points simultaneously. Because simultaneity is a relative concept, length measurements will also depend on the reference frame and be relative. Furthermore, we find that the rates at which clocks run also depend on the reference frame. This can be illustrated as follows. Consider two clocks, one on a train and one on the ground, and assume that at the moment they pass one another (i.e., the instant that they are coincident) they read the same time (i.e., the hands of the clocks are in identical positions). Now, if the clocks continue to agree, we can say that they go at the same rate. But, when they are a great distance apart, we know from the preceding discussion that their hands cannot have identical positions simultaneously as measured both by the ground observer and the train observer. Hence, time interval measurements are also relative, that is, they depend on the reference frame of the observer. As a result of the relativity of length and time interval measurements it is perhaps possible to reconcile ourselves to the experimental fact that observers who are moving relative to each other measure the speed of light to be the same (see Question 20). In succeeding sections, we shall look more carefully into these matters.

## 2.2 *Derivation of the Lorentz Transformation Equations*

We have seen that the Galilean transformation equations must be replaced by new ones consistent with experiment. Here we shall derive these new equations, using the postulates of special relativity theory. To show the consistency of the theory with the discussion of the previous section, we shall then derive all the special features of the new transformation equations again from the more physical approach of the measurement processes discussed there.

We observe an event in one inertial reference frame $S$ and characterize its location and time by specifying the coordinates $x$, $y$, $z$, $t$ of the event. In a second inertial frame $S'$, this *same event* is recorded as the space-time coordinates $x'$, $y'$, $z'$, $t'$. We now seek the functional relationships $x' = x'(x,y,z,t)$, $y' = y'(x,y,z,t)$, $z' = z'(x,y,z,t)$, and $t' = t'(x,y,z,t)$. That is, we want the equations of transformation which relate one observer's space-time coordinates of an event with the other observer's coordinates of the same event.

We shall use the fundamental postulates of relativity theory and, in addition, the assumption that space and time are homogeneous. This homogeneity assumption (which can be paraphrased by saying that all points in space and time are equivalent) means, for example, that the results of a measurement of a length or time interval of a specific event should not depend on where or when the interval happens to be in our reference frame. We shall illustrate its application shortly.

We can simplify the algebra by choosing the relative velocity of the $S$ and $S'$ frames to be along a common $x$-$x'$ axis and by keeping corresponding planes parallel (see Fig. 1-1). This does not impose any fundamental restrictions on our results for space is isotropic—that is, has the same properties in all directions. Also, at the instant the origins $0$ and $0'$ coincide, we let the clocks there read $t = 0$ and $t' = 0$, respectively. Now, as explained below, the homogeneity assumption requires that transformation equations must be linear (i.e., they involve only the first power in the variables), so that the most general form they can take (see Question 5) is

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' &= a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' &= a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' &= a_{41}x + a_{42}y + a_{43}z + a_{44}t. \end{aligned} \tag{2-1}$$

Here, the subscripted coefficients are constants that we must determine to obtain the exact transformation equations. Notice that we do not exclude the possible dependence of space and time coordinates upon one another.

If the equations were not linear, we would violate the homogeneity assumption. For example, suppose that $x'$ depended on the square of $x$, that is, as $x' = a_{11}x^2$. Then the distance between two points in the primed frame would be related to the location of these points in the unprimed frame by $x_2' - x_1' = a_{11}(x_2^2 - x_1^2)$. Suppose now that a rod of unit length in S had its end points at $x_2 = 2$ and $x_1 = 1$; then $x_2' - x_1' = 3a_{11}$. If, instead, the same rod happens to be located at $x_2 = 5$ and $x_1 = 4$, we would obtain $x_2' - x_1' = 9a_{11}$. That is, the measured length of the rod would depend on where it is in space. Likewise, we can reject any dependence on $t$ that is not linear, for the time interval of an event should not depend on the numerical setting of the hands of the observer's clock. The relationships must be linear then in order not to give the choice of origin of our space-time coordinates (or some other point) a physical preference over all other points.

