IE630: Simulation Modelling & Analysis
Output Analysis

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Quick Recap

Introduction

We have a simulator that generates data. What should we do with it?

First Possibility:

- Run the simulation with typical input settings.
- If the results look good → Recommend the system or suggest improvements.
- If the results look bad → Ask the system designers to make changes and try again.
- Many research papers stop at this step.

Second Possibility:

- A single simulation run doesn't tell us much.
- Just like in gambling, where the house usually wins, one lucky or unlucky run doesn't prove anything.
- We need **many** runs to understand real patterns and trends.

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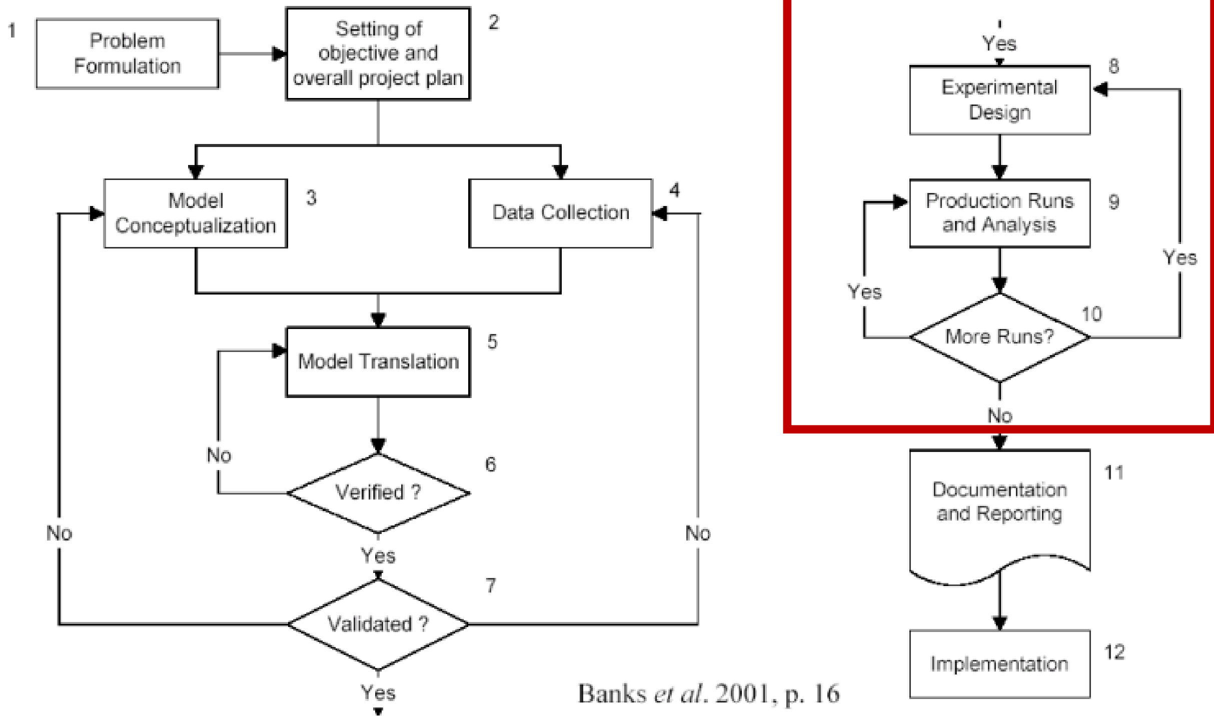
Introduction

