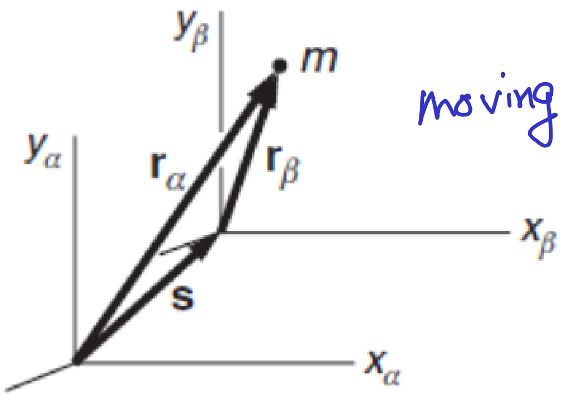
Noninertial frames



moving frame β

lab frame α

$\boxed{\text{Position of a mass } m \quad \bar{r}_\beta = \bar{r}_\alpha - \bar{S}}$

$\boxed{\begin{array}{c} \dot{\bar{r}}_\beta = \dot{\bar{r}}_\alpha - \dot{\bar{S}} \\ \text{(vel)} \\ \ddot{\bar{r}}_\beta = \ddot{\bar{r}}_\alpha - \ddot{\bar{S}} \\ \text{(accl}^{\text{n}}\text{)} \end{array}}$

Ext. Force measured in the 2-frames,

$$\bar{F}_\alpha = m\bar{a}_\alpha \quad , \quad \bar{F}_\beta = m\bar{a}_\beta \quad \text{- - - - } (1)$$

If Newton's law valid in both frames.

Typically F^{ext} will not depend on frame.

$$\bar{F}_\alpha = \bar{F}_\beta$$

(eg, gravity, coulomb)

$\boxed{\text{except } \bar{F} = q\,\bar{v} \times \bar{B} \text{ for example}}$

But then both eqns in (1) can't be true.

Because, $\bar{a}_\beta = \ddot{\bar{r}}_\beta = \ddot{\bar{r}}_\alpha - \ddot{\bar{S}} = \bar{a}_\alpha - \ddot{\bar{S}}$

$\Rightarrow$ Newton's law not valid in accl-rating frame β (if it's valid in lab/rest frame α)

But if $\ddot{\bar{S}} = 0$, $\dot{\bar{S}} = \bar{v}_0$ (const) then $\bar{a}_\beta = \bar{a}_\alpha$

Newton's law valid in both frames α & β

Newton's law holds in frames with const vel

But

Not in frames with accl^n (Noninertial frame)

If $\bar{F}_\beta \neq \bar{F}_\alpha$?

Can we make Newton's law hold in frame β

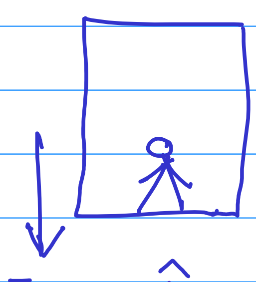
$$\bar{F}_\beta = m\bar{a}_\beta = m\left(\bar{a}_\alpha - \ddot{\bar{S}}\right) = \bar{F}_\alpha - m\ddot{\bar{S}}$$

$$\boxed{\bar{F}_\beta = \bar{F}_\alpha - m\ddot{\bar{S}}}$$

pseudo force in accelerating fram

Ex-1 In a descending elevator

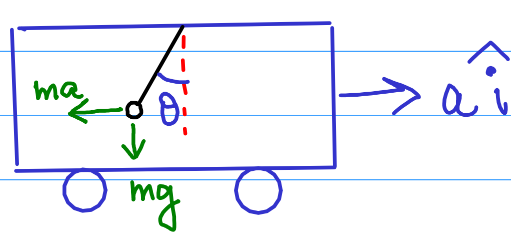
$$\underbrace{\bar{F}_\beta}_{\text{measured force}} = \underbrace{-mg\hat{k}}_{\text{ext. force}} - \underbrace{(-a\hat{k})}_{\text{pseudo force}}$$

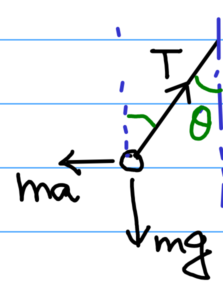


$$= \boxed{-m(g-a)\hat{k}}$$

we feel/measure reduced wt.

