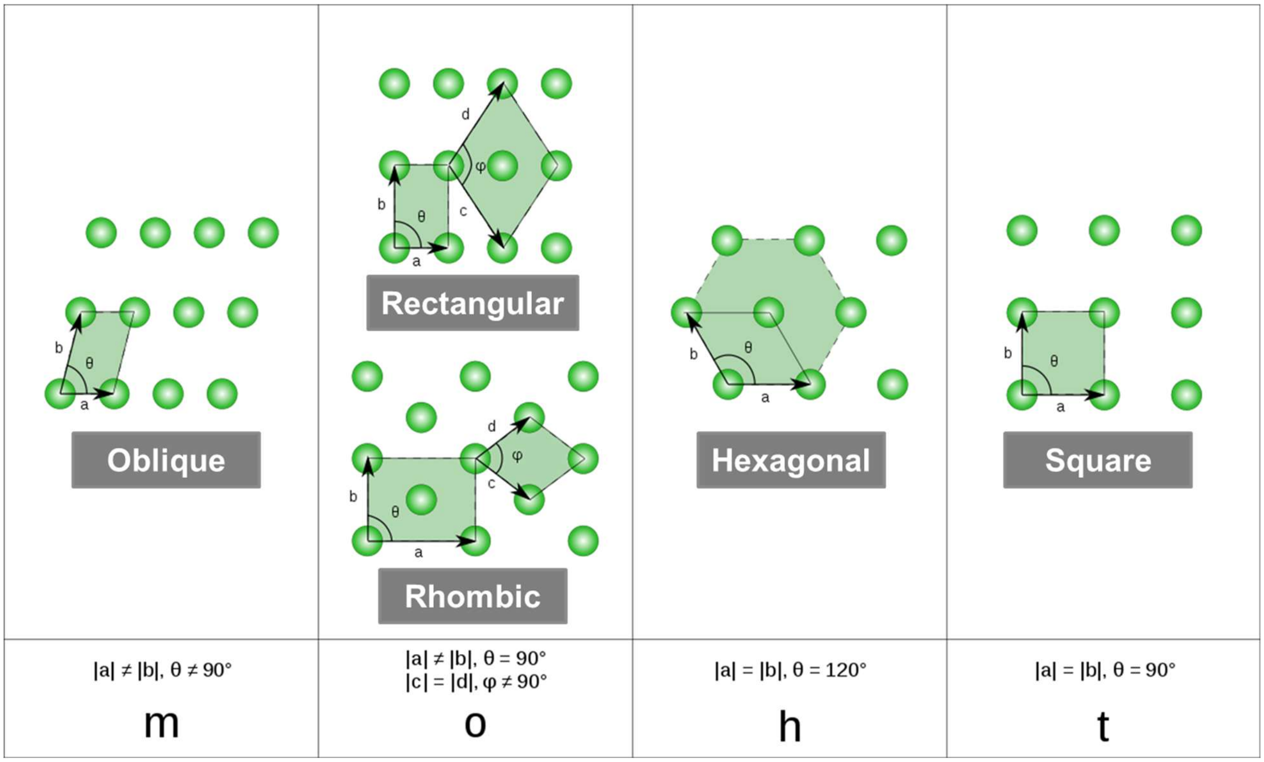
EE 207 EXAM-I (Mid-sem) Solutions 22nd February, 2025

1. The following are the 5 Bravais lattices possible in two-dimensions (2D). Evaluate the highest packing fraction possible for a 2D Bravais lattice. (You may carefully eliminate options by qualitative arguments where possible, instead of calculating the packing fraction for each type.) [5]



Packing fraction for lattice type:

(i) Square: $(\pi\, a^2/4)/a^2 = \pi/4$ [1]

(ii) Hexagonal: $(\pi\, a^2\, /\, 4)\, /\, \left(\sqrt{3}a^2/2\right) = \pi\, /\, 2\sqrt{3}$ [1]

(iii) Rhombic: Can be close packed only when $\varphi = 60^0$, viz. the hexagonal case [1]

Similarly, rectangular and oblique can be close packed only in the cases where they reduce to the square or hexagonal case. [1+1]

Therefore, the maximal packing fraction is for the hexagonal.

2. Consider a particle of energy $E > 0$ impinging as a plane wave on a delta-function potential barrier $V = \alpha \cdot \delta(x)$, where $\alpha \in \mathbb{R}$ and $\alpha > 0$.

