Notes on pg-2

J. Peraire
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Lecture D30 - Orbit Transfers

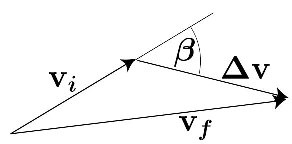
In this lecture, we will consider how to transfer from one orbit, or trajectory, to another. One of the assumptions that we shall make is that the velocity changes of the spacecraft, due to the propulsive effects, occur instantaneously. Although it obviously takes some time for the spacecraft to accelerate to the velocity of the new orbit, this assumption is reasonable when the burn time of the rocket is much smaller than the period of the orbit. In such cases, the Δv required to do the maneuver is simply the difference between the velocity of the final orbit minus the velocity of the initial orbit.

When the initial and final orbits intersect, the transfer can be accomplished with a singe impulse. For more general cases, multiple impulses and intermediate transfer orbits may be required.

Given initial and final orbits, the objective is generally to perform the transfer with a minimum Δv. In some situations, however, the time needed to complete the transfer may also be an important consideration.

Most orbit transfers will require a change in the orbit's total specific energy, E. Let us consider the change in total energy obtained by an instantaneous impulse Δv. If $\boldsymbol{v}_i$ is the initial velocity, the final velocity, $\boldsymbol{v}_f$, will simply be,

$$\boldsymbol{v}_f = \boldsymbol{v}_i + \Delta\boldsymbol{v} \ .$$



If we now look at the magnitude of these vectors, we have,

$$v_f^2 = v_i^2 + \Delta v^2 + 2 v_i \Delta v \cos\beta,$$

where β is the angle between $\boldsymbol{v}_i$ and Δv. The energy change will be

$$\Delta E = \frac{1}{2}\Delta v^2 + v_i \Delta v \cos\beta \ .$$

From this expression, we conclude that, for a given Δv, the change in energy will be largest when:

- $\boldsymbol{v}_i$ and $\Delta\boldsymbol{v}$ are co-linear ($\beta = 0$), and,
- v_i is maximum.

For example, to transfer a satellite on an elliptical orbit to an escape trajectory, the most energy efficient impulse would be co-linear with the velocity and applied at the instant when the satellite is at the elliptical orbit's perigee, since at that point, the velocity is maximum. Of course, for many required maneuvers, the applied impulses are such that they cannot satisfy one or both of the above conditions. For instance, firing at the perigee in the previous example may cause the satellite to escape in a particular direction which may not be the required one.

Hohmann Transfer

A Hohmann Transfer is a two-impulse elliptical transfer between two co-planar circular orbits. The transfer itself consists of an elliptical orbit with a perigee at the inner orbit and an apogee at the outer orbit.

Derivatn of Eq.1,2 : $E_T = \frac{\mu}{2}\dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{k}{r}$ - - - (1.1)

On C_1, C_2 & at π, α pnts of E_l we hv $\dot{r} = 0$

Also $l = \mu v r$ (ang. mom) on C_1, C_2, π, α

$l_{C_1} \neq l_{C_2} \neq l_{E_l}$ but $l_\pi = l_\alpha$; $v_\pi r_1 = v_\alpha r_2$

Note: At α, π $\bar{r} \times \bar{v} = rv$ ∵ ⊥ (on the ellipse)

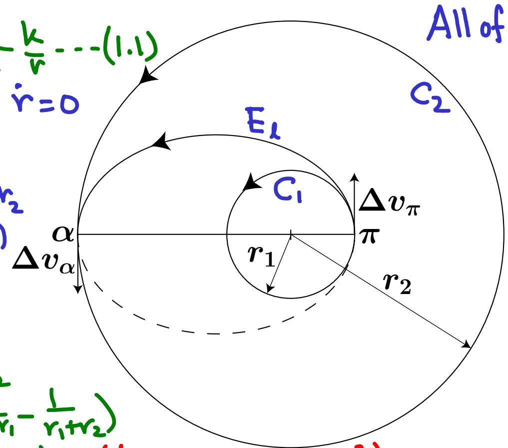
So, $E_T = \frac{\mu v^2}{2} - \frac{k}{r}$ (from 1.1)