Ex-2

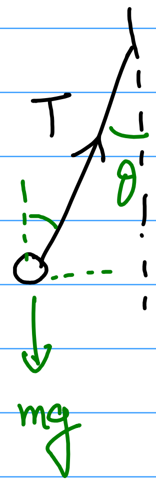




$$\left.\begin{aligned} T\cos\theta &= mg \\ T\sin\theta &= ma \end{aligned}\right\}$$ In the Non-Inertial frame (at rest) ie, no accl^n

$$\boxed{\tan\theta = a/g}$$

In the lab (inertial) frame



$$\left.\begin{aligned} T\sin\theta &= ma_x = ma \\ T\cos\theta - mg &= ma_y = 0 \end{aligned}\right\}$$ Same eqns.

$\Rightarrow$ Any motion can be described from the rest/Inertial frame using Newton's laws.

$\Rightarrow$ It can be described from the noninertial frame also if we add pseudo force to Newton's law.

Introduction

- Question: How do laws of physics change, when we change the frame of reference (coordinate system)?
- Are laws of physics same in all inertial frames of reference, i.e., frames moving with constant velocities?
- What if the frames of reference are accelerating?
- Underlying assumption will be that frames are moving with nonrelativistic velocities ($v \ll c$)
- For relativistic velocities, correct theory is Einstein's Special theory of relativity.

Rotating frames of reference

- Rotating frames of references are non-inertial
- Because any article executing circular motion experiences centripetal acceleration $\Rightarrow$ So the frame accelerates
- Next, we develop the theory of rotating frames of references
- But, before that, we illustrate the vector nature of angular velocity

Vector nature of angular velocity

- We denoted the position of a particle as a vector

$$\mathrm{r} = x\hat{\mathrm{i}} + y\hat{\mathrm{j}} + z\hat{\mathrm{k}}$$

- Can we similarly specify the angular position of a particle

$$\theta = \theta_x\hat{\mathrm{i}} + \theta_y\hat{\mathrm{j}} + \theta_z\hat{\mathrm{k}}\,?$$

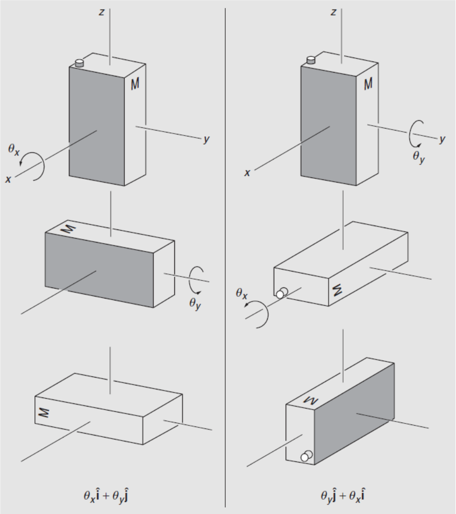
- The answer is no because such an expression does not satisfy commutative law of vector addition

$$\theta_1 + \theta_2 \neq \theta_2 + \theta_1$$

- Let us rotate a block first around the x axis, and then around the y axis. Compare that to the same operations performed in the reverse order

Non-commutative nature of finite rotations

- Consider those two rotations, with each one of them being $\pi/2$



- Clearly $\theta_x\hat{\imath} + \theta_y\hat{\jmath} \neq \theta_y\hat{\jmath} + \theta_x\hat{\imath}$

- On the other hand, one can verify that infinitesimal rotations commute to first order terms

$$\boxed{\Delta\theta_x\hat{\imath}+\Delta\theta_y\hat{\jmath}\approx\Delta\theta_y\hat{\jmath}+\Delta\theta_x\hat{\imath}}$$

Needs more work to prove (Later)

- Thus, infinitesimal rotations can be represented as vectors
- Because angular velocity is defined in terms of infinitesimal rotations

$$\boxed{\omega=\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t},}$$

- Angular velocity can be denoted as a vector

$$\omega=\omega_x\hat{\imath}+\omega_y\hat{\jmath}+\omega_z\hat{k}$$

However static rotations are not unique

- And, in general,

$$\boxed{\omega=\omega\hat{n},}$$

where $\hat{n}$ is the direction of the axis of rotation, and ω is the magnitude of the angular velocity.

