1 Perfect Foresight Liquidity Constrained Solution

This appendix taxonomizes the characteristics of the limiting consumption function $\grave{\mathrm{c}}(m)$ under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$ under various conditions. Results are summarized in table 1.

1.1 If PF-GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{PF-GIC}}$, $1 < \text{\textbf{Þ}}/\Gamma$). Under $\cancel{\text{PF-GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[1]

$$1 < (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \tag{1}$$

$$1 < \mathsf{R}\beta\Gamma^{-\rho} \tag{2}$$

$$\mathrm{u}'(1) < \mathsf{R}\beta\mathrm{u}'(\Gamma). \tag{3}$$

Similar logic shows that under these circumstances the constraint will never bind for a constrained consumer with a finite horizon of n periods, so such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

If the RIC fails ($1 < \text{\textbf{Þ}}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{4}$$

If the RIC fails and the FHWC fails, human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[2]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying $\cancel{\text{PF-GIC}}$ we must impose the RIC (and the FHWC can be shown to be a consequence of $\cancel{\text{PF-GIC}}$ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from (**??**):

$$m_{\#} = (m_{\#} - 1 + h)\underline{\kappa} \tag{5}$$

$$m_{\#}(1 - \underline{\kappa}) = (h - 1)\underline{\kappa} \tag{6}$$

$$m_{\#} = (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \tag{7}$$

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[1]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[2]The knife-edge case is where $\text{\textbf{Þ}} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[3] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[4] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

1.2 If PF-GIC Holds

Imposition of the PF-GIC reverses the inequality in (3), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^{n}$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^{n}$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b_{\#}^{0}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^{1}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the PF-GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^{0} = 1 + b_{\#}^{0} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^{0} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{\th}_{\Gamma}$, so consumption n periods in the past must have been

$$c_{\#}^{n} = \text{\th}_{\Gamma}^{-n} c_t = \text{\th}_{\Gamma}^{-n}. \tag{8}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned} \mathbb{C}_{t-n}^{t} &= c_{t-n}(1 + \text{\th}/\mathsf{R} + ... + (\text{\th}/\mathsf{R})^{n}) \\ &= c_{\#}^{n}(1 + \text{\th}_{\mathsf{R}} + ... + \text{\th}_{\mathsf{R}}^{n}) \\ &= \text{\th}_{\Gamma}^{-n}\left(\frac{1 - \text{\th}_{\mathsf{R}}^{n+1}}{1 - \text{\th}_{\mathsf{R}}}\right) \end{aligned} \tag{9}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1 + ... + \mathcal{R}^{-n} \tag{10}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} \quad = b_{\#}^{n} + h_{\#}^{n}$$

[3] Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~PF-GIC~~.

[4] As an illustration, consider a consumer for whom $\text{\th} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

from which we can solve for the $b^n_{\#}$ such that the consumer with $b_{t-n} = b^n_{\#}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b^n_{\#} = \qquad \mathbb{C}^t_{t-n} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h^n_{\#}}. \tag{11}$$

Defining $m^n_{\#} = b^n_{\#} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m^n_{\#}, c^n_{\#}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes, because for infinitesimal ϵ the MPC of a consumer with assets $m = m^n_{\#} - \epsilon$ is discretely higher than for a consumer with assets $m = m^n_{\#} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (11) for the entire domain of positive real values of b, we need $b^n_{\#}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b^n_{\#} = \infty. \tag{12}$$

1.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (11) limits to a constant as $n \uparrow \infty$, and (12) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^n \text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty.
\end{aligned}$$

Given the PF-GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the PF-GIC is stronger (harder to satisfy) than the RIC; thus, FHWC and the PF-GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because as the date at which the constraint binds becomes arbitrarily distant, the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{c}(m) - \bar{c}(m) = 0. \tag{13}$$

1.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex. Given failure of FHWC, (12) requires

$$\lim_{n\to\infty}\left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}-\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)+\left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right)=\infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1}-\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)=\infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}-\frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)=\infty. \tag{14}$$

If RIC Holds. When the RIC holds, rearranging (14) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)=\infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1}\\ \Gamma/\text{Þ} &> \Gamma/\mathsf{R}\\ 1 &> \text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n}=\lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{15}$$

which with a few lines of algebra can be shown to asymptote to the MPC in the perfect foresight model:[5]

$$\lim_{m\to\infty}\grave{\boldsymbol{\kappa}}(m)=1-\text{Þ}_{\mathsf{R}}. \tag{16}$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\text{Þ}_{\mathsf{R}}>1$. In this case the constant multiplying $\mathcal{R}^{-n}$ in (14) will be positive if

$$\begin{aligned}\text{Þ}_{\mathsf{R}}\mathcal{R}^{-1}-\text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}}-\mathcal{R}^{-1}\\ \mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}}\\ \Gamma &> \text{Þ}\end{aligned}$$

which is merely the PF-GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned}\mathcal{R}^{-1} &> \text{Þ}_{\Gamma}^{-1}\\ \Gamma/\mathsf{R} &> \Gamma/\text{Þ}\\ \text{Þ}/\mathsf{R} &> 1\end{aligned}$$

[5]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the software archive.