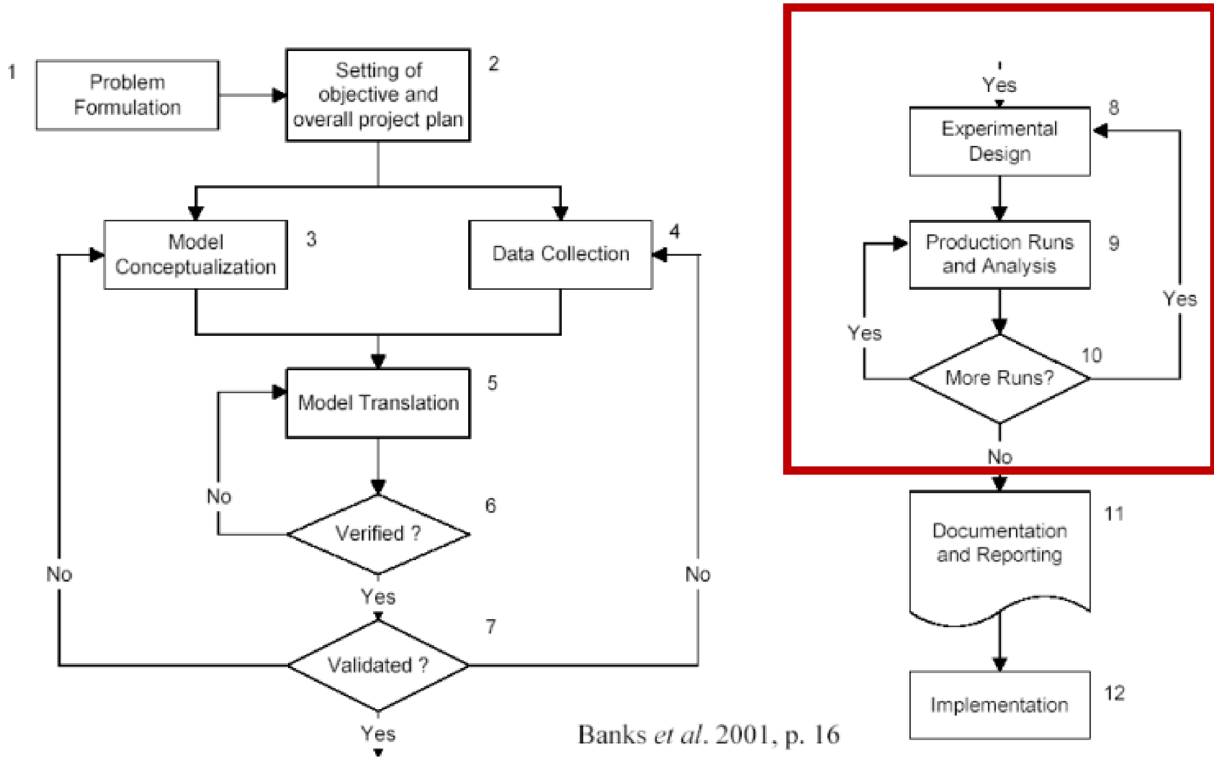
- **Output Analysis –**
  Analysis of the data generated by simulation
- **Why?**
  - Performance of the system
  - Compare the performance of multiple systems
- **Objective of Statistical Analysis**
  - Estimation of confidence interval
  - Number of observations required to achieve the desired confidence



Banks *et al*. 2001, p. 16

Introduction

- Issues
  - Autocorrelation
  - Initial Conditions
  - Computation costs to collect necessary amount of data
  - Lack of Expertise



