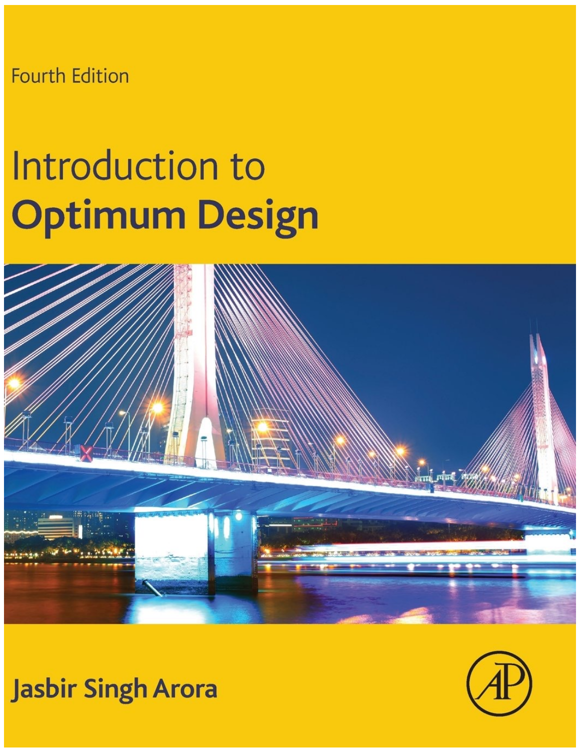
MS 101 Spring 2024
Engineering Optimization

Dnyanesh Pawaskar

References



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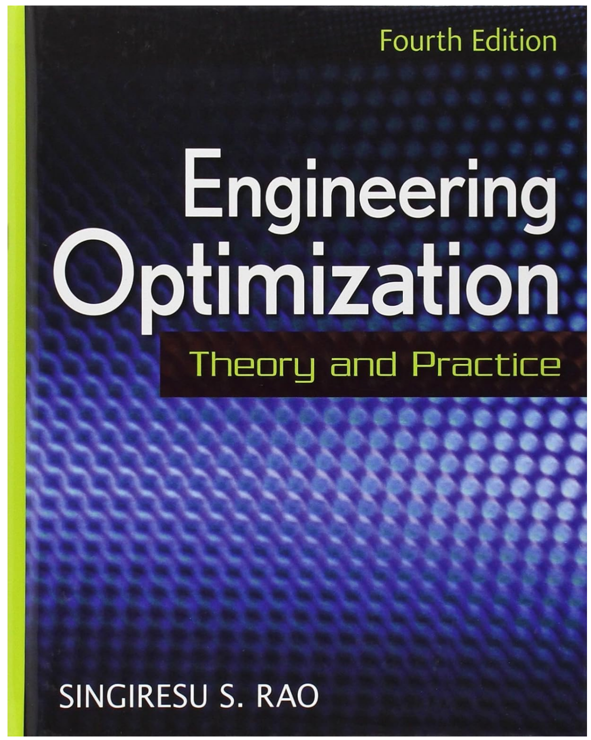

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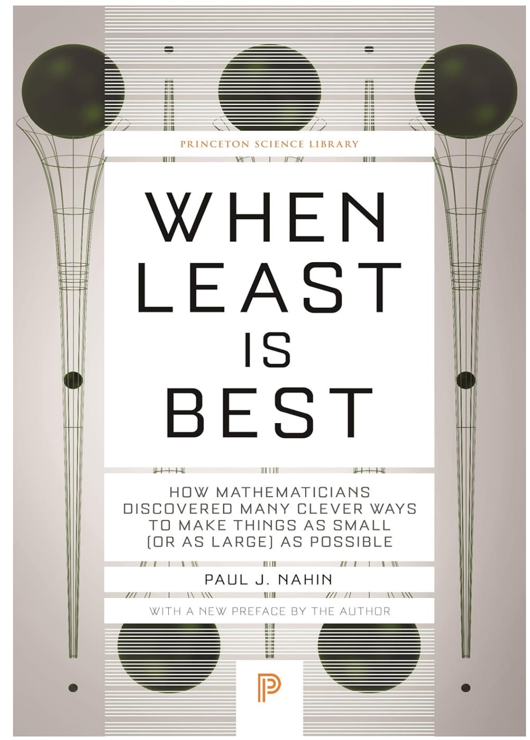
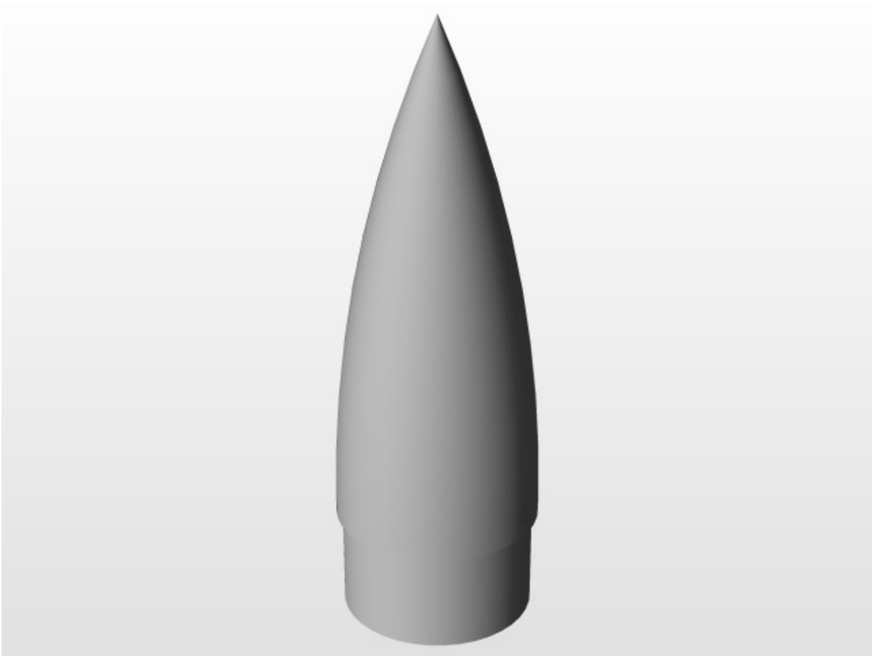


PRINCETON SCIENCE LIBRARY
WHEN
LEAST
IS
BEST
HOW MATHEMATICIANS
DISCOVERED MANY CLEVER WAYS
TO MAKE THINGS AS SMALL
(OR AS LARGE) AS POSSIBLE
PAUL J. NAHIN
WITH A NEW PREFACE BY THE AUTHOR

Engineering Questions
Example: Nose cone of a rocket

- Analysis: Given a shape of the nose cone, what is the drag?
- Design: Given the drag, what is the shape of the nose cone?
- Manufacturing: How do we manufacture the shape?
- Design Optimization: What is the shape of the nose cone that will minimize the drag?



Engineering Design

Analysis to Design to Manufacturing

- Design is a decision-making process. All engineering analysis should lead to design which should eventually lead to manufacturing.
- Engineering is essentially about improving our lives by building better, safer, & cheaper products and services.
- Often must choose between several competing or conflicting alternatives
- Optimization is a formal (mathematical) way to carry out this decision making
- Finds the “best possible” solution to a problem. “Best” + “possible”.
- Often, have to choose the “least bad option”.
- “There are no solutions, only trade-offs.”
- Maximize/minimize requirement(s): safety, cost, reliability, stiffness, utility, strength, profit, etc.

Engineering Optimization for different purposes

Optimization is dependent on the end use of the product.

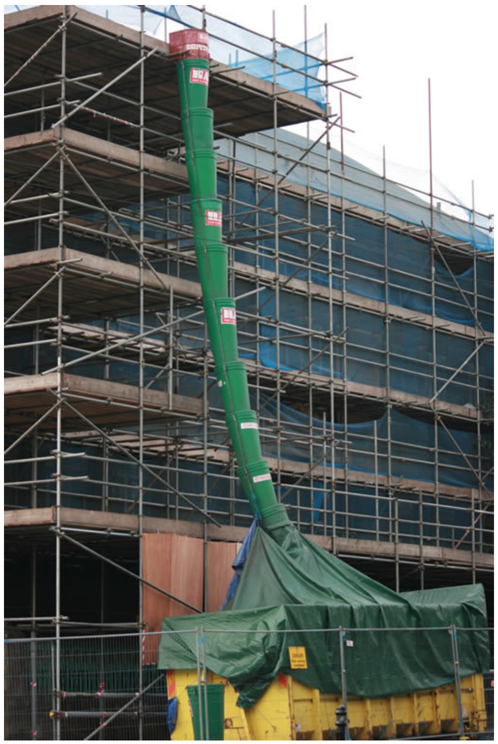
Boeing 747-400 passenger aircraft

SU-27 multirole fighter aircraft

Some Examples of Design Optimization

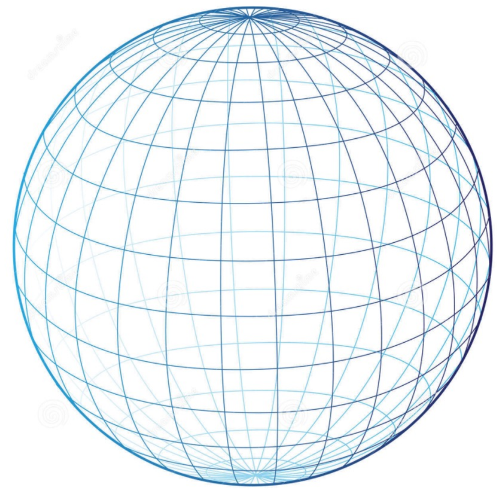
- Path optimization
- Shape optimization
- Structural optimization
- And many more

Path Optimization



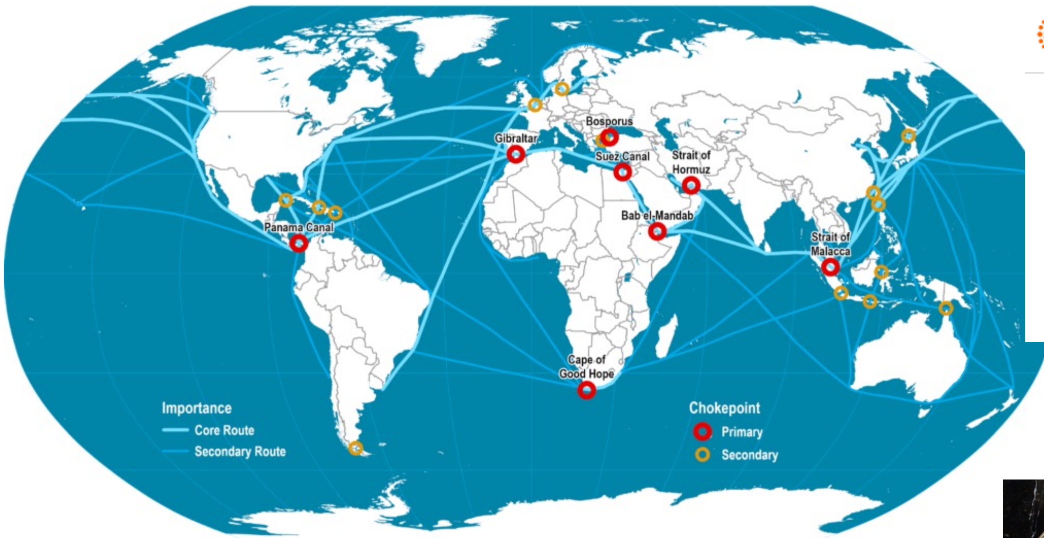
Debris chute

Rollercoaster

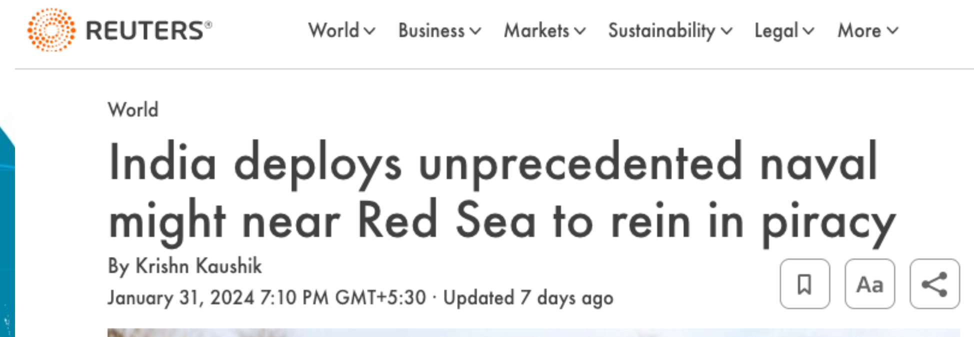


Hyperloop

Path Optimization



Shipping routes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World

India deploys unprecedented naval might near Red Sea to rein in piracy

By Krishn Kaushik

January 31, 2024 7:10 PM GMT+5:30 · Updated 7 days ago

Rescue operations

Slow steaming

Astern = Backwards movement

Optimization?

Path Optimization

- "Understanding the process of fluid mixing is a long-standing challenge in fluid dynamics research. One of the problems that lies at the heart of the study of mixing is the broad concept of 'efficiency'. For example, what is an optimal mixing strategy that produces, at a given time, the best possible mixture, subject perhaps to some constraints?"

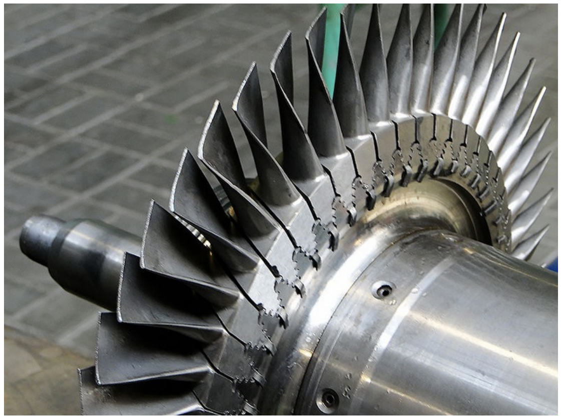
Journal of Fluid Mechanics , Volume 850 , 10 September 2018 , pp. 875 - 923

Structural Optimization

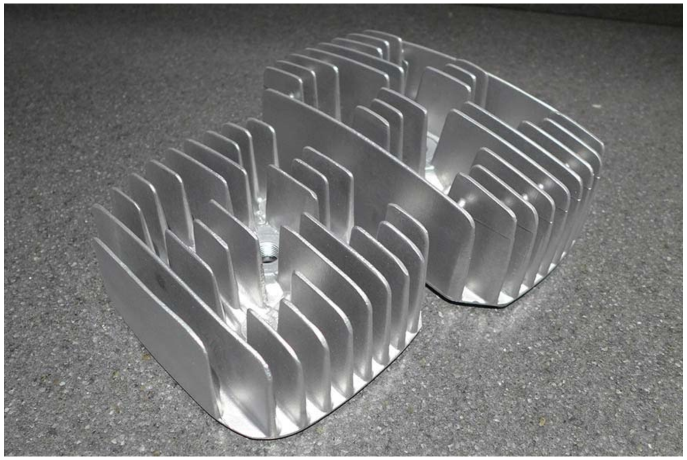
- Maximize buckling load, resistance to wind/wave loads, earthquake loads, stiffness, reliability, fire resistance, visual appeal, etc.
- Minimize weight, cost, construction time, environmental damage, etc.