Relation between linear velocity and angular velocity

- It is obvious that angular velocity ω, will give rise to linear velocity v
- What is the mathematical relation between the two?

Rate of change of a general rotating vector

- Previous discussion was about when position vector $\mathbf{r}$ was precessing with a constant angular velocity $\boldsymbol{\omega}$, about the axis in direction $\hat{\mathbf{n}}$.
- But the same arguments will hold if, instead of $\mathbf{r}$, some general vector $\mathbf{A}$, was precessing with a constant angular velocity $\boldsymbol{\omega}$, about the axis in direction $\hat{\mathbf{n}}$.
- Then, we will have

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}.$$

Can have a more general proof

- This is a very important general relation about the rate of change of rotating vectors.
- Let $\mathbf{A} = \mathbf{v}$, then using above, we get the expression for acceleration of a rotating particle

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = \boldsymbol{\omega} \times \mathbf{v} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \text{ (centripetal acceleration)}$$

Physics in Rotating Reference Frames

- Consider a general vector A which is changing with time
- When observed from an inertial frame, its rate of change is $\left(\frac{dA}{dt}\right)_{in}$
- Suppose we have a non-inertial frame of reference which is rotating with a constant angular velocity $\boldsymbol{\Omega}$
- What is the rate of change of A, i.e., $\left(\frac{dA}{dt}\right)_{rot}$, with respect to the rotating frame?
- Let $\hat{i}, \hat{j}, \hat{k}$ be the basis vectors of the inertial frame
- And $\hat{i}', \hat{j}', \hat{k}'$ be the basis vectors of the rotating frame

- This means that $\hat{\mathrm{i}}', \hat{\mathrm{j}}', \hat{\mathrm{k}}'$ are rotating w.r.t. the inertial frame with angular velocity $\boldsymbol{\Omega}$.
- At a given point in time, A in two frames can be expressed as

$$\mathrm{A} = A_x\hat{\mathrm{i}} + A_y\hat{\mathrm{j}} + A_z\hat{\mathrm{k}}\ (\text{inertial})$$

$$\mathrm{A} = A'_x\hat{\mathrm{i}}' + A'_y\hat{\mathrm{j}}' + A'_z\hat{\mathrm{k}}'\ (\text{rotating})$$

- Therefore, in inertial frame

$$\left(\frac{d\mathrm{A}}{dt}\right)_{in} = \frac{dA_x}{dt}\hat{\mathrm{i}} + \frac{dA_y}{dt}\hat{\mathrm{j}} + \frac{dA_z}{dt}\hat{\mathrm{k}}$$

Physics in rotating frames contd.

- We can also compute the time derivative of the second expression of A
- Keeping in mind that not only the components of A, but the basis vectors $\hat{\mathrm{i}}', \hat{\mathrm{j}}', \hat{\mathrm{k}}'$ are changing with time due to rotation
- Therefore, the rate of change of A will be

$$\boxed{\left(\frac{d\mathrm{A}}{dt}\right)_{in} = \frac{dA'_x}{dt}\hat{\mathrm{i}}' + \frac{dA'_y}{dt}\hat{\mathrm{j}}' + \frac{dA'_z}{dt}\hat{\mathrm{k}}' + A'_x\frac{d\hat{\mathrm{i}}'}{dt} + A'_y\frac{d\hat{\mathrm{j}}'}{dt} + A'_z\frac{d\hat{\mathrm{k}}'}{dt}}$$

