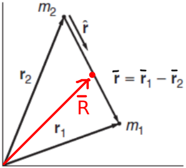
Decoupling equations of motion

- Both the equations above are coupled, because both depend upon r_1 and r_2.
- In order to decouple them, we replace $\bar{r}_1$ and $\bar{r}_2$ by $\bar{r} = \bar{r}_1 - \bar{r}_2$ (called relative coordinate), and center of mass coordinate R

$$\bar{R} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{m_1 + m_2}$$



- Now

$$\ddot{R} = \frac{m_1\ddot{r}_1 + m_2\ddot{r}_2}{m_1 + m_2} = \frac{f\hat{r} - f\hat{r}}{m_1 + m_2} = 0$$
$$\implies \bar{R} = \bar{R}_0 + \bar{V}t,$$

above R_0 is the initial location of center of mass, and V is the center of mass velocity.

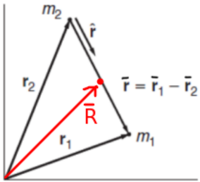
Decoupling equations of motion...

- This equation physically means that the center of mass of this two-body system is moving with constant velocity, because there are no external forces on it.
- We also obtain

$$\ddot{\mathrm{r}}_1 - \ddot{\mathrm{r}}_2 = f(r)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{\mathrm{r}}$$

$$\implies \ddot{\mathrm{r}} = \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(r)\hat{\mathrm{r}}$$

$$\mu\ddot{\mathrm{r}} = f(r)\hat{\mathrm{r}},$$



where $\mu = \frac{m_1 m_2}{m_1 + m_2}$, is called reduced mass.

We keep track of only one variable $\bar{\mathrm{r}}$, instead of $\bar{\mathrm{r}}_1$ and $\bar{\mathrm{r}}_2$, separately.

Reduction of two-body problem to one body problem

- Note that this final equation is entirely in terms of relative coordinate r
- It is an effective equation of motion for a single particle of mass μ, moving under the influence of force $f(r)\hat{r}$.
- There is just one coordinate (r) involved in this equation of motion
- Thus the two body problem has been effectively reduced to a one-body problem
- This separation was possible only because the two-body force is central, i.e., along the line joining the two particles
- In order to solve this equation, we need to know the nature of the force, i.e., $f(r)$.

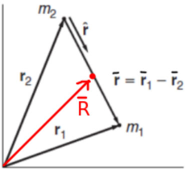
Two-body central force problem continued

- We have already solved the equation of motion for the center-of-mass coordinate R
- Therefore, once we solve the "reduced equation", we can obtain the complete solution by solving the two equations

$$\mathrm{R} = \frac{m_1 \mathrm{r}_1 + m_2 \mathrm{r}_2}{m_1 + m_2}$$

$$\mathrm{r} = \mathrm{r}_1 - \mathrm{r}_2$$

- Leading to



In rest frame

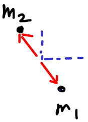
$$\bar{\mathrm{r}}_1 = \bar{\mathrm{R}} + \left(\frac{m_2}{m_1 + m_2}\right)\bar{\mathrm{r}}$$

$$\bar{\mathrm{r}}_2 = \bar{\mathrm{R}} - \left(\frac{m_1}{m_1 + m_2}\right)\bar{\mathrm{r}}$$

In the center of mass frame $\bar{r}_j' = \bar{r}_j - \bar{R}$

$$\bar{r}_1' = \frac{\mu}{m_1}\bar{r}$$

$$\bar{r}_2' = -\frac{\mu}{m_2}\bar{r}$$



- Next, we discuss how to approach the solution of the reduced equation

General Features of Central Force Motion

- Before attempting to solve $\mu \ddot{\mathrm{r}} = f(r)\hat{\mathrm{r}}$, we explore some general properties of central force motion
- Let $\bar{\mathrm{L}} = \bar{\mathrm{r}} \times \bar{\mathrm{p}}$ be angular momentum corresponding to the relative motion
- Then clearly

$$\frac{d\bar{\mathrm{L}}}{dt} = \frac{d\bar{\mathrm{r}}}{dt} \times \bar{\mathrm{p}} + \bar{\mathrm{r}} \times \frac{d\bar{\mathrm{p}}}{dt} = \bar{\mathrm{v}} \times \bar{\mathrm{p}} + \bar{\mathrm{r}} \times \bar{\mathrm{F}}$$

