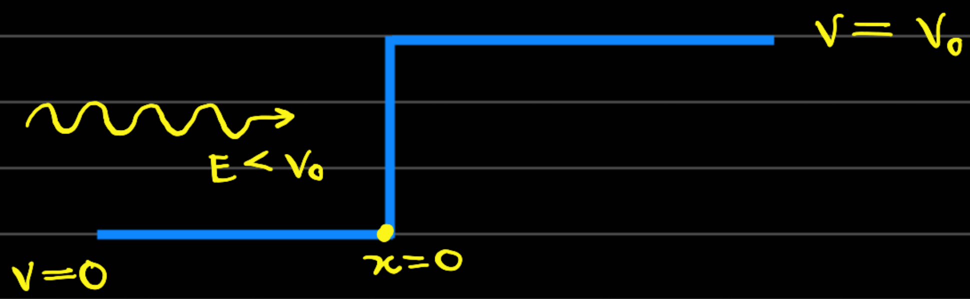
# Tutorial – 5

## ☆ Scattering and tunnel barrier –

1.) a.)




$$\psi_L = A e^{ik_L x} + B e^{-ik_L x} \qquad \left[ k_L = \frac{\sqrt{2mE}}{\hbar} \right]$$

$$\psi_R = C e^{-k_R x} \qquad \left[ k_R = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \right]$$

Now, $\dfrac{|\psi(x_0)|^2}{|\psi(0)|^2} = \dfrac{1}{e} \qquad (x_0 > 0)$

$$\Rightarrow \frac{|C e^{-k_R x_0}|^2}{|C|^2} = \frac{1}{e}$$

$$\Rightarrow e^{-2k_R x_0} = e^{-1}$$

$$\Rightarrow x_0 = \frac{1}{2k_R} = \frac{\hbar}{2\sqrt{2m(V_0 - E)}}$$

b.) $\Delta x = x_0 = \dfrac{\hbar}{2\sqrt{2m(V_0 - E)}}$

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \sqrt{2m(V_0 - E)}$$

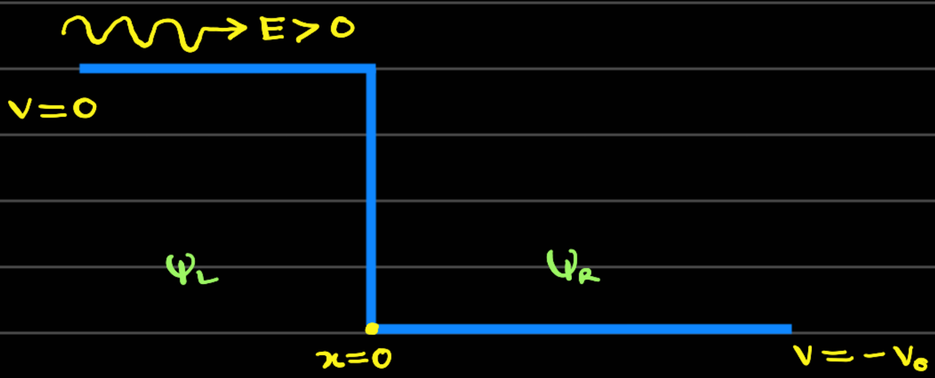
Put $E = V$

For $E = K$

$$\Rightarrow \Delta E = \Delta K$$
$$= \Delta\left(\frac{p^2}{2m}\right)$$
$$= \frac{p\Delta p}{m}$$
$$\geq \frac{\sqrt{2mK}\sqrt{2m(V_0-E)}}{m}$$
$$\geq 2\sqrt{E(V_0-E)}$$

Clearly for RHS to be real (as LHS $= \Delta E \in \mathbb{R}$)
$V_0 > E$

2.)




we have, $$\frac{-\hbar^2}{2m}\frac{d^2\psi_L}{dx^2} = E\psi_L$$

$$\Rightarrow \frac{d^2\psi_L}{dx^2} = -\frac{2mE}{\hbar^2}\psi_L = -k_L^2\psi_L$$

$$\Rightarrow \psi_L = Ae^{ik_Lx} + Be^{-ik_Lx} \quad \left[k_L = \frac{\sqrt{2mE}}{\hbar}\right]$$

And, $$\frac{-\hbar^2}{2m}\frac{d^2\psi_R}{dx^2} - V_0\psi_R = E\psi_R$$

$$\Rightarrow \frac{d^2\psi_R}{dx^2} = -\frac{2m(E+V_0)}{\hbar^2}\psi_R = -k_R^2\psi_R$$

$$\Rightarrow \psi_R = Ce^{ik_Rx} + De^{-ik_Rx} \quad \left[k_R = \frac{\sqrt{2m(E+V_0)}}{\hbar}\right]$$

$D = 0$ as there is no 'reflecting source' at $x=\infty$.