(a) Derive the discontinuity in the first-derivative of the wavefunction at $x = 0$. [1]

The Schrödinger equation in this case is:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\delta(x)\psi = E\psi$$

Let us integrate this in neighborhood of $x = 0$ from $x = -\varepsilon$ to $x = +\varepsilon$.

$$\int_{-\varepsilon}^{+\varepsilon} \frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} dx + \int_{-\varepsilon}^{+\varepsilon} \alpha\delta(x)\psi\, dx = E \int_{-\varepsilon}^{+\varepsilon} \psi\, dx$$
[0.5]

Now as $\varepsilon \to 0$, we have:

$$\frac{-\hbar^2}{2m} \left[\Delta\left(\frac{d\psi}{dx}\right)\right]_{x=0} + \alpha\psi(0) = 0$$

$$\left[\Delta\left(\frac{d\psi}{dx}\right)\right]_{x=0} = \frac{2m\alpha}{\hbar^2}\psi(0)$$
[0.5]

(b) Derive the probabilities for transmission through, and reflection from, the barrier. [4]

The solutions to the Schrodinger Equation are plane- waves of the form $e^{\pm ikx}$, we assume a plane wave Ae^{ikx} incident from the left and no incident wave from the right. Let the reflected and transmitted wave be Be^{-ikx} and Fe^{ikx} respectively.

Now, continuity of the wave function at the origin implies:

$A + B = F$ [0.5]

The above calculated discontinuity in the derivative of the wavefunction implies:

$ikF - ikA + ikB = (2m\alpha/\hbar^2)F$ [0.5]

Putting these together and eliminating B:

$F\left(1 - (m\alpha/ik\hbar^2)\right) = A$ [1]

The transmission amplitude is:

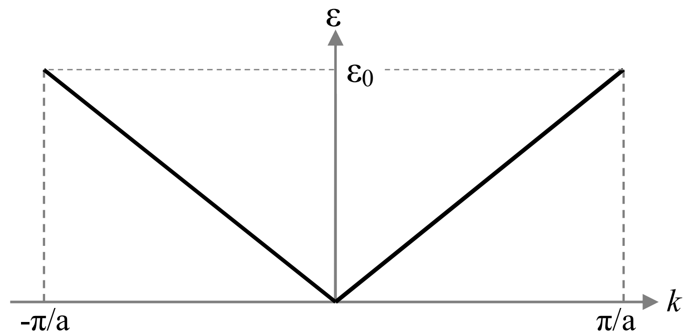
$t = F/A = 1/\left(1 - (m\alpha/ik\hbar^2)\right)$

Therefore, the transmission and reflection probabilities are:

$T = |t|^2 = 1/(1 + (m\alpha/k\hbar^2)^2) = 1/\left(1 + (m\alpha^2/2E\hbar^2)\right)$ [1]

$R = 1 - T = 1/\left(1 + (2E\hbar^2/m\alpha^2)\right)$ [1]

3. Consider the linear dispersion relation for a one-dimensional crystalline solid shown below, shown in the reduced zone scheme. Symbols have their usual meaning. Ignore all other energy bands.



(a) Write down the ε-*k* relation for all *k*. [1]

$$\varepsilon = \left(\frac{a\varepsilon_0}{\pi}\right)|k|,\ \text{for } |k| \le \frac{\pi}{a}$$
$$\varepsilon\left(k + \frac{2\pi}{a}\right) = \varepsilon(k) \qquad [0.5+0.5]$$

(b) Derive the expression for the group velocity of electrons in this band as a function of *k*. [2]

The following semiclassical equation relates the group velocity of the electron wave in the periodic crystal to the slope of the dispersion relation:

$$v = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial k}$$

$$v = \frac{1}{\hbar}\frac{\partial}{\partial k}\left(\frac{a\varepsilon_0}{\pi}\right)|k| = \left(\frac{a\varepsilon_0}{\pi\hbar}\right) sgn(k) \cong v_0\, sgn(k)\,,\ \text{for } |k| \le \frac{\pi}{a}$$

[1+1]

4. Consider the quantum well for electrons, with two bound states, formed in the heterostructure shown below. In the X-Y plane, electrons obey the dispersion relation:

$$E = \frac{\hbar^2 k_x^2}{2m_1} + \frac{\hbar^2 k_y^2}{2m_2}$$

(a) The 'density of states (DOS) effective mass' (viz. averaged effective mass to be used for a DOS calculation) is then given by $m_{dos}^* = \sqrt{m_1 m_2}$. Give a concise justification for this. [1]

