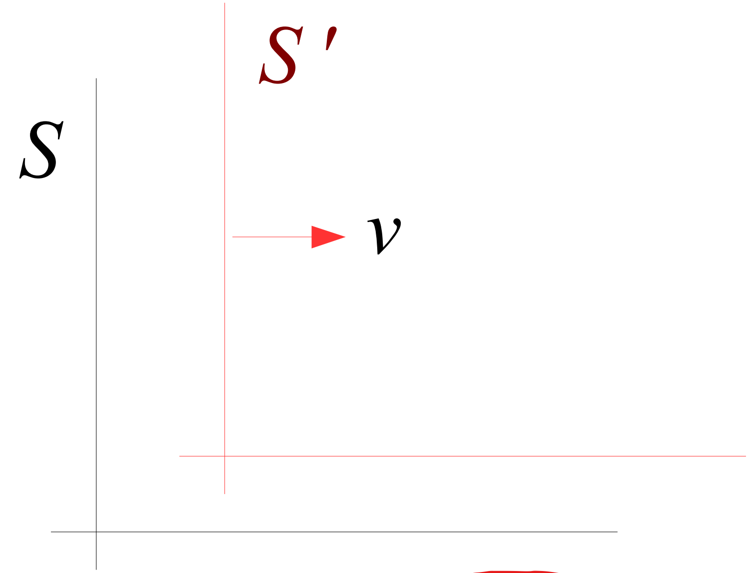
Energy , momentum, wave propagation

The classical connection between Energy conservation and momentum



Seen from S: A and B collide

$$\vec{u}_{Ai}, \vec{u}_{Bi} \rightarrow \vec{u}_{Af}, \vec{u}_{Bf}$$

KE conservation $\Rightarrow$

$$K(u_{Ai}) + K(u_{Bi}) = K(u_{Af}) + K(u_{Bf})$$

S measures a velocity as $\vec{u} \Rightarrow S'$ will measure it as $\vec{u} - \vec{v}$

$$\frac{m_A(\vec{u'}_{Ai} + \vec{v})^2}{2} + \frac{m_B(\vec{u'}_{Bi} + \vec{v})^2}{2} = \frac{m_A(\vec{u'}_{Af} + \vec{v})^2}{2} + \frac{m_A(u'_{Bf} + \vec{v})^2}{2}$$

$$\left[K(u'_{Ai}) + K(u'_{Bi}) - K(u'_{Af}) - K(u'_{Bf})\right] +$$

$$\left[m_A \vec{u'}_{Ai} + m_B \vec{u'}_{Bi} - m_A \vec{u'}_{Af} - m_B \vec{u'}_{Bf}\right] . \vec{v} = 0$$

$$\vec{P}_i = \vec{P}_f$$

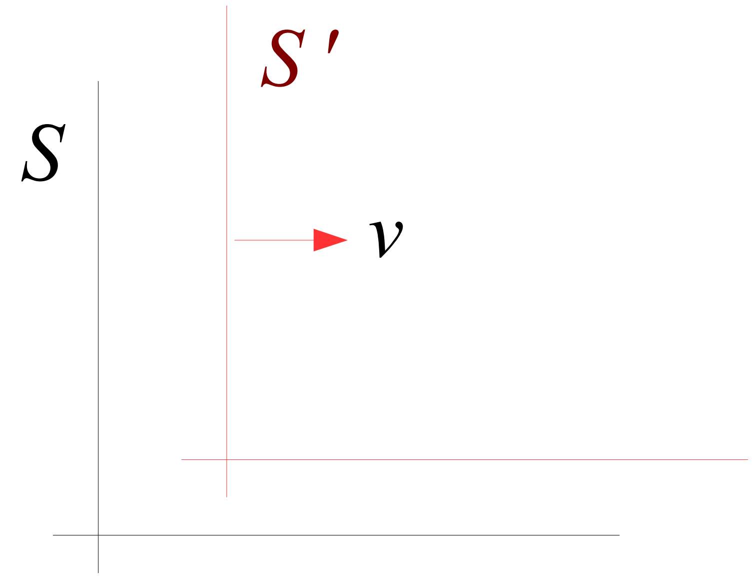
KE conservation across intertial frames $\rightarrow$ momentum conservation.