Now, regarding these sixteen coefficients, it is expected that their values will depend on the relative velocity $v$ of the two inertial frames. For example, if $v = 0$, then the two frames coincide at all times and we expect $a_{11} = a_{22} = a_{33} = a_{44} = 1$, all other coefficients being zero. More generally, if $v$ is small compared to $c$, the coefficients should lead to the (classical) Galilean transformation equations. We seek to find the coefficients for *any* value of $v$, that is, as functions of $v$.

How then do we determine the values of these sixteen coefficients? Basically, we use the postulates of relativity, namely (1) The Principle of Relativity—that no preferred inertial system exists, the laws of physics being the same in all inertial systems—and (2) The Principle of the Constancy of the Speed of Light—that the speed of light in free space has the same value $c$ in all inertial systems. Let us proceed.

The $x$-axis coincides continuously with the $x'$-axis. This will be so only if for $y = 0$, $z = 0$ (which characterizes points on the $x$-axis) it always follows that $y' = 0$, $z' = 0$ (which characterizes points on the $x'$-axis). Hence, the transformation formulas for $y$ and $z$ must be of the form

$$y' = a_{22}y + a_{23}z \qquad \text{and} \qquad z' = a_{32}y + a_{33}z$$

That is, the coefficients $a_{21}$, $a_{24}$, $a_{31}$, and $a_{34}$ must be zero. Likewise, the $x$-$y$ plane (which is characterized by $z = 0$) should transform over to the $x'$-$y'$ plane (which is characterized by $z' = 0$); similarly, for the $x$-$z$ and $x'$-$z'$ planes, $y = 0$ should give $y' = 0$. Hence, it follows that $a_{23}$ and $a_{32}$ are zero so that

$$y' = a_{22}y \qquad \text{and} \qquad z' = a_{33}z.$$

These remaining constant coefficients, $a_{22}$ and $a_{33}$, can be evaluated using the relativity postulate. We illustrate for $a_{22}$. Suppose that we have a rod lying along the $y$-axis, measured by $S$ to be of unit length. According to the $S'$ observer, the rod's length will be $a_{22}$, (i.e., $y' = a_{22} \times 1$). Now, suppose that the very same rod is brought to rest along the $y'$ axis of the $S'$-frame. The primed observer must measure the same length (unity) for this rod when it is at rest in his frame as the unprimed observer measures when the rod is at rest with respect to him; otherwise there would be an asymmetry in the frames. In this case, however, the $S$-observer would measure the rod's length to be $1/a_{22}$ [i.e., $y = (1/a_{22})y' = (1/a_{22}) \times 1$]. Now, because of the reciprocal nature of these length measurements, the first postulate requires that these measurements be identical, for otherwise the frames would not be equivalent physically. Hence, we must have $a_{22} = 1/a_{22}$ or $a_{22} = 1$. The argument is identical in determining that $a_{33} = 1$. Therefore, our two middle transformation equations become

$$y' = y \quad \text{and} \quad z' = z. \tag{2-2}$$

There remain transformation equations for $x'$ and $t'$, namely,

$$x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t$$

and

$$t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t.$$

Let us look first at the $t'$-equation. For reasons of symmetry, we assume that $t'$ does not depend on $y$ and $z$. Otherwise, clocks placed symmetrically in the $y$-$z$ plane (such as at $+y$, $-y$ or $+z$, $-z$) about the $x$-axis would appear to disagree as observed from $S'$, which would contradict the isotropy of space. Hence, $a_{42} = a_{43} = 0$. As for the $x'$-equation, we know that a point having $x' = 0$ appears to move in the direction of the positive $x$-axis with speed $v$, so that the statement $x' = 0$ must be identical to the statement $x = vt$. Therefore, we expect $x' = a_{11}(x - vt)$ to be the correct transformation equation. (That is, $x = vt$ always gives $x' = 0$ in this equation.) Hence, $x' = a_{11}\, x - a_{11}\, vt = a_{11}\, x + a_{14}\, t$. This gives us $a_{14} = -va_{11}$, and our four equations have now been reduced to

$$\begin{aligned} x' &= a_{11}(x - vt) \\ y' &= y \\ z' &= z \\ t' &= a_{41}x + a_{44}t. \end{aligned} \tag{2-3}$$

