IE630: Simulation Modelling & Analysis

Random Number Generation
Random Variate Generation

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Quick Recap

Part I
Random Number Generation

Introduction

- Data, parameters are not deterministic
- Example
  - Inter-arrival time, service time
  - Number of students attending the class
  - Balking behaviour
  - Machine break down frequency, machine fix frequency
  - and many more
- Challenges
  - Replication of the Process
- From this statistical distribution (PDF or PMF), how do we get different parameters (variates) each time?

Basic Concept

- We have the parameters represented as statistical distributions –
  - UNIF(1,4)
  - NORMAL(25, 1)
  - TRIANGLE (4,5,6)
  - EXPO(0.2)

Big Question – How do we draw the appropriate value from these input distributions?

- Random Numbers
- Random Variates

Steps - Generate the Random Variates

- Generate the INDEPENDENT & IDENTICALLY DISTRIBTUED (I.I.D) random numbers following UNIF (0,1)

- Transform the I.I.D UNIF (0,1) random number into __________ from the defined input distribution

Generate the Pseudo Random Number Generation (RNG)

↓

Generate the Random Variates

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Generation of the Random Numbers

- Physics Processes –
  - Noise from the analog circuits, Thermal/atmospheric noise
  - Hardware Random Number Generator
- Computational Methods –
  - Pseudo-random number generator
  - Recursive methods to generate numbers
  - Output is predictable, given the seeds
- Hence, we need to find out -
  - Truly random number generator
  - Computational schemes to generate RN
  - Statistically random

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Pseudo Random Number Generator: Properties

- Possess the ideal statistical properties
  - Uniformity
  - Independence
- Should be reproducible the given stream of random numbers exactly or vice versa
- Reasonably fast
- Have sufficiently long cycle
- Produce the separate streams of random numbers

Linear Congruential Method

- To produce a sequence of integers $Z_1, Z_2, Z_3, \ldots..$ Between 0 and m-1 by following a recursive relationship:

$$\boxed{Z_i = (aZ_{i-1} + c) mod\, m,\; i = 1, 2, 3, \ldots.}$$

- Assumption: m > 0 - Modulus
  a < m - Multiplier
  c < m - Increment
  Z_i - Seed
- Values for a, c, m, and X_1 affects the ________________ & ____________

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Linear Congruential Method

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- Define the U_n as shown below –

$$U_i = Z_i/m \text{ for } i = 1, 2, 3, \ldots.$$

- ( ) < U_i < ( )

Linear Congruential Method

- LCM parameters used by Arena until version 3.0

$m = 2^{31} – 1 = 2147483647$

$a = 7^5 = 16807$

$c = 0$

$Z_0 = 14561, \ldots.., 32535$

- Excel - $Z_i=(1140671485 \cdot Z_{i-1}+12820163) \bmod 2^{24}$

- Cycle, Full Cycle
  - Period (p): the length of a cycle (p m)
  - Generator is called “full period” if p == m

General Congruence

- General Congruences
  - $\boxed{Z_i = g(Z_{i-1}, Z_{i-2}, \ldots \ldots ..) \, mod \, m}$
- Composite Generators
  - Combination to produce a separate random number generators
  - Let Z_{2i} and Z_{1i} be two separate generators having two different LCGs
  - Then,
  - $\boxed{Z_i = (Z_{2i} - Z_{1i}) mod \, m}$
  - Then, $U_i = Z_i/m$, if $Z_i > 0$

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Composite Generators

- L'Ecuyer (1988)
  - $Z_{1,i} = (40{,}014)\ Z_{1,i-1} \bmod (2^{31} - 85)$
  - $Z_{2,i} = (40{,}692)\ Z_{2,i-1} \bmod (2^{31} - 249)$
  - $Y_i = (Z_{1,i} - Z_{2,i}) \bmod (2^{31}-86)$
  - $U_i = Y_i / (2^{31} - 85)$

Combined period
$\sim 2 \times 10^{18}$

- L'Ecuyer (1999)
  - $Z_{1,i} = [1{,}403{,}580\ Z_{1,i-2} - 810{,}728\ Z_{1,i-3}] \bmod (2^{32} - 209)$
  - $Z_{2,i} = [527{,}612\ Z_{2,i-1} - 1{,}370{,}589\ Z_{2,i-3}] \bmod (2^{32} - 22853)$
  - $Y_i = (Z_{1,i} - Z_{2,i}) \bmod (2^{32} - 209)$
  - $U_i = Y_i / (2^{32} - 209)$

