PH 112 : Introduction to Quantum Physics : General policies

Division D1
April – June 2023
Lecture timings @ LA201
Mon 930 -- 1025
Tue 1035 -- 1130
Thu 1135 – 1230

Evaluation (approx) :

Tutorial attendance	**20%**
Moodle Quizzes	**20%**
Quiz 1	**20%**
Endsem	**40%**
No DX grade	

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Consultation Hours: Lecture hours, any other time by e-mail

What was understood and what was not : circa ~1900

Successes	Failures
Newtonian Mechanics : Motion of planets etc	Radiation from hot objects (Blackbody radiation)
Electromagnetic basis of light/optics with some phenomenological input	Photoelectric effect : why it has a threshold ?
Gas Laws and fluid mechanics	Specific heats of materials at low temperatures
Thermodynamic principles	Scattering of X-ray by electrons (Compton scattering)
Engineering of ships, trains, engines, motors...which utilised all these.	Stability of the atom...

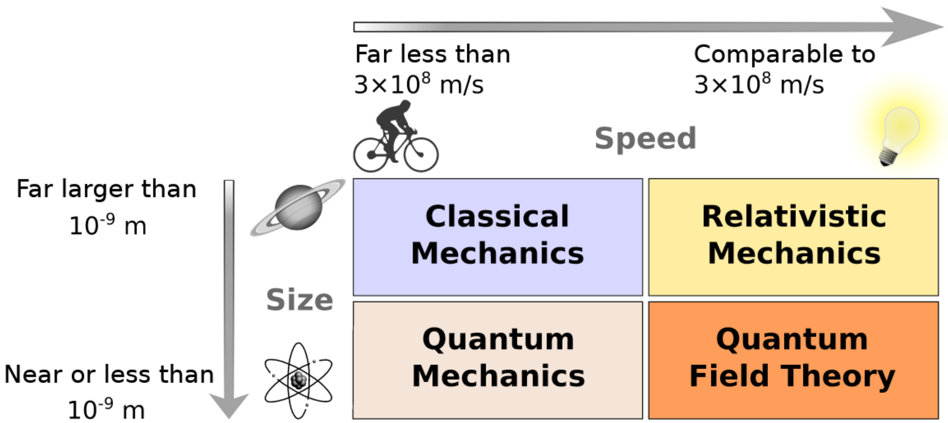
The atomistic nature of matter was NOT established ! Einstein's "dissertation" is on Brownian motion, which he used to argue for existance of atoms as building blocks.

Topics we are going to do & References

[1] Brief historical description about the need for quantum theory
[2] Compton scattering : Xray photon & electron
[3] Wave-particle duality, de Broglie hypothesis
[4] Concept of wave packets, group velocity, phase velocity, etc
[5] Mathematical interlude: discussion of Fourier transform, Gaussian wave packets etc.
[6] Heisenberg uncertainty principle
[7] Schrödinger equation and its applications to simple problems such as free particle, particle in a box, potential barriers and wells, and 1D simple harmonic oscillator
[8] Brief discussion of how to solve problems in 2D & 3D

- **Modern Physics:** R. A.Serway, C. J. Moses, C. A. Moyer, Thomson Learning Inc. 2005
- **Concepts of Modern Physics:** A. Beiser, S. Mahajan, S. Rai Choudhury; McGraw Hill International,1987 4th Ed...

The new developments in ~ 1900-1930 are still called Modern Physics !!



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Waves and particles

Photoelectric effect , Compton scattering are examples where lightwaves behave like a particle. We treat them using some Energy-momentum conservation rules

There are situations where particles (like electrons) behave like a wave. These are called matter waves.

A critical question: What wave equation can correcty describe the behaviour of waves of matter? We will return to this key question later

What are the expectations and observations ?

Classcial EM can desribe the scattering of light from things like ~1 micron dust particle quite well.

Xray are also EM waves. So think of an electron as a very small particle and repeat the calculation.

One finds that the expectation is NOT met. This is the key problem.

The solution was found by independently by Compton and Debye (~1922-23)

Compton scattering
Particle like behaviour of EM wave

The momentum of light (any EM radiation)

Classical EM theory predicts that electromagnetic field has a certain energy (U) and momentum (P) per unit volume. For a field corresponding to an EM wave U = Pc

Light exerts pressure. If I is the energy density then the radiation pressure is I/c. It is fairly easy to show this !

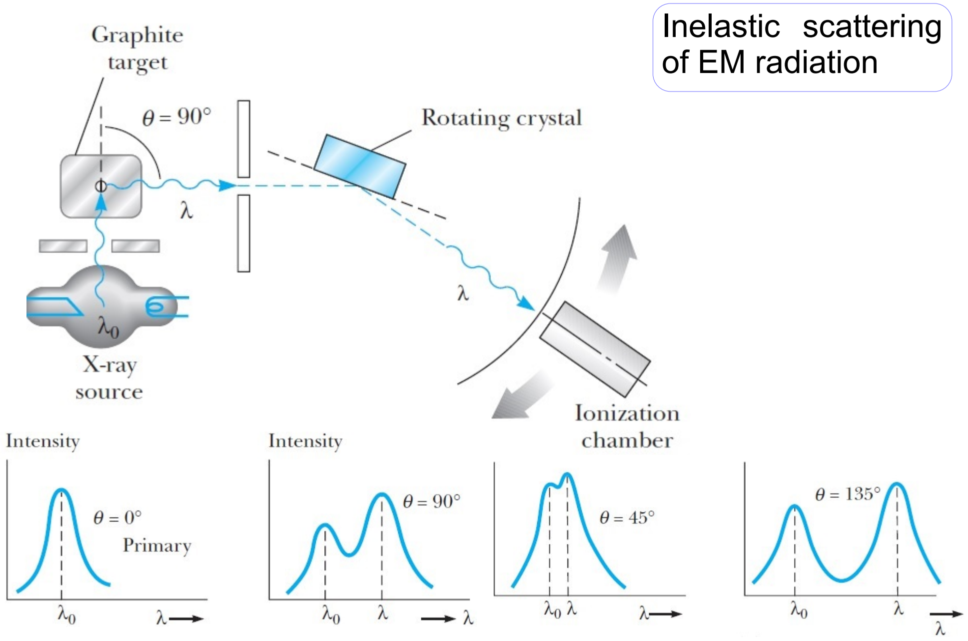
If the EM field is replaced by "quanta" of light, then those must carry energy and Momentum. What experiment demonstrates the momentum of light?