- Because vectors $\hat{\mathrm{i}}', \hat{\mathrm{j}}', \hat{\mathrm{k}}'$ are rotating with angular velocity $\mathbf{\Omega}$, w.r.t. to the inertial frame

$$\boxed{\begin{aligned}\frac{d\hat{\mathrm{i}}'}{dt} &= \mathbf{\Omega}\times\hat{\mathrm{i}}' \\ \frac{d\hat{\mathrm{j}}'}{dt} &= \mathbf{\Omega}\times\hat{\mathrm{j}}' \\ \frac{d\hat{\mathrm{k}}'}{dt} &= \mathbf{\Omega}\times\hat{\mathrm{k}}'\end{aligned}}$$

Rotating frames

- So that

$$\left(\frac{dA}{dt}\right)_{in} = \frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}' + \boldsymbol{\Omega} \times \left(A'_x\hat{i}' + A'_y\hat{j}' + A'_z\hat{k}'\right)$$

- First three terms denote the rate of change of A as seen in the rotating frame, i.e.,

$$\frac{dA'_x}{dt}\hat{i}' + \frac{dA'_y}{dt}\hat{j}' + \frac{dA'_z}{dt}\hat{k}' = \left(\frac{dA}{dt}\right)_{rot}$$

- Leading to

$$\left(\frac{dA}{dt}\right)_{in} = \left(\frac{dA}{dt}\right)_{rot} + \boldsymbol{\Omega} \times A$$

- Because A is a general vector, the previous formula can be symbolically expressed as

$$\left(\frac{d}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} + \boldsymbol{\Omega} \times$$

Velocity and acceleration in a rotating frame

- Taking $\mathbf{A} = \mathbf{r}$, we have

$$\left(\frac{d\mathbf{r}}{dt}\right)_{in} = \left(\frac{d\mathbf{r}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{r}$$
$$\Longrightarrow \boxed{\mathbf{v}_{in} = \mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}}$$

if $\bar{A} = \bar{\Omega}$

$$\left(\frac{d\Omega}{dt}\right)_{in} = \left(\frac{d\Omega}{dt}\right)_{rot} + \text{zero}$$

- On taking $\mathbf{A} = \mathbf{v}_{in}$, we get

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{rot} + \boldsymbol{\Omega} \times \mathbf{v}_{in}$$
$$\Longrightarrow \left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d}{dt}\right)_{rot} (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\mathbf{v}_{rot} + \boldsymbol{\Omega} \times \mathbf{r})$$

Acceleration in Rotating Frame

- Or

$$\left(\frac{d\mathbf{v}_{in}}{dt}\right)_{in} = \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot} + \left(\frac{d(\boldsymbol{\Omega}\times\mathbf{r})}{dt}\right)_{rot} + \boldsymbol{\Omega}\times\mathbf{v}_{rot} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

$$= \left(\frac{d\mathbf{v}_{rot}}{dt}\right)_{rot} + \boldsymbol{\Omega}\times\mathbf{v}_{rot} + \boldsymbol{\Omega}\times\mathbf{v}_{rot} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

$$\Longrightarrow \boxed{\mathbf{a}_{in} = \mathbf{a}_{rot} + 2\boldsymbol{\Omega}\times\mathbf{v}_{rot} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})}$$

Above, we used that condition that $\boldsymbol{\Omega} =$ constant, so that $\dot{\boldsymbol{\Omega}} = 0$.

- Thus, the acceleration as seen in the rotating frame is

$$\boxed{\mathbf{a}_{rot} = \mathbf{a}_{in} - 2\boldsymbol{\Omega}\times\mathbf{v}_{rot} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})}$$

Pseudo Forces in Rotating Frames

- Multiplying the previous equation by m on both sides, and using notations $\mathbf{F}_{rot} = m\mathbf{a}_{rot}$ and $\mathbf{F} = m\mathbf{a}_{in}$, we have