Buckling is a sudden lateral failure of an axially loaded member in compression, under a load value less than the compressive load-carrying capacity of that member.

Shape Optimization

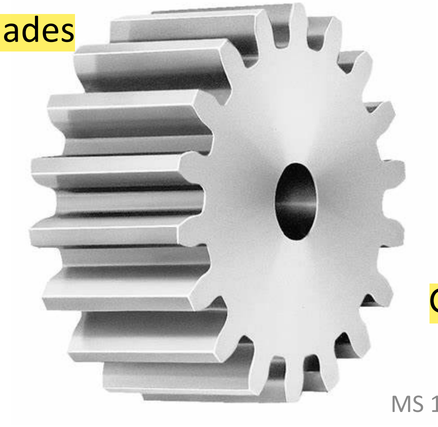


Gas turbine blades



Heat fins

Cooling towers



Gear teeth

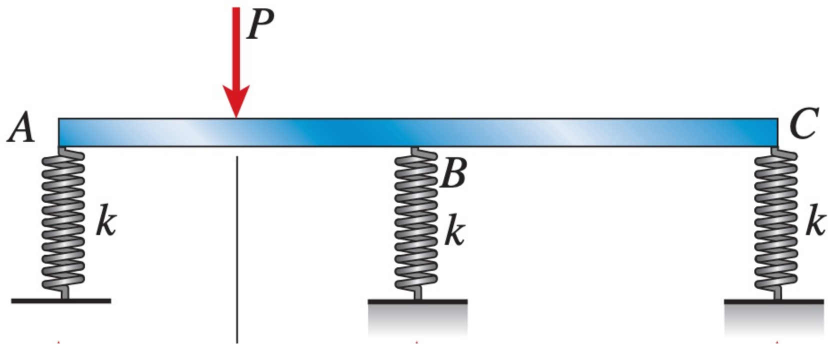
Fins are surfaces that extend from an object to increase the rate of heat transfer to or from the environment by increasing convection.

Optimization Principles in Nature

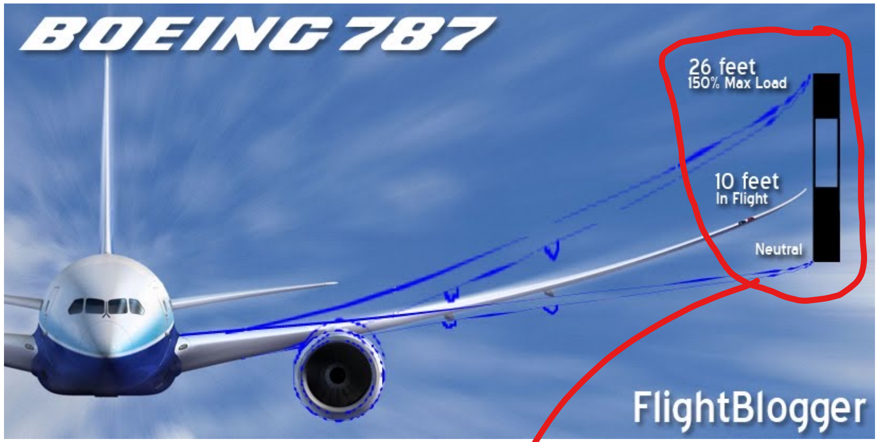
- Statics: Principle of Minimum Potential Energy
- Dynamics: Principle of Least Action
- Ray Optics: Principle of Least Time
- Thermodynamics: Maximum entropy
- Biology: Survival of the fittest

Many scientific/engineering problems can be posed as optimization problems. Many computational methods (like FEM finite element methods) are essentially optimization problems.

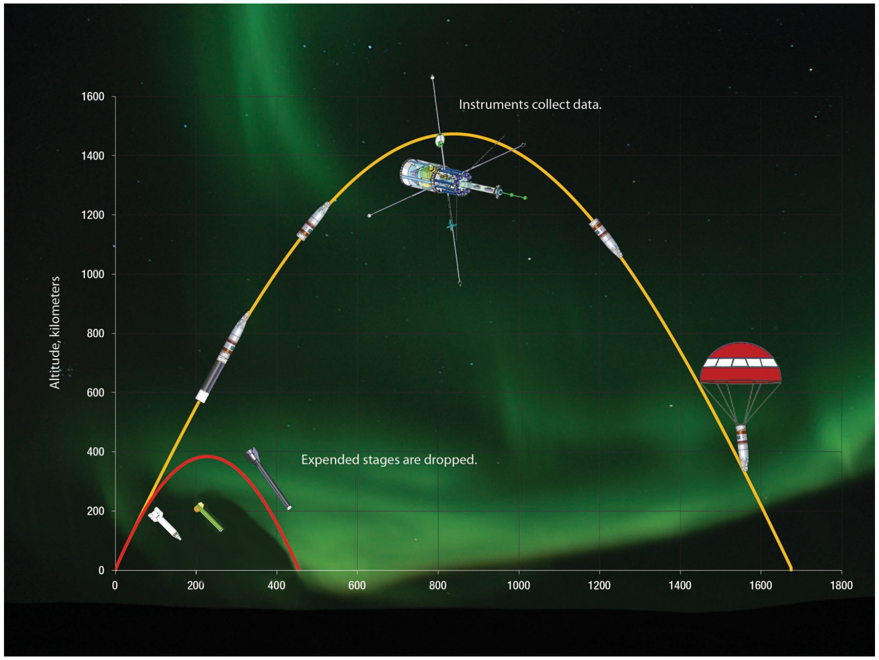
Statics: Principle of Minimum Potential Energy



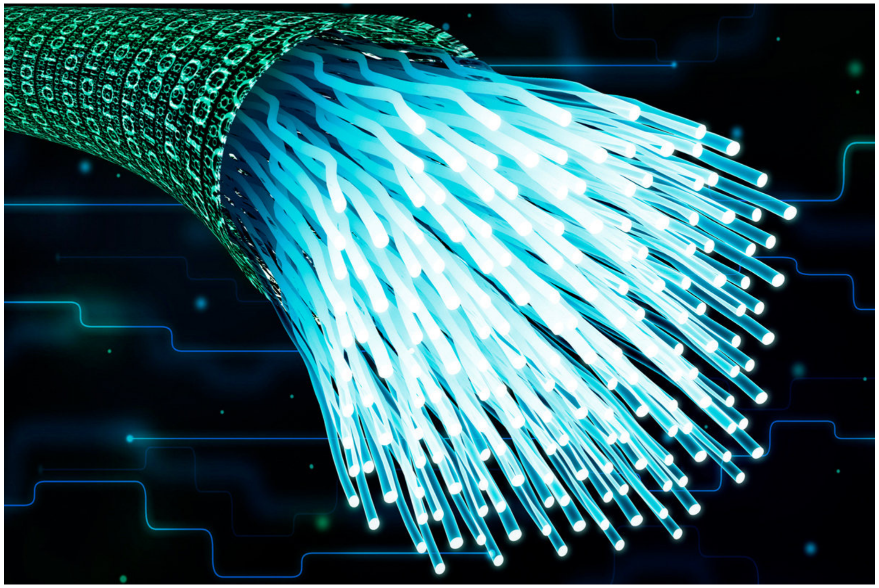
Dynamics: Principle of Least Action



Huh?



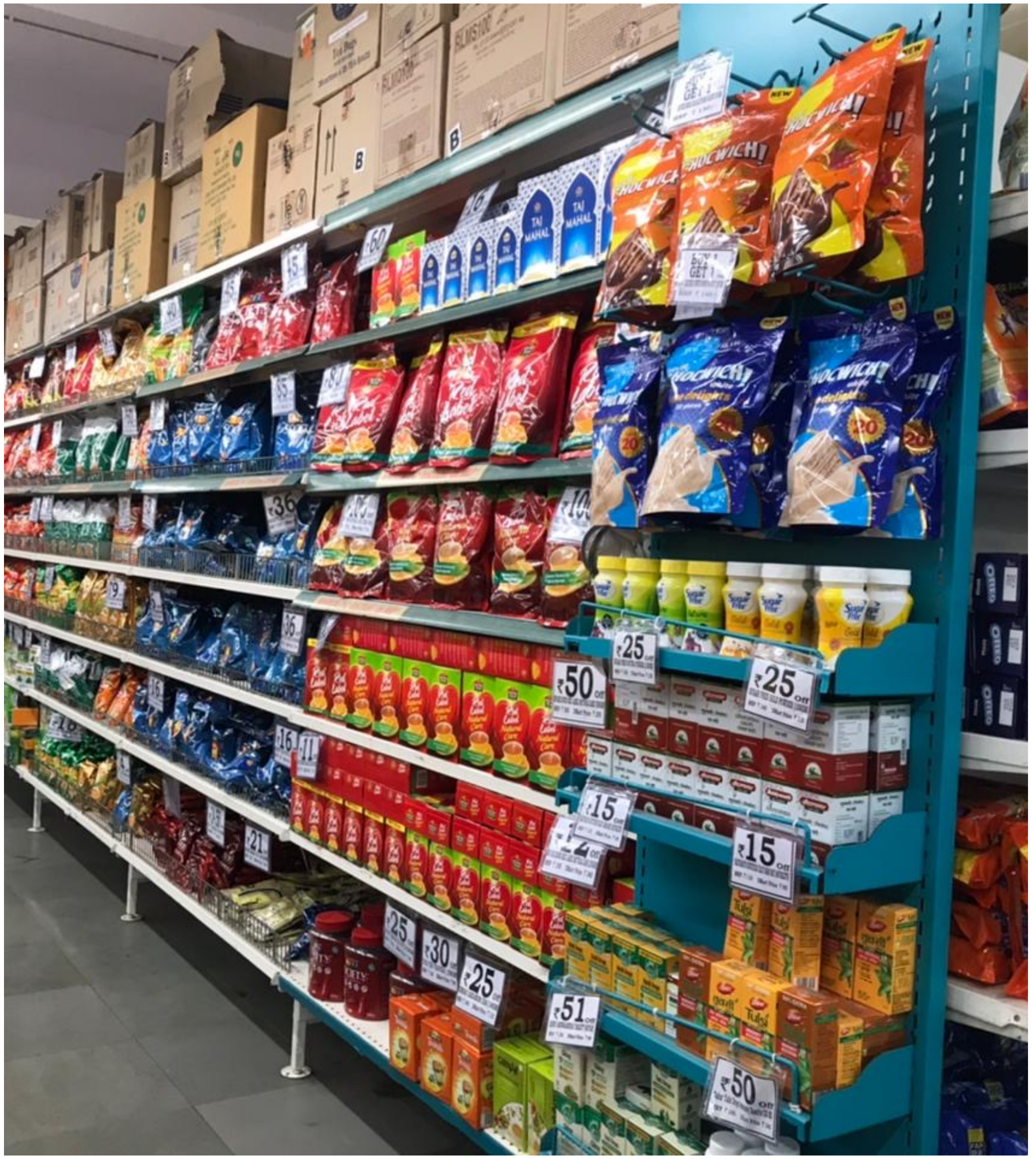
Optics: Fermat Principle of <u>Least</u> Time



Biology: "Nature is the best teacher"
Can we copy/use ideas from the natural world to develop optimization algos?

Genetic Algorithms
Survival of the fittest

Particle Swarm Optimization
Flock/herd behavior



Economic Optimization

- Optimal production problem
- Optimal consumption problem
- Optimal investment/capital allocation problem
- Optimal tax minimization problem

Objectives of Engineering Optimization

- Given an engineering problem
- Some objective function F (in reality, more than one)
- Dependency on certain underlying variables x1, x2, x3, … xN
- This dependency exists through the underlying mathematical model
- How do we choose x1, x2, x3, ..., xN such that F is maximized/minimized?
- Are there any inter-dependencies (underlying relationships) between variables x1, x2, ..., xN that must be obeyed? Constraints
- Note that the variables x1, x2, ..., xN must be experimentally measurable, quantifiable, or implementable in practice.

Problem Statement of Engineering Optimization

- Given an objective function F(x1,x2,…,xN) of N variables
- (x1, x2, …, xN) are called design/decision variables (can be functions also but more on this later).
- Find (x1,x2,…,xN) such that F is a minimum
- It is possible that x1,x2,...xN may be inter-related through equations/inequalities called constraints of the sort:
- G(x1,x2,...xN) = 0 equality constraint
- H(x1,x2,…,xN) <= 0 inequality constraint
- In real life, many objective functions, many constraints.
- F is called objective function / cost function.
- We will talk about minimization. If maximize F, then minimize?

Mathematical Modeling

- Universe is governed by rules/laws and that these rules can be figured out by humans. The way to figure out these rules is to:
- Observe a phenomenon in nature or in a controlled setting aka laboratory
- Propose a mathematical model based on physical laws and reasonable assumptions. Models are typically ODEs/PDEs in engineering. Note that we analyze only the mathematical abstraction of the real system
- Analytically solve / computer simulate the model
- Compare model predictions with experimental results or observations.
- If agreement, keep the model and use it to improve our lives by building better, safer, and cheaper products, and services.
- If no agreement, improve model or discard it and start afresh.
- "All models are wrong, but some are useful."

Optimization Methods

- *Trial-and-error through experimentation*. Learn from your own/others' mistakes.
- *Graphical methods*. Based on visualization. Good for up to 2D problems.
- *Analytical methods*. Calculus based approach i.e. first derivative (gradient) test, second derivative (Hessian) test. Elegant but not suitable for real world applications with complex geometries, loading, boundary conditions, and multi-physics scenarios (say thermal + mechanical + electrical). But physically insightful and gives scaling laws.
- *Computational methods*. Most useful but not as straightforward as thought to be. Output needs to be properly interpreted. Which method best for your problem? Schemes/parameters/starting points in the method need to be chosen with care.
- Even if analytical/computational methods used, experimental validation is always necessary. Build prototype and show in tests that it performs better than other competing designs.