Continuity at $x=0$: $A+B=C$

continuity of $\left.\frac{d\psi}{dx}\right|_{x=0}$ : $ik_L(A-B)=ik_R C$

$$\Rightarrow k_L(A-B)=k_R C$$

$$\Rightarrow \frac{k_L}{k_R}(A-B)=A+B$$

$$\Rightarrow \frac{k_L}{k_R}=\frac{A+B}{A-B}$$

$$\Rightarrow \frac{A}{B}=\frac{k_L+k_R}{k_L-k_R}$$

$$R=\frac{|B|^2\frac{\hbar k_L}{m}}{|A|^2\frac{\hbar k_L}{m}}=\left(\frac{k_L-k_R}{k_L+k_R}\right)^2$$

$$=\left(\frac{1-k_R/k_L}{1+k_R/k_L}\right)^2$$

$$=\left[\frac{1-\sqrt{1+V_0/E}}{1+\sqrt{1+V_0/E}}\right]^2$$

$$=\left[\frac{\sqrt{1+V_0/E}-1}{\sqrt{1+V_0/E}+1}\right]^2$$

Now, $A+B=C$

$$\Rightarrow 1+\frac{B}{A}=\frac{C}{A}$$

$$\Rightarrow \frac{C}{A}=1+\frac{k_L-k_R}{k_L+k_R}$$

$$=\frac{2k_L}{k_L+k_R}$$

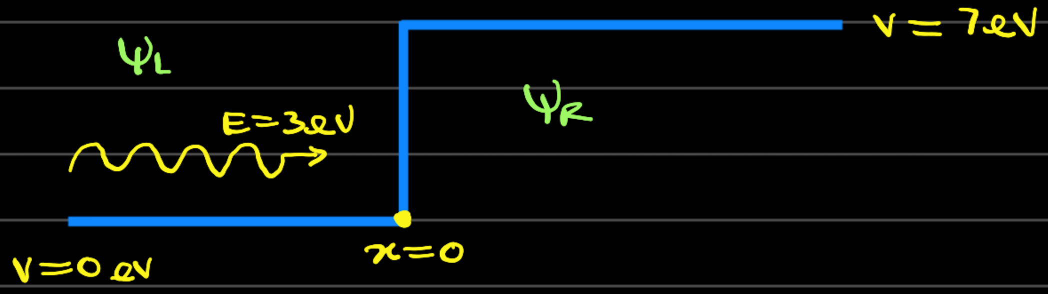
$$\therefore T = \frac{|C|^2 k_R}{|A|^2 k_L} = \frac{4 k_L k_R}{(k_L + k_R)^2}$$

$$= \frac{4 \frac{\sqrt{2mE}}{\hbar} \frac{\sqrt{2m(E+V_0)}}{\hbar}}{\left[\frac{\sqrt{2mE} + \sqrt{2m(E+V_0)}}{\hbar}\right]^2}$$

$$= \frac{4 \times \frac{2m}{\hbar^2} \sqrt{E(E+V_0)}}{\frac{2m}{\hbar^2} E \left[1 + \sqrt{1 + \frac{V_0}{E}}\right]^2}$$

$$= \frac{4\sqrt{1 + V_0/E}}{\left[1 + \sqrt{1 + \frac{V_0}{E}}\right]^2}$$

3)




$$\psi_L = A e^{i k_L x} + B e^{-i k_L x} \qquad \left[k_L = \frac{\sqrt{6m}}{\hbar} \, m^{-1}\right]$$

$$\psi_R = C e^{-k_R x} \qquad \left[k_R = \frac{\sqrt{8m}}{\hbar} \, m^{-1}\right]$$

Now, $\dfrac{|\psi(x)|^2}{|\psi(0)|^2} = \dfrac{1}{2}$

$x > 0$ as $\psi$ is sinusoidal for $x < 0$ and thus particle is equiprobable (w.r.t. location) in the region $x<0$.