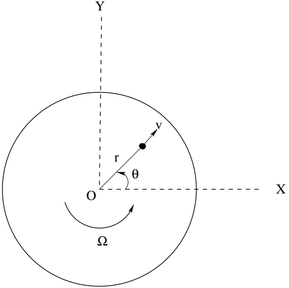
$$
\begin{aligned}
\mathbf{F}_{rot} &= \mathbf{F} - 2m\boldsymbol{\Omega} \times \mathbf{v}_{rot} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= \mathbf{F} + \mathbf{F}_{coriolis} + \mathbf{F}_{centrifugal} \\
&= \mathbf{F} + \mathbf{F}_{fict}
\end{aligned}
$$

- Where $\mathbf{F}$ is the real force acting on the particle, while $\mathbf{F}_{coriolis}$ and $\mathbf{F}_{centrifugal}$ are pseudo (fictitious) forces

$$
\begin{aligned}
\mathbf{F}_{coriolis} &= -2m\boldsymbol{\Omega} \times \mathbf{v}_{rot} \\
\mathbf{F}_{centrifugal} &= -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})
\end{aligned}
$$

Probing Centrifugal and Coriolis Forces

- Consider a circular plank (say a merry-go-round) rotating with an angular velocity $\mathbf{\Omega} = \Omega\hat{k}$, with a mass m as shown



- Mass m is moving with a velocity v which is in the radial direction w.r.t. plank
- Location of the particle at a given instant is r, w.r.t. to rotating coordinate system
- What are the magnitudes and directions of pseudo forces?

Centrifugal Force

- Centrifugal force can be computed as

$$\begin{aligned}
\mathbf{F}_{centrifugal} &= -m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= -m\Omega^2 r\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{r}}) \\
&= -m\Omega^2 r\hat{\mathbf{k}} \times \hat{\theta} \\
&= -m\Omega^2 r(-\hat{\mathbf{r}}) \\
&= m\Omega^2 \mathbf{r}.
\end{aligned}$$

Thus centrifugal force has the same magnitude as the centripetal force, but opposite direction, as expected of a pseudo force.

Coriolis Force

- Coriolis force exists only when the particle moves with respect to the rotating frame. Here

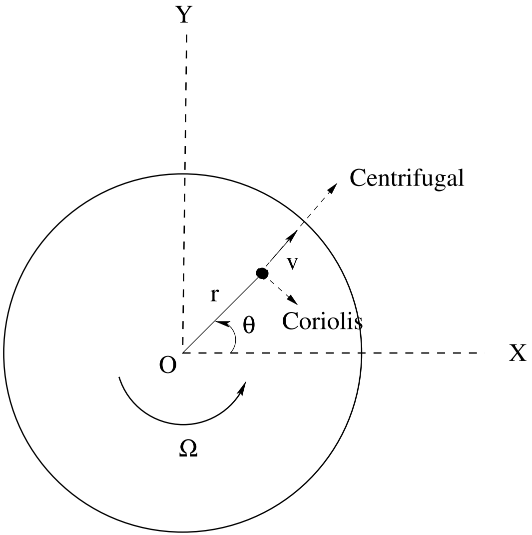
$$\mathbf{v}_{rot} = v\hat{\mathbf{r}}.$$

- Therefore,

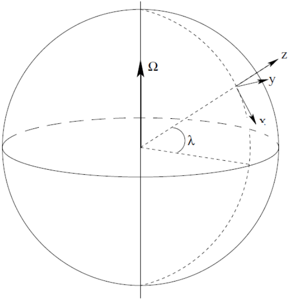
$$\begin{aligned}\mathbf{F}_{coriolis} &= -2m\boldsymbol{\Omega} \times \mathbf{v}_{rot} \\ &= -2m\Omega v(\hat{\mathbf{k}} \times \hat{\mathbf{r}}) \\ &= -2m\Omega v\hat{\theta}\end{aligned}$$

Coriolis and Centrifugal Forces

Thus, finally the direction of the forces



- Earth's Angular Velocity in a Non-Intertial Frame



- Here x points to south, y to east, and z is radially outwards (vertically above from earth), and λ is latitude angle
- In this frame

$$\mathbf{\Omega} = -\Omega\cos\lambda\hat{\mathrm{i}} + \Omega\sin\lambda\hat{\mathrm{k}}$$