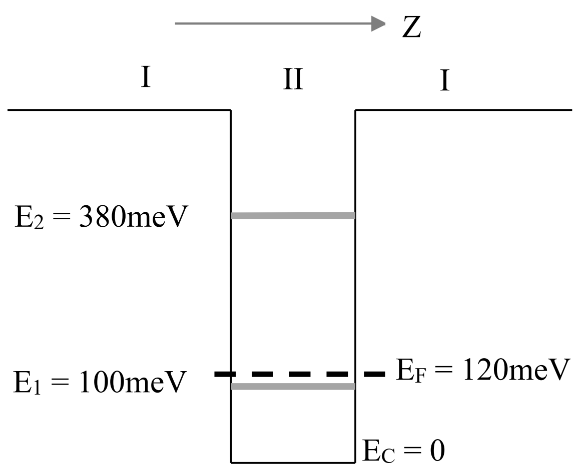
If we consider the circle of equal area in k-space, we will have the same number of k-states therein, and therefore we will arrive at the same DOS. Thus, the DOS effective mass may be obtained from:

$$\pi k_{eff}^2 = \pi k_1 k_2 \Rightarrow k_{eff}^2 = k_1 k_2$$

$$\Rightarrow 2m_{eff}\varepsilon/\hbar^2 = \sqrt{2m_1\varepsilon/\hbar^2}\cdot\sqrt{2m_2\varepsilon/\hbar^2} \qquad [1]$$

$$\Rightarrow m_{eff} = \sqrt{m_1 m_2}$$

(b) Suppose the 2D DOS $g \cong m_{dos}^*/\pi\hbar^2 = 10^{14} cm^{-2}eV^{-1}$. Calculate the 2D carrier density at 300K. Clearly justify any assumptions you make. [4]



The carrier density is given by:

$$n_{2D} \approx \int_{E_1}^{\infty} g_{2D}(E) f_{FD}(E) dE \qquad [1]$$

$$n_{2D} \approx g\int_{E_1}^{\infty} f_{FD}(E) dE$$

Now,

$$\int_{E_1}^{\infty} f_{FD}(E)dE = \int_{E_1}^{\infty} \frac{1}{e^{(E-E_F)/kT}+1} dE$$

It turns out the integral can be evaluated simply for this 2D case, using:

$$\int \frac{d\zeta}{e^{\zeta}+1} = \zeta - ln(1+e^{\zeta}) + C$$

[1]

This yields:

$$\int_{E_1}^{\infty} f_{FD}(E)dE = \int_{E_1}^{\infty} \frac{1}{e^{(E-E_F)/kT}+1} dE = kT\left[ln\left(1+e^{(E_1-E_F)/kT}\right) - (E_1-E_F)/kT\right]$$

For the given values:

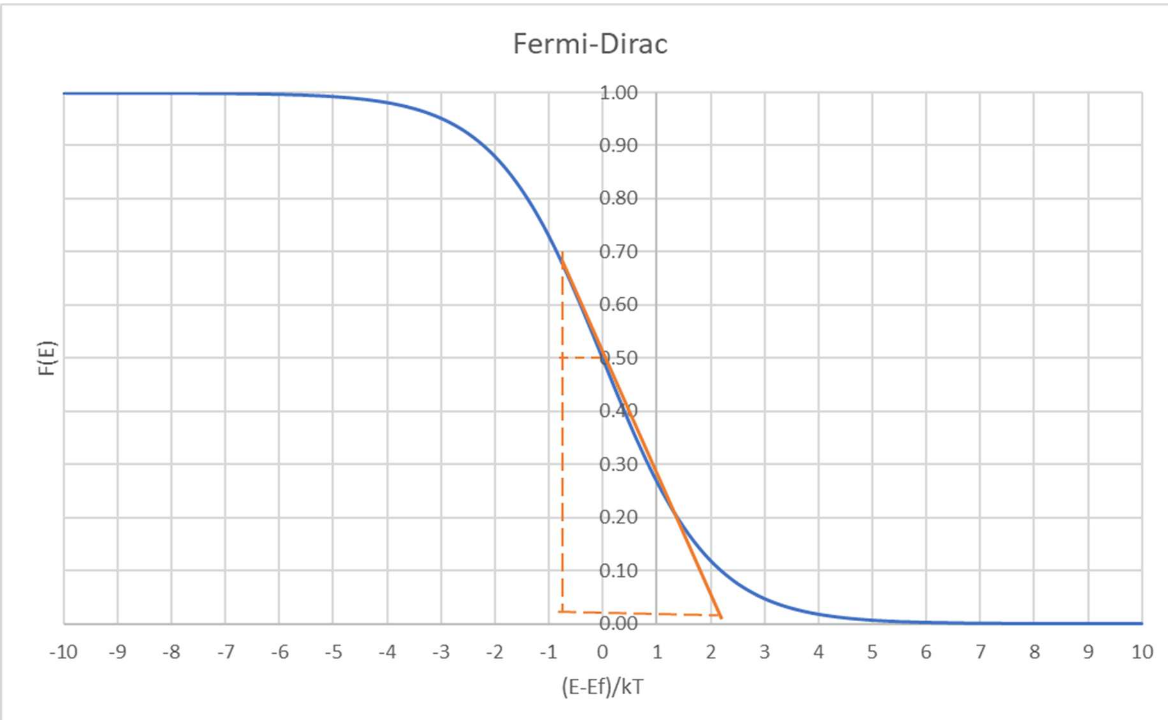
$$\int_{E_1}^{\infty} f_{FD}(E)dE = 1.15kT$$