$\therefore$ we have

$$\frac{|C e^{-K_R x}|^2}{|C|^2} = \frac{1}{2}$$

$$\Rightarrow e^{-2K_R x} = e^{-\ln 2}$$

$$\Rightarrow x = \frac{\ln 2}{2K_R} = \frac{\ln 2}{2\frac{\sqrt{8m}}{\hbar}}$$

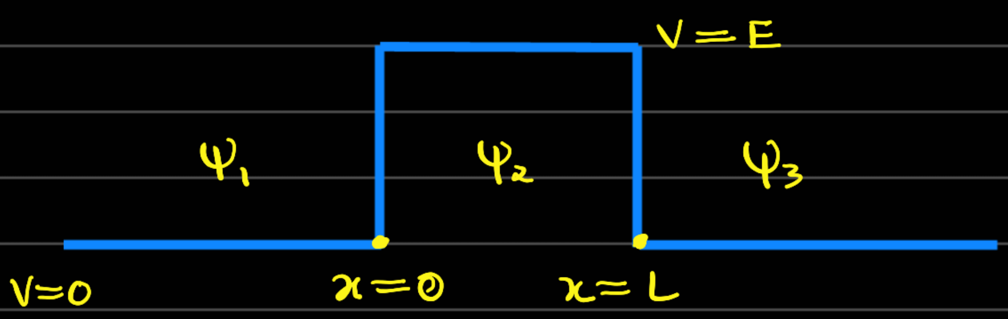
$$= \frac{\hbar \ln 2}{4\sqrt{2m}}$$

$$= \frac{6.626\times10^{-34}\ln 2}{4\times2\pi\sqrt{2\times9.1\times10^{-31}}}\ m$$

$$= \frac{3.313\ln 2}{4\pi\sqrt{1.82}}\times10^{-18}\ m$$

$$\simeq 0.1355\times10^{-18}\ m$$
$$= 1.355\times10^{-19}\ m$$

4.)




$$\frac{-\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} = E\psi_1$$

$$\Rightarrow \psi_1 = Ae^{ik_1x} + Be^{-ikx} \qquad \left[k = \frac{\sqrt{2mE}}{\hbar}\right]$$

Similarly, $\psi_3 = Ee^{ikx}$

Now, $\dfrac{-\hbar^2}{2m}\dfrac{d^2\psi_2}{dx^2} + E\psi_2 = E\psi_2$

$$\Rightarrow \frac{d^2\psi_2}{dx^2} = 0 \qquad \Rightarrow \psi_2 = Cx + D$$

a.) Continuity at $x=0$: $\quad A+B = D$

Continuity at $x=L$: $\quad Ee^{ikL} = CL + D$

Continuity of $\left.\dfrac{d\psi}{dx}\right|_{x=0}$ : $\quad ik(A-B) = C$

Continuity of $\left.\dfrac{d\psi}{dx}\right|_{x=L}$ : $\quad C = ikEe^{ikL}$

Now, $ik(A-B) = ikEe^{ikL}$

$$\Rightarrow 2A - D = Ee^{ikL}$$

$$\Rightarrow 2A = 2Ee^{ikL} - CL$$

$$= 2Ee^{ikL} - ikLEe^{ikL}$$

$$= Ee^{ikL}(2-ikL)$$

$$\Rightarrow \frac{E}{A} = \frac{2e^{-ikL}}{2-ikL}$$

Now, $T = \dfrac{|E|^2\frac{\hbar k}{m}}{|A|^2\frac{\hbar k}{m}} = \left|\dfrac{2e^{-ikL}}{2-ikL}\right|^2$

$$= \frac{4}{4+k^2L^2}$$

$$= \frac{4}{4+\frac{2mEL^2}{\hbar^2}}$$

$$= 1$$

$$1+\frac{mEL^2}{2\hbar^2}$$

b.) We have $A-B = Ee^{ikL}$

$$\Rightarrow 1-\frac{B}{A} = \frac{E}{A}e^{ikL}$$

$$\Rightarrow \frac{B}{A} = 1-\frac{E}{A}e^{ikL}$$

$$= 1-\frac{2}{2-ikL}$$

$$= \frac{-ikL}{2-ikL}$$

$$R = \frac{|B|^2\hbar k/m}{|A|^2\hbar k/m} = \left|\frac{-ikL}{2-ikL}\right|^2$$