Coriolis Force on a Falling Object

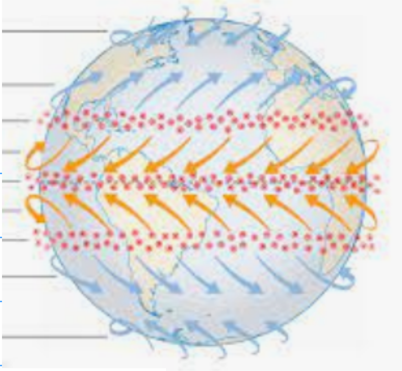
- If a particle of mass m is falling vertically down, at a given instant with velocity v, then

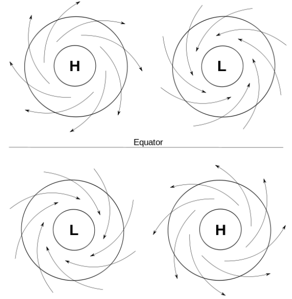
$$\mathrm{v} = -v\hat{\mathrm{k}}$$

- Then Coriolis force on it due to Earth's rotation is

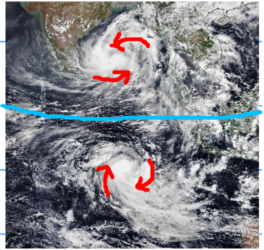
$$\mathrm{F}_c = -2m(\boldsymbol{\Omega} \times \mathrm{v}) = -2m\Omega \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ -\cos\lambda & 0 & \sin\lambda \\ 0 & 0 & -v \end{vmatrix} = 2mv\Omega\cos\lambda\hat{\mathrm{j}}$$

- Thus, the object will experience a force towards east, and will get deviated in that direction
- Another example: away from equator, wind flow becomes circular due to Coriolis force
- Note $\mathrm{F}_c \perp \mathrm{v}_{rot}$, so it will lead to a circular motion





Note: the dir of rotatn depend on the dir of wind flow. On the left, H=high prsr zone L=low prsr zone
On the right the centre of the cyclones have L
Note, India on the top left & equator in blue



$$\boxed{\bar{r}' = \bar{r}\cos\phi + \hat{n}(\hat{n}\cdot\bar{r})(1-\cos\phi) + (\hat{n}\times\bar{r})\sin\phi}$$ proved below

Infinitesimal Rotn $\phi = d\phi \to 0$. $\cos(d\phi) = 1 - \frac{(d\phi)^2}{2}$, $\sin(d\phi) \simeq d\phi$.

$$\bar{r}' = \bar{r} + (\hat{n}\times\bar{r})\,d\phi.$$

$$d\bar{r} = \bar{r}' - \bar{r} = (d\phi\hat{n})\times r = d\bar{\Omega}\times\bar{r}$$ Active Rotn

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{\Omega}}{dt}\times\bar{r} = \bar{\omega}\times\bar{r},\quad \boxed{\bar{\omega} = \frac{d\bar{\Omega}}{dt}}$$ Angular velocity

From the $\tilde{R}$ matrix $\bar{r}' = \tilde{R}\bar{r}$ we get (for $d\phi \to 0$)

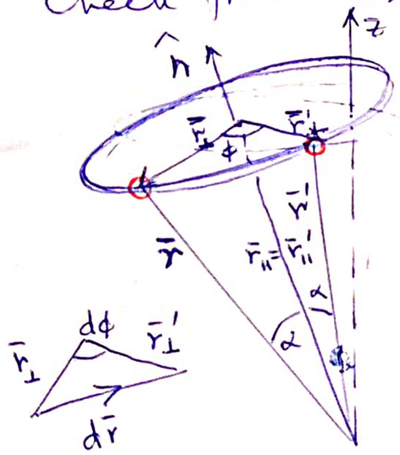
$$\tilde{R} \simeq \begin{pmatrix} 1 & -n_z d\phi & n_y d\phi \\ n_z d\phi & 1 & -n_x d\phi \\ -n_y d\phi & n_x d\phi & 1 \end{pmatrix} = \underbrace{\mathbb{1}}_{\text{Unity}} + d\phi\underbrace{\begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix}}_{\text{Antisym}}$$