Plan

- We will look at the mathematical formulation of several optimization problems.
- Important to understand where engineering optimization problems come from.
- Try to identify:
- 1. The objective function
- 2. The design variables
- 3. Constraints on the design variables

Plan – Formulation of “Simple” Optimization Problems in 1D, 2D, InfiniteD

- 1D: Rescue problem, Speed limit on highway, Structural optimization
- 2D: Rescue problem, Least squares, Shape optimization
- InfiniteD: Rescue problem, Path optimization, Shape optimization

Note: Keep an eye on the statement of the problem and the underlying assumptions of the mathematical model

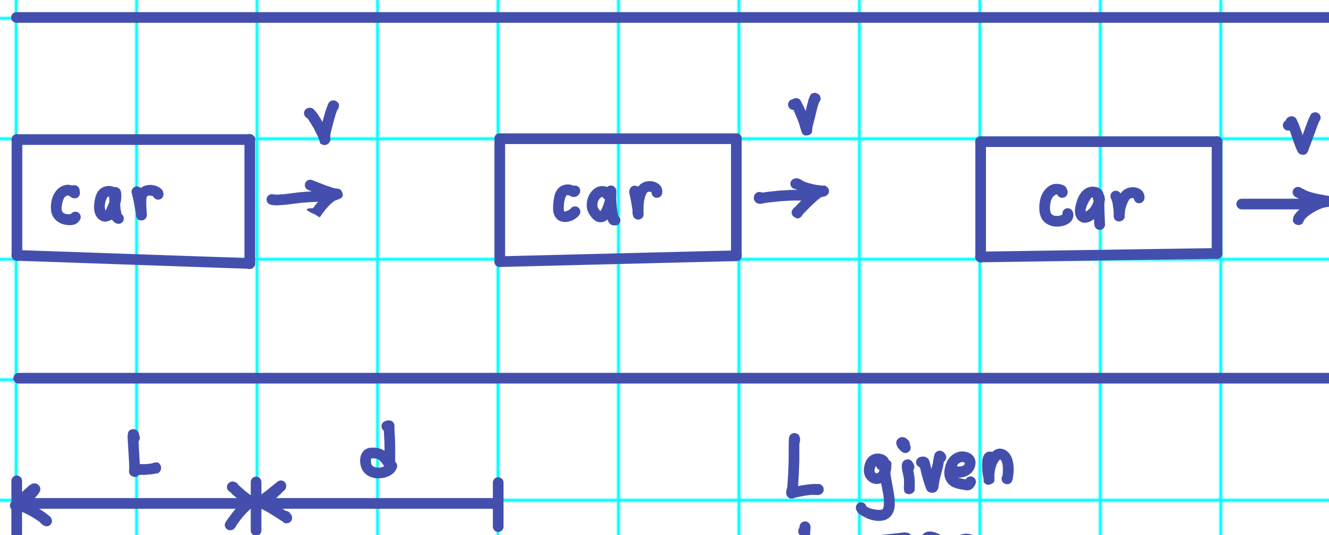
Problem 1 Traffic Congestion (1D)

Find the optimum safe speed v at which cars should travel on a single lane highway to minimize traffic congestion i.e. to maximize the flux of cars.

$\min(A) \Rightarrow$ maximizing (B)

1D OPTIMIZATION TRAFFIC PROBLEM

Optimum speed v
single-lane highway



L given
d TBD

Maximize flux / throughput $\varphi(v)$

φ Flux = cars passing a point in unit time

If v too high $\Rightarrow$ accidents

v too low $\Rightarrow$ delays / frustration

Let t_r = driver reaction time $\simeq 0.5$ s, given

μ = friction coefficient bt tires and road surface, given

Safe stopping distance $\boxed{d = t_r v + \frac{v^2}{2\mu g}}$

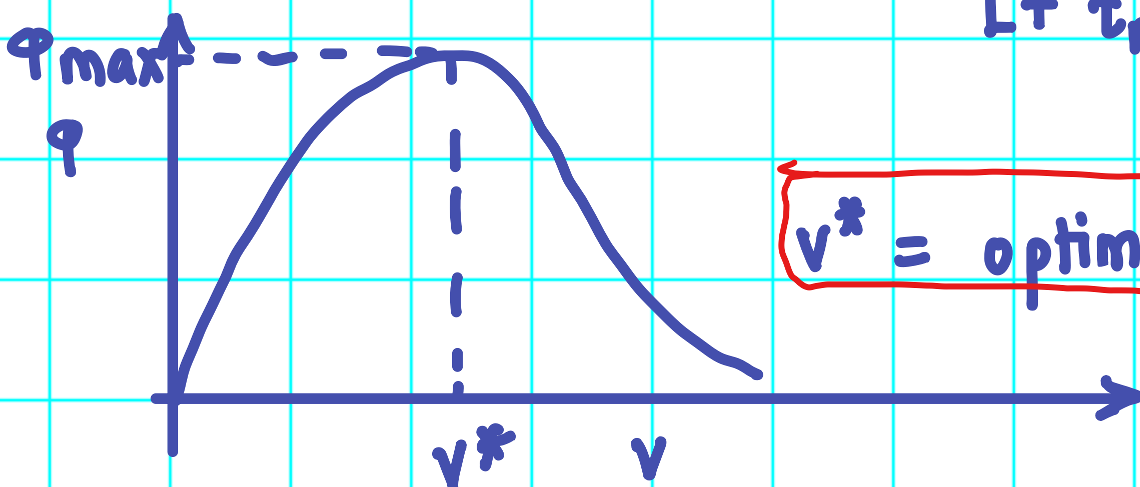
1 car takes $\frac{L+d}{v}$ seconds to pass a point

N cars take $N\left(\frac{L+d}{v}\right)$ seconds to pass a point

$= t$

Flux $\varphi = \frac{N}{t} = \frac{v}{L+d}$, $\boxed{v = \text{design variable}}$

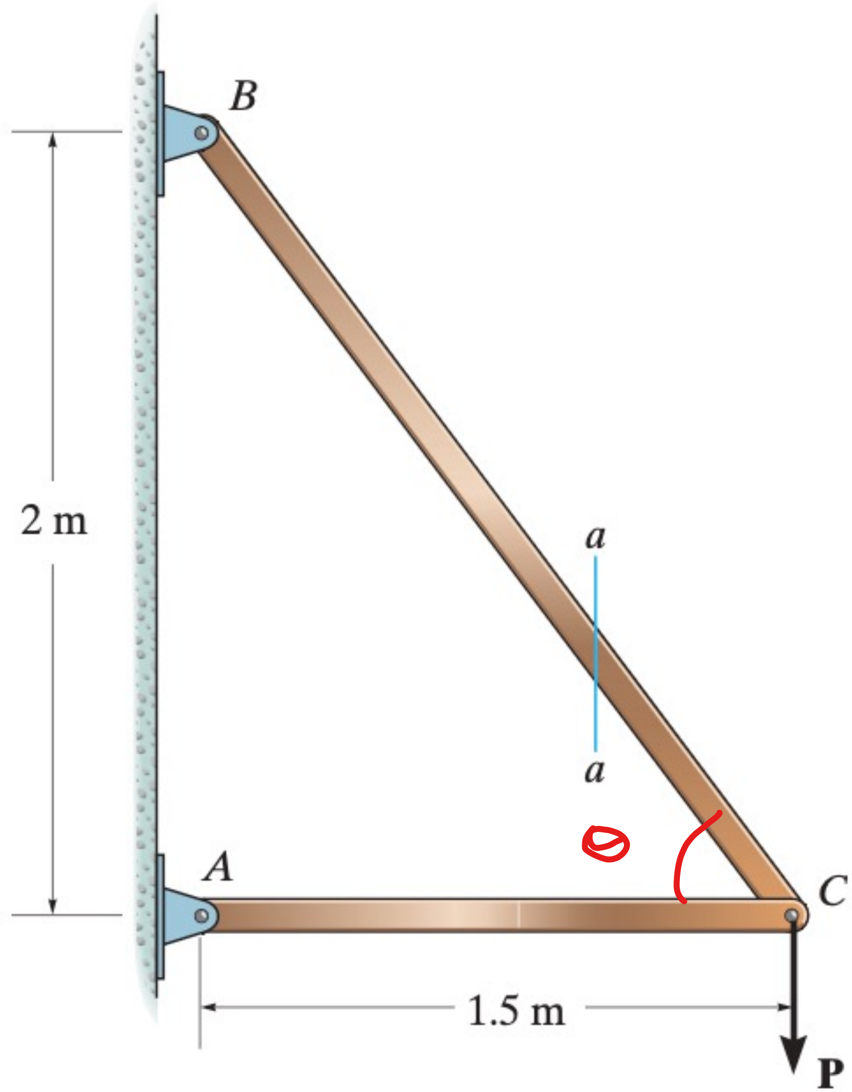
Objective function $\varphi(v) = \dfrac{v}{L + t_r v + \dfrac{v^2}{2\mu g}}$



$\boxed{v^* = \text{optimum speed}}$

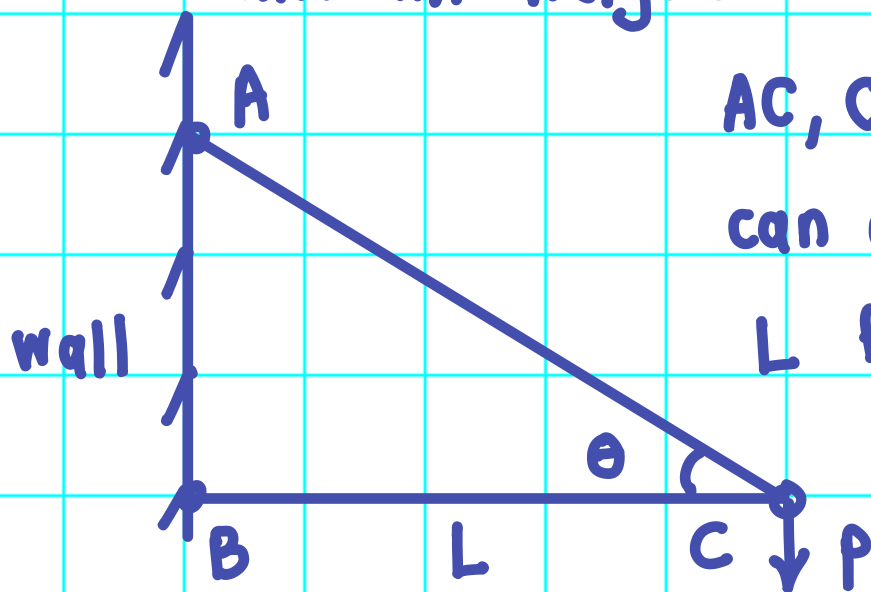
Problem 2 Structure (1D

For the truss shown, find the angle of the inclined structural member to minimize the total weight of the structure while ensuring that the stress in each member does not exceed the allowable stress.



1D STRUCTURAL OPTIMIZATION

Minimum weight design



AC, CB bars / two force members

can carry only tensile/compressive force

L fixed = L_{BC} given

Find θ for min weight

Weight of structure $\boxed{W = \rho g (L_{AC} A_{AC} + L_{BC} A_{BC})}$

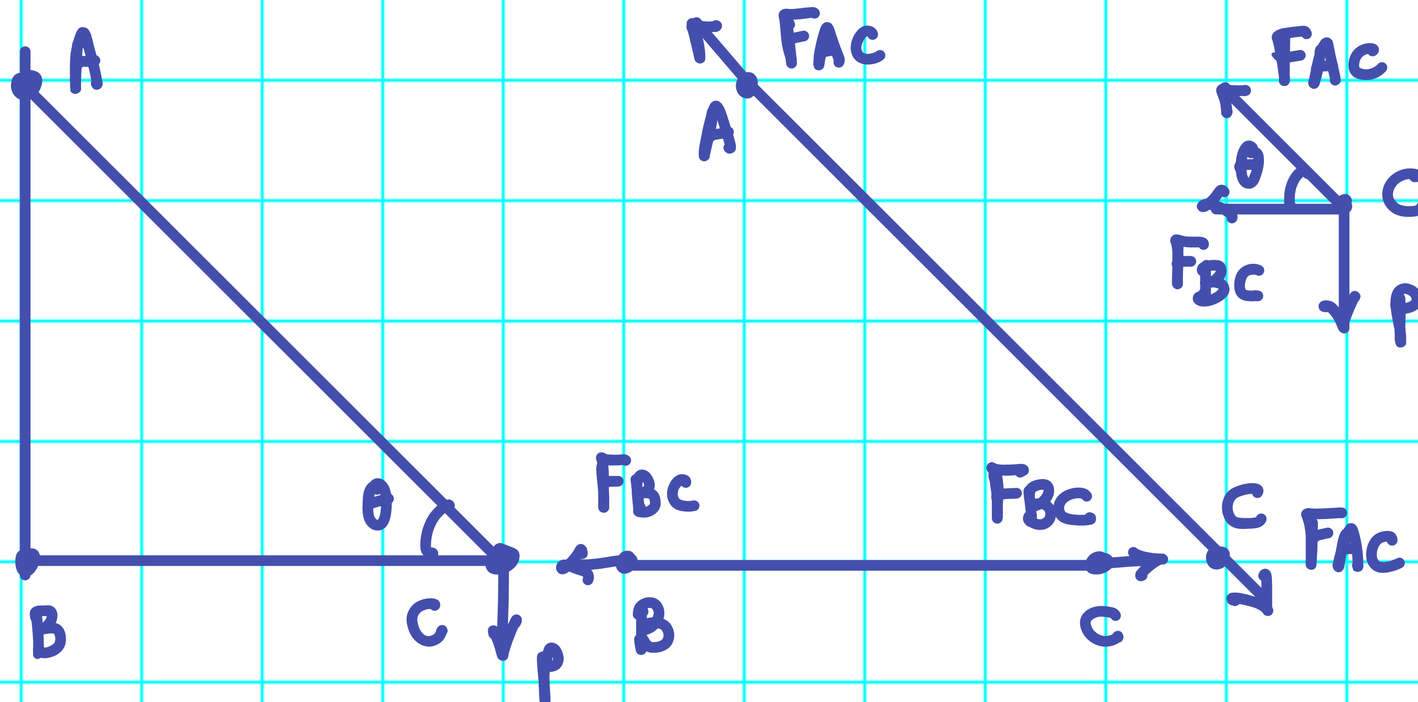
ρ mass density of bar material kg/m^3, given

g gravitational acceleration N/kg, given

$$A_{AC} = \frac{F_{AC}}{\sigma}, \quad A_{BC} = \frac{F_{BC}}{\sigma}, \quad L_{AC} = \frac{L}{\cos\theta}$$

