Eqn. of orbits from $E = \frac{\mu \dot{r}^2}{2} + \frac{l^2}{2\mu r^2} - \frac{c}{r}$ ⇐ grav. potn. $V(r) = -\frac{c}{r}$

Magic transform $u = 1/r$. $\Rightarrow \frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\left(\frac{l}{\mu}u^2\right) \leftarrow \text{from } \mu r^2\dot{\theta} = l$$

$$= -\frac{l}{\mu}\frac{du}{d\theta}$$

Using, $u = 1/r$ & $\dot{r} = -\frac{l}{\mu}\frac{du}{d\theta}$,

$$E = \frac{l^2}{2\mu}\left(\frac{du}{d\theta}\right)^2 + \frac{l^2}{2\mu}u^2 - cu = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - cu \quad \text{--- (1)}$$

E - conserved, $\frac{dE}{dt} = 0$, $\Rightarrow \frac{dE}{d\theta}\frac{d\theta}{dt} = 0$, but $\dot{\theta} \neq 0$ in general, $\because r \neq 0$ in $\mu r^2\dot{\theta} = l$

$$\Rightarrow \frac{dE}{d\theta} = 0.$$

Apply $\frac{d}{d\theta}$ on (1) $\Rightarrow \frac{l^2}{2\mu}\left[2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta}\right] - c\frac{du}{d\theta} = 0$

Note: (r,θ) no more indp. & hence $\frac{du}{d\theta} \neq 0$

we already used one EoM ; $\mu r^2\dot{\theta} = l$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{\mu c}{l^2}} \quad \text{--- (2)}$$

this is equ. to a shifted oscillator, but ~~not~~ not in 't' but in θ.
so the oscillatn is not $u(t)$ but $u(\theta)$

Check: if ~~θ=t, then~~ $c = 0$ then $\frac{d^2u}{d\theta^2} = -u \Rightarrow u = \pm u_0 \cos(\theta - \theta_0)$ ± depends on initial Cond.

for $c \neq 0$, define $u_1 = u - \frac{\mu c}{l^2}$, $\frac{d^2u_1}{d\theta^2} = \frac{d^2u}{d\theta^2}$

Then we hv, $\frac{d^2u_1}{d\theta^2} = -u_1 \Rightarrow u_1 = \pm\bar{u}_1 \cos(\theta - \theta_0)$
const.

$$\Rightarrow u = \frac{\mu c}{l^2} \pm \bar{u}_1 \cos(\theta - \theta_0)$$

$$u = \frac{\mu c}{\ell^2} \pm \bar{u}_1 \cos(\theta - \theta_0)$$

$$\frac{1}{r} = \frac{1}{r_0} \pm \bar{u}_1 \cos(\theta - \theta_0) \quad , \quad \text{define } \bar{u}_1 = \frac{\epsilon}{r_0}$$

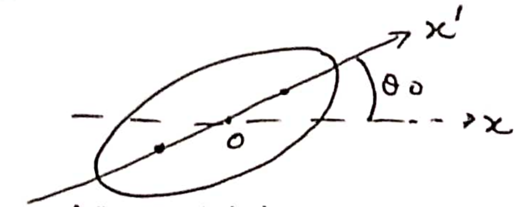
$$\frac{1}{r} = \frac{1}{r_0}\left[1 \pm \epsilon \cos(\theta - \theta_0)\right]$$

{ henceforth we choose the convention $1 - \epsilon \cos(\theta - \theta_0)$ & see the meaning of it later

$$\boxed{r = \frac{r_0}{1 - \epsilon \cos(\theta - \theta_0)}} \quad \text{--- (2)}$$

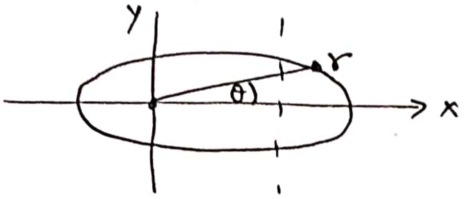
−ve sign chosen

We can rotate co-ord frame (r, θ) to get rid of θ_0



ϵ turns out to be eccentricity of the conic section which $r = \frac{r_0}{1 - \epsilon \cos\theta}$ represent.

for $1 > \epsilon > 0$ it's an ellipse



Use $r = \sqrt{x^2 + y^2}$, $\cos\theta = \frac{x}{r}$

(3) $\Rightarrow$ $r = \frac{r_0}{1 - \epsilon\cos\theta} \Rightarrow r = \frac{r_0}{1 - \epsilon\frac{x}{r}}$, $r = \frac{r_0 r}{r - \epsilon x}$, $r \neq 0$

$\downarrow$

~~$x^2(1-\epsilon^2)$~~ $\cancel{\Leftarrow r^2 - 2r\epsilon x + \epsilon^2 x^2 = r_0^2} \Leftarrow \cancel{(r - \epsilon x)^2 = r_0^2}$