$$U=\frac{1}{2}\int_{\substack{all\\space}}\left(\epsilon_0 E^2+\frac{B^2}{\mu_0}\right)d^3\vec{r}$$

$$\vec{P}=\frac{1}{c^2}\int_{\substack{all\\space}}\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)d^3\vec{r}$$

What did Compton observe ?

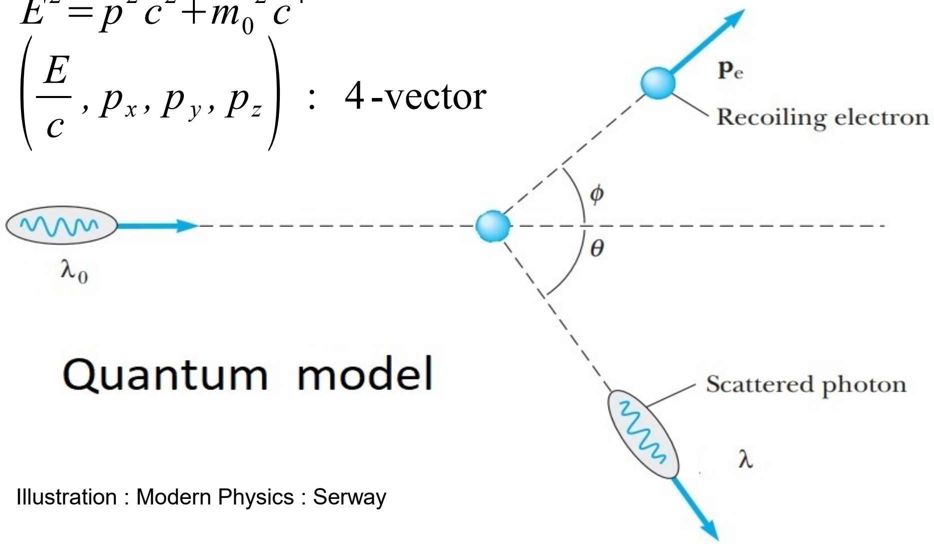
Inelastic scattering of EM radiation



The relativistic quantum view of e^--photon scattering

$$E^2 = p^2 c^2 + m_0^2 c^4$$

$$\left(\frac{E}{c}, p_x, p_y, p_z\right) \quad : \quad 4\text{-vector}$$



Quantum model

Illustration : Modern Physics : Serway

Each component of the 4-vector is conserved....

What did Compton observe ?

A few non-trivial details....

Why is the target Graphite ?

The loosely bound electrons are almost free with a work function of ~ 4eV

What is the role of the crystal?

The crystal is like a diffraction grating for X-rays.
The lattice of atoms work like opaque-transparent-opaque periodicity of a grating.
Different wavelengths will have different angular deviations.
Notice that here the X-rays are behaving like EM waves!

What do the core-electrons of the atoms do ?

The core electrons are very tightly bound. They give characteristic lines. That is not elastic either, but that is not scattering by a free electron at rest

Using the relativistic Energy-momentum relation : $E^2 = p^2 c^2 + m_0^2 c^4$ .....

$$E = h\nu_0 + m_0 c^2 = h\nu + \sqrt{p_e^2 c^2 + m_0^2 c^4}$$

$$P_x = \frac{h\nu_0}{c} = \frac{h\nu}{c}\cos\theta + p_e\cos\phi$$

$$P_y = 0 = \frac{h\nu}{c}\sin\theta - p_e\sin\phi$$

$$p_e \sin\phi = \frac{h\nu}{c}\sin\theta$$

$$p_e \cos\phi = \frac{h\nu_0}{c} - \frac{h\nu}{c}\cos\theta$$

$$\Rightarrow p_e^2 c^2 = (h\nu)^2 + (h\nu_0)^2 - 2h^2\nu\nu_0\cos\theta$$

We can calculate the same quantity in another way →

What does Energy conservation tell us for e^--Xray scattering ?

$$p_e^2 c^2 + m_0^2 c^4 = \left[h(\nu_0 - \nu) + m_0 c^2\right]^2$$
$$\Rightarrow \quad p_e^2 c^2 = h^2(\nu_0-\nu)^2 + 2h(\nu_0-\nu) m_0 c^2$$

$$(h\nu)^2 + (h\nu_0)^2 - 2h^2 \nu\nu_0 \cos\theta = h^2(\nu_0-\nu)^2 + 2h(\nu_0-\nu)m_0 c^2$$

$$\left.\lambda - \lambda_0 = \frac{h}{m_0 c}(1-\cos\theta)\right\} \begin{array}{l} \Delta\lambda = 0 \quad \text{for} \quad \theta = 0 \\ \text{largest} \quad \text{for} \quad \theta \to \pi \end{array}$$
$$\frac{h}{m_0 c} = 2.43\times 10^{-12}\, m \Rightarrow \frac{\Delta\lambda}{\lambda} \approx 0.02 \text{ for } 1\, A^o \text{ X-ray}$$

Does an analogue of Compton effect exist for visible light... ?

Unless the wavelength was already small the fractional change would be too small to detect.

Also a heavier particle (like the nucleus) would give a smaller shift...

Compton scattering of visible light by free electrons would be too small to detect. BUT....could there be other "objects"‘ causing ineleastic scattering of visible light ?