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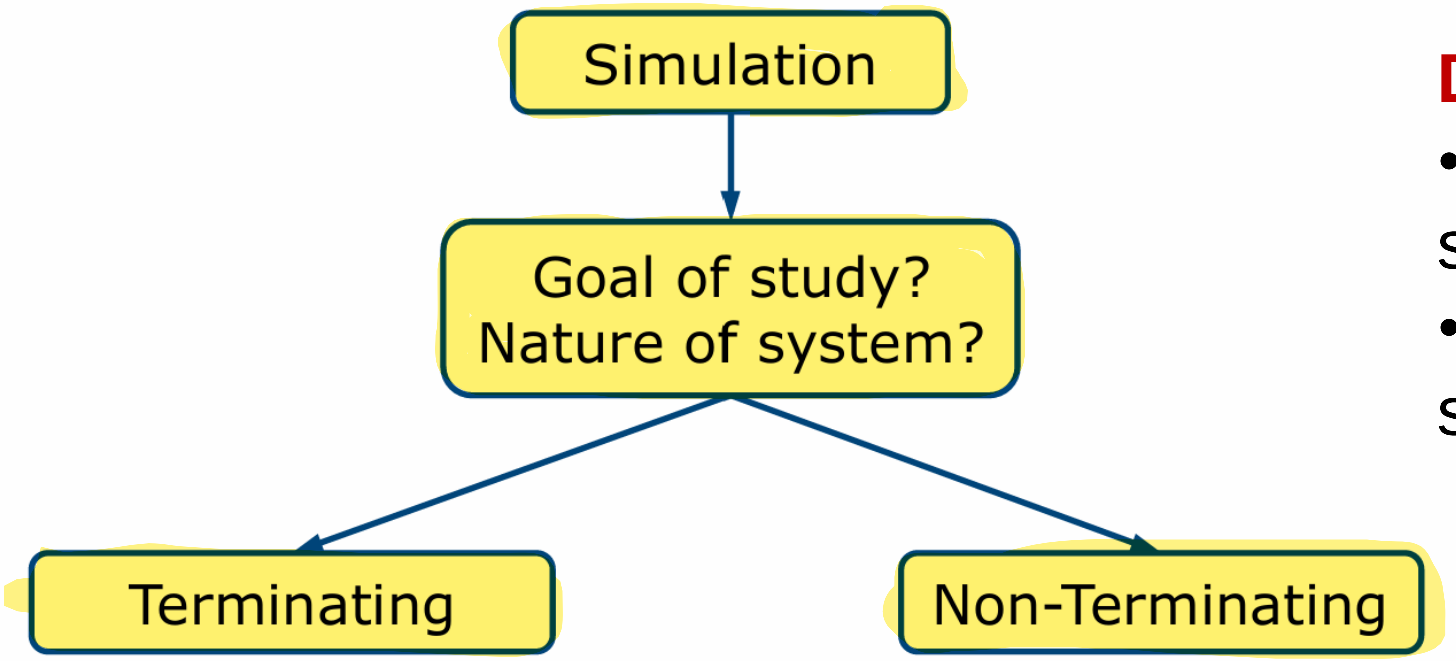
Types of the Simulations



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Depends on:

- The objectives of the simulation study
- The nature of the system

Types of the Simulations

- **Terminating Simulation**
  - Interested in modeling a specific interval of a system (i.e., specific starting and stopping conditions)
  - Examples: restaurant, service shops, etc.
  - At the beginning of simulation, empty and idle => this is part of system characteristics
  - Warm up period (from beginning till it reaches a steady state, also called transient state) is part of the system characteristic
  - Two ways to put stopping conditions Replication time: the office closes every 5 O'clock Counters of entities that arrive at the system or depart from the system (e.g., I am interested in the store with the first 100 customers)

Types of the Simulations

- **Non-Terminating Simulation**
  - Quantities to be estimated are defined in the long run
  - Stable system => e.g., queue is not built up continuously, in other words, resources are not busy all the time
  - The initial conditions (empty and idle state) should not be part of simulation results
  - Method to determine a warm up period will be covered later "Replicate" element: "warm up period" section
  - Peak time (11~1 O'clock) in Mcdonalds, results of queueing systems => it should not start from the empty and idle state
- **Why differentiate these two systems?**

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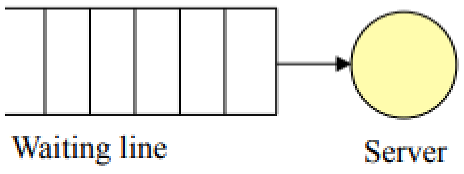
Nature of Output Data

- **Key Point:** Model output consists of random variables due to input randomness.
- **M/G/1 Queueing Example:**
  - Poisson arrival rate: 0.1 per time unit
  - Service time follows $N(\mu = 9.5, \sigma^2 = 1.75^2)$
  - Long-run queue length, $L_Q(t)$
  - **Simulation setup:**
    - Run for 5000 time units
    - Divide into 5 equal intervals
    - Y_j: Average number of customers in queue per interval



$$\boxed{L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\rho^2}{1-\rho}}$$