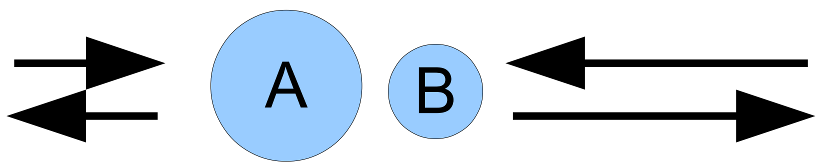
Relativistic velocity addition : $P = mv$ will not be conserved

S observes the collision

object	mass	v_i	v_f
A	$2m$	v	$-v$
B	m	$-2v$	$2v$

$$\boxed{P_i = P_f = 0}$$





This is what "should" happen if Newtonian momentum is conserved

Take S' as the rest frame of A before collision

$$\left.\begin{aligned} u_x' &= \frac{u_x - v}{1 - u_x v/c^2} \\ u_x &= \frac{u_x' + v}{1 + u_x' v/c^2} \end{aligned}\right\}$$

Use this to predict what S' should see

Relativistic velocity addition : $P = mv$ will not be conserved

S' observes the collision

object	mass	v_i	v_f
A	$2m$	0	$-\dfrac{2v}{1+v^2/c^2}$
B	m	$-\dfrac{3v}{1+2v^2/c^2}$	$\dfrac{v}{1-2v^2/c^2}$

$$P_i = -\frac{3mv}{1+2v^2/c^2}$$

$$P_f = -\frac{4mv}{1+v^2/c^2} + \frac{mv}{1-2v^2/c^2}$$

Total momentum
$P_i \neq P_f$

Q: How to redefine energy and momentum ?
→ Conservation must work for all inertial frames.
→ & also be consistent with Lorenz velocity addition

The connection between Energy conservation and momentum

KE conservation $\Rightarrow K(u_{Ai})+K(u_{Bi}) = K(u_{Af})+K(u_{Bf})$

S measures a velocity (x-component) as u

$$\Rightarrow S' \text{ will measure } u' = \frac{u-v}{1-uv/c^2} \equiv f(u,v)$$

For S' values of $u'_{Ai}, u'_{Bi}, u'_{Af}, u'_{Af}$ will be different

- ✔ But the same conservation law should hold
- ✔ Functional dependence of KE on velocities should not change
- ✔ Expect KE to depend on "speed" only and not direction

$$u' = u + \left.\frac{\partial f}{\partial v}\right|_{v=0} v + O(v^2)$$

momentum P

$$K(u) = K(u' - f'v) = K(u') \boxed{-\frac{\partial K}{\partial u} \times f'} \times v$$

$$\Rightarrow \delta K = -\frac{p\,\delta u}{f'}$$

f' denotes the derivative of f(u,v) w.r.t v
u' denotes the velocity in S'

The connection between Energy conservation and momentum

Work done = increase in KE should continue to hold

$$\delta K = \vec{F}\,.\,\delta\vec{r} = \frac{\delta\vec{p}}{\delta t}\delta\vec{r} = \delta\vec{p}.\vec{u}$$

$$= u\ \delta p \qquad (\text{in 1D})$$

Equate the two expressions for incremental KE

$$\delta K = -\frac{p\,\delta u}{f\,'} = u\,\delta p$$

$$\Rightarrow \frac{\delta p}{p} = -\frac{\delta u}{u f\,'}$$

Solving this differential equation should give p(u)

The connection between Energy conservation and momentum

$$f(u,v) = \frac{u-v}{1-uv/c^2} \Rightarrow f' = \frac{-1}{1-uv/c^2} - (u-v)\left(\frac{1}{1-uv/c^2}\right)^2\left(-\frac{u}{c^2}\right)$$

$$-f'(u,0) = 1-\frac{u^2}{c^2}$$

Do the integral by using partial fractions

$$\ln\frac{p}{p_0} = \boxed{\int\frac{du}{u\left(1-u^2/c^2\right)}} = \ln(u/c) - \ln\sqrt{1-u^2/c^2}$$

p_0 : const of integration $\equiv m_0 c$

Momentum becomes infinite as u → c

$$p = m_0 c\left(\frac{u}{c}\right)\left(\frac{1}{\sqrt{1-u^2/c^2}}\right) = \frac{m_0 u}{\sqrt{1-u^2/c^2}}$$

$$\boxed{\vec{p} = \gamma\left(m_0\vec{u}\right)}$$

$$\beta = u/c \;\;\&\;\; \gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$

commonly used notation

m_0 : mass measured in rest frame

How much work is done to speed up the particle ?