$$= \frac{k^2L^2}{4+k^2L^2}$$

$$= \frac{1}{4/k^2L^2+1}$$

$$= \frac{1}{1+2\hbar^2/mEL^2}$$

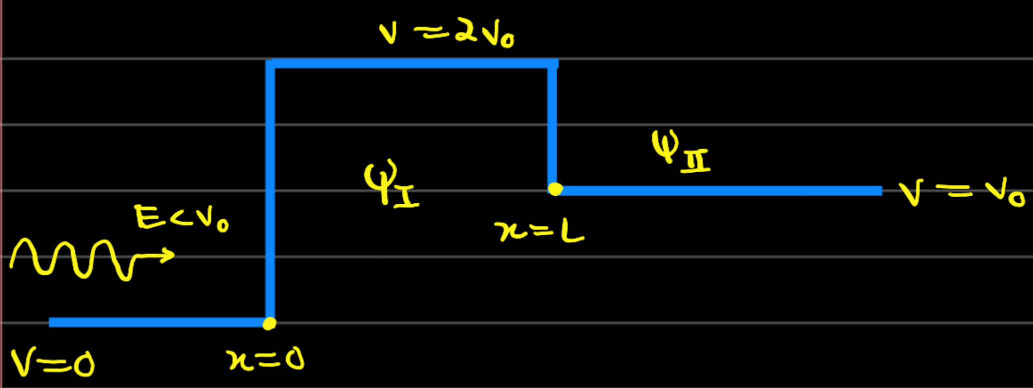
For $R = 1/2 \Rightarrow \frac{2\hbar^2}{mEL^2} = 1$

$$\Rightarrow L = \sqrt{\frac{2}{mE}}\,\hbar$$

$$= \sqrt{\frac{2}{m\times 2\pi\hbar c/\lambda}}\,\hbar$$

$$= \sqrt{\frac{\hbar\lambda}{\phantom{x}}}$$

5.)




$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_I + 2V_0\psi_I = E\psi_I$$

$$\Rightarrow \frac{d^2\psi_I}{dx^2} = \frac{2m(2V_0 - E)}{\hbar^2}\psi_I$$

$$= k_1^2\psi_I$$

So $\psi_I = A_1 e^{k_1 x} + A e^{-k_1 x}$

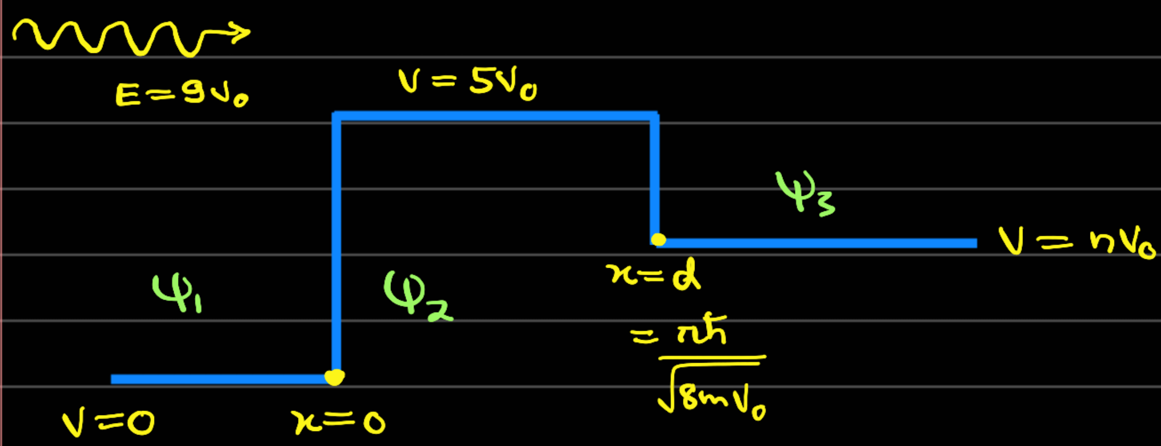
Now, same differential eq$^n$ for $\psi_{II}$ (with $2V_0$ replaced by $V_0$).

