It becomes increasingly difficult to increase speed as $u \to c$

Q: At what speed does the kinetic energy of a particle become equal to its rest energy ?

$$\text{Total Energy} = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} = KE + m_0 c^2 = 2 m_0 c^2$$

$$\Rightarrow 1 - v^2/c^2 = 1/4 \Rightarrow v/c = \sqrt{3}/2$$

$\approx 87\%$ of speed of light

Useful to remember

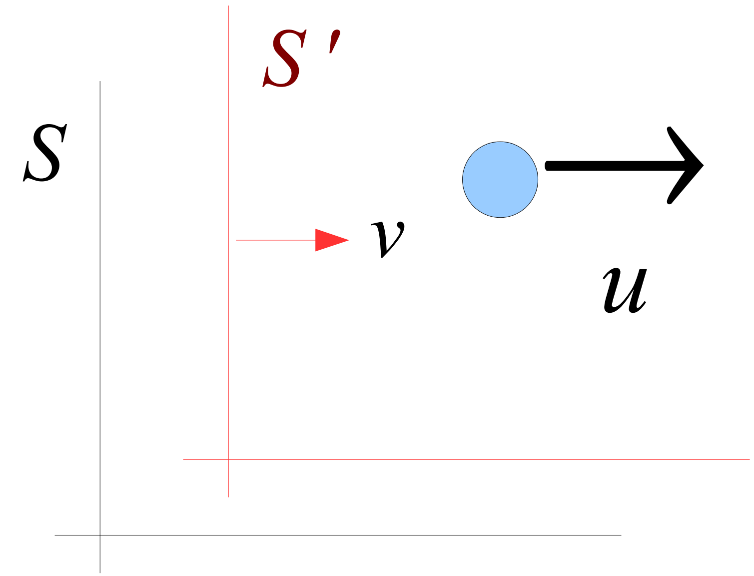
electron $m_0^{(e)} c^2 = 0.51$ MeV

proton $m_0^{(p)} c^2 = 939$ MeV ≈ 1 GeV

Often we say a proton's mass is 1 GeV...dropping the c^2 lazily!

How does S and S' see the p, E of the same particle?

$$S \text{ sees } \begin{cases} p_x = \dfrac{m_0 u}{\sqrt{1-u^2/c^2}} \\ E = \dfrac{m_0 c^2}{\sqrt{1-u^2/c^2}} \end{cases}$$



Q : What does S' see ?

S and S' are intertial observers. The "physical laws“ must be same & they should be able to predict what the other will see.

$$u' = \frac{u-v}{1-uv/c^2} \quad \& \quad S' \text{ must see } \begin{cases} p'_x = \dfrac{m_0 u'}{\sqrt{1-u'^2/c^2}} \\ E' = \dfrac{m_0 c^2}{\sqrt{1-u'^2/c^2}} \end{cases}$$

How does S and S' see the p, E of the same particle?

$$p'_x = \frac{m_0 u'}{\sqrt{1-u'^2/c^2}} = \frac{m_0\left(\dfrac{u-v}{1-uv/c^2}\right)}{\sqrt{1-\left(\dfrac{u-v}{1-uv/c^2}\right)^2/c^2}} = \frac{m_0(u-v)}{\sqrt{1-u^2/c^2}\sqrt{1-\beta^2}}$$

$$E' = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \frac{m_0 c^2}{\sqrt{1-\left(\dfrac{u-v}{1-uv/c^2}\right)^2/c^2}} = \frac{m_0 c^2(1-uv/c^2)}{\sqrt{1-u^2/c^2}\sqrt{1-\beta^2}}$$