Nature of Output Data

- Batching Average Queue Length for 3 Replications
- Data Table: Batching intervals vs. queue length across replications
- **Key Insights:**
  - Variability within single replications
  - Variability across different replications
  - Averages across replications can be treated as ( )
  - Within-replication averages are not ( )

Basic Statistical Concepts

- Covariance: measure of dependency
  - Cov (X, Y) = E [(X − E(X)) (Y − E(Y))] = E (XY) − E(X)E(Y)
  - Cov (X, X) = ( ) = Var ( )
  - Cov (X, Y) > 0 : positively correlated
  - Cov (X, Y) < 0 : negatively correlated
  - Cov (X, Y) = 0 : uncorrelated (= independent)
- Correlation: normalized value (-1 < correlation < 1)

$$\boxed{Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}}$$

- Mean or expected value of a RV: μ or E(X)
- E(cX) = c*E(X)
- E(X+Y) = E(X) + E (Y)
- Variance of RV: σ^2 or Var (X)
- $\sigma^2 = E(X^2) - \mu^2$
- Var (cX) = ( )
- Var (X + Y) = ( )

Basic Statistical Concepts

- $X_{1,}\ X_2, \ldots, X_n$ ~ iid samples. from population with mean μ and variance σ^2
- Sample mean is centered about population mean but the spread (variance) reduces as sample size increases

$$\boxed{E[\bar{X}] = E\left[\frac{\sum_{i=1}^{n} X_i}{n}\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}(n\mu) = \mu}$$

$$\boxed{\text{Var}[\bar{X}] = \frac{\sigma^2}{n}}$$

Basic Statistical Concepts

- Let X_1 , X_2 , …, X_n be a sequence of IID random variables each with mean μ and variance σ^2 . Then for large n, the distribution of (X_1 + X_2 + … + X_n) is approximately Normal with mean nμ and variance $n\sigma^2$
- That is, the sum of the random variables:

$$\boxed{(X_1 + X_2 \ldots + X_n) \;\sim\; \text{Normal}(n\mu, n\sigma^2)}$$

Or

$$\boxed{\left(\frac{X_1 + X_2 \ldots + X_n}{n}\right) = \bar{X} \;\sim\; \text{Normal}(\mu, \frac{\sigma^2}{n})}$$

Or

$$\boxed{\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right) \;\sim\; \text{Normal}(0, 1) \;\sim \textit{Standard Normal}}$$

Basic Statistical Concepts

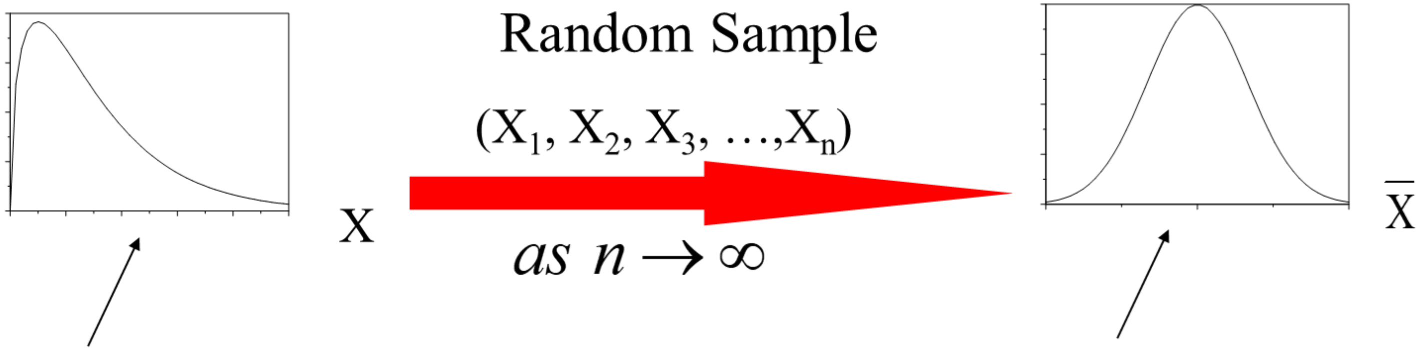
Suppose X_1 , X_2 , …, X_n be IID sample from Normal population having mean μ and variance σ^2