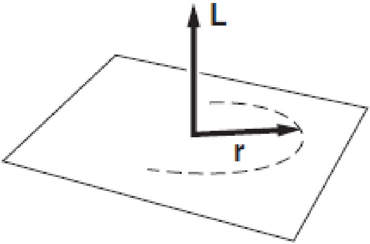
- But v and $\mathrm{p} = \mu \mathrm{v}$ and parallel, so that $\mathrm{v} \times \mathrm{p} = 0$
- And for the central force case, $\mathrm{r} \times \mathrm{F} = f(r)\mathrm{r} \times \hat{\mathrm{r}} = 0$, so that

$(\because \; \bar{F} = f(r)\hat{r})$

$$\frac{d\mathrm{L}}{dt} = 0$$
$$\implies \mathrm{L} = \text{constant}$$

- Thus, in case of central force motion, the angular momentum is conserved, both in direction, and magnitude

- Conservation of angular momentum implies that the relative motion occurs in a plane



Motion with const. $\bar{L}$

$\bar{L} = \bar{r} \times \bar{p} = \mu \bar{r} \times \bar{v}$

$\bar{r}$ & $\bar{v}$ must remain in the same plane

- Direction of L is fixed, and because r $\perp$ L, so r must be in the same plane
- So, we can use plane polar coordinates (r, θ) to describe the motion

Equations of motion in plane-polar coordinates

- We know that in plane polar coordinates

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

- Therefore, the equation of motion $\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}$, becomes

$$\mu(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \mu(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = f(r)\hat{\mathbf{r}}$$

- On comparing both sides, we obtain following two equations

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$\mu(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

- By multiplying second equation on both sides by r, we obtain

$$\frac{d}{dt}(\mu r^2\dot{\theta}) = 0 \qquad \Leftarrow \mu(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}) = 0$$

$r \neq 0$, so cancels

Equations of motion

- This equation yields

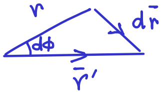
Recall
$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
$L = \bar{r}\times\bar{p} = \mu r\hat{r}\times(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$
$= 0 + \mu r^2\dot{\theta}\,\hat{r}\times\hat{\theta} = \mu r^2\dot{\theta}\,\hat{z}$

$$\boxed{\mu r^2\dot{\theta} = L\,(\text{constant})},$$

we called this constant L because it is nothing but the angular momentum of the particle about the origin. Note that $L = I\omega$, with $I = \mu r^2$.

- As the particle moves along the trajectory so that the angle θ changes by an infinitesimal amount $d\theta$, the area swept with respect to the origin is

$\phi = \theta$



$d\bar{A} = \frac{1}{2}\bar{r}\times\bar{r}' = \frac{1}{2} r r' \sin d\phi$
$\simeq \frac{1}{2} r r\, d\phi$
$d\phi \to 0$

$$dA = \frac{1}{2}r^2 d\theta$$

$$\Longrightarrow \frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2\mu} = \text{constant},$$

Note $r' = |\bar{r} + d\bar{r}|$
$= (r^2 + dr^2 + 2\bar{r}\cdot d\bar{r})^{1/2}$
$\simeq r\left[1 + 2\frac{dr}{r}\cos\alpha + O(dr^2)\right]^{1/2}$
$\simeq r\left[1 + O(dr)\right]$
order

if we included the dr term then we get a term $d\phi\, dr$ which we ignore.

because L is constant.