σ max allowable stress in each bar N/m^2
given

F_{AC}, F_{BC}, P related through static equilibrium



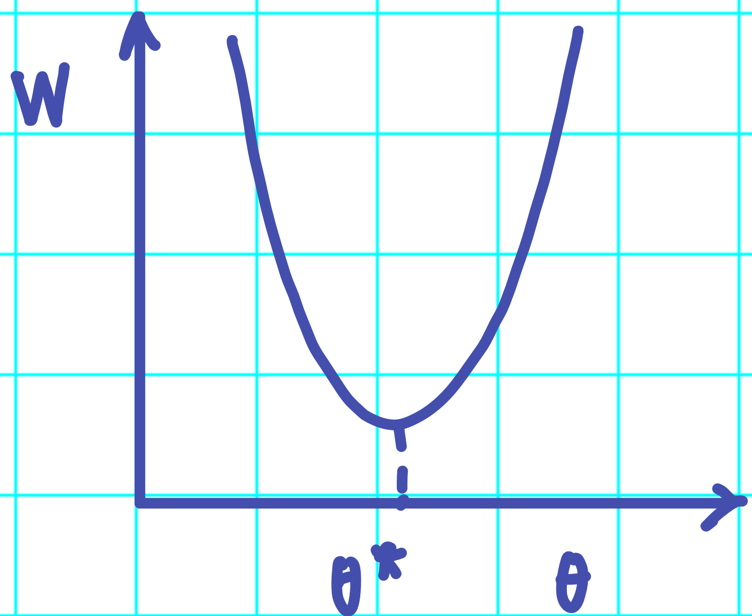
static equilibrium at C $\Rightarrow \Sigma F_x = 0, \ \Sigma F_y = 0$

$-P + F_{AC} \sin\theta = 0, \quad F_{BC} + F_{AC} \cos\theta = 0$

$$F_{AC} = \frac{P}{\sin\theta}, \quad F_{BC} = \frac{P}{\tan\theta}$$

plug back in expression for W

$$W = \frac{\rho g L}{\sigma}\left(\frac{1+\cos^2\theta}{\sin\theta\,\cos\theta}\right)$$ objective function



$\simeq 0.955$ rad $\simeq 54.71^\circ$

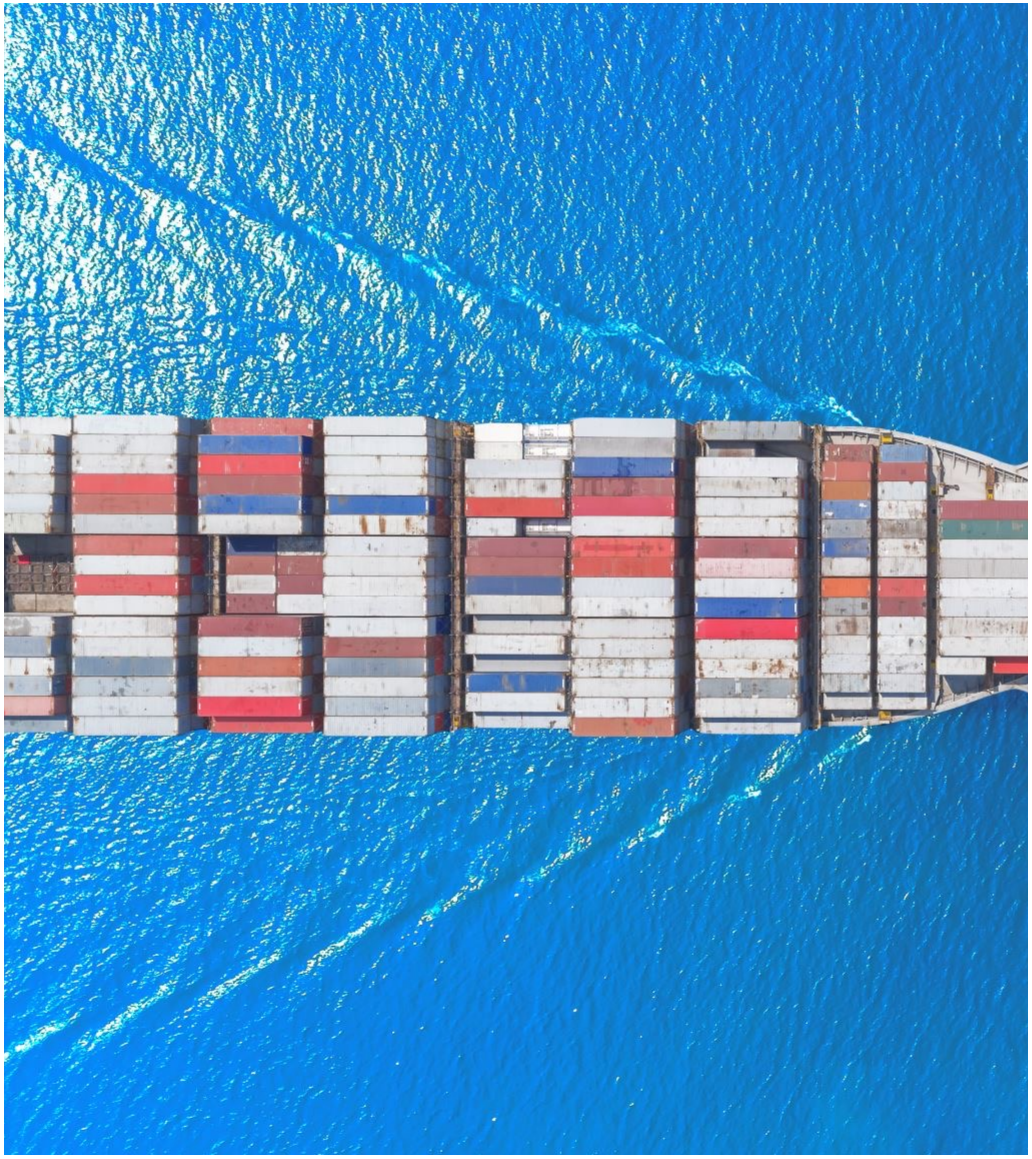
or $\dfrac{dW}{d\theta} = 0$, θ = design variable

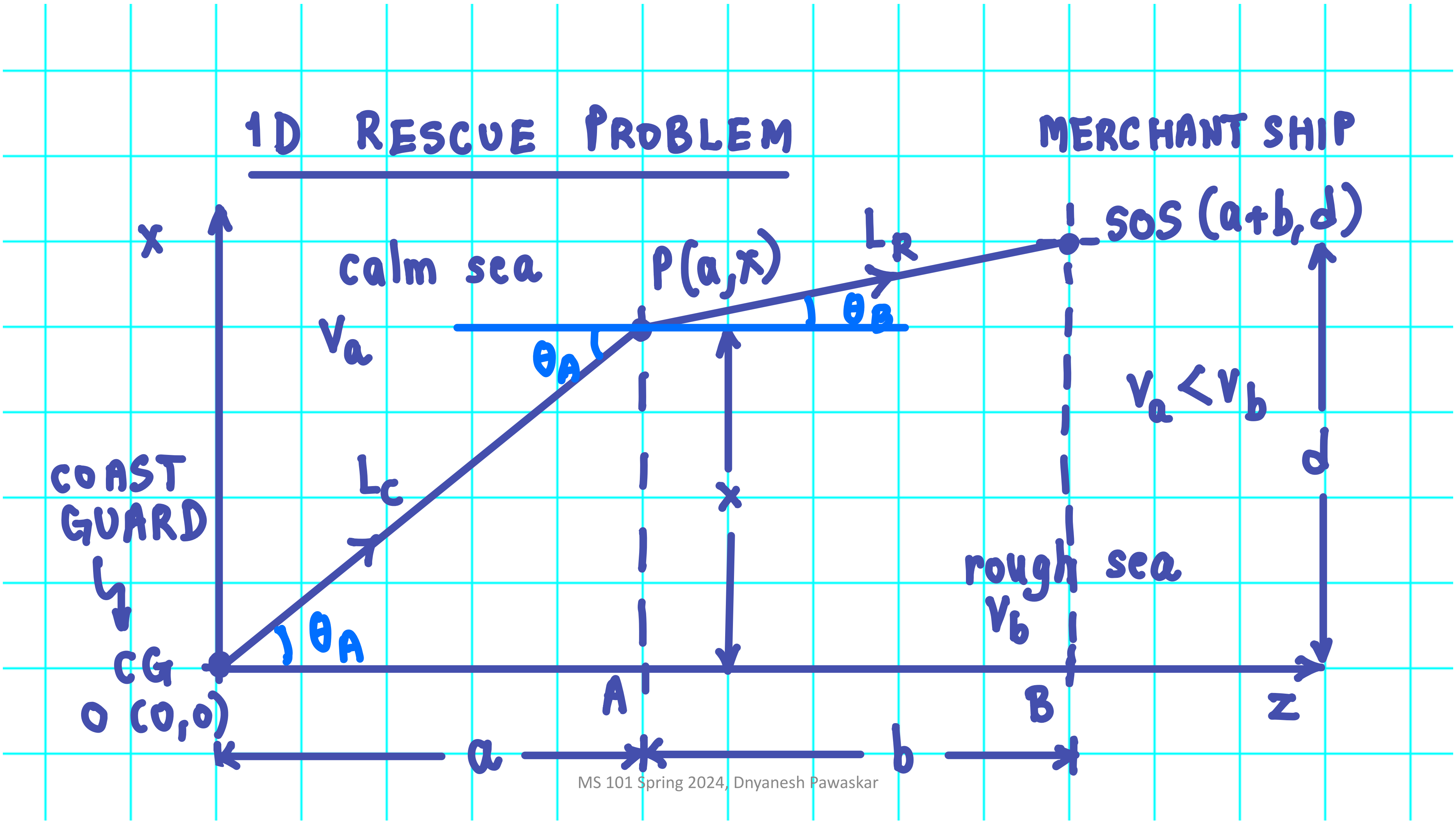
$\Rightarrow \quad 3\sin^2\theta = 2$

$\Rightarrow \quad \theta = 0.955 \text{ rad} \simeq 54.71^\circ$

Problem 3 Rescue (1D)

A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best piecewise linear route for the coast guard vessel to reach the merchant ship in minimum time.





1D RESCUE PROBLEM
MERCHANT SHIP
SOS (a+b, d)
x
calm sea
P(a, x)
L_R
θ_B
V_a
θ_A
$V_a < V_b$
COAST GUARD
L_c
d
x
rough sea
V_b
θ_A
CG
O (0,0)
A
B
z
a
b

Find x such that

Rescue (coast guard) ship reaches SOS in **minimum** time

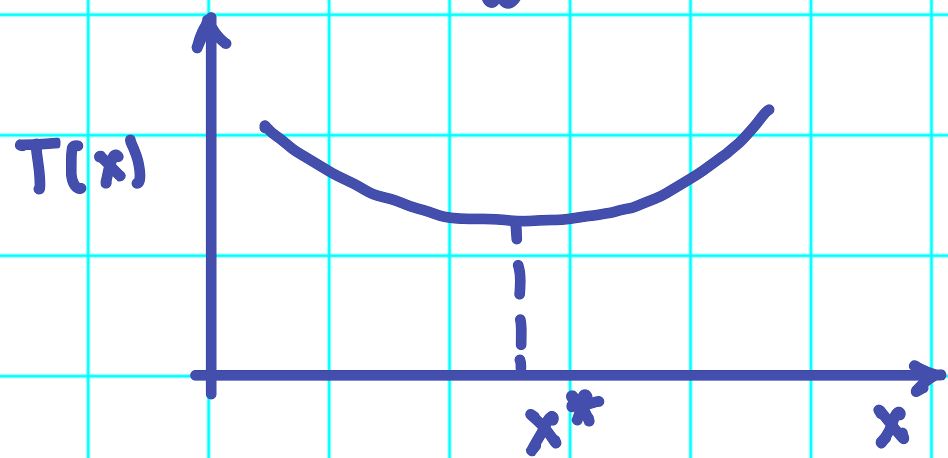
$$\text{Total time} = \frac{L_a}{v_a} + \frac{L_b}{v_b} \quad \text{objective function}$$

v_a max ship speed in calm sea, given

v_b max ship speed in rough sea, given

$$L_a = \sqrt{a^2 + x^2}, \quad L_b = \sqrt{b^2 + (d-x)^2}$$

$$T(x) = \frac{\sqrt{a^2 + x^2}}{v_a} + \frac{\sqrt{b^2 + (d-x)^2}}{v_b}$$



x = design variable

$$\frac{dT}{dx} = \frac{1}{v_a}\frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}} + \frac{1}{v_b}\frac{\cancel{2}(x-d)}{\cancel{2}\sqrt{b^2+(x-d)^2}} = 0$$

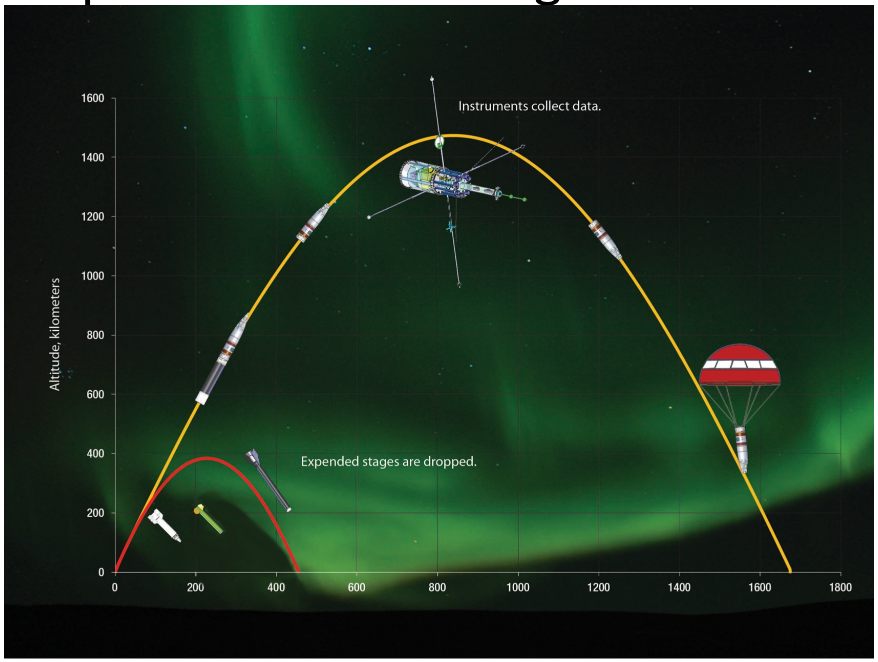
$$\Rightarrow \frac{1}{v_a}\sin\theta_A - \frac{1}{v_b}\sin\theta_B = 0$$

$$\Rightarrow \frac{\sin\theta_A}{v_a} = \frac{\sin\theta_B}{v_b}$$ Snell's Law

Fermat's Principle of Least Time

Problem 4 Projectile Range (1D)

Find the optimum launch angle for a projectile to maximize the horizontal range in the presence of air drag.

