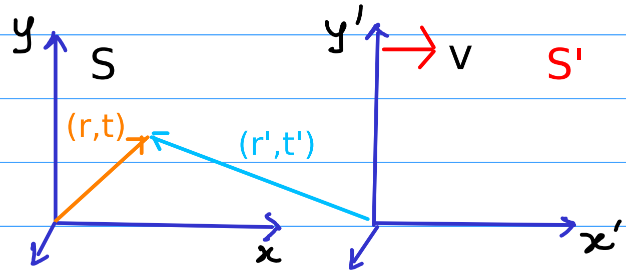
Lorentz transformation formulae



Event occuring in the rest frame S at (x,y,z,t) is viewed by the moving observer (who is at rest in S' frame) as (x',y',z',t')

If the event lies on the x axis, i.e., $\bar{r} = (x,0,0)$ and $\bar{v} = (v,0,0)$

then,

$$\boxed{\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}} \quad \text{Eq.1}$$

In general, $\bar{r}' = \dfrac{\bar{r} - \bar{v}t}{\sqrt{1 - v^2/c^2}}$

Now imagine how the event occuring in the moving S' frame at (x', y', z', t') will appear from S frame.

Note that with respect to S', the frame S is moving at $-v$ i.e, an observer who is at rest in S' sees S as moving at $-v$

But now (x', y', z', t') are in the rest frame of observer.

Thus from the moving frame S (which moves at $-v$) the event will occur at (x, y, z, t)

Applying Eq.1 to this situation,

$$x = \frac{x' - (-v)t'}{\sqrt{1 - v^2/c^2}} = \frac{x' + vt'}{\sqrt{}}; \quad y = y', \quad z = z';$$

and,

$$t = \frac{t' - (-v)x'/c^2}{\sqrt{}} = \frac{t' + x'v/c^2}{\sqrt{}}$$

The same result follows if you invert Eq.1. But here we derived it by applying Eq.1 to the relevant situation. So it's an independent check of Eq.1

Thumb Rule for using the formula in Eq. 1

Put on the lhs what a moving observer sees, with its appropriate speed put on the rhs.

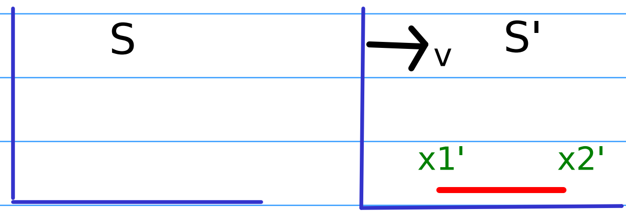
In the 1st case:

S' has the moving observer with speed $+V$,
He observes $x', y', z', t' = f(\cdots, v)$

In the 2nd case:

S has the moving observer with speed $-v$,
He observes $x, y, z, t = f(\cdots, -v)$

Length Contraction



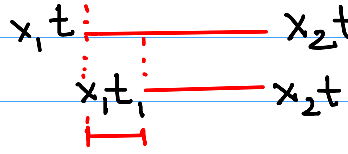
$\boxed{L_0 = x_2' - x_1'}$ (rest length)

Since the rod is at rest in frame S', at all times its ends are at x_1' & x_2', even if x_1' & x_2' are observed at diff times t_1' & t_2' (for example).

But for observer in S it matters when he locates the 2 ends of the moving rod at x_1 & x_2, because x_1 & x_2 keep changing with time.

From S : $\boxed{L = x_2 - x_1}$

It's also important that the observer in S measures the locations of the ends at the same time, $t_1 = t_2 = t$ (say). For example, if $t_1 > t_2 = t$ the point x_1 would have moved forward in that interval $(t_1 - t)$,



and the rod will appear even shorter.

(But this won't change the rest length in S'.)

Easy way :

$$x_1' = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}, \quad x_2' = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}}$$

$$L_0 = x_2' - x_1'$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}} \left[(x_2 - x_1) - v(t_2 - t_1) \right]$$

$$= \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}} \quad \text{Since } t_1 = t_2$$

$$= L / \sqrt{1 - v^2/c^2} \Rightarrow \boxed{L < L_0}$$

Harder Way involving t_1', t_2' which are not equal (symultaneity broken in S')

$$L = x_2 - x_1 = \left[\frac{(x_2' + vt_2') - (x_1' + vt_1')}{\sqrt{1 - v^2/c^2}} \right]$$

$$= \frac{1}{\sqrt{\ }} \left[(x_2' - x_1') + v(t_2' - t_1') \right] = \frac{1}{\sqrt{\ }} \left[L_0 - \frac{v^2 L}{c^2 \sqrt{\ }} \right]$$

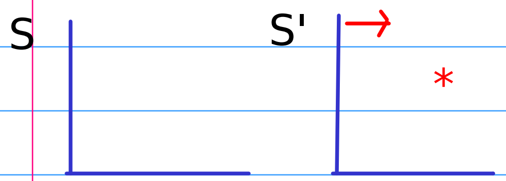
Use, $t' = \frac{t - xv/c^2}{\sqrt{\ }}$ to show, $t_2' - t_1' = -\frac{v^2}{c^2} \frac{(x_2 - x_1)}{\sqrt{\ }} = -\frac{v^2}{c^2} \frac{L}{\sqrt{\ }}$

$$L \left(1 + \frac{v^2}{c^2 - v^2} \right) = \frac{L_0}{\sqrt{\ }}$$

and finally,

$$L = L_0 \sqrt{1 - v^2/c^2}$$

Time dialation



Two events occur in S' at the same spot x'1=x'2

but at two different times t'1 and t'2

$\boxed{\text{How will the time interval } t_2' - t_1' = \Delta T_0 \text{ appear from frame S ? There it will be observed at } (x_1, t_1) \ \& \ (x_2, t_2)}$

Again note that from S these 2 events will not appear at the same spot, so $x_1 \neq x_2$

Let, $t_2 - t_1 = \Delta T$

Easy way without involving x_1, x_2 :

$$\Delta T = t_2 - t_1 = \frac{(t_2' + v x_2'/c^2) - (t_1' + v x_1'/c^2)}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{(t_2' - t_1') + \frac{v}{c^2}\underbrace{(x_2' - x_1')}_{\text{zero since } x_2' = x_1'}}{\sqrt{1 - v^2/c^2}} = \frac{\Delta T_0}{\sqrt{1 - v^2/c^2}} > T_0$$

Time Dialation

Check for yourself that you get the same answer if you start with $\Delta T_0 = t_2' - t_1'$ on the l.h.s.

You will encounter $(x_2 - x_1)$ which is not zero, but substituting them in terms of x_2', x_1', t_2', t_1' you will recover this

Time interval measured by a watch sitting at the same spot in S' (ie, the watch is at rest wrt S') appears longer from the frame S which is moving (at speed $-v$) relative to the watch.

Equivalently, time interval in the rest frame of an object is the shortest