

## 0.1 Convergence of $v_t$ in Euclidian Space

Boyd's theorem shows that  $\mathcal{T}$  defines a contraction mapping in a  $F$ -bounded space. We now show that  $\mathcal{T}$  also defines a contraction mapping in Euclidian space.

Calling  $v^*$  the unique fixed point of the operator  $\mathcal{T}$ , since  $v^*(m) = \mathcal{T}v^*(m)$ ,

$$\|v_{T-n+1} - v^*\|_F \leq \alpha^{n-1} \|v_T - v^*\|_F. \quad (1)$$

On the other hand,  $v_T - v^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$  and  $\kappa = \|v_T - v^*\|_F < \infty$  because  $v_T$  and  $v^*$  are in  $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ . It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa \alpha^{n-1} |F(m)|. \quad (2)$$

Then we obtain

$$\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v^*(m). \quad (3)$$

Since  $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$ ,  $v_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < v_T(m)$ . On the other hand,  $v_{T-1} \leq v_T$  means  $\mathcal{T}v_{T-1} \leq \mathcal{T}v_T$ , in other words,  $v_{T-2}(m) \leq v_{T-1}(m)$ . Inductively one gets  $v_{T-n}(m) \geq v_{T-n-1}(m)$ . This means that  $\{v_{T-n+1}(m)\}_{n=1}^\infty$  is a decreasing sequence, bounded below by  $v^*$ .

## 0.2 Convergence of $c_t$

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions  $\{c_{T-n+1}(m)\}_{n=1}^\infty$ .

We start by showing that

$$c(m) = \arg \max_{c_t \in [\underline{\kappa}m, \bar{\kappa}m]} \{u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1})]\} \quad (4)$$

is uniquely determined. We show this by contradiction. Suppose there exist  $c_1$  and  $c_2$  that both attain the supremum for some  $m$ , with mean  $\tilde{c} = (c_1 + c_2)/2$ .  $c_i$  satisfies

$$\mathcal{T}v(m) = u(c_i) + \underbrace{\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, c_i))]}_{\equiv \mathbf{v}} \quad (5)$$

where  $m_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$  and  $i = 1, 2$ .  $\mathcal{T}v$  is concave for concave  $v$ . Since the space of continuous and concave functions is closed,  $\mathbf{v}$  is also concave and satisfies

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, c_i))] \leq \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, \tilde{c}))]. \quad (6)$$

On the other hand,  $\frac{1}{2} \{u(c_1) + u(c_2)\} < u(\tilde{c})$ . Then one gets

$$\mathcal{T}v(m) < u(\tilde{c}) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, \tilde{c}))]. \quad (7)$$

Since  $\tilde{c}$  is a feasible choice for  $c_i$ , the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of  $c(m)$  we can now show

$$\lim_{n \rightarrow \infty} c_{T-n+1}(m) = c(m). \quad (8)$$

Suppose this does not hold for some  $m = m^*$ . In this case,  $\{c_{T-n+1}(m^*)\}_{n=1}^\infty$  has a subsequence  $\{c_{T-n(i)}(m^*)\}_{i=1}^\infty$  that satisfies  $\lim_{i \rightarrow \infty} c_{T-n(i)}(m^*) = c^*$  and  $c^* \neq c(m^*)$ . Now define  $c_{T-n+1}^* = c_{T-n+1}(m^*)$ .  $c^* > 0$  because  $\lim_{i \rightarrow \infty} v_{T-n(i)+1}(m^*) \leq \lim_{i \rightarrow \infty} u(c_{T-n(i)}^*)$ . Because  $a(m^*) > 0$  and  $\psi \in [\underline{\psi}, \bar{\psi}]$  there exist  $\{\underline{m}_+^*, \bar{m}_+^*\}$  satisfying  $0 < \underline{m}_+^* < \bar{m}_+^*$  and  $m_{T-n+1}(m^*, c_{T-n+1}^*) \in [\underline{m}_+^*, \bar{m}_+^*]$ . It follows that  $\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v(m)$  and the convergence is uniform on  $m \in [\underline{m}_+^*, \bar{m}_+^*]$ . (Uniform convergence is obtained from Dini's theorem.<sup>1</sup>) Hence for any  $\delta > 0$ , there exists an  $n_1$  such that

$$\beta \mathbb{E}_{T-n} [\Gamma_{T-n+1}^{1-\rho} |v_{T-n+1}(m_{T-n+1}(m^*, c_{T-n+1}^*)) - v(m_{T-n+1}(m^*, c_{T-n+1}^*))|] < \delta$$

for all  $n \geq n_1$ . It follows that if we define

$$w(m^*, z) = u(z) + \beta \mathbb{E}_{T-n} [\Gamma_{T-n+1}^{1-\rho} v(m_{T-n+1}(m^*, z))] \quad (9)$$

then  $v_{T-n}(m^*)$  satisfies

$$\lim_{n \rightarrow \infty} |v_{T-n}(m^*) - w(m^*, c_{T-n+1}^*)| = 0. \quad (10)$$

On the other hand, there exists an  $i_1 \in \mathbb{N}$  such that

$$|v(m_{T-n(i)}(m^*, c_{T-n(i)}^*)) - v(m_{T-n(i)}(m^*, c^*))| \leq \delta \text{ for all } i \geq i_1 \quad (11)$$

because  $v$  is uniformly continuous on  $[\underline{m}_+^*, \bar{m}_+^*]$ .  $\lim_{i \rightarrow \infty} |c_{T-n(i)}(m^*) - c^*| = 0$  and

$$|m_{T-n(i)}(m^*, c_{T-n(i)}^*) - m_{T-n(i)}(m^*, c^*)| \leq \frac{R}{\Gamma \underline{\psi}} |c_{T-n(i)}^* - c^*|. \quad (12)$$

This implies

$$\lim_{i \rightarrow \infty} |w(m^*, c_{T-n(i)+1}^*) - w(m^*, c^*)| = 0. \quad (13)$$

From (10) and (13), we obtain  $\lim_{i \rightarrow \infty} v_{T-n(i)}(m^*) = w(m^*, c^*)$  and this implies  $w(m^*, c^*) = v(m^*)$ . This implies that  $c(m)$  is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

---

<sup>1</sup>[Dini's theorem] For a monotone sequence of continuous functions  $\{v_n(m)\}_{n=1}^\infty$  which is defined on a compact space and satisfies  $\lim_{n \rightarrow \infty} v_n(m) = v(m)$  where  $v(m)$  is continuous, convergence is uniform.