δ Delta.
impulse

impulse respn

EE 103 - Control Systems Module

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δ_d — discrete

δ_c — conti

eigenfns

$\mathcal{L}$-trm ??

3rd Nov 2023 (Lecture 3)

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Stable/unstable

open/closed loop

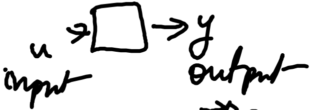
Today

$f(t) = \sin t$

$f(20) = \sin(20)$

- More Convolution
- Differential equations: feedback
- Laplace transform (and Fourier transform): we take for signals
- Systems have inputs and outputs as signals
- Ratio of Laplace transforms (output/input):

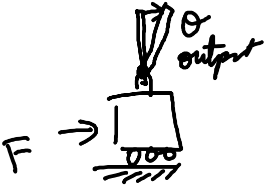
$\frac{d}{dt} y - 6y = 0 \qquad Ce^{6t} = y(t)$



$$\frac{d}{dt} y - 6y = 3u$$

unforced system

$u = 0$



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- Differential equations: feedback
- Laplace transform (and Fourier transform): we take for signals
- Systems have inputs and outputs as signals
- Ratio of Laplace transforms (output/input):
  gain: 'transfer function'
- Poles, zeros: for systems
- But transfer function $G(s)$ of system: also Laplace transform of a
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- But transfer function $G(s)$ of system: also Laplace transform of a
  signal: the system's 'impulse response'

Convolution

- $y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau,$

Convolution

- $y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$, denoted by $y = u * h$
  $y(t) = (u * h)(t)$

Convolution

- $y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$, denoted by $y = u * h$
  $y(t) = (u * h)(t)$ (Check that interchange of role of u and h is fine.)

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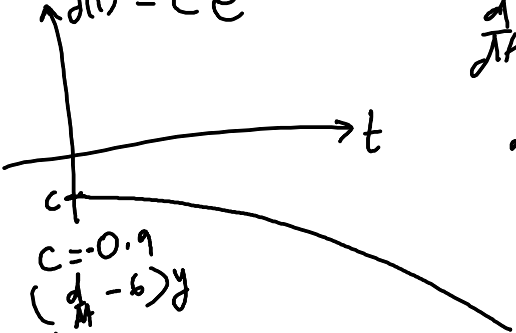
$y(t) = Ce^{6t}$



$\frac{d}{dt}y - 6y = 0$

$\mathcal{L}(\qquad) = \mathcal{L}(0)$

$(s-6)Y(s) = 0$

$(s-6)$

$c = -0.9$

$(\frac{d}{dt} - 6)y$

$\frac{d}{dt}y - 6y = u$

$u(t) = 8e^{-3t}$ ≡ open loop

$u(t) := 9y(t)$

$\frac{d}{dt}y - 6y = -9y$

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$$\left(\frac{d}{dt} - 6 + 9\right) y(t) = 0$$

$$(\cancel{s - 6 + 9})\, Y(s) = 0$$

$$(s + 3)\, Y(s) = 0$$

open loop pole (exponent in unforced system) $= +6$ unstable

using feedback law $u = -9y$

closed loop pole at -3 stable.

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$\frac{d}{dt} y - ky = u$

$u(t)$

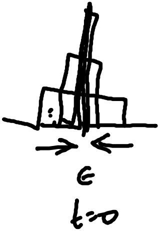
$t \in \mathbb{R}$

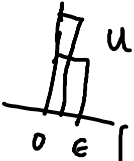
continuous time

$u(n)$, $n \in \mathbb{Z}$

discrete time

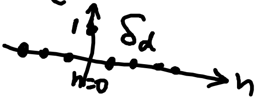
δ_c





Area under $\delta = 1$

$u(t) < 0$



$\delta_d(0) = 1$, $\delta_d(n) = 0$ for $n \neq 0$

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$u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$

Special input $u(t) = \delta_c$

$h(t) \leftarrow$ imp

that corresponding output

impulse response

For all other inputs

$u(t) \leftarrow$

$\longrightarrow \boxed{} \longrightarrow y(t) = h * u = \int_{-\infty}^{\infty} h(t-\tau)\, u(\tau)\, d\tau$

$c_1 e^{c_2 t} \longrightarrow \boxed{} \longrightarrow c_5 e^{c_2 t}$

$\frac{c_5}{c_1}$

$f(c_2)$

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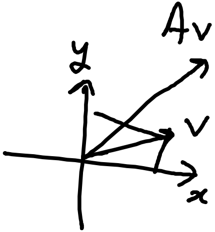
$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \qquad A \begin{bmatrix} v \end{bmatrix}$$



$$e^{5t} \qquad e^{5t+3jt}$$

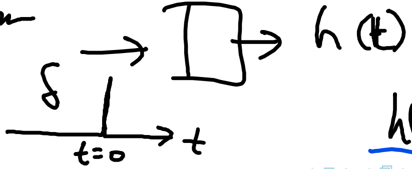
$$\sin t = \frac{e^{jt} - e^{-jt}}{2j}$$