[1]

$$n_{2D} \approx g \int_{E_1}^{\infty} f_{FD}(E) dE = 10^{14} \times 1.15 \times 0.026\ cm^{-2} = 3 \times 10^{12} cm^{-2}$$

[1]

Alternatively, an approximate geometrical evaluation of the Fermi integral could also have been done as shown below, for full points. Extending the shown line to 2kT (or 3kT) on the X-axis introduces an underestimate of 14% (or, overestimate of 12%).



5. We have a thin sample of a semiconductor with bandgap 1eV, an intrinsic carrier concentration of 10^{10} cm^{-3} at room temperature, n-doping of $10^{15} cm^{-3}$, and electron (hole) mobility of 1000 (500) $cm^2V^{-1}s^{-1}$, and lifetime of 1ms. Suppose it is irradiated uniformly with light such that the steady-state electron-hole pair generation rate everywhere is $10^{20} s^{-1} cm^{-3}$.

(a) Draw a quantitatively labeled band-diagram. Include the equilibrium Fermi level therein. [5]

First we need to calculate the excess carrier concentrations.

The low-level injection formula would yield a minority carrier concentration: $\delta n = \delta p = G\tau_p = 10^{17} cm^{-3}$. This is clearly not a low-level injection scenario, but it would beget full points. [1]

It is possible to determine the minority carrier concentration for this high level injection case as follows.

$R = \alpha(np - n_i^2)$

For low-level injection: $G = R_{low} = \alpha n_0 \delta p = \delta p/\tau$

Therefore, $\alpha = 1/(n_0\tau) = 10^{-1}\ cm^3 s^{-1}$

For high-level injection: $G = R_{high} = \alpha(\delta p)^2$

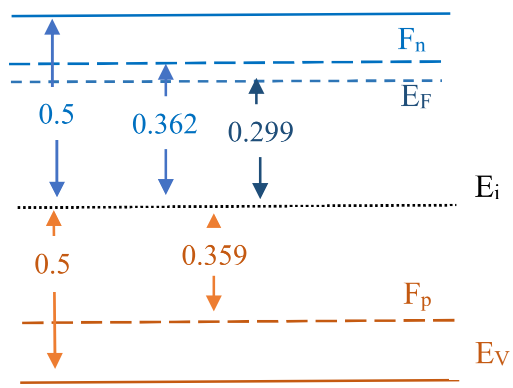
Therefore, $\delta n = \delta p = \sqrt{G/\alpha} = 10^{16} cm^{-3}$

The semiconductor has a bandgap of 1eV.

Using the formulas:

$n_0 = n_i \exp((E_F - E_i)/kT)$ for the equilibrium electron concentration, and

$n = n_i \exp((F_n - E_i)/kT)$ and $p = n_i \exp\left(\left(E_i - F_p\right)/kT\right)$ for the non-equilibrium distributions characterized by quasi-Fermi levels, the band diagram is as follows (with the high level injection formula; with the low-level injection formula, the energy difference between quasi-Fermi level and intrinsic Fermi level of 0.36eV would be about 0.42eV).



