Tutorial – 5

☆ Simple Harmonic Oscillator –

1.) The oscillator oscillates between $x = -\frac{A}{2}$ to $x = \frac{A}{2}$

$$\Rightarrow \Delta x = \frac{A}{2} - \left(-\frac{A}{2}\right) = A$$

Now, $\Delta p \, \Delta x \geq \frac{\hbar}{2}$

$$\Rightarrow \Delta p \geq \frac{\hbar}{2A}$$

$$\Rightarrow \sqrt{\langle p^2 \rangle} \geq \frac{\hbar}{2A} \qquad \langle p \rangle = 0$$

$$\Rightarrow \langle p^2 \rangle \geq \left(\frac{\hbar}{2A}\right)^2$$

$$\Rightarrow K_{min} = \frac{1}{2m}\left(\frac{\hbar}{2A}\right)^2$$

Now, $E(A) = \frac{1}{2} m\omega^2 A^2 + \frac{\hbar^2}{8mA^2}$

Now, $\left.\frac{dE}{dA}\right|_{A=A_0} = m\omega^2 A_0 - \frac{\hbar^2}{4mA_0^3} = 0$

$$\Rightarrow A_0^4 = \frac{\hbar^2}{4m^2\omega^2} \Rightarrow A_0^2 = \frac{\hbar}{2m\omega}$$

$$\therefore E(A_0) = \frac{m\omega^2}{2} \times \frac{\hbar}{2m\omega} + \frac{\hbar^2}{8m} \times \frac{2m\omega}{\hbar}$$

$$= \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4}$$

$$= \frac{\hbar\omega}{2} \longrightarrow \text{min. energy}$$

2.) $\psi_0(x) = C_0 e^{-m\omega x^2/2\hbar}$

$$V = \frac{m\omega^2 x^2}{2}$$

$$\langle V \rangle = \int_{-\infty}^{\infty} \psi_0^* \hat{V} \psi_0 \, dx$$

$$= \frac{C_0^2 m\omega^2}{2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega x^2}{\hbar}} dx$$

$$= \frac{C_0^2 m\omega^2}{2} \int_{-\infty}^{\infty} x^2 e^{-\beta x^2} dx$$

T.E.

$$= -\frac{m\omega^2}{2}\sqrt{\frac{\beta}{\pi}} \frac{d}{d\beta}\sqrt{\frac{\pi}{\beta}} = \frac{m\omega^2}{4\beta} = \frac{\hbar\omega}{4}$$

$$K.E = (n+\tfrac{1}{2})\hbar\omega - \frac{1}{2}m\omega^2 x^2$$

$$\langle K \rangle = \int_{-\infty}^{\infty} \psi_0^* \hat{K} \psi_0 \, dx$$

$$= (n+\tfrac{1}{2})\hbar\omega \int_{-\infty}^{\infty} \psi_0^* \psi_0 \, dx - \frac{\hbar\omega}{4}$$

n = 0 is ground state

$$= (0+\tfrac{1}{2})\hbar\omega - \frac{\hbar\omega}{4} = \frac{\hbar\omega}{4}$$

3.) a.) $\frac{hc}{\lambda} = (4+\tfrac{1}{2})\hbar\omega - (3+\tfrac{1}{2})\hbar\omega$ $(= E_{4\to 3})$

$$= \hbar\omega = \frac{h}{2\pi}\sqrt{\frac{K}{m}}$$

$$\Rightarrow \lambda = 2\pi c\sqrt{\frac{m}{K}}$$

$$= 2\pi \times 3\times10^{8}\sqrt{\frac{5.6\times10^{-26}}{12}}\ m$$

$$= 6\pi\times10^{8}\sqrt{\frac{1.4}{3}\times10^{-13}}\ m$$

$$= 6\pi\sqrt{\frac{7}{15}}\times10^{-5}\ m$$

$$\simeq 12.877\times10^{-5}\ m$$
$$= 1.2877\times10^{-4}\ m$$

b.) $$E_{n=0} = \frac{\hbar\omega}{2} = \frac{h}{4\pi}\sqrt{\frac{K}{m}}$$

$$= \frac{6.626\times10^{-34}}{4\pi}\sqrt{\frac{12}{5.6\times10^{-26}}}\ J$$

$$= \frac{3.313}{2\pi}\sqrt{\frac{15}{7}}\times10^{-21}\ J$$

$$\simeq 0.772\times10^{-21}\ J$$
$$= 7.72\times10^{-22}\ J$$

4.) a) $$\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{1.13\times10^{3}}{1.67\times10^{-27}}}\ s^{-1}$$

$$\simeq 0.823\times10^{15}\ s^{-1}$$
$$= 8.23\times10^{14}\ s^{-1}$$

b.) $$E_{4\to3} = \hbar\omega = \frac{6.626\times10^{-34}}{2\pi}\times 8.23\times10^{14}\ J$$