If so extra lines would appear in the spectrum of a monochromatic light when scattered by atoms/molecules of a transparent medium.

This indeed happens. This is the Raman Effect.

Can the photon disappear (get absorbed) in the scattering process?

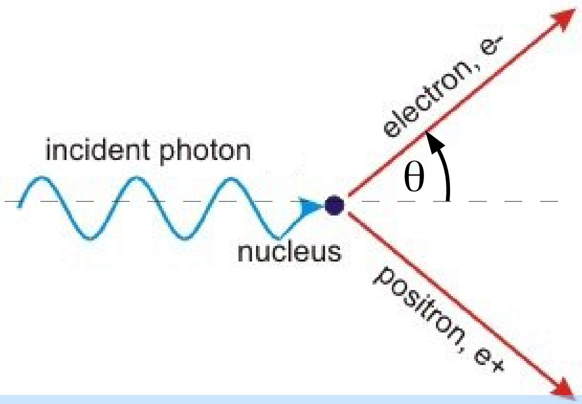
$$\left.\begin{array}{rcl} \dfrac{h\nu_0}{c} + 0 &=& p_e \\ h\nu_0 + m_0 c^2 &=& \sqrt{p_e^2 c^2 + m_0^2 c^4} \end{array}\right\} \Longrightarrow h\nu = 0$$

We assumed the electron was at rest. That is sufficient. WHY? We just go to the rest frame of the electron...the conclusions will not change.

In any “light-matter” interaction, both energy and momentum MUST to be conserved. Then how does an atom go to an excited state ?

The electron needs to be bound. The photon momentum can be absorbed by the much heavier system (atom) with negligible KE. The energy goes to raise the electron to a higher energy state.

When can a photon produce an electron-positron pair ? $h\nu \to e^+ + e^-$?



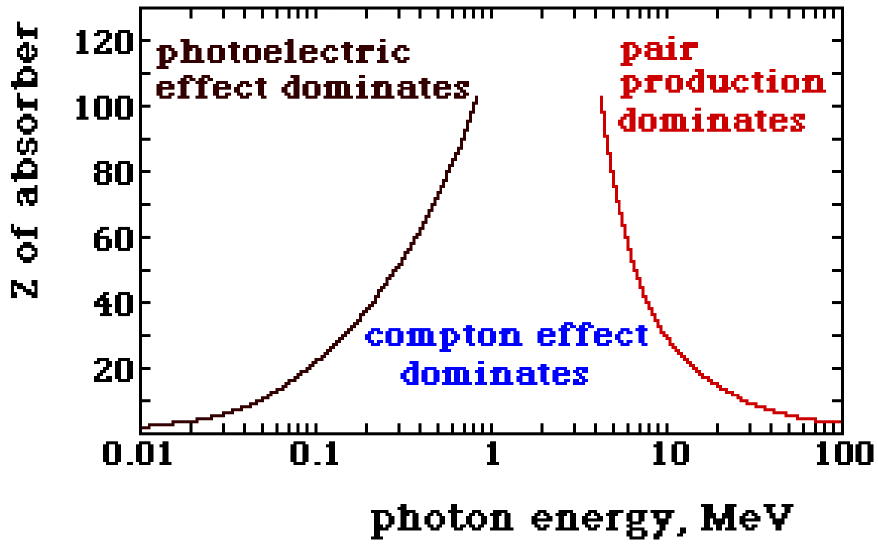
$$\frac{h\nu_0}{c} = 2 p_e \cos\theta$$
$$h\nu_0 = 2\sqrt{p_e^2 c^2 + m_0^2 c^4}$$

Why is the presence of a heavy object (nucleus) needed for this process ?

The E,p equations have no solution unless something can take away the momentum.

The heavy nucleus can take away the momentum but very little Kinetic energy

The various energy regimes of light-matter interaction



De-Broglie hypothesis (1923)
wave like behaviour of particles

de-Broglie hypothesis

de-Broglie combined the arguments from special relativity and Planck's hypothesis.

Light quanta has $E = h\nu$, speed c, rest mass $m_0 = 0$

Relation between E and p is given by $p = \dfrac{E}{c} = \dfrac{h}{\lambda}$

What if this is applied to objects that we see ?

$$\lambda = \frac{h}{p} \quad \text{OR} \quad p = \frac{h}{\lambda} = \hbar|\vec{k}|$$

For everyday objects what are $\lambda = h/p$ like ?

400 gms, ≈ 200 kmph (fastest free kick)

$$\lambda = \frac{6.63 \times 10^{-34}}{0.4 \times 55} \approx 10^{-35} \text{ meter}$$

150 gms, ≈ 160 kmph (fastest delivery)

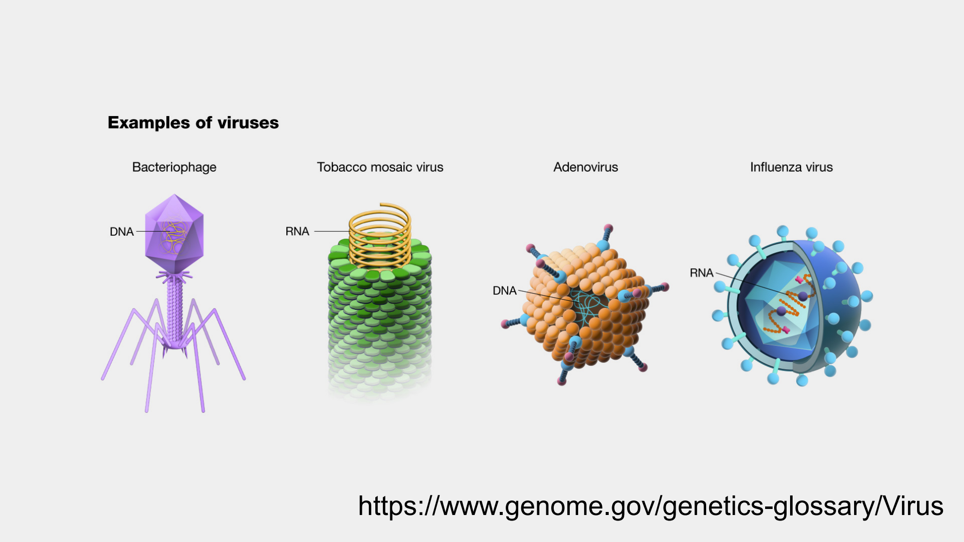
$$\lambda = \frac{6.63 \times 10^{-34}}{0.15 \times 40} \approx 10^{-34} \text{ meter}$$

These lengths are so small that they have no observational consequence.

