Indian Institute of Technology Bombay

Department of Electrical Engineering

Handout CSP-2 EE 103 Introduction to Electrical Engineering

Homework 2 Oct 12, 2023

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Question 1) Show that the solution shown in the first equation of Slide 4 indeed solves the differential equation in Slide 3. Note that c_m, $m \in \mathbb{Z}$ (i.e. m takes values in integers) are different constants, which may depend on the initial position of the string.

Question 2) In the second equation of Slide 4, $\tilde{a}_m$ and $\tilde{b}_m$ are also constants. Can you express them in terms of $c_m, m \in \mathbb{Z}$ from the first equation of Slide 4.

Question 3) Suppose the initial velocity of the string at $t = 0^+$ is taken as zero. What can you then say about the coefficients $\tilde{b}_m$ in Slide 4.

Question 4) A function is called *odd*, if $f(x) = -f(-x)$ for all $x \in \mathbb{R}$. It is called *even* if $f(x) = +f(-x)$ for all $x \in \mathbb{R}$. Show that any function can be written as a sum of an even function and an odd function.

Though I asked the questions above to improve your familiarity with these concepts, the key point from the class you should remember is that an odd function $f_o(x)$ defined in $-L \le x \le L$ can be expressed as

$$\boxed{f_o(x) = \sum_{m=1}^{\infty} b_m \sin\left(2\pi \frac{m}{2L} x\right),} \tag{1}$$

$2\pi \frac{m}{2L} = \omega$

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for the interval $-L \le x \le L$. To be correct, we should say almost all odd functions, as we are basing the sanctity of the above result on Fourier's prediction that his formula is correct. Such qualification should be applied to all statements in these notes, though we now know Fourier is right for almost all functions we encounter. So assume nice functions throughout this discussion.

Remember the sinusoidal components of the movements we simulated on the screen when the string was tied at 0 and L. Notice that the first harmonic (or the fundamental sine wave) has a period of $T = 2L$. Such sinusoids will correspond to an odd function in $-\frac{T}{2} \le x \le \frac{T}{2}$, clearly so will behave their weighted sum. On the other hand there also exists even functions in $-\frac{T}{2} \le x \le T2$. An even function $f_e(x)$ defined in $-\frac{T}{2} \le x \le \frac{T}{2}$ can be represented as

$$\boxed{f_e(x) = \sum_{m=0}^{\infty} a_m \cos(2\pi \frac{m}{2L} x)} \tag{2}$$

Question 5) Show that any function $g(t)$ defined in $-\frac{T}{2} \le t \le \frac{T}{2}$ can be represented as

$$g(t) = \sum_{m \in \mathbb{Z}} \alpha_m \exp(j 2\pi \frac{m}{T} t), \tag{3}$$

for some $\alpha_m, m \in \mathbb{Z}$ (these can be complex numbers).

$g(t) = \sum_{m \in \mathbb{Z}} \alpha_m e^{j 2\pi \frac{m}{T} t}$

Notice that the last equation is exactly in a form

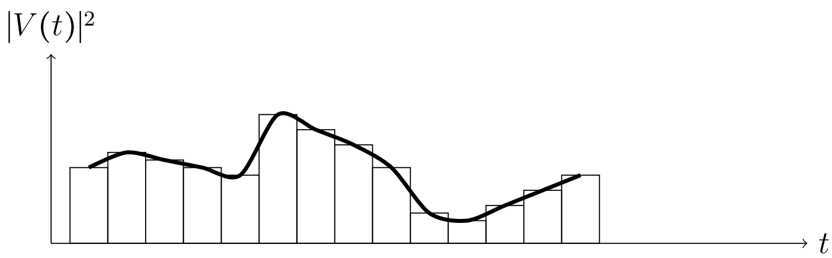
$$x(t) = \sum_{m \in \mathbb{Z}} \alpha_m \Phi_m(t), \tag{4}$$

where the coefficients α_m need to be evaluated for the given function $x(t)$. There is a simple trick to get there, and this is why we talked about projections or dot-products. FYI, different functions in the set $\{\Phi_m(t), m \in \mathbb{Z}\}$ are called basis functions (for reasons that will become apparent after the algebra course).

Given a signal $V(t)$, we can treat this as a voltage signal. We can consider complex-signals, which can be thought of as two separable simultaneous real components. The energy of the signal can then be defined as

$$\int_{\mathbb{R}} |V(t)|^2 dt = \int_{\mathbb{R}} V(t)V^*(t)dt. \tag{5}$$

In here $V^*(t)$ denotes the complex conjugate of $V(t)$, i.e. if $V(t) = a_t + jb_t$, then $V^*(t) = a_t - jb_t$ for each value of t. The energy can be thought of as the total energy dissipated by a voltage source $|V(t)|$ over a unit resistance. Let us take a closer look at the above energy term, by considering the previous integral as a Riemann Integral. Recollect the Riemann strategy is to divide the integrand to rectangles of small width and add the areas, as demonstrated in the picture below. You can take the function to have zero value outside the shown interval.