Ignored 2nd order terms, see on the right

- Thus constancy of areal velocity is a property of all central forces, not just the gravitational forces.
- And it holds due to conservation of angular momentum

- Kinetic energy in plane polar coordinates can be written as

$$
\begin{aligned}
K &= \frac{1}{2}\mu \mathrm{v}\cdot\mathrm{v} \\
&= \frac{1}{2}\mu\left(\dot{r}\hat{\mathrm{r}} + r\dot{\theta}\hat{\theta}\right)\cdot\left(\dot{r}\hat{\mathrm{r}} + r\dot{\theta}\hat{\theta}\right) \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2
\end{aligned}
$$

or, simply $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$
$\because \hat{r} \perp \hat{\theta}$ in $\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

- Potential energy $V(r)$ can be obtained by the basic formula

$$f(r) = -\frac{dV(r)}{dr} \Rightarrow \quad V(r) - V(r_O) = -\int_{r_0}^{r} f(r)dr,$$

where r_O denotes the location of a reference point.

$\bar{F}(r) \sim -\frac{\hat{r}}{r^2} \Rightarrow \text{curl}\,\bar{F} = 0 \Rightarrow V(r)$ conservatv. potential

Conservation of Energy...

- Total energy E from work-energy theorem

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\theta}^2 + V(r) = \text{constant}$$

(∵ Conservative force)

Substitute this

- We have

using, $\boxed{L = \mu r^2 \dot{\theta}}$

$$\implies \frac{1}{2}\mu r^2 \dot{\theta}^2 = \frac{L^2}{2\mu r^2}$$

- So that

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$$

- We can write

$$E = \frac{1}{2}\mu \dot{r}^2 + V_{eff}(r)$$

$$\text{with } V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$$

Conservation of energy contd.

- This energy is similar to that of a 1D system, with an effective potential energy $V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$
- In reality $\frac{L^2}{2\mu r^2}$ is kinetic energy of the particle due to angular motion
- But, because of its dependence on position, it can be treated as an effective potential energy

Integrating the equations of motion

$E = \frac{\mu}{2}\dot{r}^2 + \frac{L^2}{2\mu r^2} + V(r)$

Can they collide?

Not unless $L=0$ in I.C.

- Energy conservation equation yields

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}$$

- Leading to the solution

$$\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}} = t - t_0, \tag{1}$$

which will yield r as a function of t, once $f(r)$ is known, and the integral is performed

- Once $r(t)$ is known, to obtain $\theta(t)$, we use conservation of angular momentum

$\mu r^2 \dot{\theta} = L \Rightarrow$

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\theta - \theta_0 = \frac{L}{\mu} \int_{t_0}^{t} \frac{dt}{r^2}$$

- We can obtain the shape of the trajectory $r(\theta)$, by combining these two equations

$$\frac{d\theta}{dr} = \left(\frac{\frac{d\theta}{dt}}{\frac{dr}{dt}} \right) = \frac{\frac{L}{\mu r^2}}{\sqrt{\frac{2}{\mu}\left(E - V_{eff}(r)\right)}}$$

- Leading to

$$\theta - \theta_0 = L \int_{r_0}^{r} \frac{dr}{r^2 \sqrt{2\mu(E - V_{eff}(r))}} \tag{2}$$

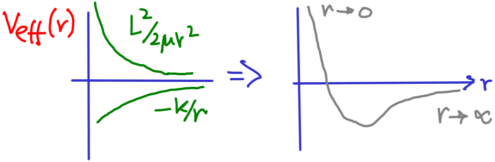
Integration of equations of motion contd.

- Thus, by integrating these equations, we can obtain $r(t)$, $\theta(t)$, and $r(\theta)$
- This will complete the solution of the problem
- But, to make further progress, we need to know what is $f(r)$
- Next, we will discuss the case of gravitational problem such as planetary orbits

But just from conserved $E = \frac{\mu}{2}\dot{r}^2 + \boxed{\frac{L^2}{2\mu r^2} + V(r)} = V_{eff}(r)$

& form of $V(r)$ we can infer a lot

$V(r) = -\frac{k}{r}$ (Gravity pot)



$$E = \mu \dot{r}^2 + \underbrace{\frac{L^2}{2\mu r^2} + V(r)}_{V_{eff}(r)}$$