$$\int dK = \int u\,dp = up\Big|_0^u - \int_0^u p\,du$$

$$K = \frac{m_0 u^2}{\sqrt{1-u^2/c^2}} - \frac{m_0 c^2}{2}\int \frac{d\left(u^2/c^2\right)}{\sqrt{1-u^2/c^2}}$$

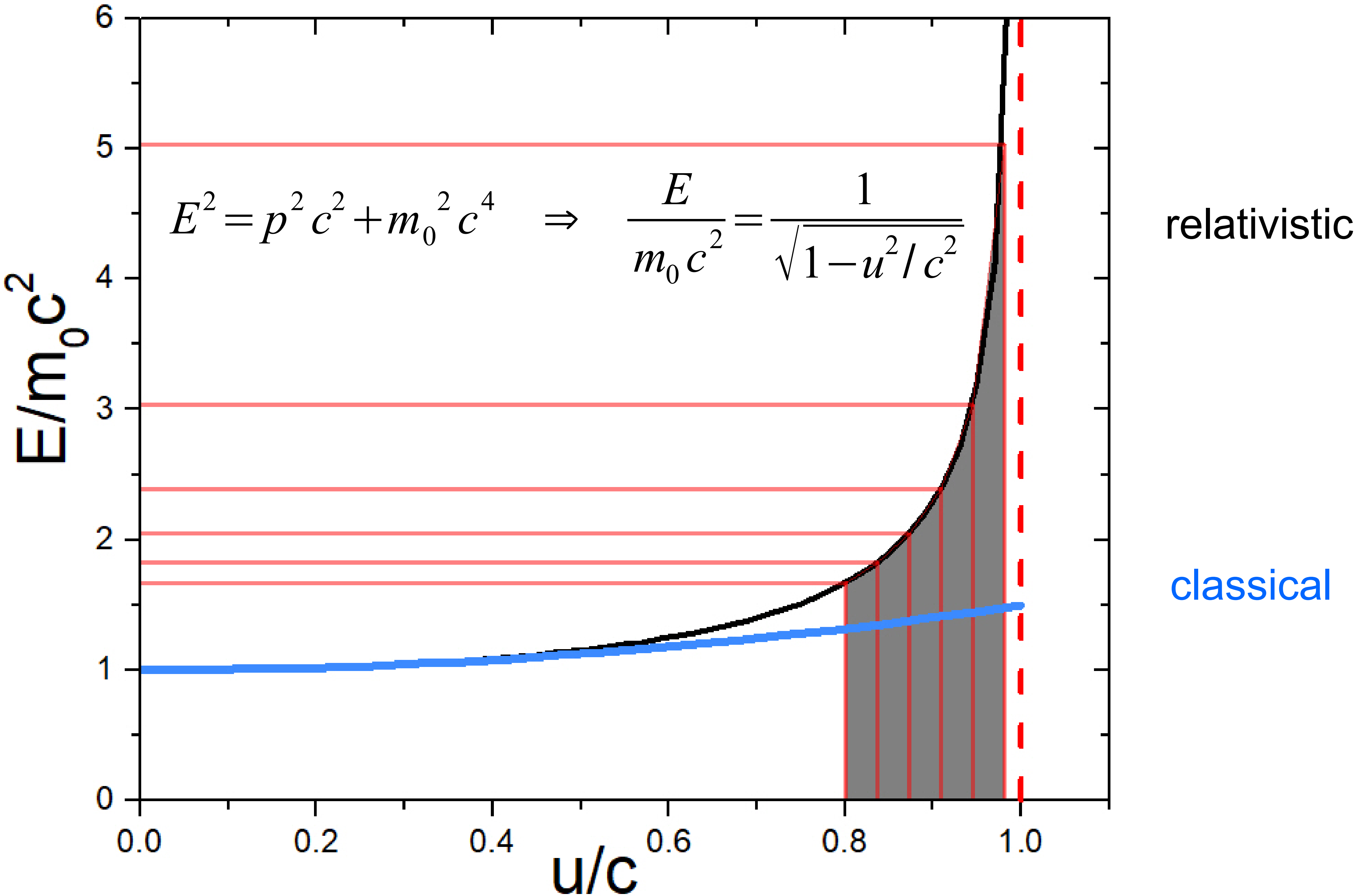
$$= \frac{m_0 u^2}{\sqrt{1-u^2/c^2}} + m_0 c^2\left[\sqrt{1-u^2/c^2} - 1\right]$$