Unless the mass is extremely small the hypothesis has litte effect

Classical behaviour of everyday objects is preserved.
Recall the "geometric optics" condition

For everyday objects what are $\lambda = h/p$ like ?



A virus weighs about 10^{-20} *kg*. At what speed does it have to move such that its de-Broglie wavelength is 1 angstrom *?*

An electron accelerated by $100\,eV$: What is λ *?*

$$\lambda = \frac{h}{p} \quad \& \quad \frac{p^2}{2m} = |eV| \quad \Rightarrow \lambda = \frac{h}{\sqrt{2m|eV|}} \approx 1.3 \text{ Å}$$

Where would this low voltage be useful ?

Does relativistic correction matter in electron diffraction and electron microscopes ?

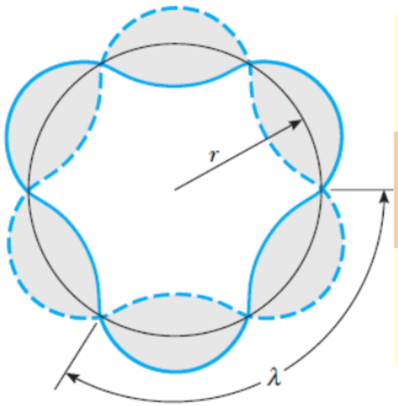
	Wavelength λ (nm)	
V (kV)	Uncorrected	Relativistically corrected
20	0·0086	0·0086
40	0·0061	0·0060
60	0·0050	0·0049
80	0·0043	0·0042
100	0·0039	0·0037
200	0·0027	0·0025
300	0·0022	0·0020
400	0·0019	0·0016
500	0·0017	0·0014
1000	0·0012	0·0009

But when applied to electrons it gives something meaningful

First Bohr orbit of hydrogen atom

$$m_0 = 9.1 \times 10^{-31} \text{ kg} : v = 2.19 \times 10^6\, ms^{-1} \Rightarrow \lambda = 3.3 \text{ Å}$$

$$\text{Bohr condition : } mvr = \frac{nh}{2\pi} \Rightarrow n\left(\frac{h}{mv}\right) = 2\pi r$$

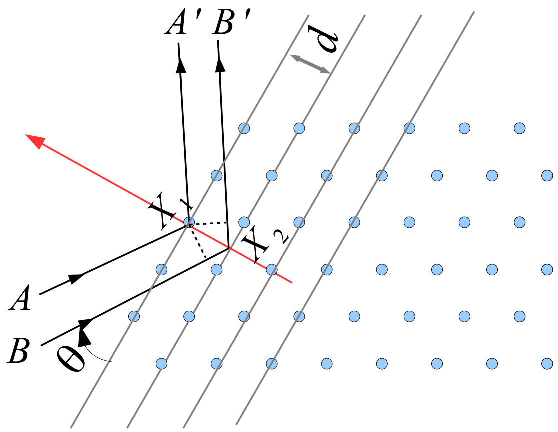


An integer number of wavelenghts would fit in the path of the electron. Like standing waves on a string

Ultimately quantum mechanics would drop the notion of path completely.....but this is a good start

If this is true then electrons must show the signature property of a wave : interference and diffraction

What can we use as a grating for these electrons ?

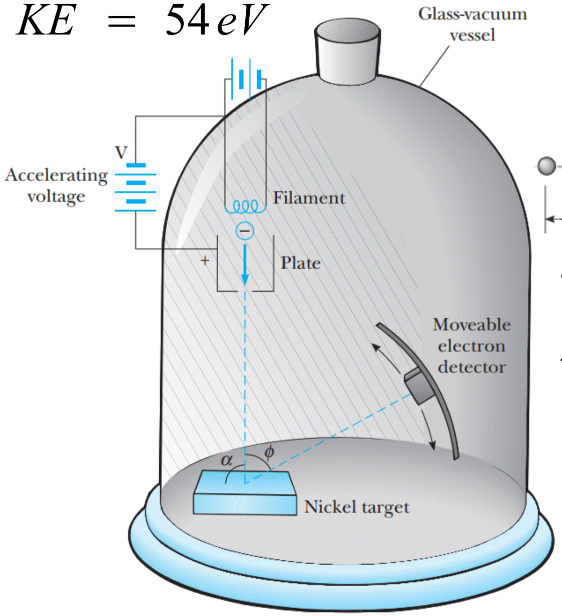


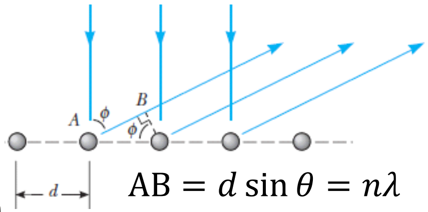
$$BX_2B' - AX_1A' = 2d\sin\theta = n\lambda$$

Bragg (1913) & von Laue had already shown this diffraction using X-ray. Typical lattice constants are 2-3 Å

The Davison-Germer experiment (1923-27)

$$KE = 54\,eV$$



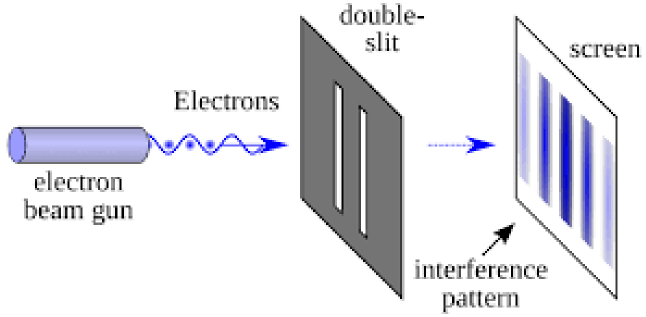


$$AB = d \sin\theta = n\lambda$$

The result matches the lattice constant obtained from X-ray diffraction

Q: Why is a high vacuum needed?