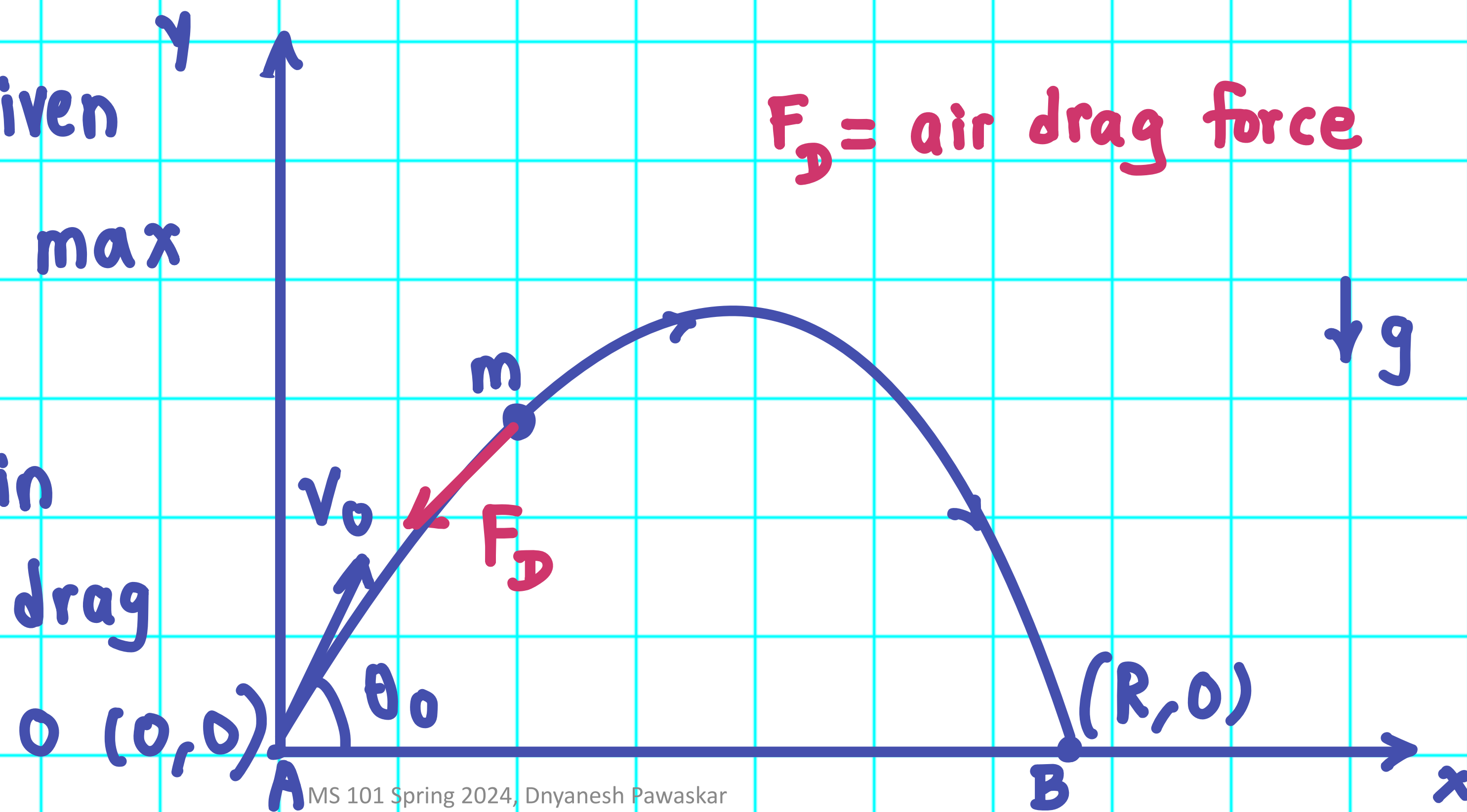


1D OPTIMIZATION MAX RANGE OF PROJECTILE

v_0 fixed/given

Find θ_0 for max range R

$\theta_0 = 45^\circ$ in absence of drag



Recap, $\underline{F} = m\underline{a} \Rightarrow \quad m\ddot{x} = 0, \quad m\ddot{y} = -mg$

4 initial conditions

$x(0) = 0, \quad y(0) = 0, \quad \dot{x}(0) = v_0 \cos\theta_0, \quad \dot{y}(0) = v_0 \sin\theta_0$

$= \dot{x}_0 \qquad = \dot{y}_0$

$x(t) = (v_0 \cos\theta_0) t$

$y(t) = (v_0 \sin\theta_0) t - \frac{1}{2} g t^2$

t_{max} = total flight time $\Rightarrow y(t_{max}) = 0$

$$\Rightarrow \quad v_0 \sin\theta_0 = \frac{1}{2} g\, t_{max} \Rightarrow t_{max} = \frac{2 v_0 \sin\theta_0}{g}$$

$$R = \text{range} = x(t_{max}) = \frac{2 v_0^2 \cos\theta_0 \sin\theta_0}{g}$$

$R = R_{max}$ when $\theta_0 = 45^\circ$

$$\Rightarrow \quad R_{max} = \frac{v_0^2}{g}$$

Linear Air Resistance

$F_D = m\beta v$, drag force

$$m\ddot{x} = -m\beta\dot{x}\,, \quad m\ddot{y} = -m\beta\dot{y} - mg$$

β = from expt measurements, given

$$\dot{x} = \dot{x}_0 e^{-\beta t}\,, \quad \dot{y} = \dot{y}_0 e^{-\beta t} - \frac{g}{\beta}\left(1 - e^{-\beta t}\right)$$

by integrating once

where $\dot{x}_0 = v_0 \cos\theta_0$, $\dot{y}_0 = v_0 \sin\theta_0$

Integrate once more, with $x(0)=0,\ y(0)=0$

$$x(t) = \frac{\dot{x}_0}{\beta}\left(1-e^{-\beta t}\right),$$

$$y(t) = \left(\frac{\dot{y}_0}{\beta}+\frac{g}{\beta^2}\right)\left(1-e^{-\beta t}\right) - \frac{g}{\beta}t$$

For horizontal range $R = x_{max}$, set $y(t)=0$

and plug in $R = \frac{\dot{x}_0}{\beta}\left(1-e^{-\beta t}\right),\ t = -\frac{1}{\beta}\ln\left(1-\frac{\beta R}{\dot{x}_0}\right)$

$$\Rightarrow \left(\frac{\dot{y}_0}{\beta} + \frac{g}{\beta^2}\right)\frac{\beta R}{\dot{x}_0} + \frac{g}{\beta^2}\ln\left(1 - \frac{\beta R}{\dot{x}_0}\right) = 0$$

which is an implicit expression for R.

Approximation $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots$

$|u| < 1$

$u = \frac{\beta R}{\dot{x}_0}$, small β small wrt ?

i.e. $\beta \ll \frac{g}{v_0}$ $\cancel{m}\beta v_0 \ll \cancel{m}g$

$$R = \frac{2\dot{x}_0 \dot{y}_0}{g} - \frac{8\dot{x}_0 \dot{y}_0^2}{3g^2}\beta + \ldots$$

$$\dot{x}_0 = v_0 \cos\theta_0, \quad \dot{y}_0 = v_0 \sin\theta_0$$

$$R = \underbrace{\frac{v_0^2 \sin 2\theta_0}{g}}_{\text{without air resistance}} - \underbrace{\frac{4 v_0^3 \sin 2\theta_0 \sin\theta_0}{3g^2}\beta}_{\text{effect of linear air resistance}} + \ldots$$

, β small

Quadratic Drag / Air Resistance

$F_D = m\beta v^2$ more realistic, agrees better with experiments

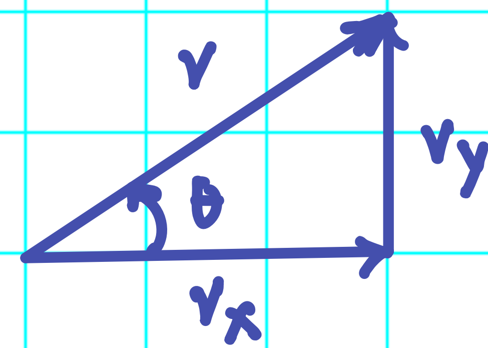
$F_D = \frac{1}{2} C_D \rho A v^2$ for ball

air density

$C_D \simeq 0.45$, $\rho = 1.23 \text{ kg/m}^3$ at sea-level

$A = \frac{\pi}{4} d^2$, $d =$ diameter of ball

$$v^2 = v_x^2 + v_y^2$$



$$v_x = \dot{x}, \quad v_y = \dot{y}$$

$$\sin\theta = \frac{v_y}{v}, \quad \cos\theta = \frac{v_x}{v}, \quad \underline{F} = m\underline{a} \Rightarrow$$

$$m\ddot{x} = -F_D\cos\theta = -m\beta\left(\dot{x}^2+\dot{y}^2\right)\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$m\ddot{y} = -F_D\sin\theta - mg = -mg - \frac{m\beta\left(\dot{x}^2+\dot{y}^2\right)\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

Get two 2^{nd} order nonlinear coupled ODEs,

$$\ddot{x} = -\beta \dot{x} \sqrt{\dot{x}^2 + \dot{y}^2}$$

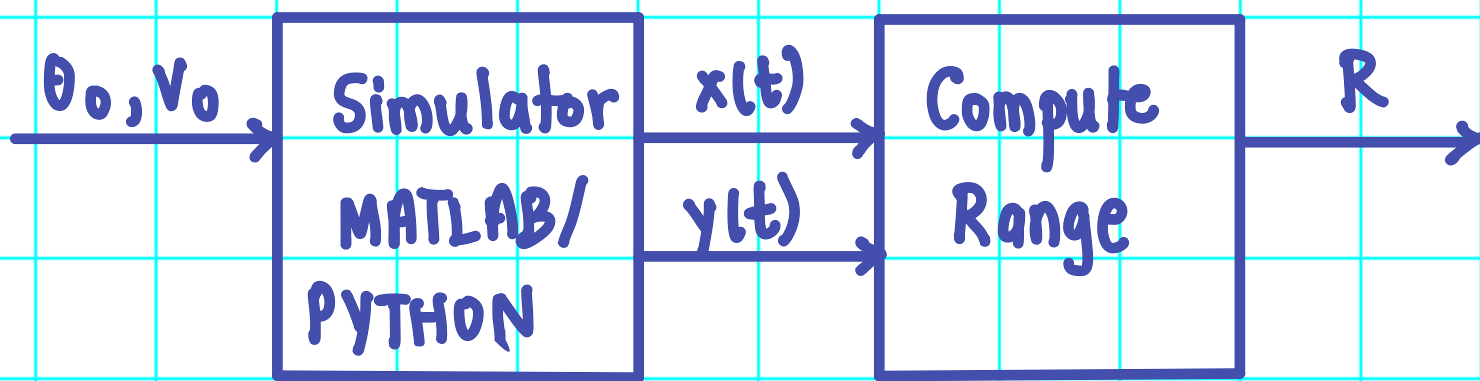
$$\ddot{y} = -g - \beta \dot{y} \sqrt{\dot{x}^2 + \dot{y}^2}$$

Simulate using computer. MATLAB, Python, etc

with initial conditions

$x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = v_0 \cos\theta_0$, $\dot{y}(0) = v_0 \sin\theta_0$

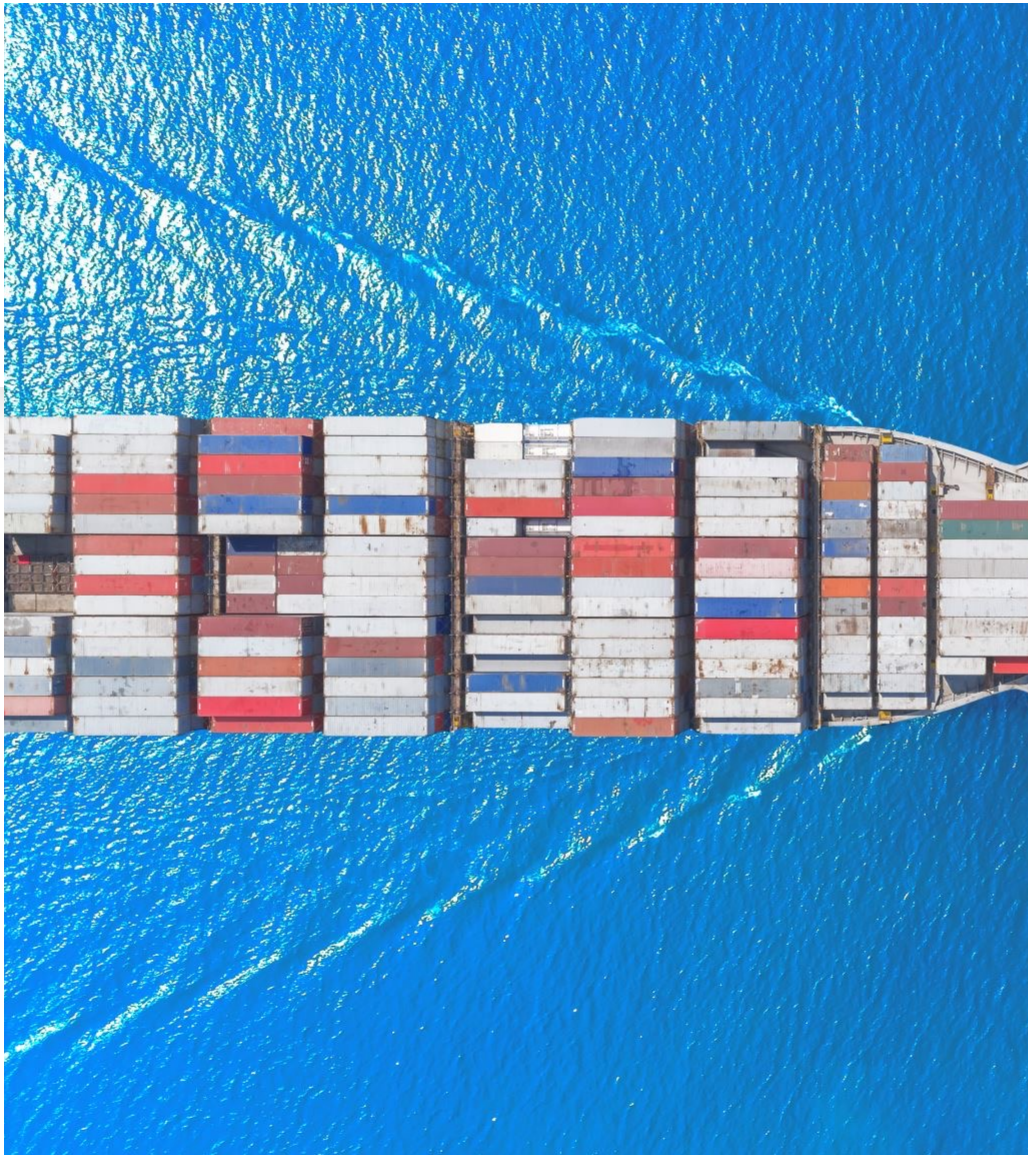
θ_0 design variable



numerical integrator like Runge-Kutta 4 (RK4) to generate time series $x(t), y(t)$

Problem 5 Rescue (2D)

A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best piecewise linear route for the coast guard vessel to reach the merchant ship in minimum time.