$$K = m_0\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]c^2 = \Delta m\, c^2$$

$$E^2 = \left(K + m_0 c^2\right)^2 = \left(m_0 c^2\right)^2 \gamma^2 = m_0^{\ 2} c^4\left[\left(\frac{p}{m_0 c}\right)^2 + 1\right]$$

$$E^2 = p^2 c^2 + m_0^{\ 2} c^4 \quad \text{if} \quad m_0 = 0 \Rightarrow E = pc\,\text{(photons)}$$

It becomes increasingly difficult to increase speed as $u \to c$



It becomes increasingly difficult to increase speed as $u \rightarrow c$

Q: At what speed does the kinetic energy of a particle become equal to its rest energy ?

$$\text{Total Energy} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = KE + m_0 c^2 = 2 m_0 c^2$$

$$\Rightarrow 1 - v^2/c^2 = 1/4 \Rightarrow v/c = \sqrt{3}/2$$

$\approx 87\%$ of speed of light

Useful to remember

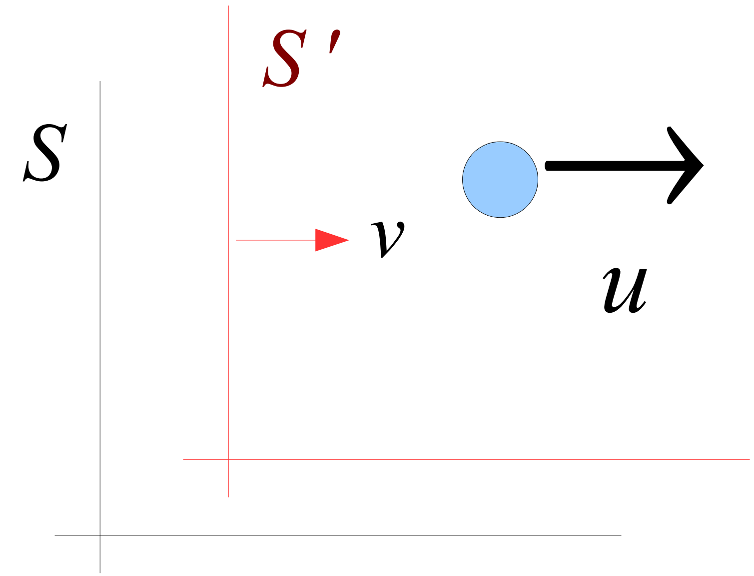
electron $m_0^{(e)} c^2 = 0.51 \text{ MeV}$

proton $m_0^{(p)} c^2 = 939 \text{ MeV} \approx 1 \text{ GeV}$

Often we say a proton's mass is 1 GeV...dropping the c^2 lazily!

How does S and S' see the p, E of the same particle?

$$S \text{ sees } \begin{cases} p_x = \dfrac{m_0 u}{\sqrt{1-u^2/c^2}} \\ E = \dfrac{m_0 c^2}{\sqrt{1-u^2/c^2}} \end{cases}$$



Q : What does S' see ?

S and S' are intertial observers. The "physical laws“ must be same & they should be able to predict what the other will see.

$$u' = \frac{u-v}{1-uv/c^2} \quad \& \quad S' \text{ must see } \begin{cases} p'_x = \dfrac{m_0 u'}{\sqrt{1-u'^2/c^2}} \\ E' = \dfrac{m_0 c^2}{\sqrt{1-u'^2/c^2}} \end{cases}$$

How does S and S' see the p, E of the same particle?