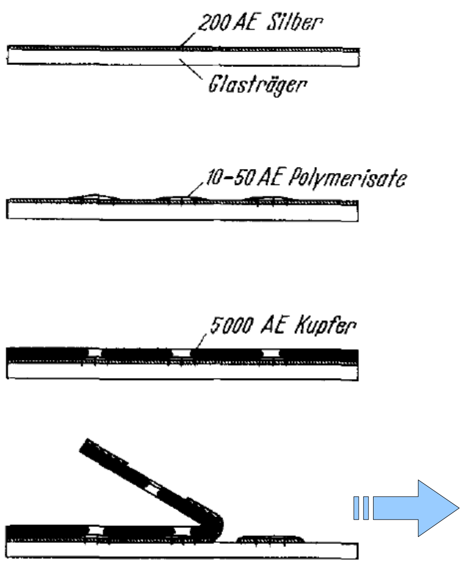
Young's double slit experiment : Points to think about



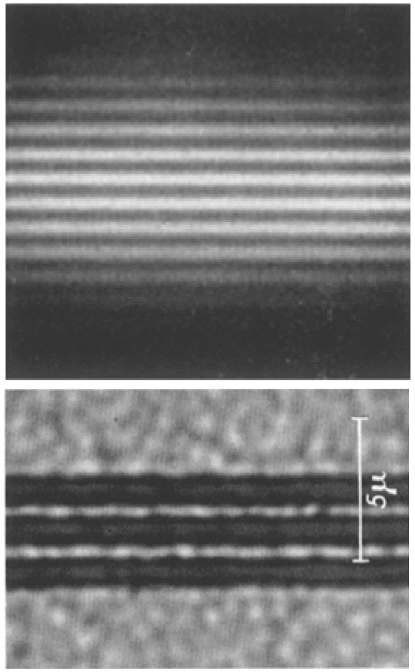
The interference pattern is NOT lost even when the intensity is so low that at any time only ONE electron is there on the average.

The electron does not need to interact with other electrons. It interferes with itself!

Clauss Joensson, Zeitschrift fur Physik, 161, 454-474 (1961)



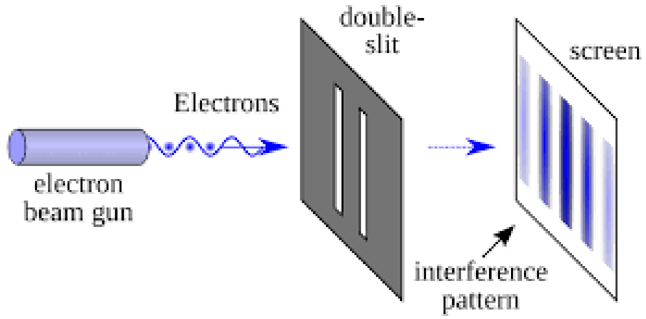
Lithographically patterned slits in a Copper foil ~ 0.3 microns



Intensity pattern

Slits

Young's double slit experiment : Points to think about



If we try to determine which slit the electron passed through by placing a detector near one of the slits, the interference pattern is lost.

The "which path" information and "interference" are mutually exclusive. We cannot determine the path and also see interference. We have to choose one.

Some references you MUST read

[1] Feynman Lectures vol 3 (Quantum Mechanics), chapter 1

[2] Demonstration of single-electron buildup of an interference pattern
A. Tonomura, J. Endo, T. Matsuda ... et. al
American Journal of Physics **57**, 117–120 (1989)

Fourier series and transform:
Adding up waves to get what you want

OR

The mathemtical basis of representing *localized pulses* as superposition of many waves.

Any periodic function : $f(x+L)=f(x)$ is a sum of simple waves

$$\text{If} \quad f(x+L) = f(x)$$

$$f(x) = \sum_{n=0}^{\infty}\left(A_n \cos\frac{2\pi n}{L}x + B_n \sin\frac{2\pi n}{L}x\right)$$

But it is useful only if we know what A_n and B_n are ....

Fourier's remarkable discovery

Multiply both sides by $\cos\frac{2\pi m}{L}x$ OR $\sin\frac{2\pi m}{L}x$

Integrate over one period

The solution of either A_n OR B_n will emerge : Why?

Recall a simple property of product of $\sin$ and $\cos$.......

$$\int_{-L/2}^{L/2} \cos\frac{2\pi n}{L}x \cos\frac{2\pi m}{L}x\,dx$$

$$=\frac{1}{2}\int_{-L/2}^{L/2}\left(\cos\frac{2\pi(n-m)}{L}x + \cos\frac{2\pi(n+m)}{L}x\right)dx$$