$x^2 + y^2 - 2\epsilon x r_0 - \epsilon^2 x^2 = r_0 \Leftarrow r^2 = (r_0 + \epsilon x)^2 \Leftarrow r - \epsilon x = r_0.$

$\hookrightarrow x^2(1 - \epsilon^2) - 2 r_0 \epsilon x + y^2 = r_0^2$

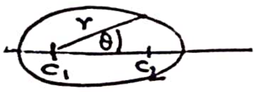
Case-1 : $\epsilon = 1 \rightarrow$ parabola

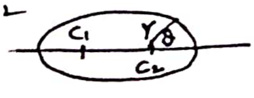
Case-2 $\epsilon > 1 \Rightarrow$ hyperbola

Case 3 $\epsilon = 0 \Rightarrow$ Circle

Case 4 $0 < \epsilon < 1 \Rightarrow$ ellipse $\rightarrow$ for −ve sign in Eq. 2 the origin at C_1

for +ve ------ the origin at C_2





Kepler's 3rd law $A = \pi a b$ (ellipse)

$\underbrace{\frac{dA}{dt} = \ell/2\mu}_{\text{using 2nd law}} \Rightarrow \int dA = \frac{\ell}{2\mu}\int dt \Rightarrow A = \frac{\ell}{2\mu}T$, $T = \frac{2\mu}{\ell}\pi a b.$

Now $r_0 = \frac{\ell^2}{\mu c}$, $a = \frac{r_0}{1-\epsilon^2}$, $b = \frac{r_0}{\sqrt{1-\epsilon^2}}$, $r_{min} = \frac{r_0}{1+\epsilon}$, $r_{max} = \frac{r_0}{1-\epsilon}$

~~$E = \frac{\mu c^2}{2\ell^2}(\epsilon^2 - 1) < 0$ ellipse~~

P.T.O.

$$T = \cancel{\frac{2\pi}{\ell}}\ \frac{2\mu}{\ell}\,\pi a b = \frac{2\mu}{\ell}\,\pi\,\frac{r_0^2}{(1-\epsilon^2)^{3/2}}$$

$$\left.\begin{aligned} \text{semi major } a &= \frac{r_0}{1-\epsilon^2} \\ \text{minor } b &= \frac{r_0}{\sqrt{1-\epsilon^2}} \end{aligned}\right\}$$

$$r_0 = \frac{\ell^2}{\mu C}$$

$$T^2 = \left(\frac{2\mu}{\ell}\right)^2 \pi^2\, r_0 \left(\frac{r_0}{1-\epsilon^2}\right)^3$$

$$= \frac{4\mu^2\pi^2}{\ell^2}\left(\frac{\ell^2}{\mu C}\right) a^3 = \frac{4\mu\pi^2}{C}\, a^3$$

$$= \frac{\mu}{2C}\,\pi^2\,\underbrace{(2a)}_{\text{major axis}}{}^3$$

Kepler's 3rd law $\Rightarrow$ $\boxed{T = \sqrt{\frac{\mu}{2C}}\,\pi\,(2a)^{3/2}}$ for elliptical orbit

proof for $\boxed{E = \frac{\mu C^2}{2\ell^2}(\epsilon^2-1)}$

Recall $E = \frac{\ell^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - Cu$

Use $r = \frac{r_0}{1-\epsilon\cos\theta}$, $\Rightarrow u = \frac{1-\epsilon\cos\theta}{r_0} \Rightarrow \frac{du}{d\theta} = \frac{\epsilon\sin\theta}{r_0}$

$$E = \frac{\ell^2}{2\mu}\left[\frac{\epsilon^2\sin^2\theta}{r_0^2} + \frac{1}{r_0^2}\left(1 - \frac{2\epsilon\cos\theta}{r_0} + \epsilon^2\cos^2\theta\right)\right] - Cu$$

$$= \frac{\ell^2}{2\mu}\left[\frac{\epsilon^2}{r_0^2}\right] + \underbrace{\frac{\ell^2}{2\mu}\frac{1}{r_0^2}}_{C/r_0}(1-2\epsilon\cos\theta) - \frac{C}{r_0}(1-\epsilon\cos\theta)$$

using $r_0 = \frac{\ell^2}{\mu C}$

$$= \frac{\epsilon^2}{2}\frac{\mu C^2}{\ell^2} + \frac{C}{r_0}\left(\frac{1}{2} - \epsilon\cos\theta\right) - \frac{C}{r_0}(1-\epsilon\cos\theta)$$

$$= \frac{\epsilon^2}{2}\frac{\mu C^2}{\ell^2} - \frac{C}{2r_0} = \frac{1}{2}\left[\frac{\epsilon^2\mu C^2}{\ell^2} - \frac{\mu C^2}{\ell^2}\right] = \boxed{\frac{\mu C^2}{2\ell^2}(\epsilon^2-1)}$$ for any orbit