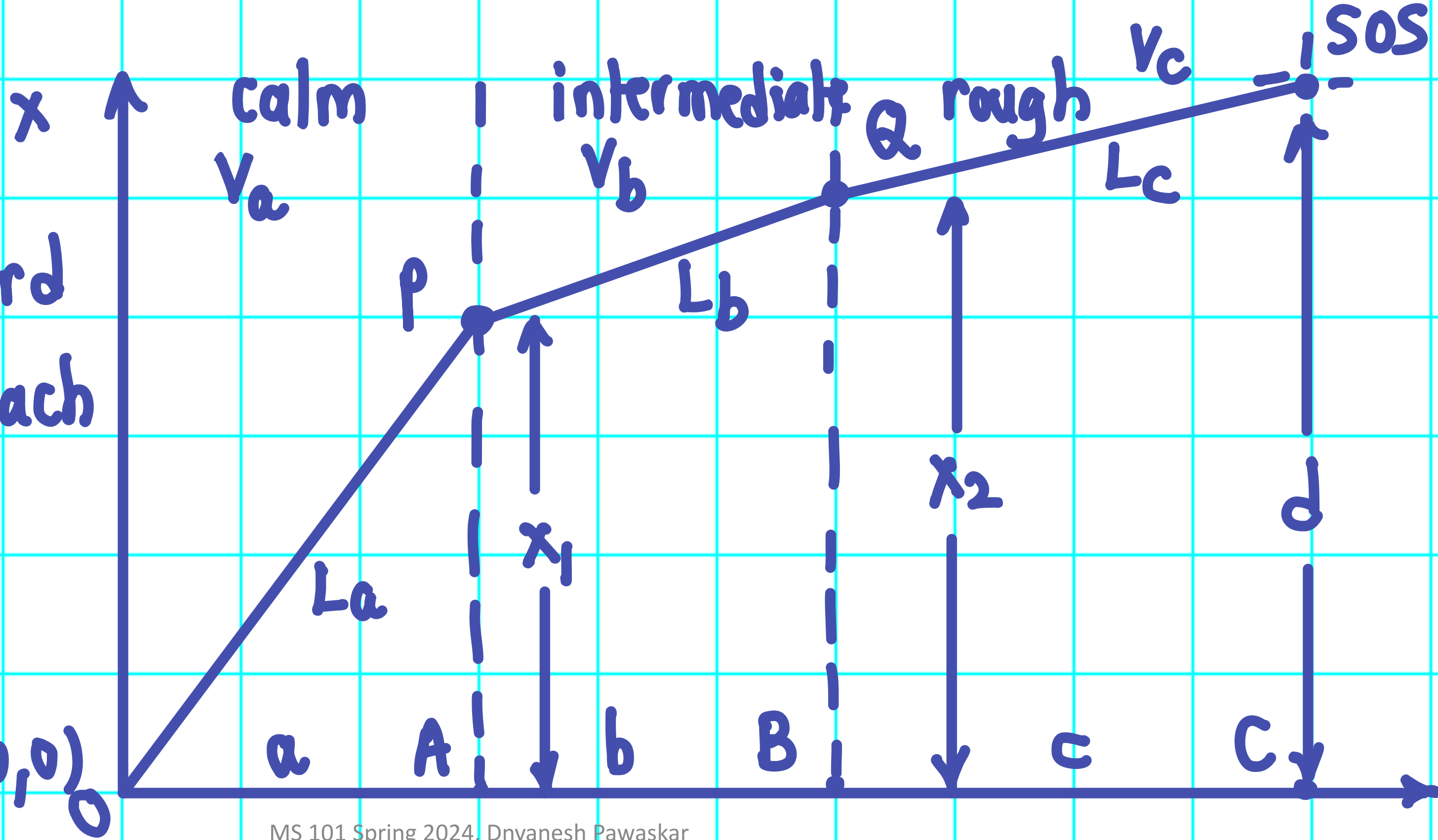


2D OPTIMIZATION RESCUE PROBLEM

Find x_1, x_2 for coast guard at (0,0) to reach rescue

$V_a > V_b > V_c$

COAST GUARD(0,0)



$$T(x_1, x_2) = \frac{L_a}{v_a} + \frac{L_b}{v_b} + \frac{L_c}{v_c}$$, objective function

$$L_a = \sqrt{a^2 + x_1^2}, \quad L_b = \sqrt{b^2 + (x_2 - x_1)^2}$$

$$L_c = \sqrt{c^2 + (d - x_2)^2}$$

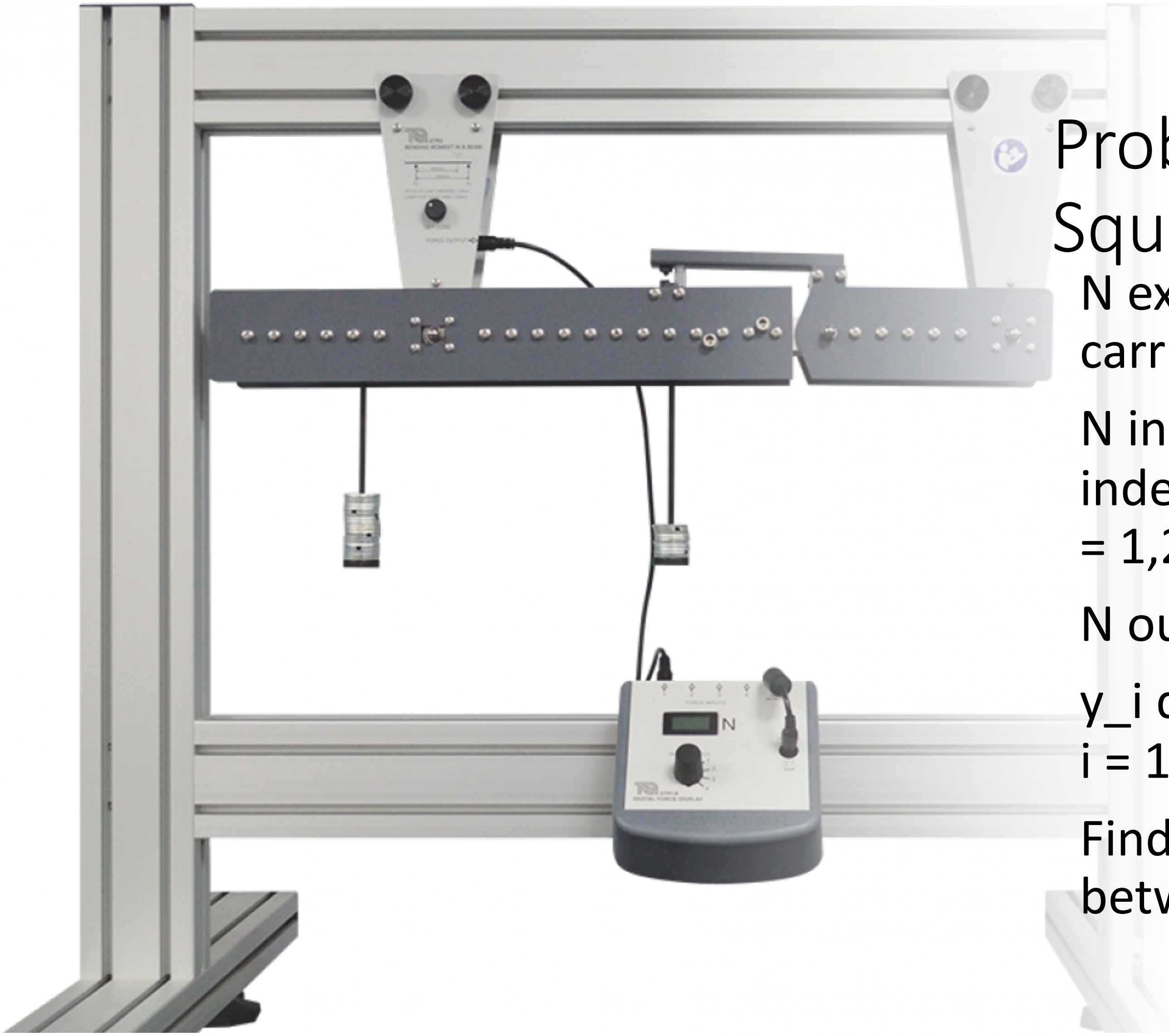
$$T(x_1, x_2) = \frac{\sqrt{a^2 + x_1^2}}{v_a} + \frac{\sqrt{b^2 + (x_2 - x_1)^2}}{v_b} + \frac{\sqrt{c^2 + (d - x_2)^2}}{v_c}$$

x_1, x_2 design variables

Locate optimum using $\underbrace{\frac{\partial T}{\partial x_1} = 0, \frac{\partial T}{\partial x_2} = 0}$

to solve for x_1, x_2

$$\underline{\nabla} T = \underline{0}$$



Problem 6 Least Squares (2D)

N experiments are carried out with

N inputs: x_i independent variables, i = 1,2,3,…, N

N outputs:

y_i dependent variables, i = 1,2,3,…, N

Find the best fit line between y and x

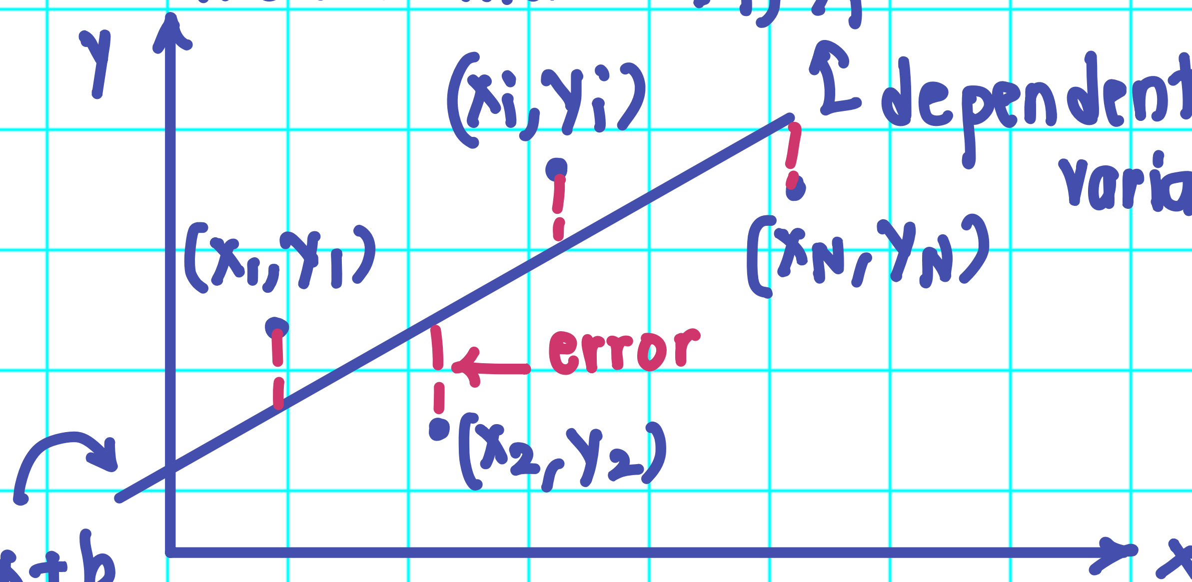
2D OPTIMIZATION LEAST SQUARES

independent variable

N experimental measurements x_i, y_i

dependent variable

x_1	y_1
x_2	y_2
...	...
x_N	y_N



$y = ax + b$

Expect linear relationship $y = ax+b$ from physical/ logical arguments

Find (a,b) for "best fit" line,

error $e_i = ax_i + b - y_i$

Σ error^2 $= e_1^2 + e_2^2 + \ldots + e_N^2$ = objective function

$$F(a,b) = (ax_1+b-y_1)^2 + (ax_2+b-y_2)^2 + \ldots + (ax_N+b-y_N)^2$$

a, b design variables.

Linear regression as $y = ax+b$ linear in a, b

Locate optimum using $\underbrace{\frac{\partial f}{\partial a} = 0, \frac{\partial f}{\partial b} = 0}$

to solve for a and b

$$\underline{\nabla} f = \underline{0}$$

Problem 7 Utility Max (2D)

Find the optimum quantities of apples (x1) and oranges (x2) that a person should eat in a month to maximize his/her satisfaction/utility, while operating on a fixed monthly budget.