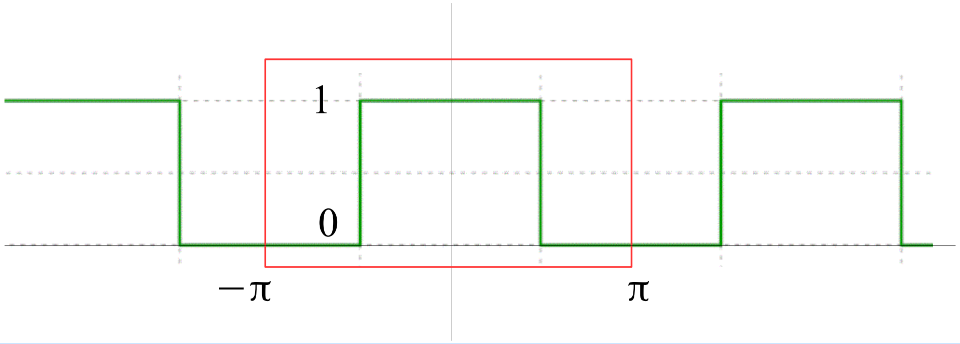
ZERO except when $n = m$ it is $\frac{L}{2}$

$$\int_{-L/2}^{L/2} \sin\frac{2\pi n}{L}x \cos\frac{2\pi m}{L}x\,dx$$

$$=\frac{1}{2}\int_{-L/2}^{L/2}\left(\sin\frac{2\pi(n-m)}{L}x + \sin\frac{2\pi(n+m)}{L}x\right)dx$$

ALWAYS ZERO

Expanding a square wave in harmonics



$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{2p+1}\cos(2p+1)x$$

We will approximate the infinite series successively by 1,2,5,10....100.......500 terms and see how it matches with the original

run Scilab script....square-wave-fourier.sce

Expanding a triangular wave in harmonics

$$f(x)=\frac{1}{2}+\frac{4}{\pi^2}\sum_{p=0}^{\infty}\frac{1}{(2p+1)^2}\cos(2p+1)x$$

run Scilab script....traingular-wave-fourier.sce

The triangular wave was easier. Why ?

In general if a pattern has discontinuities, jumps, sharp peaks etc it will require larger number of fourier wave components to be well represented.

But we need to represent a pulse or a localized packet. Not a repeating pattern.

The trick is to allow the period to go to infinity i.e. L → infinity limit has to be taken

Rewrite the sin & cos using $e^{i\theta} = \cos\theta + i\sin\theta$

$$f(x) = \sum_{n=0}^{\infty}\left(A_n\cos\frac{2\pi n}{L}x + B_n\sin\frac{2\pi n}{L}x\right)$$

$$= \sum_{n=-\infty}^{\infty} a_n e^{i\frac{2\pi n}{L}x}$$ Notice the summation range

$$a_n=\frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-i\frac{2\pi n}{L}x}dx$$ Why ? Justify

We need a formulation to represent a pulse or a localized disturbance.

It is a reasonable guess that the mathematical basis of representing a particle with waves would be like that.
A pattern that repeats itself is not localized

A way to go from a periodic function to an aperiodic one is to let the period go to infinity. This means taking a limit....

How to take the $L\to\infty$ limit ? From Fourier series to Fourier Transform.

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{i\frac{2\pi n}{L}x} = \left(\frac{L}{2\pi}\right)\sum a_n\, e^{i\frac{2\pi n}{L}x}\frac{2\pi}{L}\Delta n$$

Note : this works because $\Delta n=1$ in the sum

$$=\frac{1}{2\pi}\sum(a_n L)e^{ikx}\Delta k \qquad \text{where } k\equiv\frac{2\pi n}{L}$$

$$\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty} a(k)e^{ikx}dk$$

where $a(k) \equiv \lim_{L\to\infty} a_n L = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$

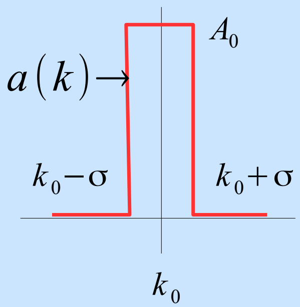
But...what about the convergence/divergence of these integrals ?

Construction of a wave packet using wavevectors $k_0-\sigma < k < k_0+\sigma$

$$f(x) = \frac{A_0}{2\pi}\int_{k_0-\sigma}^{k_0+\sigma} e^{ikx}\,dk$$

$$= \frac{A_0}{2\pi} e^{ik_0x}\int_{-\sigma}^{\sigma} e^{iqx}\,dq$$

$$= \frac{A_0}{\pi} e^{ik_0x}\,\frac{\sin\sigma x}{x}$$



$$|f(x)| \sim \left|\frac{\sin\sigma x}{x}\right|$$

peak at $x=0$ | peak height $\propto\sigma$ | peak width $\approx\dfrac{2\pi}{\sigma}$

$\Rightarrow$ spread in x $\times$ spread in k $\approx 4\pi$

Where will the wave packet be, a little later ?

$$f(x,0) = \frac{A_0}{2\pi}\int_{k_0-\sigma}^{k_0+\sigma} e^{ikx}\,dk = \frac{A_0}{2\pi} e^{ik_0x}\boxed{\int_{-\sigma}^{\sigma} e^{iqx}\,dq}$$