So $\psi_{II} = B_1 e^{k_2 x} + B e^{-k_2 x}$

Now, $\lim_{x \to \infty} \psi_{II}(x) = 0 \Rightarrow B_1 = 0$

The claim made for $\psi_{II}$ is correct but for $\psi_I$ it is incorrect.

6.)



$T = 0.75$ (from $x<0$ to $x>d$)

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \quad \left[k_1 = \frac{\sqrt{18mV_0}}{\hbar}\right]$$

$$\psi_2 = Ce^{ik_2x} + De^{-ik_2x} \quad \left[k_2 = \frac{\sqrt{8mV_0}}{\hbar} = \frac{\pi}{d}\right]$$

Clearly, $n<9$ [You check, it won't work for $n \geq 9$]

$$\psi_3 = Ee^{ik_3x} \quad \left[k_3 = \sqrt{\frac{2m(9-n)V_0}{\hbar}}\right]$$

Continuity at $x=0$:

$$A+B = C+D$$

Continuity at $x=d$:

$$-C-D = Ee^{ik_3d}$$

Continuity of $\left.\frac{d\psi}{dx}\right|_{x=0}$:

$$ik_1(A-B) = ik_2(C-D)$$

$$\Rightarrow k_1(A-B) = k_2(C-D)$$

Continuity of $\left.\frac{d\psi}{dx}\right|_{x=d}$:

$$ik_2(-C+D) = ik_3Ee^{ik_3d}$$

$$\Rightarrow k_2(-C+D) = k_3Ee^{ik_3d}$$

Now, $A+B = -Ee^{ik_3d}$

$$-k_1(A-B) = k_3Ee^{ik_3d}$$

$$\Rightarrow A-B = \frac{k_3}{k_1}\left(-Ee^{ik_3d}\right)$$

Add: $2A = -Ee^{ik_3d}\left(1+\frac{k_3}{k_1}\right)$

$$\Rightarrow \frac{E}{A} = \frac{-2e^{-ik_3d}}{1+\frac{k_3}{k_1}}$$

$$T = 0.75 = \frac{|E|^2 \frac{\hbar k_3}{m}}{|A|^2 \frac{\hbar k_1}{m}}$$

$$\Rightarrow \frac{3}{4} = \underbrace{\frac{k_3}{k_1}}_{t} \frac{4}{\left(1+\frac{k_3}{k_1}\right)^2}$$

$$= \frac{4t}{(1+t)^2}$$

$$\Rightarrow 3+6t+3t^2 = 16t$$

$$\Rightarrow 3t^2 - 10t + 3 = 0$$

$$\Rightarrow t = 3, \ 1/3$$

$$\Rightarrow \sqrt{1-\frac{n}{9}} = \frac{1}{3} \text{ or } 3$$

$$\Rightarrow 1-\frac{n}{9} = \frac{1}{9} \text{ or } 9$$

$$\Rightarrow n = 8 \text{ or } -72$$

so $k_3 = \frac{\sqrt{2mV_0}}{\hbar}$ or $\frac{9\sqrt{2mV_0}}{\hbar}$

$$k_2 = \frac{2\sqrt{2mV_0}}{\hbar}$$

$$k_1 = \frac{3\sqrt{2mV_0}}{\hbar}$$

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}$$
$$\psi_2 = Ce^{ik_2x} + De^{-ik_2x}$$
$$\psi_3 = Ee^{ik_3x}$$

$-ik_3d$

$$\frac{E}{A} = \frac{2e}{1+\frac{k_3}{k_1}}$$

$$A+B = -E e^{ik_3 d}$$

$$\Rightarrow 1+\frac{B}{A} = -\frac{E}{A} e^{ik_3 d}$$

$$= \frac{2}{1+\frac{k_3}{k_1}} = \frac{2k_1}{k_1+k_3}$$

$$\Rightarrow \frac{B}{A} = \frac{k_1-k_3}{k_1+k_3}$$

$$A+B = C+D$$

$$\Rightarrow 1+\frac{B}{A} = \frac{C}{A}+\frac{D}{A}$$

$$\Rightarrow \frac{2k_1}{k_1+k_3} = \frac{C}{A}+\frac{D}{A}$$

$$k_1(A-B) = k_2(C-D)$$

$$\Rightarrow \frac{k_1}{k_2}\left(1-\frac{B}{A}\right) = \frac{C}{A}-\frac{D}{A}$$