There remains the task of determining the three coefficients $a_{11}$, $a_{41}$, and $a_{44}$. To do this, we use the principle of the constancy of the velocity of light. Let us assume that at the time $t = 0$ a spherical electromagnetic wave leaves the origin of $S$, which coincides with the origin of $S'$ at that moment. The wave propagates with a speed $c$ in all directions in each inertial frame. Its progress, then, is described by the equation of sphere whose radius expands with time at a rate $c$ in terms of either the primed or unprimed set of coordinates. That is,

$$x^2 + y^2 + z^2 = c^2t^2 \tag{2-4}$$

or

$$x'^2 + y'^2 + z'^2 = c^2t'^2. \tag{2-5}$$

If now we substitute into Eq. 2-5 the transformation equations (Eqs. 2-3), we get

$$a_{11}^2(x - vt)^2 + y^2 + z^2 = c^2(a_{41}x + a_{44}t)^2.$$

Rearranging the terms gives us

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(va_{11}^2 + c^2a_{41}a_{44})xt = (c^2a_{44}^2 - v^2a_{11}^2)t^2.$$

In order for this expression to agree with Eq. 2-4, which represents the same thing, we must have

$$\begin{aligned} c^2a_{44}^2 - v^2a_{11}^2 &= c^2 \\ a_{11}^2 - c^2a_{41}^2 &= 1 \\ va_{11}^2 + c^2a_{41}a_{44} &= 0. \end{aligned}$$

Here we have three equations in three unknowns, whose solution (as the student can verify by substitution into the three equations above) is

$$\begin{aligned} a_{44} &= 1/\sqrt{1 - v^2/c^2} \\ a_{11} &= 1/\sqrt{1 - v^2/c^2} \end{aligned} \tag{2-6}$$

and

$$a_{41} = -\frac{v}{c^2}/\sqrt{1 - v^2/c^2}.$$

By substituting these values into Eqs. 2-3, we obtain, finally, the new sought-after transformation equations,

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}, \end{aligned} \tag{2-7}$$

the so-called* *Lorentz transformation equations.*

Before probing the meaning of these equations, we should put them to two necessary tests. First, if we were to exchange our frames of reference or—what amounts to the same thing—consider the given space-time coordinates of the event to be those observed in $S'$ rather than in

*Poincaré originally gave this name to the equations. Lorentz, in his classical theory of electrons, had proposed them before Einstein did. However, Lorentz took $v$ to be the speed relative to an absolute ether frame and gave a different interpretation to the equations.

S, the only change allowed by the relativity principle is the physical one of a change in relative velocity from $v$ to $-v$. That is, from $S'$ the S-frame moves to the left whereas from $S$ the $S'$-frame moves to the right. When we solve Eqs. 2-7 for $x$, $y$, $z$, and $t$ in terms of the primed coordinates (see Problem 3), we obtain

$$\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \\ y &= y', \\ z &= z', \\ t &= \frac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}} \end{aligned} \tag{2-8}$$

which are identical in form with Eqs. 2-7 except that, as required, $v$ changes to $-v$.

Another requirement is that for speeds small compared to $c$, that is, for $v/c \ll 1$, the Lorentz equations should reduce to the (approximately) correct Galilean transformation equations. This is the case, for when $v/c \ll 1$, Eqs. 2-7 become*

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{2-9}$$

which are the classical Galilean transformation equations.

In Table 2-1 we summarize the Lorentz transformation equations.

TABLE 2-1 THE LORENTZ TRANSFORMATION EQUATIONS

| | |
|---|---|
| $x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}}$ | $x = \dfrac{x' + vt'}{\sqrt{1 - v^2/c^2}}$ |
| $y' = y$ | $y = y'$ |
| $z' = z$ | $z = z'$ |
| $t' = \dfrac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$ | $t = \dfrac{t' + (v/c^2)x'}{\sqrt{1 - v^2/c^2}}$ |

*In the time equation, $t' = (t - vx/c^2)/\sqrt{1 - v^2/c^2}$, consider the motion of the origin $O'$, for example, given by $x = vt$. Then

$$t' = (t - v^2t/c^2)/\sqrt{1 - v^2/c^2} = t\sqrt{1 - v^2/c^2}.$$

As $v/c \to 0$, $t' \to t$.