non-anticipatory

$\equiv$

causal

impulse response



$h(t) = 0$ for $t<0$

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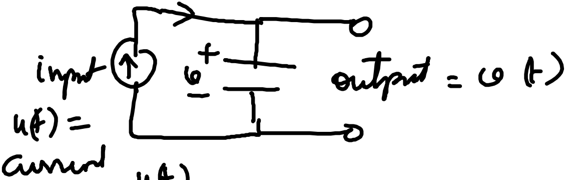
start expt to system initially at rest

call $t=0$

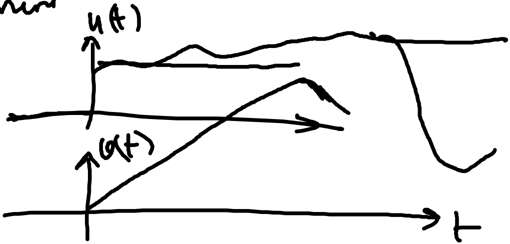
$$I(s) \longrightarrow \boxed{\frac{1}{s}} \longrightarrow V(s)$$



$$\boxed{\frac{d}{dt} v = i}$$

$$v(0) = 0$$

$$v(t) = \int_0^t u(\tau)\, d\tau$$



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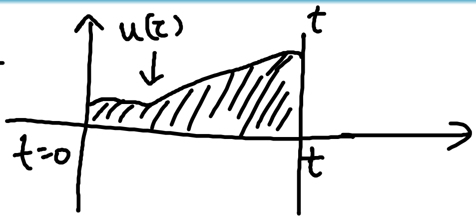
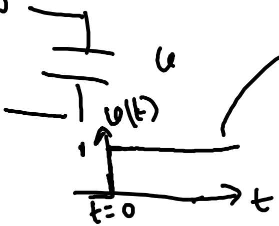
$u(t) = \delta$



$$y(t) = \int_0^t u(\tau)\,d\tau$$

$$\int^t 1 \cdot u(\tau)\,d\tau.$$

δ



$$\int_0^t h(t-\tau)\,u(\tau)\,d\tau$$

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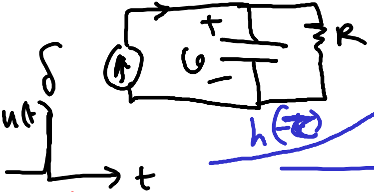
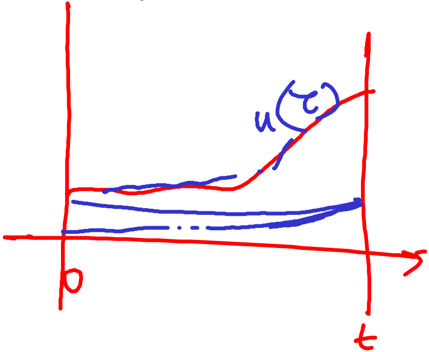
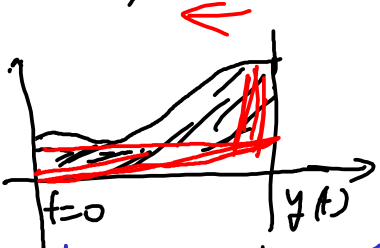
output = $v(t)$







$$\int_0^t u(\tau)\, h(t-\tau)\, d\tau$$

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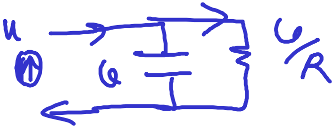

$$u = \frac{dv}{dt} + \frac{v}{R}$$

$$U(s) = sY(s) + \frac{Y(s)}{R}.$$

$$u \longrightarrow \boxed{\frac{R}{1+sR}} \longrightarrow y$$

$$\frac{Y(s)}{U(s)} = \frac{R}{1+sR}$$

$$G(s) = \frac{Y(s)}{U(s)}$$

Impulse response

$$\mathcal{L}^{-1}\left(G(s)\,U(s)\right)$$

$$U(s) = 1$$

$$\mathcal{L}^{-1}\left(G(s)\right)$$

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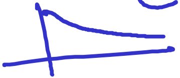
$G(s) = \frac{R}{1+sR}$

$\mathcal{L}^{-1}(G(s)) = \cancel{R}\, e^{-\frac{1}{R}t}$

$\frac{1}{1\cdot s + \frac{1}{R}}$



$R = ??$

0.01, 0.1, $R = 10$

e^{3t}

$\frac{1}{s-3}$

$\delta \longrightarrow 1$

$\int_0^\infty \delta \, dt = 1$

$= 1$

$t < 0$

$f(t) = 0$

$e^{-8t} \xrightarrow{\mathcal{L}} \frac{1}{s+8}$

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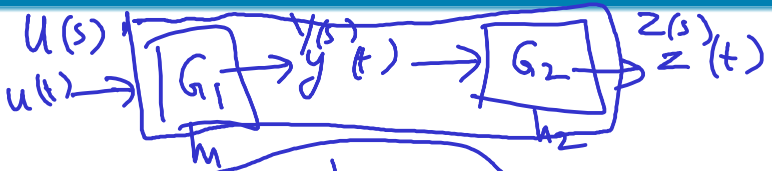