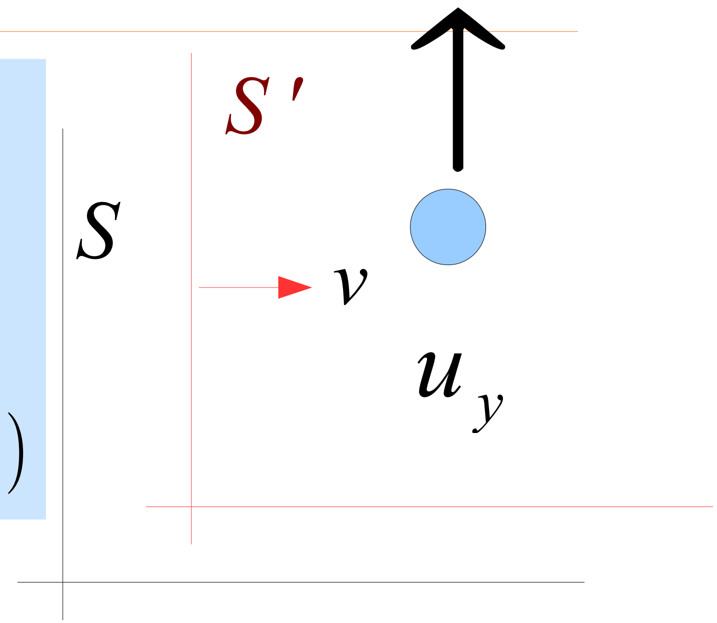
$$p'_x = \frac{p_x - v\dfrac{E}{c^2}}{\sqrt{1-\beta^2}} \quad \& \quad \frac{E'}{c^2} = \frac{\dfrac{E}{c^2} - v\dfrac{p_x}{c^2}}{\sqrt{1-\beta^2}}$$

What does this remind you of ? What about y and z momenta ?

How does S and S' see p_y of the same particle?

S sees $\qquad\qquad\qquad\qquad$ S' will see

$$\begin{cases} u_x = 0 \\ u_y = u_y \\ u^2 = u_y^{\,2} \end{cases} \Rightarrow \begin{cases} u'_x = -v \\ u'_y = u_y\sqrt{1-\beta^2} \\ u'^2 = v^2 + u_y^{\,2}\left(1-\beta^2\right) \end{cases}$$



$$p'_x = \frac{-m_0 v}{\sqrt{1-u'^2/c^2}} = \frac{-m_0 v}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{-m_0 v}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = \frac{0-v\left(E/c^2\right)}{\sqrt{1-\beta^2}}$$

$$p'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}} = \frac{m_0 u_y\sqrt{1-\beta^2}}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{m_0 u_y\sqrt{1-\beta^2}}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = p_y$$

$$\frac{E'}{c^2} = \frac{m_0}{\sqrt{1-u'^2/c^2}} = \frac{m_0}{\sqrt{1-\dfrac{v^2+u_y^{\,2}(1-\beta^2)}{c^2}}} = \frac{m_0}{\left(\sqrt{1-\dfrac{u_y^{\,2}}{c^2}}\right)\sqrt{1-\beta^2}} = \frac{E/c^2}{\sqrt{1-\beta^2}}$$

The similarity between $(p_x, p_y, p_z, E/c)$ & (x, y, z, ct)

$$p'_x = \frac{p_x - v\dfrac{E}{c^2}}{\sqrt{1-\beta^2}} \qquad x' = \frac{x - vt}{\sqrt{1-\beta^2}}$$

$$p'_y = p_y \qquad y' = y$$

$$p'_z = p_z \qquad z' = z$$

$$\frac{E'}{c^2} = \frac{\dfrac{E}{c^2} - v\dfrac{p_x}{c^2}}{\sqrt{1-\beta^2}} \qquad t' = \frac{t - v\dfrac{x}{c^2}}{\sqrt{1-\beta^2}}$$

$(p_x, p_y, p_z, E/c^2)$ transform like (x, y, z, t)

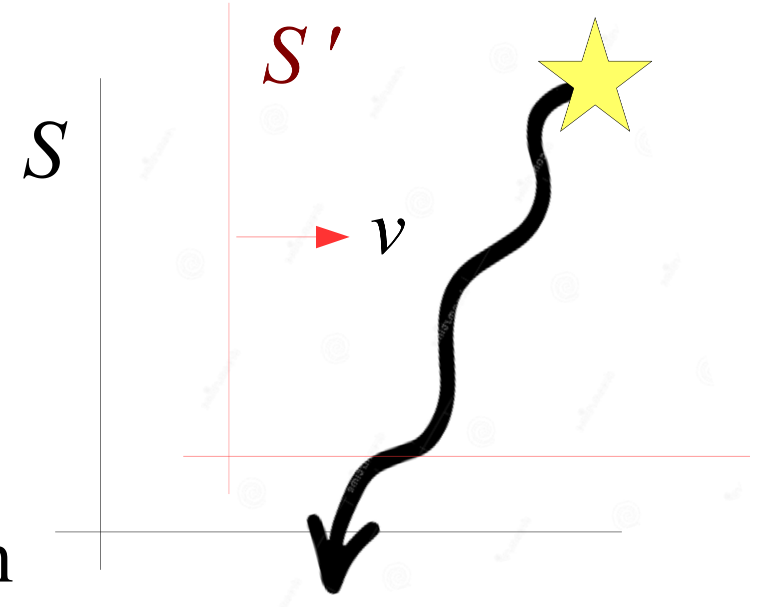
There is a little bit more to it. Can you spot it?