$$= 8.679\times10^{-20}\ J$$

$$\lambda = \frac{hc}{E} = \frac{hc}{\frac{h}{2\pi}\omega}$$

$$= \frac{2\pi \times 3 \times 10^{8}}{8.23 \times 10^{14}} \text{ m}$$

$$= \frac{6\pi}{8.23} \times 10^{-6} \text{ m}$$

$$\simeq 2.29 \ \mu\text{m}$$

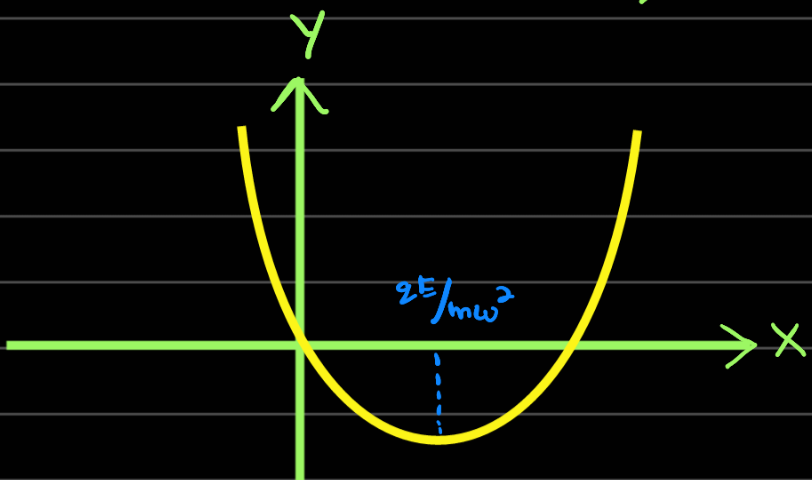
5.) a.) $V(x) = \frac{1}{2} m\omega^2 x^2 - qEx$

b.) $V(x) = \frac{1}{2} m\omega^2 \left(x^2 - \frac{2qE}{m\omega^2} x\right)$

$$= \frac{1}{2} m\omega^2 \left[\left(x - \frac{qE}{m\omega^2}\right)^2 - \left(\frac{qE}{m\omega^2}\right)^2\right]$$

$$= \frac{1}{2} m\omega^2 \left(x - \frac{qE}{m\omega^2}\right)^2 - \frac{q^2E^2}{2m\omega^2}$$

c.)



d.) $E_{n=0} = \frac{\hbar\omega}{2} - \frac{q^2E^2}{2m\omega^2}$

e.) Take $\tilde{x} = x - \frac{qE}{m\omega^2}$

$\psi_0(\tilde{x}) = C_0 e^{-\alpha \tilde{x}^2/2}$ $\left[\alpha = m\omega/\hbar, \ C_0 = \left(\frac{\alpha}{\pi}\right)^{1/4}\right]$

Now $\langle \tilde{x} \rangle = \int^{\infty} \psi^* \tilde{x} \ldots d\tilde{x}$

Now, $\langle x \rangle = \int_{-\infty} \psi_0^* \, x \psi_0 \, dx$

$$\Rightarrow \left\langle x - \frac{qE}{m\omega^2} \right\rangle = c_0^2 \int_{-\infty}^{\infty} \tilde{x} \, e^{-\alpha \tilde{x}^2} d\tilde{x}$$

$$\boxed{\Rightarrow \langle x \rangle - \frac{qE}{m\omega^2} = 0 \quad \Rightarrow \langle x \rangle = \frac{qE}{m\omega^2}}$$

6.) $\psi(x) = \frac{2\beta}{\sqrt{3}} \left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-\frac{\beta x^2}{2}}$

a.) since $-\frac{\beta x^2}{2}$ is an exponent (of e) so it has to be dimensionless.

$\Rightarrow \beta$ has dimension L^{-2}

let $\beta = \lambda m^a \omega^b \hbar^c$ ($\lambda \rightarrow$ dimensionless constant)

$$\Rightarrow [M^0 L^{-2} T^0] = [M^1 L^0 T^0]^a [M^0 L^0 T^{-1}]^b [M^1 L^2 T^{-1}]^c$$

(For M) $a + c = 0$ ——— ①

(For L) $2c = -2$ ——— ②

(For T) $-b - c = 0$ ——— ③

we have $a = 1 = b$, $c = -1$

So $\beta = \frac{\lambda m \omega}{\hbar}$

b.) $\psi(x) = a\psi_0(x) + b\psi_2(x)$

$$a = \int^{\infty} \psi_0^*(x) \, \psi(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{\beta}{\pi}\right)^{1/4} e^{-\frac{\beta x^2}{2}} \left(\frac{2\beta}{\sqrt{3}}\right)\left(\frac{\beta}{\pi}\right)^{1/4} x^2 e^{-\frac{\beta x^2}{2}} dx$$

$$= 2\beta\sqrt{\frac{\beta}{3\pi}} \int_{-\infty}^{\infty} x^2 e^{-\beta x^2} dx$$

$$= -2\beta\sqrt{\frac{\beta}{3\pi}} \frac{d}{d\beta}\sqrt{\frac{\pi}{\beta}}$$

$$= -2\beta\sqrt{\frac{\beta}{3}} \frac{d}{d\beta} \beta^{-1/2}$$

$$a^2 + b^2 = 1$$

$$= \beta\sqrt{\frac{\beta}{3}} \times \frac{1}{\beta\sqrt{\beta}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{b = \sqrt{1-a^2} = \sqrt{\frac{2}{3}}} \quad \star$$

$$\boxed{\langle E\rangle = \sum \text{probability} \times \text{eigen value}}$$

$$= \left(\frac{1}{\sqrt{3}}\right)^2 \times \left(0+\frac{1}{2}\right)\hbar\omega + \left(\sqrt{\frac{2}{3}}\right)^2 \left(2+\frac{1}{2}\right)\hbar\omega$$

$$= \frac{\hbar\omega}{6} + \frac{10}{6}\hbar\omega$$

$$= \frac{11\hbar\omega}{6}$$