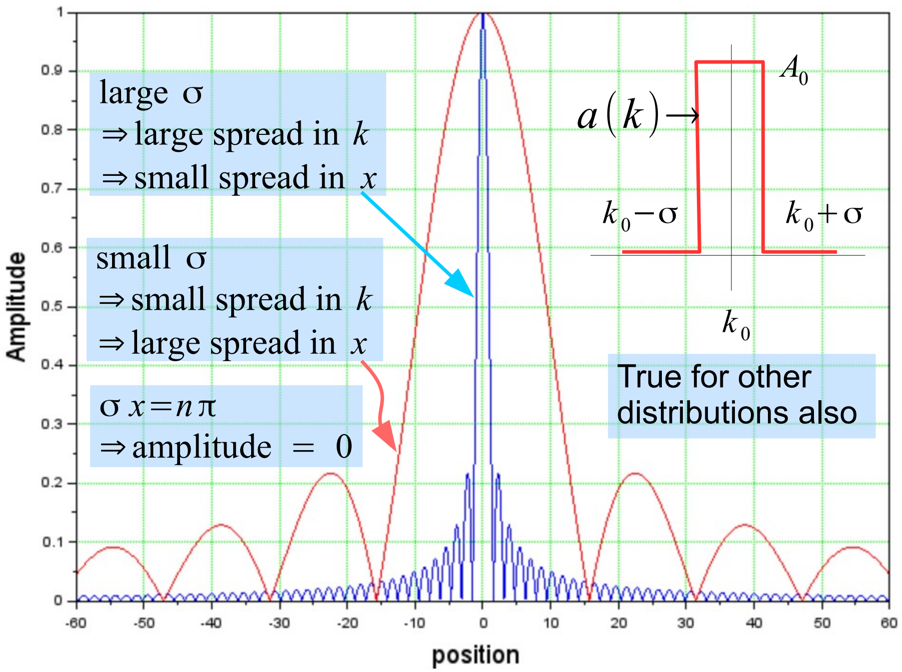
$$\Rightarrow f(x,t) = \frac{A_0}{2\pi}\int_{k_0-\sigma}^{k_0+\sigma} e^{i(kx-\omega t)}\,dk \qquad \ldots\ldots q=k-k_0$$

$$kx-\omega t = (k_0+q)x-\left(\omega_0+\frac{d\omega}{dk_0}q\right)t \qquad \ldots\ldots \omega(k_0)\equiv\omega_0$$

$$= (k_0x-\omega_0 t)+\left(x-\frac{d\omega}{dk_0}t\right)q$$

$$\Rightarrow f(x,t) = \frac{A_0}{2\pi} e^{i(k_0x-\omega_0 t)}\boxed{\int_{-\sigma}^{\sigma} e^{iq\left(x-\frac{d\omega}{dk_0}t\right)}\,dq}$$

Let us familiarise ourselves with $\sin\sigma x/\sigma x$ ....



How do we interprete this result ?

Amplitude near $x=0$ at $t=0 \rightarrow \left(x-\dfrac{d\omega}{dk_0}t\right)$ at $t=t$

The packet made of a group of waves centered at k_0 moves with a velocity $v_g = d\omega/dk_0$

The physical velocity of the particle is $\dfrac{d\omega}{dk_0}$

This is the group velocity of the wave, different from the phase velocity $v_p = \dfrac{\omega}{k}$

Energy and information travel with v_g

$v_p > c$ is possible and does not violate special relativity!

Gaussian (normal) distribution and Gaussian wave packet

Integrals of the type $I=\int_{-\infty}^{\infty} e^{-x^2}\,dx$ occur everywhere

To do this : Define $I(\lambda)=\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx$ & SQUARE it

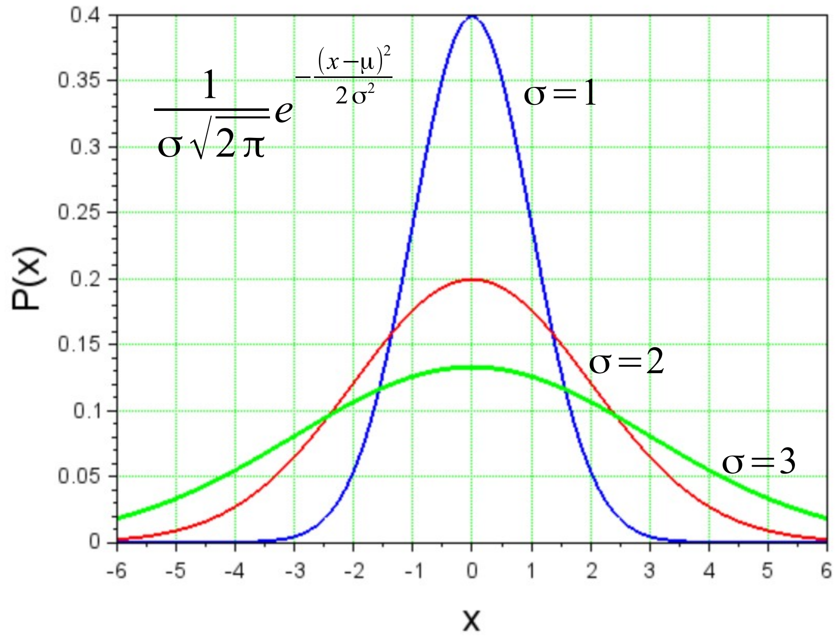
$$I^2(\lambda) = \left(\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\lambda y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boxed{dx\,dy}\, e^{-\lambda(x^2+y^2)}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} \boxed{r\,d\theta\, dr}\, e^{-\lambda r^2} \qquad (x,y)\to(r,\theta)$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} e^{-\lambda r^2}\frac{d(\lambda r^2)}{2\lambda} = 2\pi.\frac{1}{2\lambda}.1=\frac{\pi}{\lambda}$$

$$\Rightarrow I(\lambda) = \sqrt{\frac{\pi}{\lambda}}$$ ......A result we will use many times.

Gaussian (normal) distribution $\mu=0$ & $\sigma=1,2,3\ldots$



Gaussian (normal) distribution and Gaussian wave packet

$P(x) = \sqrt{\frac{\lambda}{\pi}}e^{-\lambda x^2}$ is thus normalized $\Rightarrow \int_{-\infty}^{\infty} P(x)\,dx = 1$

mean $\langle x\rangle=\int_{-\infty}^{\infty} x\,P(x)\,dx=0$ variance $\langle x^2\rangle=\int_{-\infty}^{\infty} x^2 P(x)\,dx=?$

$$\langle x^2\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2}\,dx = -\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty}\frac{d}{d\lambda}\left(e^{-\lambda x^2}\right)dx$$

$$= -\sqrt{\frac{\lambda}{\pi}}\frac{d}{d\lambda}\left(\int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx\right) = -\sqrt{\frac{\lambda}{\pi}}\frac{d}{d\lambda}I(\lambda)$$

$$= -\sqrt{\frac{\lambda}{\pi}}\left(-\frac{1}{2}\frac{\sqrt{\pi}}{\lambda\sqrt{\lambda}}\right) = \frac{1}{2\lambda} \equiv \sigma^2 \quad\rightarrow\quad \lambda=\frac{1}{2\sigma^2}$$