2D OPTIMIZATION UTILITY MAXIMIZATION

x_1 apples/month, x_2 oranges/month in kg

2 design variables

Utility/satisfaction/benefit/pleasure, objective function

$$f = x_1^2 + 8x_2^2 \leftarrow \text{say } ?$$

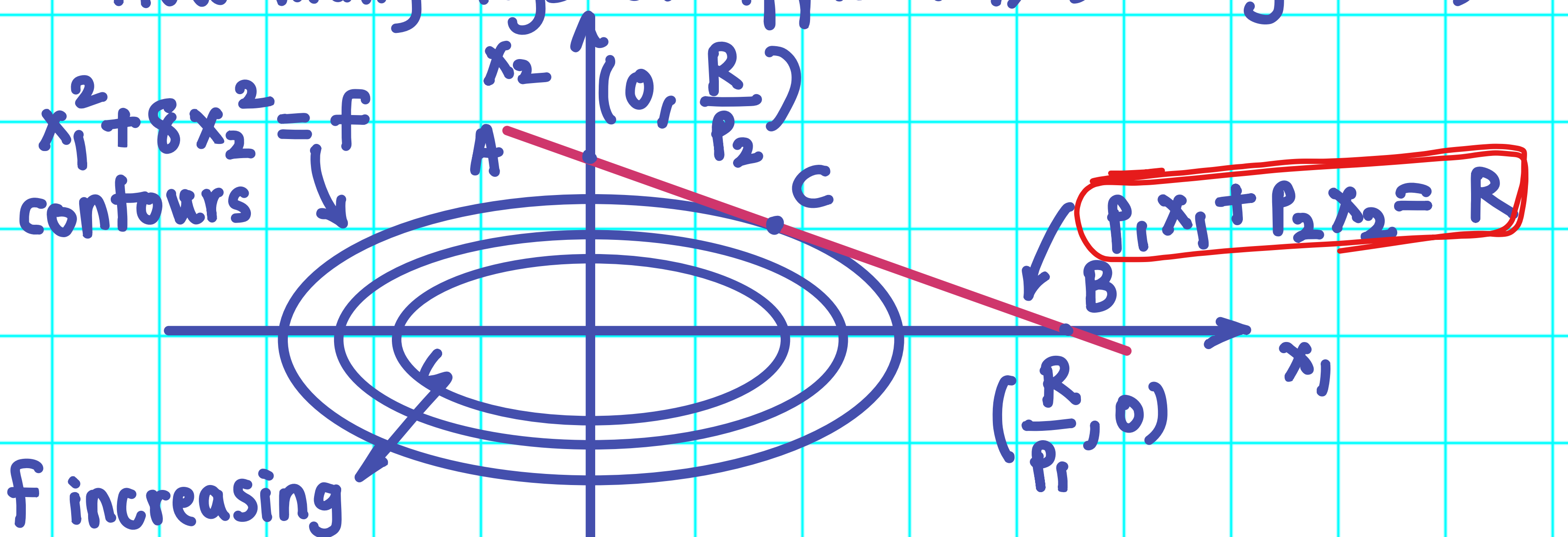
Apples cost Rs p_1/kg, Oranges Rs p_2/kg

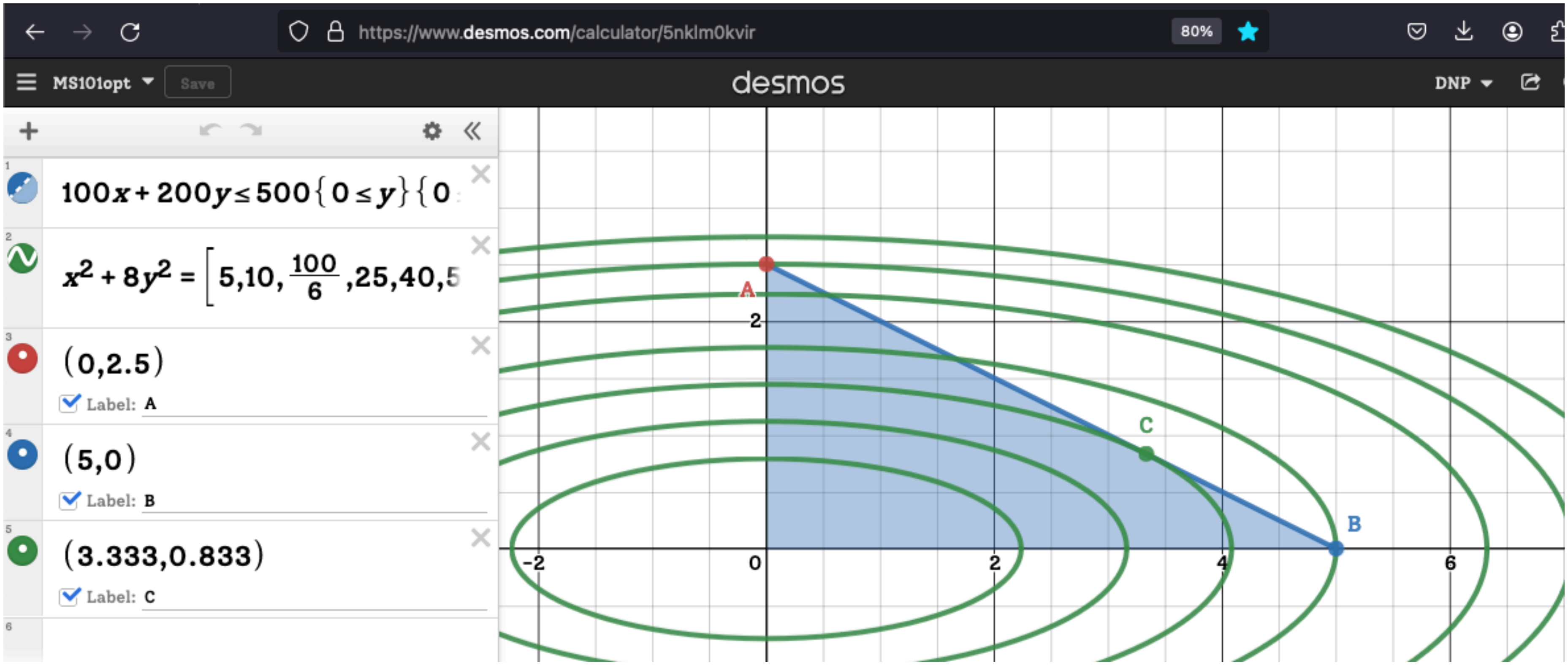
$p_1 = 100$ $\quad$ $p_2 = 200$

Monthly fruit budget R fixed, say $R = 500$ Rs

$$R = p_1 x_1 + p_2 x_2 \Rightarrow \boxed{500 = 100 x_1 + 200 x_2}$$

How many kgs of apples (x_1) & oranges (x_2) ?





https://www.desmos.com/calculator/5nklm0kvir
80%
MS101opt
Save
desmos
DNP
100x + 200y ≤ 500 {0 ≤ y} {0
x^2 + 8y^2 = [5,10, 100/6 ,25,40,5
(0,2.5)
Label: A
(5,0)
Label: B
(3.333,0.833)
Label: C
A
B
C
2
-2
0
2
4
6

$$f_A = f\left(0, \frac{R}{p_2}\right) = 0^2 + 8\frac{R^2}{p_2^2} = 8\,\frac{500^2}{200^2} = 50$$

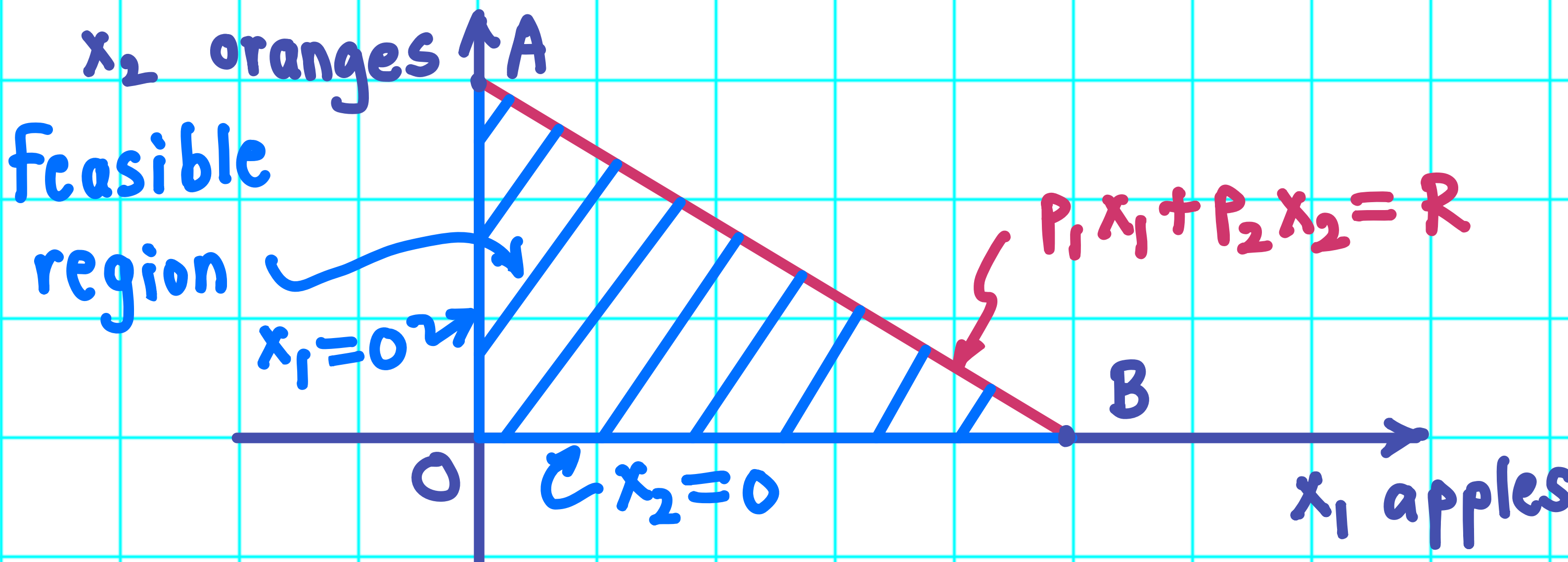
$$f_B = f\left(\frac{R}{p_1}, 0\right) = \frac{R^2}{p_1^2} + 0^2 = \frac{500^2}{100^2} = 25$$

$x_1 = 0,\ x_2 = 2.5$ kg $\Rightarrow$ 50 satisfaction units

$x_1 = 5$ kg, $x_1 = 0$ $\Rightarrow$ 25 satisfaction units

This is a 2D constrained optimization problem.

constraints: $x_1 \geq 0$, $x_2 \geq 0$, $p_1 x_1 + p_2 x_2 \leq R$



If additional constraints, they need to be considered

INFINITE DIMENSIONAL PROBLEMS

- "A function is a vector with infinite components"
- Formulation only
- Calculus of Variations / Optimal Control / Dynamic Programming

Problem 8 Rescue (Function, Inf D)

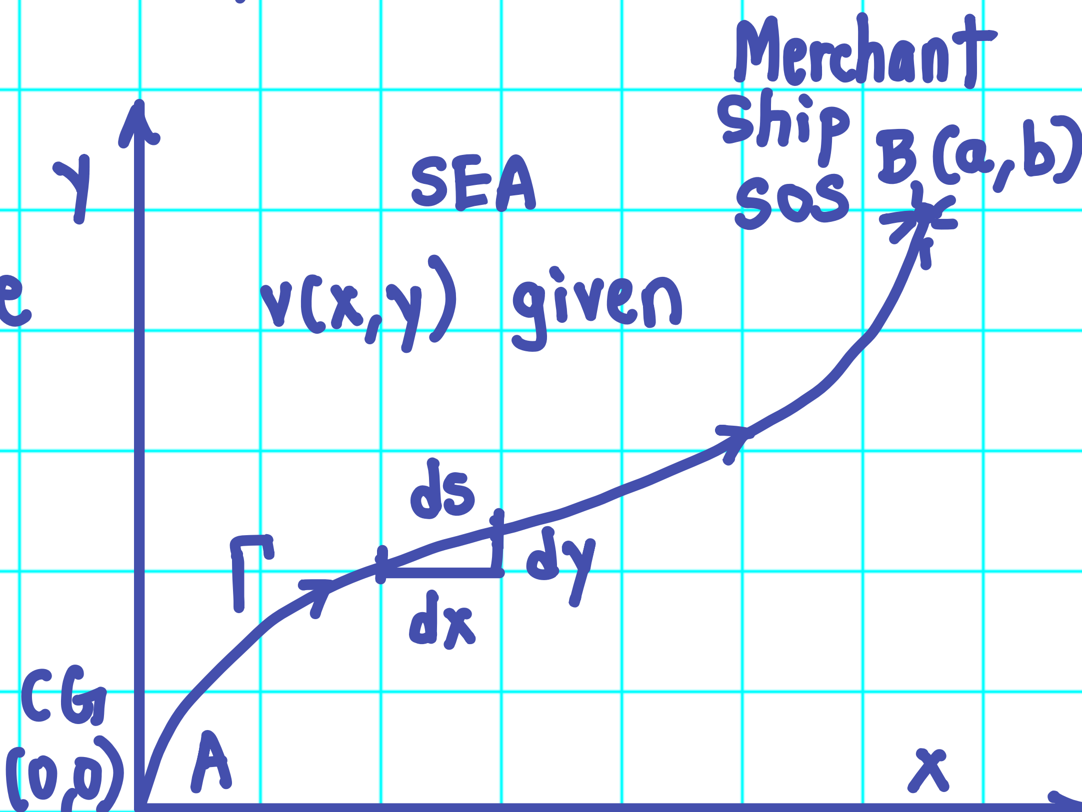
A merchant ship is under pirate attack and sends out an SOS to a coast guard vessel. Find the best route for the coast guard vessel to reach the merchant ship in minimum time.

