$\bar{r} = r\hat{r}$; Actually $\hat{r} = \hat{r}(\theta)$ but can write $\hat{r}(r,\theta)$ also & see what happens

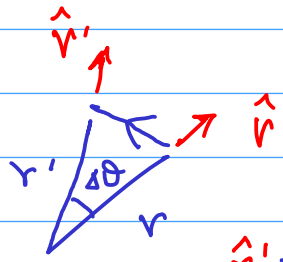
$\bar{v} = \dot{\bar{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$

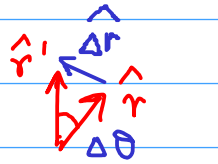
Now $\frac{d\hat{r}}{dt} = \frac{\partial \hat{r}}{\partial r}\dot{r} + \frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt}$

$= \text{zero} + \hat{\theta}\dot{\theta}$

proof next pg

proof below





this is eqv to having $\hat{r} = \hat{r}(\theta)$ from the beginning where $\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$

same

$\frac{\Delta\hat{r}}{\Delta\theta} = \frac{1.\Delta\theta\,\hat{\theta}}{\Delta\theta} = \hat{\theta}$ ans

Alternatively, can work with the vecs instead of unit vecs

Proof

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$$

$\frac{\partial \hat{r}}{\partial \theta} = \lim_{\Delta\theta\to 0}\frac{\Delta\hat{r}}{\Delta\theta} = \frac{\hat{r}' - \hat{r}}{\Delta\theta} = \frac{1}{\Delta\theta}\left(\frac{\bar{r}'}{r'} - \frac{\bar{r}}{r}\right)$

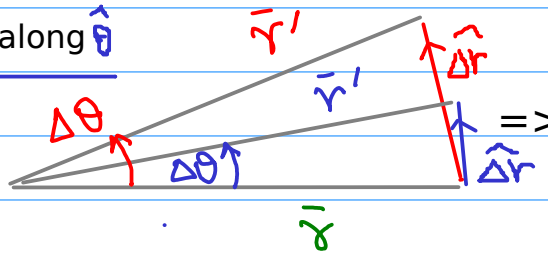
* value of r not changed, only theta changed

~~let~~ but $r' = r$ (same magnitude)

$= \frac{1}{\Delta\theta}\cdot\frac{1}{r}(\bar{r}' - \bar{r})$

$= \frac{1}{r\Delta\theta}(r\Delta\theta)\hat{\theta}$ (as $\Delta\theta \to 0$)

Why $\Delta\hat{r}$ points along $\hat{\theta}$



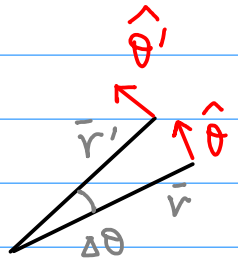
=> As $\Delta\theta \to 0$ $\Delta\hat{r}$ becomes $\perp$ to $\bar{r}$ ie, along $\hat{\theta}$

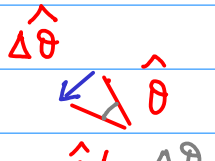
$r = r'$ (same mag.)

Similarly let's evaluate $\frac{\partial \hat{\theta}}{\partial \theta} = \lim_{\Delta\theta\to 0}\frac{\Delta\hat{\theta}}{\Delta\theta}$

Proof

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$





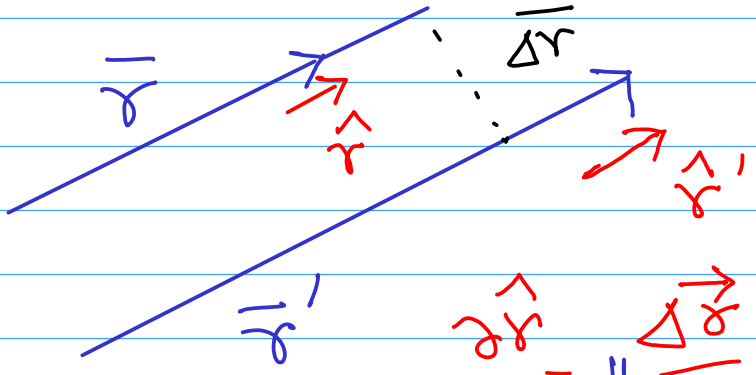
$\frac{1.\Delta\theta}{\Delta\theta}(-\hat{r})$

$= -\hat{r}$

$\vec{r}' = \vec{r} + \Delta\vec{r}$

How do $\hat{r}, \hat{\theta}$ change when only r changed

$\Delta\hat{r} = \hat{r}' - \hat{r} = 0$



$$\frac{\partial \hat{r}}{\partial r} = \lim_{\Delta r \to 0} \frac{\Delta \hat{r}}{\Delta r} = 0$$

Also $$\frac{\partial \hat{\theta}}{\partial r} = \lim_{\Delta r \to 0} \frac{\Delta \hat{\theta}}{\Delta r} = 0$$

$\hat{r}, \hat{\theta}$ do not change by change in r