then, $\bar{X}$ and S^2 are independent random variables,

distribution of sample mean, $\bar{X}$ is ( )

and

distribution of $(n–1)S^2/\sigma^2$ is ( ) with ( )



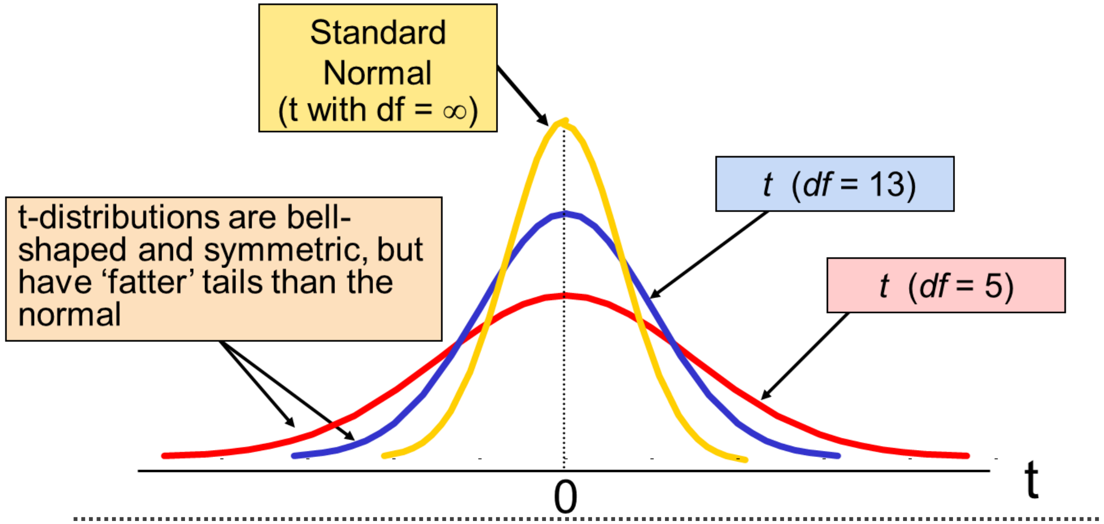
Basic Statistical Concepts

Suppose X_1 , X_2 , …, X_n be IID sample from Normal population with mean μ (σ^2 unknown)