Combined period
$\sim 3.1 \times 10^{57}$

Random Number Generators: Test

- There are several ways to test RNG:
  - Testing the goodness of fit with IID U(0,1)
  - “Die Hard Tests”
  - Basic Statistician Tests
  - Theoretical Tests
- Die-Hard Tests

- Birthday spacing
- Overlapping permutations
- Monkey tests
- Count the 1s
- Parking lot test
- The craps test
- Minimum distance test
- Random spheres test
- The squeeze test
- Overlapping sums test
- Runs test

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Random Numbers: Test

- Two properties – Uniformity & Independence
- Test for **uniformity**:
  - H_0: $R_i \sim U[0,1]$
  - H_1: R_i !~ $U[0,1]$
  - Failure to reject the H_0 means that evidence of non-uniformity has been detected
- Test for **Independence**:
  - H_0: R_i ~ Independent
  - H_1: R_i !~ Independent
  - Failure to reject the H_0 means that evidence of dependence has been detected

Chi-Square Test

- Test Statistic:

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Expected Frequency

$$E_i = n \times p_i$$

where p_i is the theoretical prob. of the i-th interval.

O_i : Observed frequency in the i-th class

- Test Conditions:

$$\chi_0^2 \le \chi_{k-1,1-\alpha}^2$$

Unable to reject Null Hypothesis

$$\chi_0^2 > \chi_{k-1,1-\alpha}^2$$

Reject Null Hypothesis

Kolmogorov-Smirnov Test (K-S Test)

- Test fitted cdf with empirical cdf
  - Step 1: Rank the data smallest to largest
  - Step 2: Compute D^+ and D^-
  - Step 3: $D = \text{Max}\{D^+, D^-\}$
  - Step 4: Test Statistics

Case	*Adjusted Test Statistics*	*1-α*				
		0.850	**0.900**	**0.950**	**0.975**	**0.990**
All parameters known	$\left(\sqrt{n}+0.12+\frac{0.11}{\sqrt{n}}\right)D$	1.138	1.224	1.358	1.480	1.628
Normal $(\hat{\mu},\hat{\sigma}^2)$	$\left(\sqrt{n}-0.01+\frac{0.85}{\sqrt{n}}\right)D$	0.775	0.819	0.895	0.955	1.035
Expo$(\hat{\lambda})$	$\left(\sqrt{n}+0.26+\frac{0.5}{\sqrt{n}}\right)\left(D-\frac{0.2}{n}\right)$	0.926	0.990	1.094	1.190	1.308

Part II

Random Variate Generation

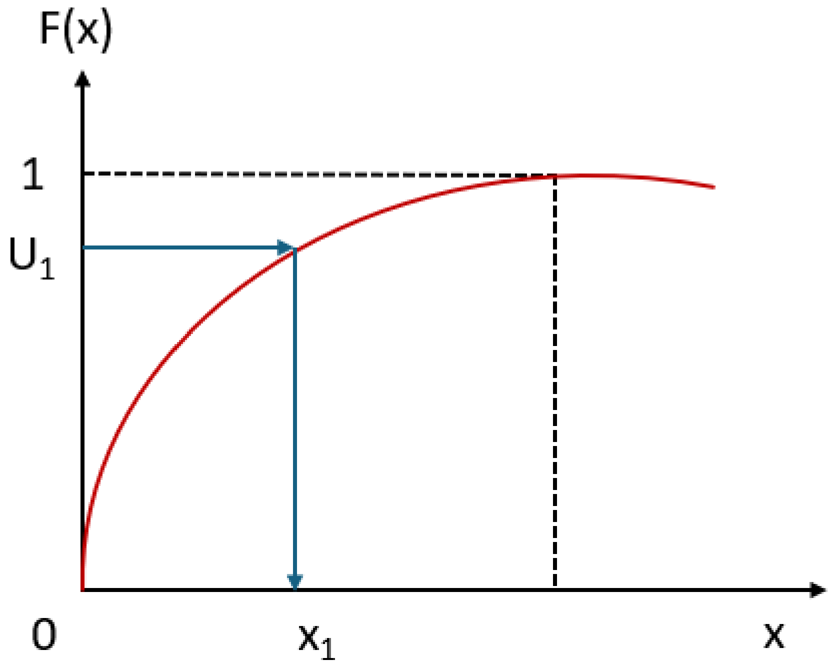
Random Variates Generation

- Generate IID U(0,1) random number
- Transformation of Random Number to Variates using Input Distribution

