s-2}{s^2+3s+2}$$

poles $= -2 \rightarrow p_1,\ -1 \rightarrow p_2$

zeros $= +2$

$e^{p_1 t} \quad e^{p_2 t}$

Let $u(t) \equiv 0$

what solution satisfy $c_1 e^{-2t} + c_2 e^{-t}$

$c_1 \in \mathbb{R},\ c_2 \in \mathbb{R}$

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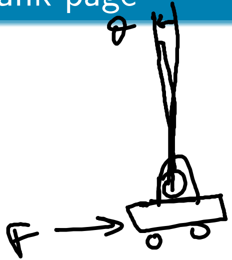


$F = M\ddot{\theta} - c\theta$ w.r.t. vertically up, $c > 0$

$\cancel{F =}\ M\ddot{\theta} + c\theta$ $\quad c > 0$

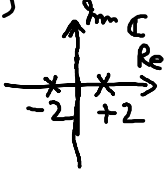
$F =$

θ w.r.t. below vertical

$\hat{F}(s) = s^2 M\hat{\theta}(s) - c\hat{\theta}(s)$

$F \to \boxed{} \to \theta$

$$\frac{\hat{\theta}(s)}{\hat{F}(s)} = \frac{1}{Ms^2 - c}$$



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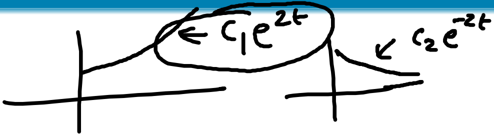
$$c_1 e^{2t} + c_2 e^{-2t}$$



$M = 1$

$c = 4$

$F = M$

$F = \ddot{\theta} - 4\theta$

$F = k\theta$

$\pm\sqrt{k+4}$ $\quad k > -4$

$k\theta = \ddot{\theta} - 4\theta$

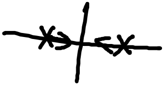
$\ddot{\theta} + (-4-k)\theta = 0.$

$s^2 + (-k-4)$

$\boxed{s^2 + (-k-4)}\ \hat{\theta}(s) = 0$

$k = -4$

$s^2 + \bigcirc\, s - k_p - 4$



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at least one pole in RHP for $k > -4$.

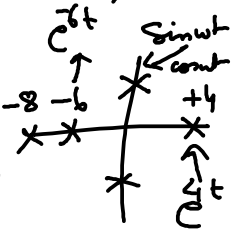
for $k < -4$ on $j\mathbb{R} \equiv$ imaginary axis

$k = -4$ 2 poles at origin

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}.$$

$\omega > 0$

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$$



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$-8+6j$

$\frac{\omega}{s^2+\omega^2}$

$\mathbb{C}$

$\times \leftarrow 10+8j$

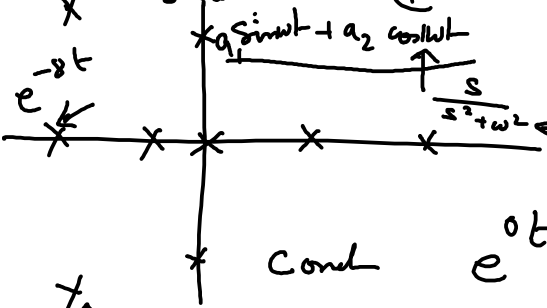
$10t$

$e \quad \sin 8j$

$a_1 \sin\omega t + a_2 \cos\omega t$

e^{-8t}

$\frac{s}{s^2+\omega^2}$



Const

e^{0t}

$\times \leftarrow 10-8j$

$-8-6j$

$\frac{6}{(s+8)^2+36}$ $\quad$ $\frac{s}{(s+8)^2+36}$

$a_1 e^{-8t} \sin 6t + a_2 e^{-8t} \cos 6t$

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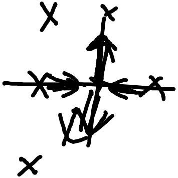
$$\ddot{\theta} - 4\theta = F$$

$$F = k_p\theta + k_d\dot{\theta}$$



$$\frac{1}{s^2 - 4}$$

can find k_p, k_d values to get closed loop poles at $-3 \pm 4j$

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