then,

$$\boxed{\left(\frac{\bar{X}-\mu}{S/\sqrt{n}}\right) \sim t_{n-1}}$$

where, t_{n-1} : t-distribution with n-1 degrees of freedom



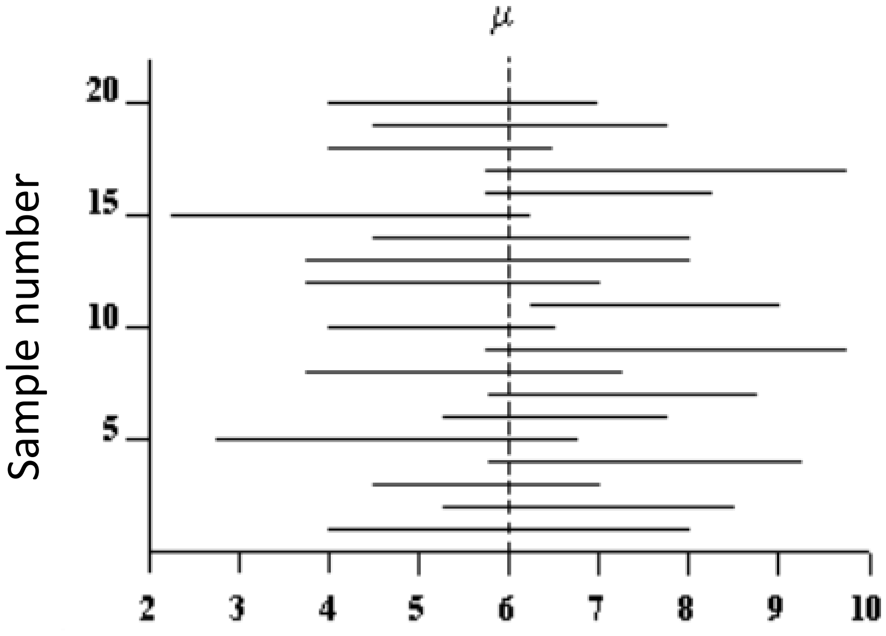
Basic Statistical Concepts

- Interval within which we have certain level of confidence that true mean (μ) falls.
- CI bounds the error between sample mean and true population mean
- $100(1 - \alpha)\%$ CI
- To understand confidence intervals fully, distinguish between measures of error and measures of risk:
  - Confidence interval
  - Prediction interval

Basic Statistical Concepts

- Confidence Interval (CI):

Distribution of Confidence intervals



Basic Statistical Concepts

- Prediction Interval (PI):

Terminating Simulations

Terminating Simulations

- **Data collection:**
  - Time-persistent statistics
  - Tally statistics
- **Multiple output data :**
  - Sample mean and variance calculations
- **Given n samples**, X_i (i=1,2,….,n)
  - Sample Mean
  - Sample Variance
- **Confidence intervals:**
  - Point estimates vs. interval estimates
  - Calculation example for 95% confidence interval

Terminating Simulations: Confidence Intervals

- **Point Estimates**
  - Single value of estimates, e.g. $\bar{X}$ and S^2 little information on how accurate it estimates the true value of the unknown parameter
- **Interval Estimates**
  - Interval estimates using $\bar{X}$ and S^2 give a better idea
  - Method to determine this is called confidence interval estimation
- **Confidence interval** – Range within which we can have a certain level of confidence that the true mean falls

Terminating Simulations: Confidence Intervals

- **Half Width:** Distance from $\bar{X}$ to either endpoint
- Obtain the 95% CI for expected time to produce 2000 parts. The data for 10 replications is shown below:

$T_1 = 32.62$, $T_2 = 32.57$, $T_3 = 33.51$, $T_4 = 33.29$, $T_5 = 32.10$, $T_6 = 34.24$, $T_7 = 32.70$, $T_8 = 33.49$, $T_9 = 33.36$, $T_{10} = 34.61$

- What is the df?
- Would the result have been different 1000 instead of 2000?
- How many data points (T_i) belong to CI? Is that to do something with 95%?

Terminating Simulations: Confidence Intervals

- **Confidence Interval:** Not the interval where 95% of the average measures from replication will fall. This is in fact called ( )

- **Interpretation**: Consider lot sets of 10 replications, about 95% of these intervals will cover μ

- ( ) will shrink a point as n increases, but ( ) won't since it needs to allow for a variation in future replications.

Terminating Simulations: Replications

- This is a function of how ( ) you want to estimate the performance of interest
- Given the current half width (h_0) with n_0, what is n to achieve the accuracy, h?

$$h = t_{n-1,1-\frac{\alpha}{2}}.\, s/\sqrt{n} \quad => \quad n =$$

- Problems:
  - Degree of freedom
  - S is function of n_0 and n samples

Terminating Simulations: Replications

- Approximation 1

- Approximation 2

Terminating Simulations: Replications

- In the first simulation of replication of 20 => $\bar{X}$ =11.3, S^2 = 3.81, when n_0 = 20, half width (h_0) = 1.78

Goal: h = 0.5, what is the required replication number?

α = 0.05

- The ( ) approximation is always wider since it uses t-value rather than z-value.