- Methods –
  - Inverse-Transform
  - Acceptance-Rejection
  - Convolution
  - Composition
  - Special

Inverse Transform Method

- Intuition:
  - Uses the inverse of the cdf $F(x)$
  - Mostly useful when F^{-1} can be computed



- Steps to find generator
  - Compute CDF of the desired random variable X
  - Set $F(X) = U$, where $U \sim$ IID Uniform(0,1)
  - Solve $F(X) = U$ for X in terms of U

- Steps to use generator
  - Generate U_i
  - Return $X_i = F^{-1}(U_i)$

Inverse Transform Method

- Can be used for sampling from continuous as well as discrete distribution
  - Uniform
  - Exponential
  - Triangular
  - Weibull
  - Log-logistic
  - Empirical Continuous
  - Discrete Uniform
  - Geometric
  - Bernoulli
  - Empirical Discrete
- Advantages:
  - Valid for mixed discrete-continuous distributions
  - Facilitates variance reduction techniques
  - Ease of use in truncated distributions
- Disadvantages:
  - Need to evaluate $F^{-1}(U)$
  - May not be the fastest way!

Inverse Transform Method – Some Distributions

Inverse Transform Method – Empirical Continuous Distribution

- Suppose the data collected for 100 broken-widget repair times are:

Interval (Hours)	Frequency	Relative Frequency	Cumulative Frequency	Slope
[0.25,0.5]	31			
[0.5, 1.0]	10			
[1.0, 1.5]	25			
[1.5, 2.0]	34			

Inverse Transform Method – Empirical Discrete Distribution

- Suppose the number of shipments, x, on the loading dock of a company is either 0, 1, or 2
- Data - Probability distribution:

x	P(x)	F(x)
0	0.5	
1	0.3	
2	0.2	

Inverse Transform Method

- The most common random number generator in use today is called a linear congruential generator. The linear congruential generator that we use is as follows:

$$Z_i = (2Z_{i-1} + 3) mod\ 5$$

Using this generator, manually generate 3 consecutive random variates (i.e. $i = 1, 2, \& \ 3$) following the Uniform distribution, Unif (4, 6). Assume your seed value is 3. Show all the steps of your answers

Inverse Transform Method

- For the same random numbers, derive three random variates following a discrete distribution: 10% of combo 1, 80% of combo 2, 10% of combo 3

Convolution method

- Convolution
  - Sum of IID random variables
  - Can be readily generated than direct generation of main Random variate

$$Y = X_1 + X_2 + X_3 + X_4 + \dots \dots + Xm$$

$$where\ Xi\ are\ independent\ R.V's\ from\ same\ distribution$$

Procedure:

- Step 1: Generate $X_1, X_2, \dots, X_m$ IID each with distribution function
- Step 2: Return

$$Y = X_1 + X_2 + X_3 + X_4 + \dots \dots + Xm$$

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Convolution method

- Y~ Erlang(k, θ)

- Y ~ Binomial (n,p)

Composition method

- Composition
  - Lets say we have a distribution function F for generating the random variate
  - F can be expressed convex combination of other distribution functions $F_1, F_2, F_3, \ldots\ldots$
  - The goal is to be able to sample more easily from F_j's compared to original F

$$F(x) = \sum_{j=1}^{\infty} p_j F_j(x)$$

$where$

$$pj \geq 0$$

$$\sum_{j=1}^{\infty} p_j = 1$$

Composition method

- Continuous Random Variate

- Discrete Random Variate

Acceptance-Rejection Technique

- Useful particularly when inverse CDF does not exist in closed form
  - Thinning
- Illustration: Generate random variates, $X \sim U(\frac{1}{4}, 1)$

Procedure:

- Step 1: Generate $R \sim U(0, 1)$
- Step 2: If $R \geq \frac{1}{4}$, accept $X = R$
- Step 3: If $R < \frac{1}{4}$, reject R, return to step 1

- Efficiency heavily depends on the ability to minimize the number of rejections

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Acceptance-Rejection Technique: Poisson Distribution

- Probability mass function of a Poisson Distribution

$$P(X = n) = \frac{\alpha^n}{n!} e^{-\alpha}$$

- Apply the A-R technique to generate R.V. for Pois(2)