$$y(t) = h_1 * u$$

$$z(t) = h_2 * h_1 * u$$

$$Y(s) = \underbrace{G_1(s)} \cdot \underbrace{U(s)}$$

$$Z(s) = G_2(s) \cdot Y(s) = G_2 \cdot G_1 \cdot U(s)$$

$$Z(s) = G_2 G_1 \cdot$$

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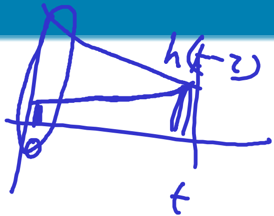
$$G(s) = \frac{1}{s-3}$$

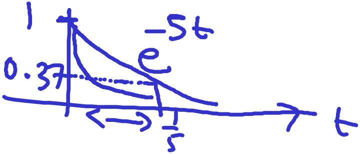


$$u = \frac{d}{dt}y - 3y.$$

$$(s+5)\,Y(s) = 0$$

$$\boxed{u = -8y}$$

$$\frac{d}{dt}y - 3y + 8y = 0$$



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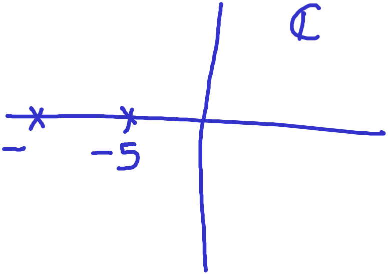

$u = -8y \longrightarrow$ cl. loop pol

-5

tim $\frac{1}{5}$

$u = -18y$

$\searrow$ cl. loop pol

$-15, \ \frac{1}{15}$

$u = -ky$

smaller time const $\Leftarrow$ large k (for many exam)

$$\frac{d^2}{dt^2} y + 3$$

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$$G(s) = \frac{1}{s^3+s^2+3s-6}$$

$$\frac{d^3}{dt^3}y + \frac{d^2}{dt^2}y + 3\frac{d}{dt}y - 6y = u$$

- how check large k makes unstable.

- open loop stable?

$$(s+1)^3 = s^3+3s^2+3s+1$$

Laplace transform

- Given $u(t)$, $U(s) := (L(u))(s) := \int_0^\infty u(t)e^{-st}dt$
- For suitable class of functions, Laplace transform is 'well-defined'.
- Look-up table, linearity, 'linear combination'
- Laplace transform of $f(t)$: exponentials, sinusoids, etc : 'strictly proper' $F(s)$.

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$$s^3+s^2+3s-6$$

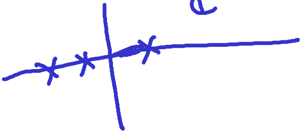
$$u = -k_p y - \cancel{k_d \dot{y}}$$

$$(s^3+s^2+3s-6+k)$$ closed loop polynomial.

Root locus

well-known that for large k, closed loop unstable

$\mathbb{C}$



Impulse

- Impulse: δ: is nonzero for very small time around $t = 0$,
- Still manages area $= 1$ (becomes unbounded)
- Has Fourier and Laplace transform as 1.
- When $u(t) = \delta$, then output $y(t) = L^{-1}G(s)$
- Examples

Laplace transform of (both sides in) a differential equation

- $\frac{d}{dt}y + y = 2\frac{d}{dt}u - y$

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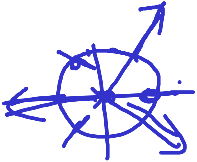
s^3

$$u \rightarrow \boxed{\frac{1}{s}} \rightarrow \boxed{\frac{1}{s}} \rightarrow \boxed{\frac{1}{s}} \rightarrow y$$

$$\frac{d^3}{dt^3} y = u .$$

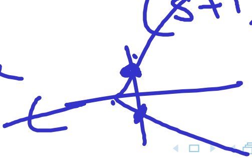


$s^3 + k$

$u = -ky$

$(s+1)^3 + k$

$s^3 + s^2 + 3s - 6 + k$



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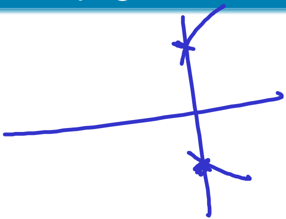

$s^3+s^2+3s-6+k$

$= (s^2+w^2)(s+a)$ with roots $\pm jw$ and $-a$

$s^3+as^2+sw^2+aw^2$

$(k-6) = 1.3.$

For $k=6$, one root at $s=0$

check for 5.9 & 6.1 also

So $k = 9$

Check that for k 8.9 & 9.1)

some change happens about RHP roots.

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More generally,

For polynomial of degree 3 with real coefficients,

$s^3 + k_2 s^2 + k_1 s + k_0$

to get factored into

$(s^2 + \omega^2)(s + a)$,

we need $\boxed{k_0 = k_1 \cdot k_2}$

(Proof: open brackets to verify that $a\omega^2 = k_0$.)