$$\bar{r}' = \mathbb{1}\,\bar{r} + d\bar{\Omega}\times\bar{r}$$ when $d\bar{\Omega} = d\phi\,\hat{n}$

$$\boxed{\delta\bar{r} = d\bar{\Omega}\times\bar{r}}$$ Infinitesimal Rotation

$$\delta x = d\Omega_y z - d\Omega_z y.$$
$$\delta y = d\Omega_z x - d\Omega_x z$$
$$\delta z = d\Omega_x y - d\Omega_y x.$$

Check from Geometry. $\boxed{\bar{r}_{\parallel} = (\hat{n}\cdot\bar{r})\hat{n},\quad \bar{r}_{\perp} = \bar{r} - \bar{r}_{\parallel}}$ - Proof



$|\bar{r}_{\perp}| = |\bar{r}_{\perp}'|$

$|d\bar{r}| = |\bar{r}_{\perp}|d\phi = r\sin\alpha\, d\phi$

$$d\bar{r} = d\phi\,\hat{n}\times\bar{r} = d\bar{\Omega}\times\bar{r}$$

this puts toward $\bar{r}'$ ~~also~~ $\bar{r}\times d\bar{\Omega}$ will ~~be toward~~ be opposite.

Magnitude of $\bar{r}_{\perp}'$

$$\bar{r}' = \bar{r}_{\parallel} + \underbrace{(\cos\phi\,\overset{\text{Unit vector}}{\hat{r}_{\perp}} + \sin\phi\,\overset{\text{unit vector}}{\hat{n}\times\hat{r}_{\perp}})}_{\text{Splitting } \bar{r}_{\perp}' \text{ into components.}} r\sin\alpha$$

$$= \bar{r}_{\parallel} + \cos\phi\,\underbrace{\hat{r}_{\perp}\, r\sin\alpha}_{\bar{r}_{\perp}} + \sin\phi\,\hat{n}\times\underbrace{\hat{r}_{\perp}\, r\sin\alpha}_{\bar{r}_{\perp}}$$

