Quantum Statistical Mechanics

Probability of finding particle in state λ

$$P_\lambda \propto \exp\left(-\frac{E_\lambda}{k_B T}\right)$$

Boltzmann factor

Average number of particles in state λ

$$\overline{n}_\lambda = \frac{\sum_{n_\lambda} n_\lambda \exp\left(-\beta n_\lambda E_\lambda\right)}{\sum_{n_\lambda} \exp\left(-\beta n_\lambda E_\lambda\right)}$$

$$\beta \simeq 1/k_B T$$

CASE – I: unlimited indistinguishable particles per state

$$n_\lambda = 0, 1, 2 \ldots$$

Bose-Einstein statistics → Bosons

CASE – II: at most one indistinguishable particle per state

$$n_\lambda = 0, 1$$

Fermi-Dirac statistics → Fermions

Planck distribution

$$\bar{n}_\lambda = \frac{\sum_{n_\lambda=0}^{\infty} n_\lambda e^{-\beta n_\lambda E_\lambda}}{\sum_{n_\lambda=0}^{\infty} e^{-\beta n_\lambda E_\lambda}} = \frac{-\frac{1}{\beta}\frac{\partial}{\partial E_\lambda}\sum_{n_\lambda=0}^{\infty} e^{-\beta n_\lambda E_\lambda}}{\sum_{n_\lambda=0}^{\infty} e^{-\beta n_\lambda E_\lambda}}$$

$$n_\lambda = 0, 1, 2 \ldots$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\ln(y)$$

$$\Rightarrow \bar{n}_\lambda = -\frac{1}{\beta}\frac{\partial}{\partial E_\lambda}\ln\left(\sum_{n_\lambda=0}^{\infty} e^{-\beta n_\lambda E_\lambda}\right)$$

$$\sum_{n_\lambda=0}^{\infty} e^{-\beta n_\lambda E_\lambda} = \frac{1}{1-e^{-\beta E_\lambda}}$$

Geometric Series

$$\Rightarrow \bar{n}_\lambda = \frac{1}{\beta}\left(\partial/\partial E_\lambda\right)\ln\left(1-e^{-\beta E_\lambda}\right) = \frac{e^{-\beta E_\lambda}}{1-e^{-\beta E_\lambda}}$$

$$\Rightarrow \bar{n}_\lambda = \frac{1}{e^{\beta E_\lambda}-1}$$

Photons, phonons…

Case – I: Bosons

Planck

$$\overline{n} = \frac{1}{e^{\beta E} - 1}$$

$$n_\lambda = 0, 1, 2 \ldots$$

Thermodynamics

$$dE = TdS - pdV + \varphi dQ + \ldots + \sum_i \mu_i dN_i$$

Non-zero ‘chemical potential’ μ:

$$\boxed{\overline{n} = \frac{1}{e^{(E-\mu)/k_B T} - 1}}$$

Bose-Einstein

n → ∞?!

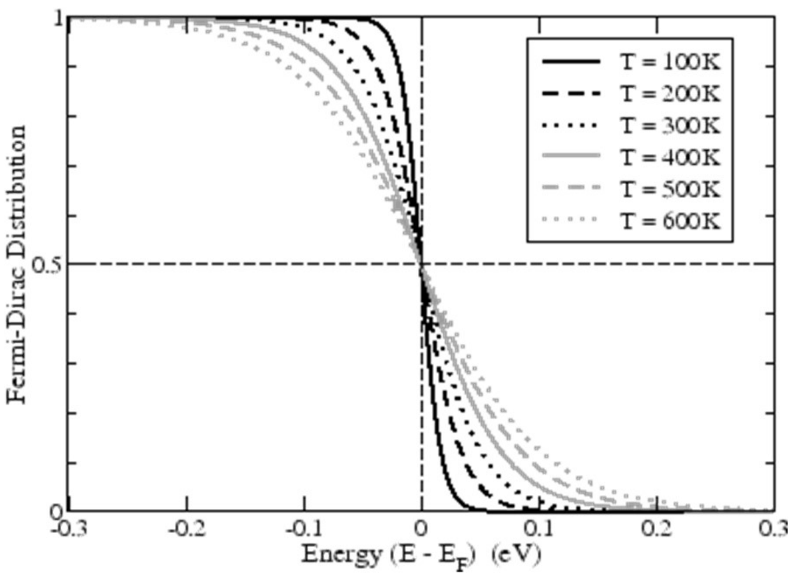
Case – II: Fermions

$$\bar{n}_\lambda = \frac{\sum_{n_\lambda=0}^{1} n_\lambda e^{-\beta n_\lambda E_\lambda}}{\sum_{n_\lambda=0}^{1} e^{-\beta n_\lambda E_\lambda}} = \frac{e^{-\beta E_\lambda}}{1+e^{-\beta E_\lambda}} = \frac{1}{e^{\beta E_\lambda}+1}$$

$$n_\lambda = 0,1$$

Non-zero 'chemical potential' μ:

Electrons?



$$\bar{n} = \frac{1}{e^{(E-\mu)/k_B T}+1}$$

Occupation probability

Fermi-Dirac

Spin-statistics

Finis

Artwork Sources:

1. www.iue.tuwien.ac.at
2. electrons.wikidot.com