Marking: Ec, Ev, Ei placement [1]; Ef placement [1]; Fn placement [1]; Fp placement [1]

(b) What is the percentage change in the conductivity of the sample upon illumination? [2]

The fractional change in the conductivity is given by:

$$\frac{\left((n_0 + \delta n)q\mu_n + (p_0 + \delta p)q\mu_p\right) - \left(n_0 q\mu_n + p_0 q\mu_p\right)}{\left(n_0 q\mu_n + p_0 q\mu_p\right)}$$

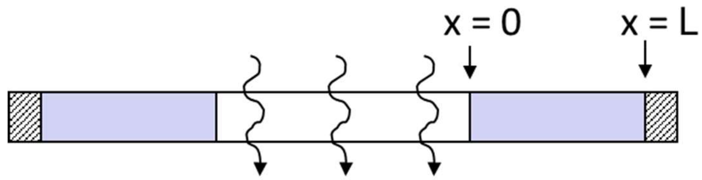
[1]

$$= \frac{(10^{16} \times 1100 + 10^{16} \times 500) - 10^{15} \times 1000}{10^{15} \times 1000} = 15$$

[1]

Therefore, the percentage change is 1400%. This is with the high-level injection formula. Going with the low-level injection formula, the fractional change would be 150, viz. 15000%.

6. Consider an n-doped semiconductor bar illuminated in the middle – as illustrated below. *Assume steady-state conditions throughout this problem.* Assume that *the* length of the unilluminated segment, L, is much *less* than the minority carrier diffusion length λ.



Suppose the generation rate everywhere in the illuminated segment is G, and the minority carrier lifetime everywhere is τ.

(a) What is the excess minority carrier concentration at $x=0$? [1]

The excess hole concentration at $x=0$ is given by:

$\delta p(0) = G \cdot \tau$...(1) [1]

(b) The contact at $x=L$ has a hole surface recombination velocity S_p, defined below. Determine the excess minority carrier concentration profile for $x \in [0, L]$ in terms of the given parameters. [4]

$$S_p \cong \frac{\text{Hole current flowing into contact}}{\text{Excess hole charge at } x = L^-}$$

The steady-state hole continuity equation is:

$-\frac{1}{q}\frac{\partial J_p}{\partial x} + G_p - R_p = 0$...(2)

In the range $x \in [0, L]$, there is no G-R. Hence we have: $-\frac{1}{q}\frac{\partial J_p}{\partial x} = 0$...(3)

The hole diffusion current is given by: $J_p = -qD_p \frac{\partial p}{\partial x} = -qD_p \frac{\partial(\delta p)}{\partial x}$...(4)

Substituting (4) in (3): $\frac{\partial^2(\delta p)}{\partial x^2} = 0$...(5) [1]

Integrating twice, we get: $\delta p(x) = Ax + B$...(6)

Using (1) for the boundary condition on the LHS, i.e. at $x=0$, we get from (6):
$B = G \cdot \tau$...(7)

Eq. (6) implies that the diffusion current is: $J_p = -qD_p \frac{\partial p}{\partial x} = -qD_p \cdot A$...(8)

The boundary condition at $x=L$ is from the surface recombination velocity on the RHS contact:

$$S_p \cong \frac{\text{Hole current flowing into contact}}{\text{Excess hole charge at } x = L^-} = \frac{-qD_p A}{\delta p(L)}$$

$$\Rightarrow A = -\frac{S_p \delta p(L)}{qD_p} = \ldots(9)$$ [1]

Substituting (9) and (7) in (6) for $x = L$: $\delta p(L) = -\frac{S_p \delta p(L)}{qD_p} L + G\tau \ldots(10)$ [1]

$$\therefore \delta p(L)\left[1 + \frac{S_p L}{qD_p}\right] = G\tau \Rightarrow \delta p(L) = \frac{G\tau}{\left(1 + S_p L / qD_p\right)} \Rightarrow A = -\frac{G\tau \cdot S_p / qD_p}{\left(1 + S_p L / qD_p\right)} \ldots(11)$$

Finally: $\delta p(x) = G\tau\left[1 - \frac{S_p x / qD_p}{\left(1 + S_p L / qD_p\right)}\right]$ [1]

{Note that for an ideal contact, $S_p \to \infty$, $\delta p(L) \to 0$ as expected. While for a contact which cannot source or sink any carriers, $S_p \to 0 \Rightarrow \delta p(x) = G\tau \ \forall x$ as expected.}