C_1: $-k/2r_1 = \frac{\mu v_1^2}{2} - \frac{k}{r_1} \Rightarrow v_1^2 = k/\mu r_1$

C_2: similarly, - - - $\Rightarrow v_2^2 = k/\mu r_2$

C_π: $-\frac{k}{r_1+r_2} = \frac{\mu}{2}v_\pi^2 - k/r_1 \Rightarrow v_\pi^2 = \frac{2k}{\mu}\left(\frac{1}{r_1} - \frac{1}{r_1+r_2}\right)$

C_α: similarly - - - $\Rightarrow v_\alpha^2 = \frac{2k}{\mu}\left(\frac{1}{r_2} - \frac{1}{r_1+r_2}\right)$ ($\frac{k}{\mu}$ replaced by gR^2)



All of them hv bound orbits, so $E_T = -$ve (Tot. Energy)

$E_T = 0$

C_2 —— $-\frac{k}{2r_2}$

E_l —— $-\frac{k}{r_1+r_2}$

C_1 —— $-\frac{k}{2r_1}$

System transits from $C_1 \to E_l \to C_2$

Since $C_1 \to E_l$ occurs at same PE $= -k/r_1$

E_l has higher KE than $C_1 \Rightarrow v_\pi > v_{C_1}$

Similarly, for $E_l \to C_2$ transfer $v_{C_2} > v_\alpha$

From $\pi \to \alpha$ vel reduced on ellipse.

Note → $v_{C_2} < v_{C_1}$ *

It turns out that this transfer is usually optimal, as it requires the minimum $\Delta v_T = |\Delta v_\pi| + |\Delta v_\alpha|$ to perform a transfer between two circular orbits. The exception for which Hohmann transfers are not optimal is for very large ratios of r_2/r_1, as discussed below.

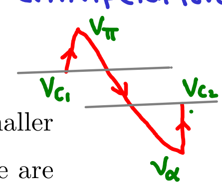
* for circular orbit larger the rads lower is the vel.

∵ $\frac{k}{r^2} = \frac{\mu v^2}{r}$ (grav. attractn = centripetal force)

$v^2 \propto 1/r$

The transfer orbit has a semi-major axis, a, which is

$$a = \frac{r_1 + r_2}{2} .$$



$\Delta V > 0$ at both π & α pnts

Hence, the energy of the transfer orbit is greater than the energy of the inner orbit ($a = r_1$), and smaller than the energy of the outer orbit ($a = r_2$). The velocities of the transfer orbit at perigee and apogee are given, from the conservation of energy equation, as

grav. accl^n g introduced to replace k/μ

Wt of mass μ : $\mu g = \frac{k}{R^2}$ (attractn of Earth.)

$\Rightarrow \frac{k}{\mu} = gR^2$

$$v_\pi^2 = gR^2\left(\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right) \qquad (1)$$

$$v_\alpha^2 = gR^2\left(\frac{2}{r_2} - \frac{2}{r_1 + r_2}\right) . \qquad (2)$$

Here, we have assumed that the orbits are around the earth and have replaced $\frac{k}{\mu}$ by gR^2, where g is the acceleration due to gravity on the earth's surface and R is the radius of the earth.

here μ is diff from the μ we used in lecture.

The velocities of the circular orbits are $v_{c1} = \sqrt{gR^2/r_1}$ and $v_{c2} = \sqrt{gR^2/r_2}$. Hence, the required impulses at perigee and apogee are,

$$\begin{aligned}
\Delta v_\pi &= v_\pi - v_{c1} = R\sqrt{\frac{g}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right) \\
\Delta v_\alpha &= v_{c2} - v_\alpha = R\sqrt{\frac{g}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right) .
\end{aligned}$$

When the initial orbit has a radius larger than the final orbit, the same strategy can be followed but in this case, negative impulses will be required, first at apogee and then at perigee, to decelerate the satellite.