$$p'_x = \frac{m_0 u'}{\sqrt{1-u'^2/c^2}} = \frac{m_0\left(\dfrac{u-v}{1-uv/c^2}\right)}{\sqrt{1-\left(\dfrac{u-v}{1-uv/c^2}\right)^2/c^2}} = \frac{m_0(u-v)}{\sqrt{1-u^2/c^2}\sqrt{1-\beta^2}}$$

$$E' = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \frac{m_0 c^2}{\sqrt{1-\left(\dfrac{u-v}{1-uv/c^2}\right)^2/c^2}} = \frac{m_0 c^2(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-\beta^2}}$$

$$\boxed{p'_x = \frac{p_x - v\dfrac{E}{c^2}}{\sqrt{1-\beta^2}} \quad \& \quad \frac{E'}{c^2} = \frac{\dfrac{E}{c^2} - v\dfrac{p_x}{c^2}}{\sqrt{1-\beta^2}}}$$

What does this remind you of ? What about y and z momenta ?

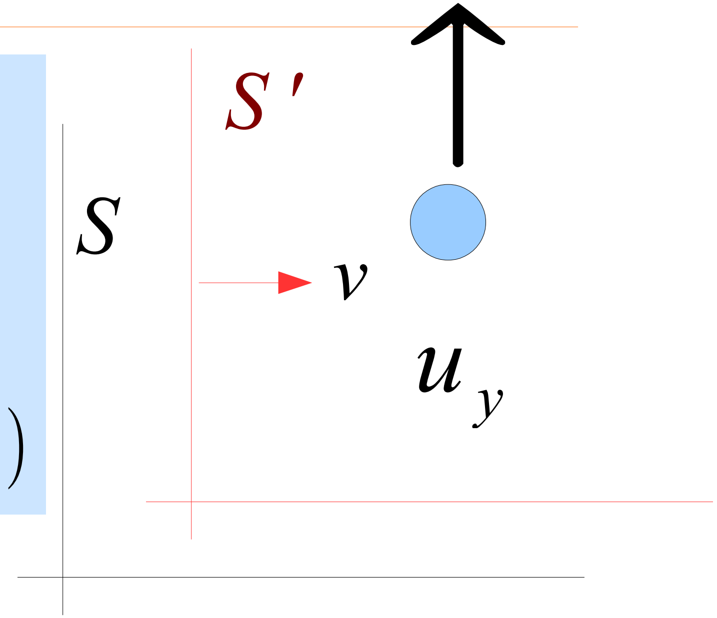
How does S and S' see p_y of the same particle?

S sees

$$\begin{cases} u_x = 0 \\ u_y = u_y \\ u^2 = u_y^{\,2} \end{cases} \Rightarrow$$

S' will see

$$\begin{cases} u'_x = -v \\ u'_y = u_y\sqrt{1-\beta^2} \\ u'^2 = v^2 + u_y^{\,2}\left(1-\beta^2\right) \end{cases}$$



$$p'_x = \frac{-m_0 v}{\sqrt{1-u'^2/c^2}} = \frac{-m_0 v}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{-m_0 v}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = \frac{0-v\left(E/c^2\right)}{\sqrt{1-\beta^2}}$$

$$p'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}} = \frac{m_0 u_y\sqrt{1-\beta^2}}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{m_0 u_y\sqrt{1-\beta^2}}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = p_y$$

$$\frac{E'}{c^2} = \frac{m_0}{\sqrt{1-u'^2/c^2}} = \frac{m_0}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{m_0}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = \frac{E/c^2}{\sqrt{1-\beta^2}}$$

The similarity between $\left(p_x, p_y, p_z, E/c\right)$ & $\left(x, y, z, ct\right)$

$$p'_x = \frac{p_x - v\dfrac{E}{c^2}}{\sqrt{1-\beta^2}} \qquad x' = \frac{x - vt}{\sqrt{1-\beta^2}}$$

$$p'_y = p_y \qquad y' = y$$

$$p'_z = p_z \qquad z' = z$$

$$\frac{E'}{c^2} = \frac{\dfrac{E}{c^2} - v\dfrac{p_x}{c^2}}{\sqrt{1-\beta^2}} \qquad t' = \frac{t - v\dfrac{x}{c^2}}{\sqrt{1-\beta^2}}$$

$\left(p_x, p_y, p_z, E/c^2\right)$ transform like $\left(x, y, z, t\right)$

There is a little bit more to it. Can you spot it?