If $y_1, y_2, \ldots, y_n$ denote the heights of the rectangles shown in the figure, then we know that approximately

$$\int |V(t)|^2 dt \approx \sum_{i=1}^{n} y_i \Delta, \tag{6}$$

where Δ is the width of each rectangle. Now if we take Δ very small and cover the graph with appropriate vertical rectangles, we can get equality in the above expression, and we then say that the Riemann Integral exists. Notice that, if the signal starts from $t = 0$, then $y_i = V(i\Delta)V^*(i\Delta)$. In effect, we have

$$\int |V(t)|^2 dt = \Delta \sum_{i=1}^{n} V(i\Delta)V^*(i\Delta). \tag{7}$$

Those who are familiar with vectors and dot products will realize that the right side summation above is a dot-product. The complex conjugate there is necessary so that the energy remains a non-negative quantity. (Think what will happen if there is no conjugate). So, in most cases we can discretize the variable and visualize the function as a vector (there are some pit-falls to this view, but ignore it for the time being, as we are Riemann's followers now). Using the bracket notation for dot-product, let us define

$$\langle x(t), p(t) \rangle := \int_{\mathbb{R}} x(t)p^*(t)dt \tag{8}$$

as the dot product of the functions $x(t)$ and $p(t)$. Then the energy of a signal $x(t)$ is

$$\langle x(t), x(t) \rangle = \int_{\mathbb{R}} x(t)x^*(t)dt = \int_{\mathbb{R}} |x(t)|^2 dt. \tag{9}$$

This will give the energy of our voltage waveform as well. Considering a large enough but finite time interval, we can find the power or rate of energy of the signal as well. We will not do it now though. Okay, where is all this story going with our need to evaluate the coefficients in (4). To connect the dots, let us limit the domain of the functions discussed below to the interval $-\frac{T}{2} \leq t \frac{T}{2}$. For the time being, think of the values as zero outside this interval. Now do the following exercise.

Question 6) Let $\Phi_m(t) = \exp(2\pi \frac{m}{T} t)$, for $m \in \mathbb{Z}$. Find the value of

$$\langle \Phi_m(t), \Phi_n(t) \rangle.$$

If you have difficulty solving this, draw the real and imaginary parts for the two signals separately for $(m, n) = (0, 1)$ and argue based on the symmetry. Now draw for any other (m, n) and extend.

Question 7) Using the last question, can you guess the answer to

$$\langle x(t), \Phi_k(t \rangle),$$

where $x(t) = \sum_{m \in \mathbb{Z}} \alpha_m \Phi_m(t)$. Let us assume a *nice* function $x(t)$ such that summation and integration can be interchanged, or be brave and try that out.

This gives us a quick way to figure out the coefficients (or constants) in the question, and we showed the dynamic Python plot using the coefficients obtained for the last equation on Slide 4. Please see a Python Notebook page shared in the email for visualization.

If you have proceeded so far, then two observations will become immensely useful. First, our expansion using sinusoids or complex exponentials for functions in $-\frac{T}{2} \leq t \leq \frac{T}{2}$ can be visualized as periodic repetitions outside this interval. This is because $\exp(2\pi \frac{m}{T} t)$ is periodic with period T. So, if we can have a series expansion for a signal limited in $-\frac{T}{2} \leq t \leq \frac{T}{2}$, the same also holds true for a $T-$periodic signal, when we consider one period. So, people say that Fourier Series expansion is for periodic signals. Now you should know what is meant is that the right side of the expansion (in Equations (1) – (3)) is in terms of periodic functions.

The second important aspect is on the dot-product. The dot-product of functions as in (8) holds even we consider the entire real axis. Like in vectors, when the dot-product of two functions is *very close* to the product of their magnitudes (magnitude of a signal is the square root of its energy), we say the functions have very good overlap. On the other hand, the dot-product is zero implies that the functions are orthogonal (just like in vectors).

Fourier's Overlap of Legacy

We call the signal $\exp(j2\pi f_0 t)$ as a pure sinusoid (complex exponential) in electrical engineering. The frequency of this pure sinusoid is f_0 Hz, and $T = \frac{1}{f_0}$seconds is its period. Given a signal $x(t)$ of finite energy, we can find its overlap with a pure exponential as

$$X(f_0) = \langle x(t), \exp(j2\pi f_0 t) \rangle = \int_{\mathbb{R}} x(t) \exp(-j2\pi f_0 t) dt. \tag{10}$$

The quantity $X(f_0)$ is called the frequency component of $x(t)$ at the frequency f_0 Hz. The function $X(f)$ for all $f \in \mathbb{R}$ is called the **Fourier Transform** of the function $x(t)$. We can generalize Fourier Transform to a very wide class of signals, but this will come in later years, not now.

Suppose $X(f_a) = 0$, we then say that $x(t)$ does not contain the frequency f_a. Suppose $X(f) = 0$ for all $|f| \geq f_0$, we will then say that the signal $x(t)$ has highest frequency content below f_0. A more popular usage is to say that the function is band-limited to frequencies below f_0. Please recollect that $X(f)$ is defined for positive and negative frequencies, so a highest frequency being f_0 has to be checked on both sides of the frequency. For such band-limited signals, we showed in the class that no information is lost if we keep samples of $x(t)$ that are separated by $\frac{1}{2f_0}$ or less. So a good ADC (analog-to-digital converter) will not loose any information, if we can store real values at a rate more than $2f_0$ samples per second. This is the celebrated Shannon's sampling theorem.

Note: You should have noticed that I used functions and signals interchangeably. This is okay, signals are a more general class than functions, and no issues in taking it one way or the other for the signals used in the current exposition.