Example ***Hohman transfer [1]***

A communication satellite was carried by the Space Shuttle into low earth orbit (LEO) at an altitude of 322 km and is to be transferred to a geostationary orbit (GEO) at $35,860$ km using a Hohmann transfer. The characteristics of the transfer ellipse and the total Δv required, Δv_T, can be determined as follows:
For the inner orbit, we have,

$$\begin{aligned}
r_1 &= R + d_1 = 6.378 \times 10^6 + 322 \times 10^6 = 6.70 \times 10^6 \,\mathrm{m} \\
v_{c1} &= \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{gR^2}{r_1}} = 7713 \,\mathrm{m/s} \\
E_1 &= -\frac{\mu}{2r_1} = -\frac{gR^2}{2r_1} = -2.975 \times 10^7 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}).
\end{aligned}$$

Similarly, for the outer circular orbit,

$$\begin{aligned}
r_2 &= R + d_2 = 42.24 \times 10^6 \,\mathrm{m} \\
v_{c2} &= 3072 \,\mathrm{m/s} \\
E_2 &= -4.718 \times 10^6 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}).
\end{aligned}$$

For the transfer trajectory,

$$\begin{aligned}
2a &= r_1 + r_2 = 48.94 \times 10^6 \,\mathrm{m} \\
E &= -\frac{\mu}{2a} = -8.144 \times 10^6 \,\mathrm{m^2/s^2}\ (\mathrm{J/kg}),
\end{aligned}$$

which shows that $E_1 < E < E_2$. The velocity of the elliptical transfer orbit at the perigee and apogee can be determined from equations 1 and 2, as,

$$\begin{aligned}
v_\pi &= 10,130 \,\mathrm{m/s} , \\
v_\alpha &= 1,067 \,\mathrm{m/s} .
\end{aligned}$$

Since the velocity at the perigee is orthogonal to the position vector, the specific angular momentum of the transfer orbit is,

$$h = r_1 v_\pi = 6.787 \times 10^{10} \,\mathrm{m^2/s} ,$$

and the eccentricity can be determined as,

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}} = 0.7265\ .$$

Finally, the impulses required are,

$$\begin{aligned}
\Delta v_\pi &= v_\pi - v_{c1} = 10,130 - 7713 = 2414\,\mathrm{m/s} \\
\Delta v_\alpha &= v_{c2} - v_\alpha = 3072 - 1607 = 1465\,\mathrm{m/s},
\end{aligned}$$

The sign of the Δv's indicates the direction of thrusting (whether the energy is to be increased or decreased) and the total Δv is the sum of the magnitudes. Thus,

$$\Delta v_T = |\Delta v_\pi| + |\Delta v_\alpha| = 2417 + 1465 = 3882\,\mathrm{m/s}.$$

Since the transfer trajectory is one half of an ellipse, the time of flight (TOF) is simply half of the period,

$$TOF = \pi\sqrt{\frac{(24.47 \times 10^6)^3}{3.986 \times 10^{14}}} = 19,050\,\mathrm{s} = 5.29\,\mathrm{h}\ .$$

In order to illustrate the optimal nature of the Hohmann transfer, we consider now an alternative transfer in which we arbitrarily double the value of the semi-major axis of the Hohmann transfer ellipse, and find the characteristics and Δv_T of the resulting fast transfer. The semi-major axis of the transfer ellipse will be $2a = 98 \times 10^6$ m, and $E = -\mu/(2a) = -4.067 \times 10^6$ m^2/s^2 (J/kg). The velocity of the transfer orbit at departure will be

$$v_\pi = \sqrt{2\left(-4.067 \times 10^6 + \frac{3.986 \times 10^{14}}{6.70 \times 10^6}\right)} = 10.530\,\mathrm{m/s},$$

and,

$$\Delta v_\pi = 10,530 - 7713 = 2817\,\mathrm{m/s}\ .$$

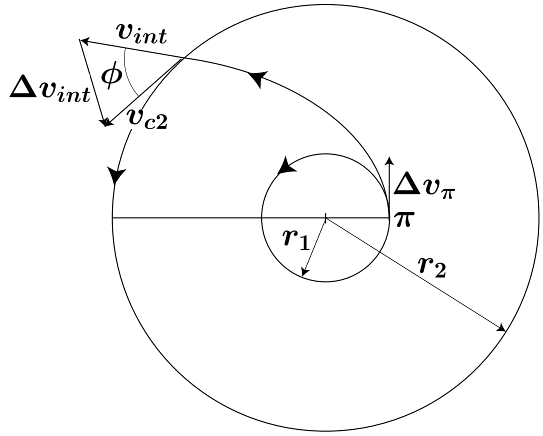
The specific angular momentum is,

$$h = 10,530 \times (6.70 \times 10^6) = 7.055 \times 10^{10}\,\mathrm{m^2/s},$$

and the orbit's eccentricity, e, is 0.863.

