0.1 Convergence of v_t in Euclidian Space

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_{\digamma} \leq \alpha^{n-1} \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma}. \tag{58}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1} |\digamma(m)|. \tag{59}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{60}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathcal{T}\mathrm{v}_{T-1} \leq \mathcal{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

0.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

We start by showing that

$$\mathrm{c}(m) = \underset{c_t \in [\underline{\kappa}m, \bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m_{t+1})\right]\right\} \tag{61}$$

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

$$\mathcal{T}\mathrm{v}(m) = \mathrm{u}(c_i) + \beta\,\underbrace{\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, c_i))\right]}_{\equiv\mathfrak{v}} \tag{62}$$

where $\mathrm{m}_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$ and $i = 1, 2$. $\mathcal{T}\mathrm{v}$ is concave for concave $\mathfrak{v}$. Since the space of continuous and concave functions is closed, $\mathfrak{v}$ is also concave and satisfies

$$\frac{1}{2}\sum_{i=1,2} \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, c_i))\right] \leq \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, \tilde{c}))\right]. \tag{63}$$

On the other hand, $\frac{1}{2}\{\mathrm{u}(c_1) + \mathrm{u}(c_2)\} < \mathrm{u}(\tilde{c})$. Then one gets

$$\mathcal{T}\mathrm{v}(m) < \mathrm{u}(\tilde{c}) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, \tilde{c}))\right]. \tag{64}$$

Since $\tilde{c}$ is a feasible choice for c_i, the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of $\mathrm{c}(m)$ we can now show

$$\lim_{n\to\infty} \mathrm{c}_{T-n+1}(m) = \mathrm{c}(m). \tag{65}$$

Suppose this does not hold for some $m = m^*$. In this case, $\{\mathrm{c}_{T-n+1}(m^*)\}_{n=1}^{\infty}$ has a subsequence $\left\{c_{T-n(i)}(m^*)\right\}_{i=1}^{\infty}$ that satisfies $\lim_{i\to\infty} \mathrm{c}_{T-n(i)}(m^*) = c^*$ and $c^* \neq \mathrm{c}(m^*)$. Now define $c^*_{T-n+1} = \mathrm{c}_{T-n+1}(m^*)$. $c^* > 0$ because $\lim_{i\to\infty} \mathrm{v}_{T-n(i)+1}(m^*) \leq \lim_{i\to\infty} \mathrm{u}(c^*_{T-n(i)})$. Because $\mathrm{a}(m^*) > 0$ and $\psi \in [\underline{\psi}, \bar{\psi}]$ there exist $\{\underline{m}^*_+, \bar{m}^*_+\}$ satisfying $0 < \underline{m}^*_+ < \bar{m}^*_+$ and $m_{T-n+1}(m^*, c^*_{T-n+1}) \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. It follows that $\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}(m)$ and the convergence is uniform on $m \in \left[\underline{m}^*_+, \bar{m}^*_+\right]$. (Uniform convergence is obtained from Dini's theorem.[1]) Hence for any $\delta > 0$, there exists an n_1 such that

$$\beta\,\mathbb{E}_{T-n}\left[\Gamma^{1-\rho}_{T-n+1}\left|\mathrm{v}_{T-n+1}(m_{T-n+1}(m^*, c^*_{T-n+1})) - \mathrm{v}(m_{T-n+1}(m^*, c^*_{T-n+1}))\right|\right] < \quad \delta$$

for all $n \geq n_1$. It follows that if we define

$$\mathrm{w}(m^*, z) = \mathrm{u}(z) + \beta\,\mathbb{E}_{T-n}\left[\Gamma^{1-\rho}_{T-n+1}\mathrm{v}(m_{T-n+1}(m^*, z))\right] \tag{66}$$

then $\mathrm{v}_{T-n}(m^*)$ satisfies

$$\lim_{n\to\infty}\left|\mathrm{v}_{T-n}(m^*) - \mathrm{w}(m^*, c^*_{T-n+1})\right| = 0. \tag{67}$$

On the other hand, there exists an $i_1 \in \mathbb{N}$ such that

$$\left|\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*, c^*_{T-n(i)})) - \mathrm{v}(\mathrm{m}_{T-n(i)}(m^*, c^*))\right| \leq \delta \text{ for all } i \geq i_1 \tag{68}$$

because v is uniformly continuous on $[\underline{m}^*_+, \bar{m}^*_+]$. $\lim_{i\to\infty}\left|\mathrm{c}_{T-n(i)}(m^*) - c^*\right| = 0$ and

$$\left|\mathrm{m}_{T-n(i)}(m^*, c^*_{T-n(i)}) - \mathrm{m}_{T-n(i)}(m^*, c^*)\right| \leq \frac{\mathsf{R}}{\Gamma\underline{\psi}}\left|c^*_{T-n(i)} - c^*\right|. \tag{69}$$

This implies

$$\lim_{i\to\infty}\left|\mathrm{w}(m^*, c^*_{T-n(i)+1}) - \mathrm{w}(m^*, c^*)\right| = 0. \tag{70}$$

From (67) and (70), we obtain $\lim_{i\to\infty} \mathrm{v}_{T-n(i)}(m^*) = \mathrm{w}(m^*, c^*)$ and this implies $\mathrm{w}(m^*, c^*) = \mathrm{v}(m^*)$. This implies that $\mathrm{c}(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

[1][**Dini's theorem**] For a monotone sequence of continuous functions $\{\mathrm{v}_n(m)\}_{n=1}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty} \mathrm{v}_n(m) = \mathrm{v}(m)$ where $\mathrm{v}(m)$ is continuous, convergence is uniform.