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Terminating Simulations: Output Analysis

- Statistical issues with simulation output (still terminating simulation)
- Three statistical assumptions must be met regarding the sample of observations used to construct the CIs:
  - Observations must be ( ) so that no correlation exists between consecutive observations
  - Observations are ( )throughout the entire duration of the process (i.e. they are time invariant)
  - Observations are ( )

Terminating Simulations: Output Analysis

- Format of the simulation output data?
- What is a replication? And why its important
- Run **n** replications with **m** observations each
  - What does this mean?
  - Do I need random numbers? If yes, How many?
  - Replication 1: Run simulation with random numbers u_{11}, u_{12} … and get y_{11}, y_{12} ... y_{1m} as output
  - Replication 2: Run simulation with (   ) and get (   ) as output
  - ….
  - Replication n: Run simulation with (   ) and get (   ) as output

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Terminating Simulations: Output Analysis

- Say, we run simulation model for R replications with m observations each.
  - Replication 1: $y_{11}, y_{12}, \dots y_{1k}, \dots y_{1m}$
  - Replication 2: $y_{21}, y_{22}, \dots y_{2k}, \dots y_{2m}$
  - …
  - Replication R: $y_{n1}, y_{n2}, \dots y_{nk}, \dots y_{nm}$
- Let, y_{ij} be time in system observations for j^{th} entity in i^{th} replication (Tally Statistic)
- Can we use **n x m** data to construct CI?

Terminating Simulations: Output Analysis

- Within each replication, the observations are _______ .
- Within each replication, the observations are _______ .
- Within each replication, are the observations Normally distributed?

Replication (i)	Within run observations, y_{ij}	Average, $\overline{y_i}$
1	$y_{11}, y_{12}, y_{13}, \quad \ldots, \quad y_{1m-1}, y_{1m}$	$\overline{y_1}$
2	$y_{21}, y_{22}, y_{23}, \quad \ldots, \quad y_{2m-1}, y_{2m}$	$\overline{y_2}$
3	$y_{31}, y_{32}, y_{33}, \quad \ldots, \quad y_{3m-1}, y_{3m}$	$\overline{y_3}$
4	$y_{41}, y_{42}, y_{43}, \quad \ldots, \quad y_{4m-1}, y_{4m}$	$\overline{y_4}$
…	……… y_{ij} ………	
n	$y_{n1}, y_{n2}, y_{n3}, \quad \ldots, \quad y_{nm-1}, y_{nm}$	$\overline{y_n}$

Terminating Simulations: Output Analysis

- Point and interval estimate will be…

Terminating Simulations: Example 1

- Given the following simulation results, what is the 95% CI & PI of the true mean of the time spent in the system (μ)?

Note: $t_{0.975}$ is 2.78 when ν = 4, and $t_{0.975}$ is 2.06 when ν = 24.

	j: index of occurrence within a replication				
i: index of replication	$y_{11} = 2$	$y_{12} = 1$	$y_{13} = 3$	$y_{14} = 5$	$y_{15} = 2$
	$y_{21} = 5$	$y_{22} = 3$	$y_{23} = 2$	$y_{24} = 1$	$y_{25} = 5$
	$y_{31} = 3$	$y_{32} = 5$	$y_{33} = 4$	$y_{34} = 2$	$y_{35} = 2$
	$y_{41} = 4$	$y_{42} = 7$	$y_{43} = 6$	$y_{44} = 3$	$y_{45} = 2$
	$y_{51} = 5$	$y_{52} = 10$	$y_{53} = 2$	$y_{54} = 6$	$y_{55} = 2$

Terminating Simulations: Example 2

- For the given time in system observations, compute the 95% CI and PI

	1	2	3	4	5
1	2.3	1.5	3.3	5.9	2.6
2	5.6	3.6	2.1	1.3	5.8
3	3.3	5.5	4.4	2.2	2
4	4.8	7.4	6.2	3.6	2.4
5	5.5	10	2.6	6.8	2.4

Terminating Simulations: Example 3

- For the given average utilization of server across replications, compute the 95% CI & PI:

0.808, 0.875, 0.708, 0.842, 0.956