Infinite Dimensional Optimization

Rescue Problem

Minimize transit time

$$T = \int_0^T dt = \int_A^B \frac{dt}{ds}\, ds$$



$\frac{dt}{ds} = \frac{1}{\dot{s}}$, $\dot{s} = v(x,y)$ tangential speed of boat, given

Note $y = y(x)$ unknown

$(ds)^2 = (dx)^2 + (dy)^2 \Rightarrow ds = dx\sqrt{1+y'^2}$, $y' = \frac{dy}{dx}$

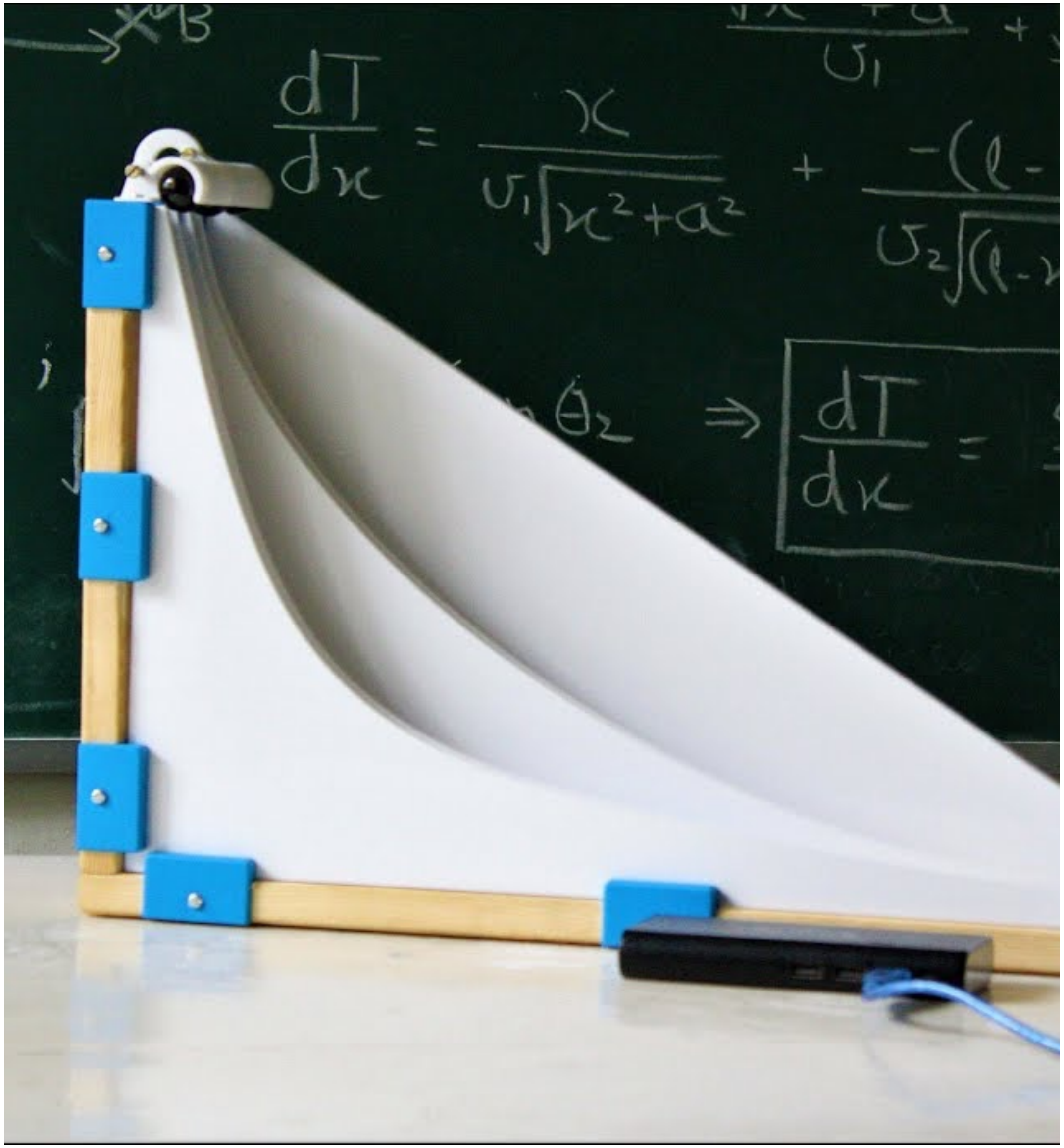
objective

$\hookrightarrow T = \int_0^a \frac{dt}{ds}\frac{ds}{dx}\,dx = \int_0^a \frac{1}{v(x,y)}\sqrt{1+y'^2}\,dx$

known dependency

Find $y(x)$ s.t. T is a min

and $y(0) = 0$, $y(a) = b$, $y(x)$ design function

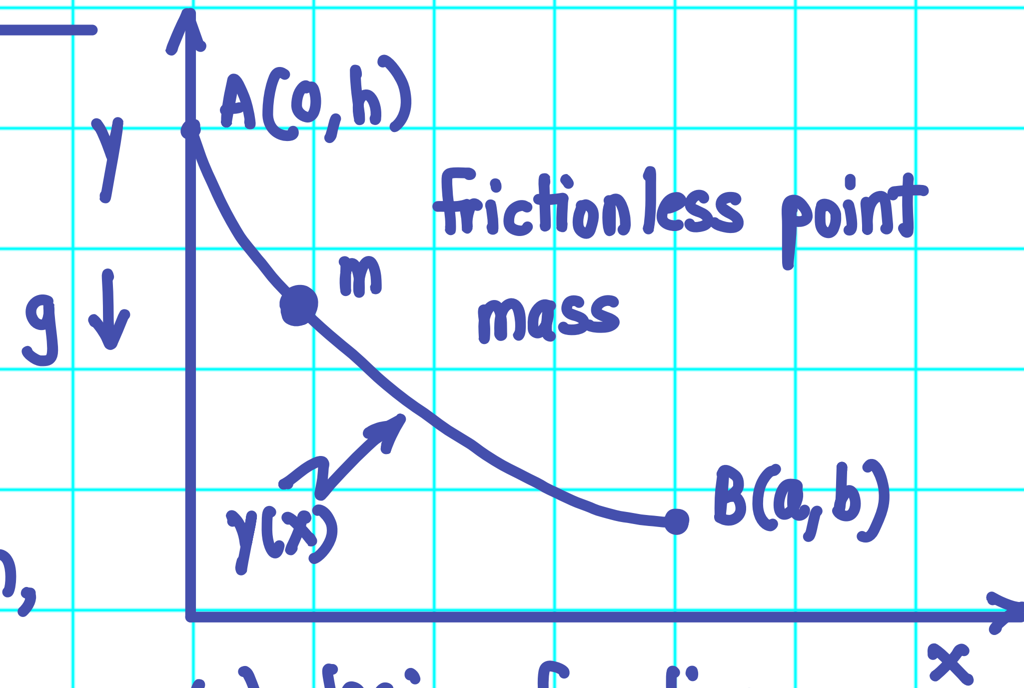


Problem 9 Descent Time/Brachistochrone (Function, Inf D)

Find the curve that minimizes the descent time for a point mass sliding down under gravity.

Minimum Descent Time

Find $y(x)$ s.t. descent time is minimized



Use energy conservation,

$$\frac{1}{2}mv^2 + mgy = mgh$$

$y(x)$ design function

$$\Rightarrow \quad v = \sqrt{2g(h-y)}$$

$$T = \int_0^T dt = \int_A^B \frac{dt}{ds}\, ds = \int_A^B \frac{1}{v}\sqrt{1+y'^2}\, dx$$

$$T = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1+y'^2}{h-y}}\, dx$$

objective function

Find $y(x)$ s.t. T is minimized

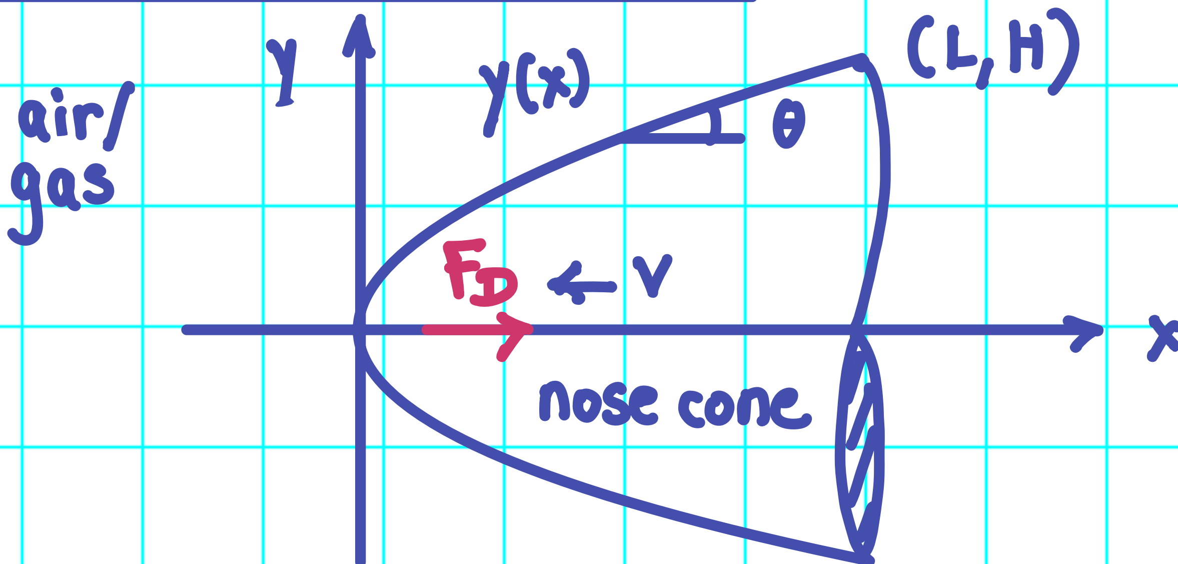
Brachistochrone

Problem 10 Minimum Drag (Function, Inf D)

Find the profile of a nose cone of an aircraft traveling through air at velocity v that minimizes the total air drag.

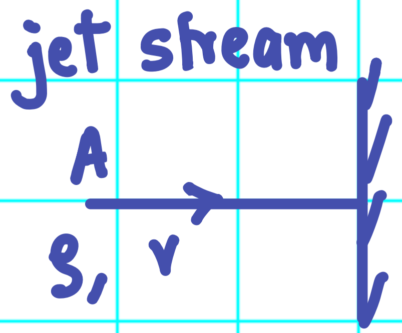
MINIMUM DRAG NOSE CONE

Find $y(x)$
s.t. drag
F_D minimum



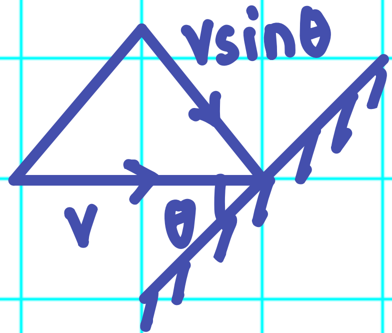
Nose cone is volume of revolution of $y(x)$ about
x-axis

Consider surface area made of infinite flat plates



Normal force on plate $= \rho A v^2$

Use impulse-momentum principle



Normal force on inclined plate

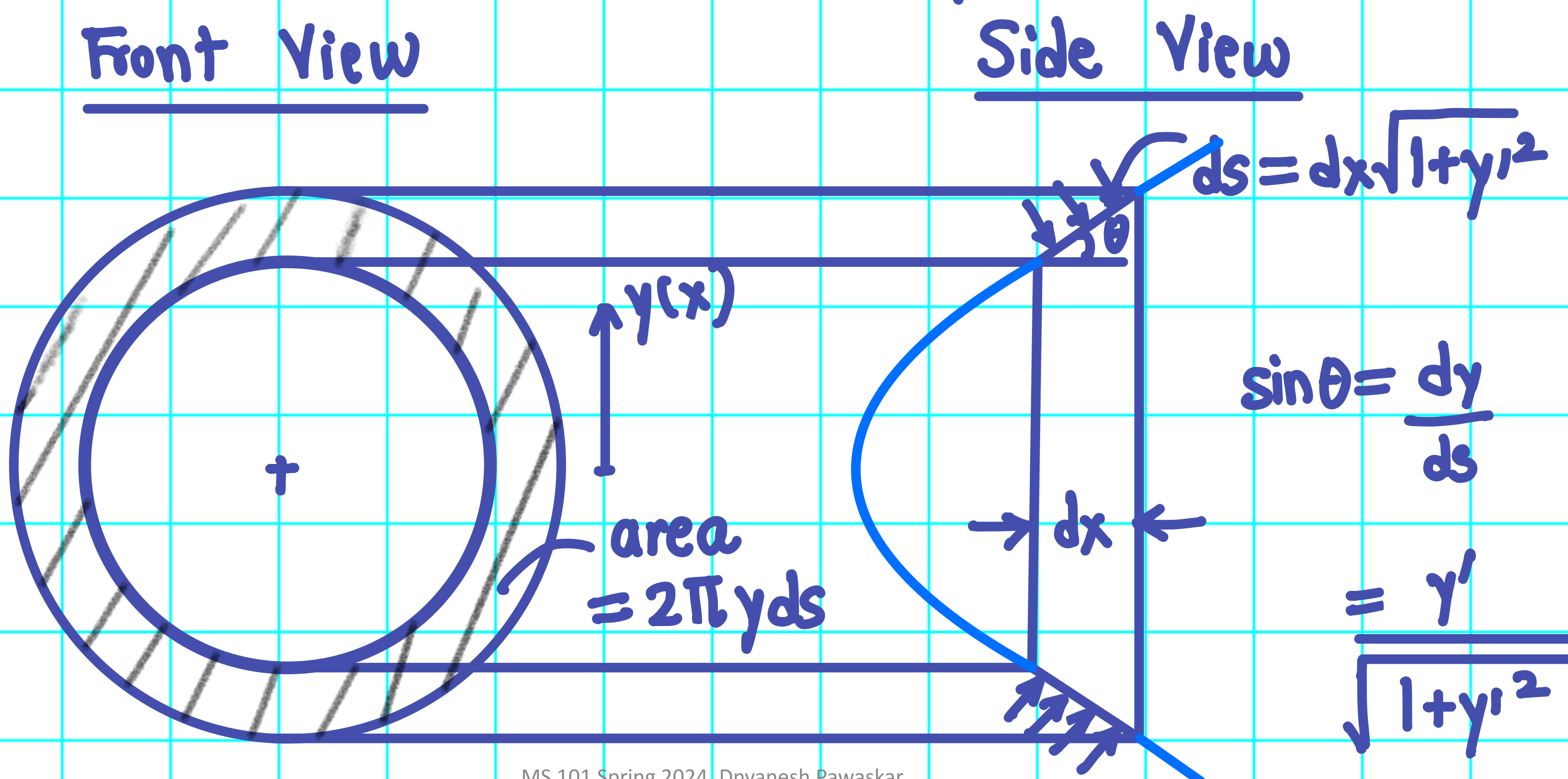
$= \rho A (v\sin\theta)^2$

Pressure drag $= \dfrac{\text{force}}{\text{area}} = \rho v^2 \sin^2\theta = p$

Horizontal component of pressure drag $= p \sin\theta$

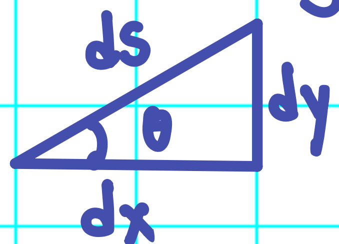
$$= \rho v^2 \sin^3\theta$$

Consider an infinitesimal strip of width dx



Horizontal force on strip $= p\, 2\pi y\, ds = \rho v^2 \sin^3\theta\, 2\pi y\, ds$

Total drag on nose cone $= 2\pi\rho v^2 \int_0^L \sin^3\theta \;\; y\sqrt{1+y'^2}\; dx$



$$F_D = 2\pi\rho v^2 \int_0^L \frac{y y'^3}{1+y'^2}\, dx$$

$\uparrow$

objective

Find $y(x)$ s.t. integral minimum