Plane electromagnetic wave : Doppler effect and aberration

S observes an EM wave in xy plane

$$A\cos\left(\vec{k}.\vec{r}-\omega t\right)$$

$$A\cos 2\pi\left(\frac{\cos\theta}{\lambda}x+\frac{\sin\theta}{\lambda}y-ft\right)$$



The wave seen by S' must have the form

$$A\cos 2\pi\left(\frac{\cos\theta'}{\lambda'}x'+\frac{\sin\theta'}{\lambda'}y'-f't'\right)$$

- ✔ S and S' must agree on the speed of the EM wave
- ✔ But they may see different wavelengths, directions & amplitudes
- ✔ We will look at the variation of wavelegth and direction

Write (x', y', t') in terms of (x, y, t)

$\lambda f = \lambda' f' = c$ must hold

Then compare the co-efficients of each variable

Plane electromagnetic wave : Doppler effect and aberration

$$\cos 2\pi\left[\frac{\cos\theta'}{\lambda'}\frac{x-vt}{\sqrt{1-\beta^2}}+\frac{\sin\theta'}{\lambda'}y-f'\frac{t-vx/c^2}{\sqrt{1-\beta^2}}\right]$$

$$=\cos 2\pi\left[\frac{\cos\theta'+\beta}{\lambda'\sqrt{1-\beta^2}}x+\frac{\sin\theta'}{\lambda'}y-\frac{1+\beta\cos\theta'}{\sqrt{1-\beta^2}}f't\right]$$

$$\therefore\left\{\begin{aligned}\frac{\cos\theta}{\lambda}&=\frac{\cos\theta'+\beta}{\lambda'\sqrt{1-\beta^2}}\\ \frac{\sin\theta}{\lambda}&=\frac{\sin\theta'}{\lambda'}\\ f&=\frac{1+\beta\cos\theta'}{\sqrt{1-\beta^2}}f'\end{aligned}\right.$$

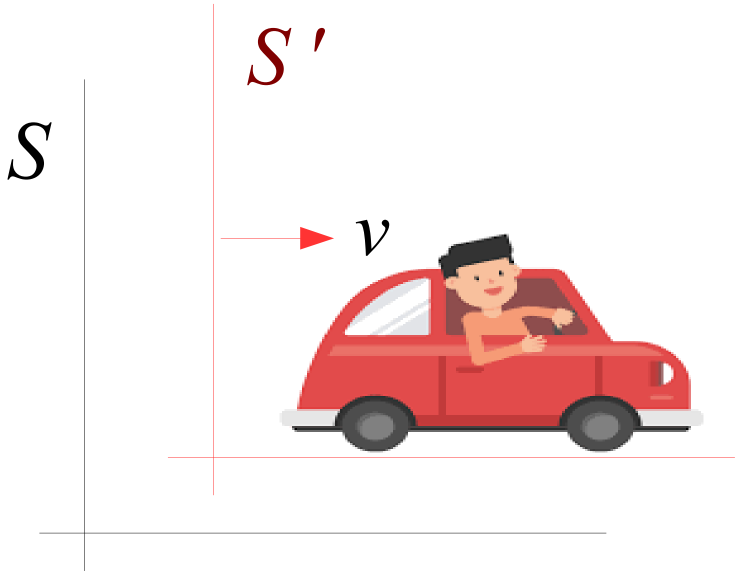
aberration :