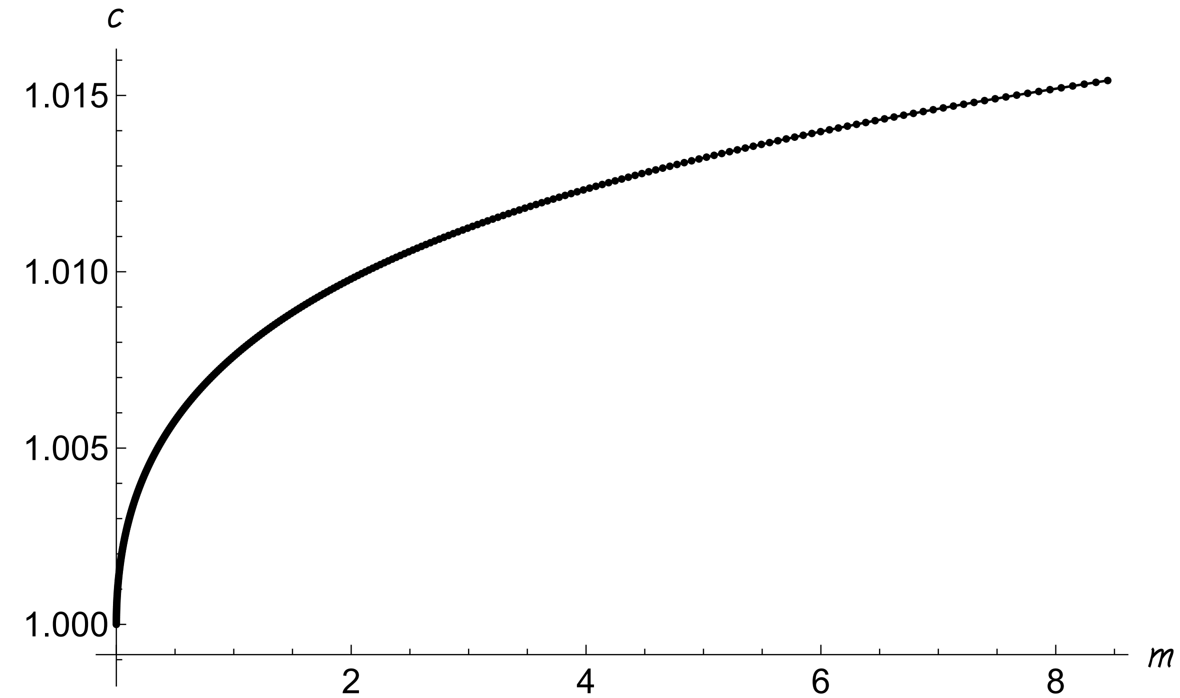

Figure 1 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

which merely confirms the starting assumption that the RIC fails. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{17}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 1 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting – $c = m$ – so not worth plotting).

We can summarize as follows. Given that the PF-GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined solution, whether or not the RIC holds.

Table 1 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math			Outcome, Comments or Results
~~PF-GIC~~		$1 <$	$\text{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R}$	< 1		FHWC holds ($\mathsf{R} > \Gamma$); $\grave{c}(m) = \bar{c}(m)$ for $m \geq 1$
and ~~RIC~~		$1 <$	$\text{Þ}/\mathsf{R}$	$\grave{c}(m)$ is degenerate: $\grave{c}(m) = 0$
PF-GIC	$\text{Þ}/\Gamma$	< 1		Constraint binds in finite time for any m
and RIC	$\text{Þ}/\mathsf{R}$	< 1		FHWC may or may not hold $\lim_{m\uparrow\infty} \bar{c}(m) - \grave{c}(m) = 0$ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~		$1 <$	$\text{Þ}/\mathsf{R}$	~~FHWC~~ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~PF-GIC~~ and RIC both hold, while the third row indicates that when the PF-GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the PF-GIC holds, the constraint will bind in finite time.