Gaussian distribution $P(x)\equiv\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ ..... $\begin{cases}\mu\rightarrow \text{mean}\\ \sigma=\text{variance}\end{cases}$

Now suppose the wave packet has $a(k)$ that is Gaussian

$$a(k) = A_0 e^{-\frac{k^2}{2\sigma^2}} \quad\Rightarrow\quad f(x) = \frac{A_0}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{k^2}{2\sigma^2}}e^{ikx}\,dk$$

$$f(x) = \frac{A_0}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{k^2}{2\sigma^2}-ikx\right)}dk \quad \text{.... complete the square}$$

$$= \frac{A_0}{2\pi}e^{-\frac{\sigma^2x^2}{2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(k-i\sigma^2x)^2}\,dk$$

$$= \frac{A_0}{2\pi}e^{-\frac{\sigma^2x^2}{2}}\sigma\sqrt{2\pi}$$

variance of $a(k)\times$ variance of $f(x) \rightarrow \sigma.\frac{1}{\sigma}=1$

Q : We have made one rather drastic mathematical assumption....what is it ?

Where does the factor of $1/2$ come from in $\Delta x \Delta(\hbar k) \geq \hbar/2$?

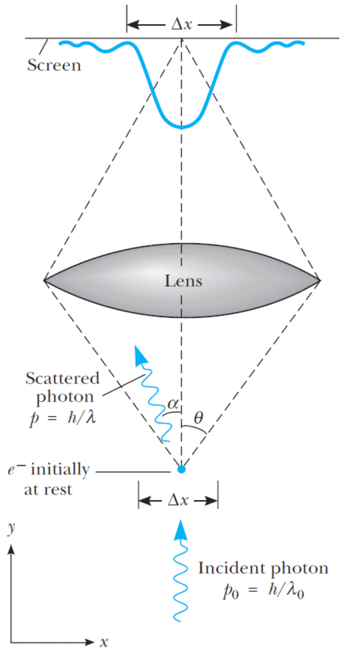
We used $a(k)$ and $f(x)$ in our arguments and got $\Delta k \Delta x \geq 1$

We should use $|a(k)|^2$ and $|f(x)|^2$: which gives $\Delta k \Delta x \geq 1/2$

Reason : position and momentum space wavefunctions need to be squared before interpreting them as probabilities

Uncertainty principle requires certain assertions about the process of *measurement* in quantum mechanics.....
We will come to those later.

Idealised microscope to view electrons: consistency with uncertainty principle



A single photon scattered from an electron is collected by a lens. Diffraction pattern forms at the focal plane.

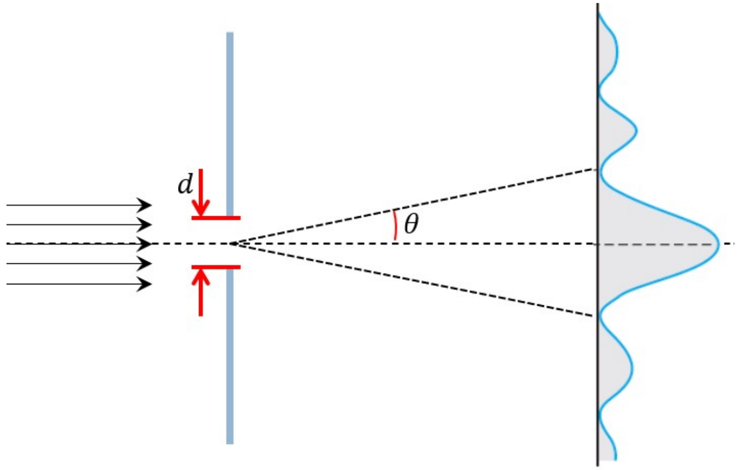
Photon scatters within $\pm\theta$

electron acquires $-\frac{h}{\lambda}\sin\theta < \Delta p_x < \frac{h}{\lambda}\sin\theta$

Spread of the diffraction maxima $\Delta x \approx \frac{\lambda}{2\sin\theta}$

$$\Delta x \Delta p_x \approx \frac{h}{2}$$

Single slit diffraction : consistency with uncertainty principle



Y co-ordinate of the photon is somewhere between the slit-edges

y-momentum must be such that it stays mostly within the two zeros shown.

$$\left.\begin{array}{l} p = \frac{h}{\lambda} \\ \sin\theta = \frac{\lambda}{d} \end{array}\right\} \Rightarrow \Delta y \Delta p_y \approx \frac{d}{2}(p\sin\theta) = \frac{d}{2}\frac{h}{\lambda}.\frac{\lambda}{d} = \frac{h}{2}$$

What is the ground state energy of a simple harmonic oscillator ?

$$E = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m} \quad \text{with} \quad \langle x \rangle = 0 \ \& \ \langle p_x \rangle = 0$$

$$\Rightarrow x^2 = (\Delta x)^2 \quad \& \quad p^2 = (\Delta p_x)^2$$

$$\Rightarrow \langle E \rangle > \frac{1}{2}m\omega^2(\Delta x)^2 + \frac{(\Delta p_x)^2}{2m} \quad \text{with} \quad \Delta x \Delta p_x \geq \frac{\hbar}{2}$$

$$\Rightarrow \langle E \rangle > \frac{1}{2}m\omega^2(\Delta x)^2 + \frac{\hbar^2}{8m(\Delta x)^2} \quad \text{...minimise w.r.t } \Delta x$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}$$

$$\Rightarrow E_{min} \geq \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

This turns out to be an exact result!