We can now calculate, from the energy conservation equation, the velocity of the transfer orbit at the point of interception with the outer orbit, v_{int},

$$v_{int} = \sqrt{2\left(-4.067 \times 10^6 + \frac{3.986 \times 10^{14}}{42.24 \times 10^6}\right)} = 3277\,\mathrm{m/s}\ .$$



Since the angular momentum, h, is conserved, we can determine the component of $\boldsymbol{v}_{int}$ in the circumferential direction

$$(\boldsymbol{v}_{int})_\theta = \frac{h}{r_2} = 1670\,\mathrm{m/s}$$

and the elevation angle, ϕ, is thus,

$$\phi = \cos^{-1}\frac{(\boldsymbol{v}_{int})_\theta}{v_{int}} = 59.36^o$$

Finally, from geometrical considerations,

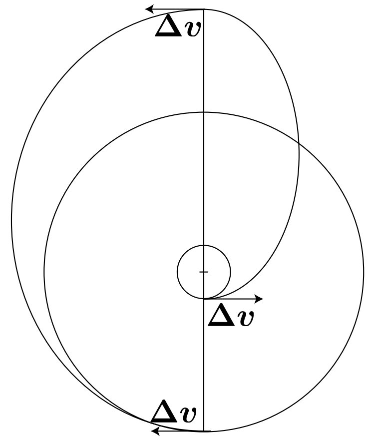
$$\Delta v_{int}^2 = v_{c2}^2 + v_{int}^2 - 2v_{c2}v_{int}\cos\phi,$$

which yields $\Delta v_{int} = 3142$ m/s, so that

$$\Delta v_T = 2817 + 3142 = 5959\,\mathrm{m/s}.$$

Comparing to the value of the Hohmann transfer Δv_T of 3875 m/s, we see that the fast transfer requires a Δv_T which is 54% higher.

It can be shown that when the separation between the inner and outer orbits is very large ($r_2 > 11.9r_1$) (a situation which rarely occurs), a three impulse transfer comprising of two ellipses can be more energy efficient than a two-impulse Hohmann transfer. This transfer is illustrated in the picture below. Notice that the distance from the origin at which the two transfer ellipses intersect is a free parameter, which can be determined to minimize the total Δv. Notice also that the final impulse is a Δv which opposes direction of motion, in order to decelerate from the large energy ellipse to the final circular orbit. Although this transfer may be more energy efficient relative to the two-impulse Hohmann transfer, it often involves much larger travel times. Try it!



Interplanetary Transfers

When more than one planet is involved, the problem is no longer a two-body problem. Nevertheless, it is common (at least to get a good approximation) to decompose the problem into a series of two body problems. Consider, for example, an interplanetary transfer in the solar system. For each planet we define the sphere of influence (SOI). Essentially, this is the region where the gravitational attraction due to the planet is larger than that of the sun.

The mission is broken into phases that are connected by patches where each patch is the solution of a two body problem. This is called the patched conic approach. Consider, for instance, a mission to Mars. The first phase will consist of a geocentric hyperbola as the spacecraft escapes from earth SOI. The second phase would start at the edge of the earth's SOI, and would be an elliptical trajectory around the sun while the spacecraft travels to Mars. The third phase would start at the edge of Mars' SOI, and would be a hyperbolic approach trajectory with the gravitational field of Mars as the attracting force.

For long missions, the situation can become very complex as one often tries to take advantage of the gravitational fields of the planets encountered on the way, by entering into their SOI's with the objective of either changing direction or gaining additional impulse. This technique is often referred to as *gravity assist*.

References

[1] F.J. Hale, *Introduction to Space Flight*, Prentice-Hall, 1994.