$$\Rightarrow \frac{k_1}{k_2}\left(\frac{2k_3}{k_1+k_3}\right) = \frac{C}{A}-\frac{D}{A}$$

$$\frac{C}{A} = \frac{k_1}{k_1+k_3}\left(1+\frac{k_3}{k_2}\right)$$

$$= \frac{k_1}{k_2}\left(\frac{k_2+k_3}{k_1+k_3}\right)$$

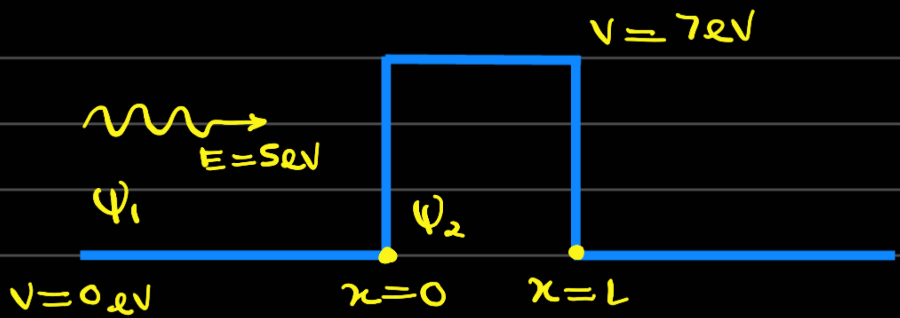
$$\frac{D}{A} = \frac{k_1}{k_1+k_3}\left[1 - \frac{k_3}{k_2}\right]$$

$$= \frac{k_1}{k_2}\left(\frac{k_2-k_3}{k_1+k_3}\right)$$

Now, we have B, C, D, E in terms of A, and $k_1, k_2, k_3$ (both for $n=8,-72$) are known, write $\psi$ on your own.

☆ 6(c) will be done at the end.

7.)




$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x} \quad \left[k_1 = \frac{\sqrt{10me}}{\hbar}\ m^{-1}\right]$$

$$\psi_2 = Ce^{k_2x} + De^{-k_2x} \quad \left[k_2 = \frac{\sqrt{4me}}{\hbar}\ m^{-1}\right]$$

$$\psi_3 = Ee^{ik_1x}$$

Here $C=0$ as question says tunnelling current reduces exponentially with distance.

Now, $$\frac{|\psi_2(3\mathring{A})|^2}{|\psi_2(6\mathring{A})|^2}$$

$$= \frac{e^{-2k_2\times3\times10^{-10}}}{e^{-2k_2\times6\times10^{-10}}}$$

$$6k_2\times10^{-10}$$

$$= e$$

$$= e^{\frac{6 \times 2\sqrt{9.1\times10^{-31} \times 1.6\times10^{-19}}}{6.626\times10^{-34}} \times 2\pi \times 10^{-10}}$$

$$= e^{\frac{12\pi}{3.313}\sqrt{91\times16\times10^{-52}} \times 2\pi\times10^{-24}}$$

$$= e^{\frac{48\pi}{3.313}\sqrt{91}\times10^{-2}}$$

$$= e^{\frac{0.48\pi\sqrt{91}}{3.313}}$$

$$\simeq e^{4.342}$$

$$\simeq 76.8611$$

(*) 6.) c.) $B \rightarrow$ coeff. of reflected wave

$A \longrightarrow$ coeff. of incident wave

$$\frac{B}{A} = \frac{k_1 - k_3}{k_1 + k_3}$$

$$= \frac{1 - k_3/k_1}{1 + k_3/k_1}$$

When $n = -72 \Rightarrow k_3 = 3k_1$

$$\frac{B}{A} = \frac{1-3}{1+3} = -\frac{1}{2} < 0$$

$\Rightarrow$ phase shift $\pi$

When $n = 8 \Rightarrow k_3 = k_1/3$

$$\frac{B}{A} = \frac{1 - 1/3}{1 + 1/3} = \frac{1}{2} > 0$$

$\Rightarrow$ No phase shift.

—x—