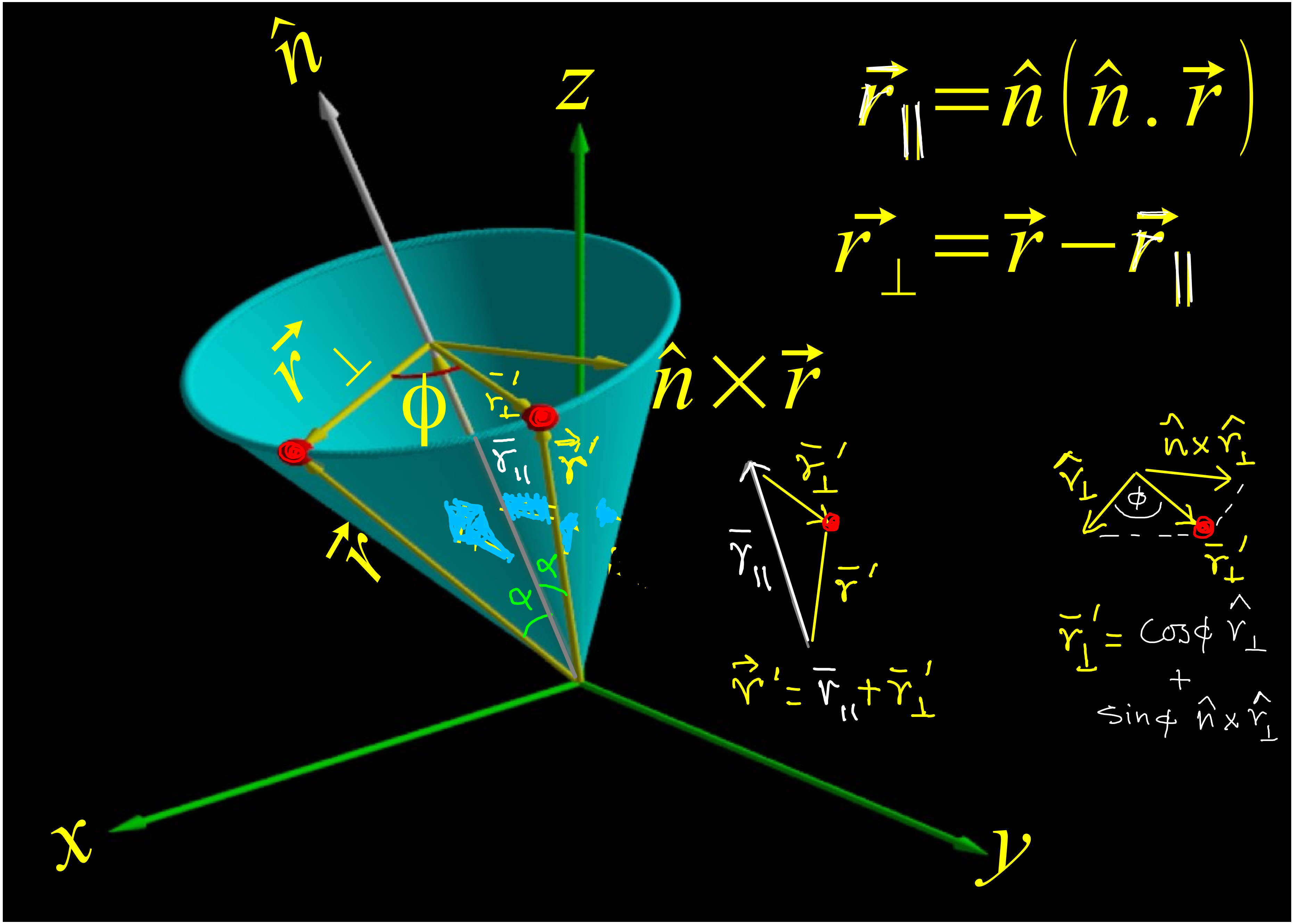
$$\cos\phi(\bar{r} - \bar{r}_{\parallel}) \qquad \sin\phi\,\hat{n}\times(\bar{r} - \bar{r}_{\parallel})$$

But $\hat{n}, \bar{r}_{\parallel}$ are $\parallel$

$$\cancel{\hat{n}(\hat{n}\cdot\bar{r})} +$$

$$= \bar{r}\cos\phi + \bar{r}_{\parallel}(1-\cos\phi) + (\hat{n}\times\bar{r})\sin\phi$$

$R(\hat{n},\phi)$:: Rotating a point about an axis $\hat{n}$ by angle ϕ



$$\vec{r}\,' \equiv R(\hat{n},\phi)\vec{r} = \vec{r}\cos\phi + \hat{n}(\hat{n}.\vec{r})[1-\cos\phi] + (\hat{n}\times\vec{r})\sin\phi$$

$R(\hat{n}, \phi)$:: Rotating a point about an axis $\hat{n}$ by angle ϕ

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi + n_x^2(1-\cos\phi) & n_x n_y(1-\cos\phi) - n_z\sin\phi & n_x n_z(1-\cos\phi) + n_y\sin\phi \\ n_x n_y(1-\cos\phi) + n_z\sin\phi & \cos\phi + n_y^2(1-\cos\phi) & n_y n_z(1-\cos\phi) - n_x\sin\phi \\ n_x n_z(1-\cos\phi) - n_y\sin\phi & n_y n_z(1-\cos\phi) + n_x\sin\phi & \cos\phi + n_z^2(1-\cos\phi) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

In the limit of $\phi \to 0$ the matrix will become an Identity matrix + infinitesimal anti-symmetric matrix. That is the part which can be represented by a cross product