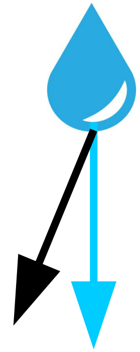
$$\Rightarrow\quad \tan\theta=\frac{\sin\theta'\sqrt{1-\beta^2}}{\cos\theta'+\beta}$$

To invert $\beta\to-\beta$

& $\theta\Leftrightarrow\theta'$

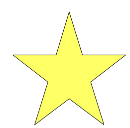
Plane electromagnetic wave : Doppler effect and aberration

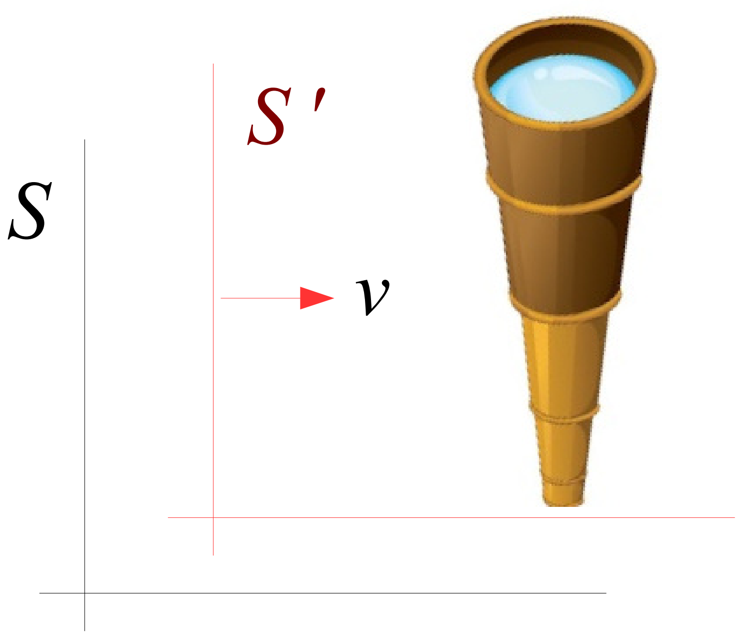




$\vec{V}_R$ is vertical seen by S

Q: What angle does S' see ?





Apparent change of direction of the light from the star due to relative motion between source and observer.

The classical analogy sees the two situations as similar.

Re-write the results in terms of $\left(k_x, k_y, k_z, \omega\right)$: Phase $= \vec{k}.\vec{r} - \omega t$

Recall : $k = \dfrac{2\pi}{\lambda}$ & $\omega = 2\pi f$

$$k'_x = 2\pi\frac{\cos\theta'}{\lambda'} = 2\pi\frac{\cos\theta - \beta}{\lambda\sqrt{1-\beta^2}} = \frac{2\pi\dfrac{\cos\theta}{\lambda} - v\dfrac{1}{c^2}\dfrac{2\pi c}{\lambda}}{\sqrt{1-\beta^2}}$$

$$k'_y = 2\pi\frac{\sin\theta'}{\lambda'} = 2\pi\frac{\sin\theta}{\lambda} = 2\pi\frac{\sin\theta}{\lambda}$$

$$\frac{\omega'}{c^2} = \frac{2\pi}{c^2}f' = \frac{2\pi}{c^2}f\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} = \frac{\dfrac{\omega}{c^2} - \dfrac{v}{c^2}\dfrac{2\pi\cos\theta}{\lambda}}{\sqrt{1-\beta^2}}$$

The similarity between $(k_x, k_y, k_z, \omega/c)$ & (x, y, z, ct)

$$k'_x = \frac{k_x - v\frac{\omega}{c^2}}{\sqrt{1-\beta^2}} \qquad x' = \frac{x - vt}{\sqrt{1-\beta^2}}$$

$$k'_y = k_y \qquad y' = y$$

$$k'_z = k_z \qquad z' = z$$

$$\frac{\omega'}{c^2} = \frac{\frac{\omega}{c^2} - v\frac{k_x}{c^2}}{\sqrt{1-\beta^2}} \qquad t' = \frac{t - v\frac{x}{c^2}}{\sqrt{1-\beta^2}}$$

$$\boxed{(k_x, k_y, k_z, \omega/c^2) \text{ transform like } (x, y, z, t)}$$