Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions. Results are summarized in table 1.

A.1 If PF-GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails (~~PF-GIC~~, $1 < \text{Þ}/\Gamma$). Under ~~PF-GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[1]

$$1 < (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \tag{1}$$

$$1 < \mathsf{R}\beta\Gamma^{-\rho} \tag{2}$$

$$\mathrm{u}'(1) < \mathsf{R}\beta\mathrm{u}'(\Gamma). \tag{3}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{4}$$

(that is, consumption is zero for any level of human or nonhuman wealth).

If the RIC fails and the FHWC fails, human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[2]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~PF-GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~PF-GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from equation (21):

$$m_{\#} = (m_{\#} - 1 + h)\underline{\kappa} \tag{5}$$

[1]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[2]The knife-edge case is where $\text{Þ} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

$$m_{\#}(1 - \underline{\kappa}) = (h - 1)\underline{\kappa} \tag{6}$$
$$m_{\#} = (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \tag{7}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[3] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{c}(m) = m$.[4] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

(**?** obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

A.2 If PF-GIC Holds

Imposition of the PF-GIC reverses the inequality in (3), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b^n_{\#}$ as the b_t such that an unconstrained consumer holding $b_t = b^n_{\#}$ would behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b^0_{\#}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b^1_{\#}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the PF-GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m^0_{\#} = 1 + b^0_{\#} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c^0_{\#} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{\textbf{Þ}}_{\Gamma}$, so consumption n periods in the past must have been

$$c^n_{\#} = \text{\textbf{Þ}}_{\Gamma}^{-n} c_t = \text{\textbf{Þ}}_{\Gamma}^{-n}. \tag{8}$$

The PDV of consumption from $t - n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}^t_{t-n} &= c_{t-n}(1 + \text{\textbf{Þ}}/\mathsf{R} + ... + (\text{\textbf{Þ}}/\mathsf{R})^n) \\
&= c^n_{\#}(1 + \text{\textbf{Þ}}_{\mathsf{R}} + ... + \text{\textbf{Þ}}^n_{\mathsf{R}}) \\
&= \text{\textbf{Þ}}_{\Gamma}^{-n}\left(\frac{1 - \text{\textbf{Þ}}^{n+1}_{\mathsf{R}}}{1 - \text{\textbf{Þ}}_{\mathsf{R}}}\right) && (9) \\
&= \left(\frac{\text{\textbf{Þ}}_{\Gamma}^{-n} - \text{\textbf{Þ}}_{\mathsf{R}}}{1 - \text{\textbf{Þ}}_{\mathsf{R}}}\right) && (10)
\end{aligned}$$

[3]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~PF-GIC~~.

[4]As an illustration, consider a consumer for whom $\text{\textbf{Þ}} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1 + \ldots + \mathcal{R}^{-n} \tag{11}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} \quad = b_{\#}^{n} + h_{\#}^{n}$$

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n} = b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h_{\#}^{n}}. \tag{12}$$

Defining $m_{\#}^{n} = b_{\#}^{n} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (12) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{13}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (12) limits to a constant as $n \uparrow \infty$, and (13) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) &= \infty.
\end{aligned}$$

Given the PF-GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the PF-GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the PF-GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{14}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (13) requires

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\mathcal{R}^{-n}\mathbf{Þ}_{\mathsf{R}} - \mathbf{Þ}_{\Gamma}^{-n}}{\mathbf{Þ}_{\mathsf{R}} - 1}\right) + \left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\mathbf{Þ}_{\mathsf{R}}}{\mathbf{Þ}_{\mathsf{R}} - 1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n} - \left(\frac{\mathbf{Þ}_{\Gamma}^{-n}}{\mathbf{Þ}_{\mathsf{R}} - 1}\right) &= \infty
\end{aligned} \tag{15}$$

If RIC Holds. When the RIC holds, rearranging (15) gives

$$\lim_{n\to\infty} \left(\frac{\mathbf{Þ}_{\Gamma}^{-n}}{1-\mathbf{Þ}_{\mathsf{R}}}\right) - \mathcal{R}^{-n}\left(\frac{\mathbf{Þ}_{\mathsf{R}}}{1-\mathbf{Þ}_{\mathsf{R}}} + \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}
\mathbf{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1} \\
\Gamma/\mathbf{Þ} &> \Gamma/\mathsf{R} \\
1 &> \mathbf{Þ}/\mathsf{R}
\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty} \kappa_{\#}^{n} = \lim_{n\to\infty} \left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{16}$$

which with a bit of algebra[5] can be shown to asymptote to the MPC in the perfect foresight model:[6]

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \mathbf{Þ}_{\mathsf{R}}. \tag{18}$$

If RIC Fails. Consider now the ~~RIC~~ case, $\mathbf{Þ}_{\mathsf{R}} > 1$. We can rearrange (15)as

$$\lim_{n\to\infty} \left(\frac{\mathbf{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\mathbf{Þ}_{\mathsf{R}}-1)} - \frac{\mathcal{R}^{-1}(\mathbf{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\mathbf{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n} - \left(\frac{\mathbf{Þ}_{\Gamma}^{-n}}{\mathbf{Þ}_{\mathsf{R}}-1}\right) = \infty. \tag{19}$$

---

[5] Calculate the limit of

$$\left(\frac{\mathbf{Þ}_{\Gamma}^{-n}}{\mathbf{Þ}_{\Gamma}^{-n}/(1-\mathbf{Þ}_{\mathsf{R}}) - (1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right) = \left(\frac{1}{1/(1-\mathbf{Þ}_{\mathsf{R}}) + \mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{17}$$

[6] For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

which makes clear that with $\cancel{\text{FHWC}} \Rightarrow \mathcal{R}^{-1} > 1$ and $\cancel{\text{RIC}} \Rightarrow \text{Þ}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (15) will be positive if

$$\begin{aligned} \text{Þ}_{\mathsf{R}}\mathcal{R}^{-1} - \text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}} - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}} \\ \Gamma &> \text{Þ} \end{aligned}$$

which is merely the PF-GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \text{Þ}_{\Gamma}^{-1} \\ \Gamma/\mathsf{R} &> \Gamma/\text{Þ} \\ \text{Þ}/\mathsf{R} &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\Gamma/\text{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/\mathsf{R} > \Gamma/\text{Þ} \tag{20}$$

because $\cancel{\text{RIC}} \Rightarrow \text{Þ} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (16) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that $\cancel{\text{RIC}}$ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{21}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 1 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting – $c = m$ – so not worth plotting).

We can summarize as follows. Given that the PF-GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

? characterize the limits of MPC in a more general framework that allows for non-

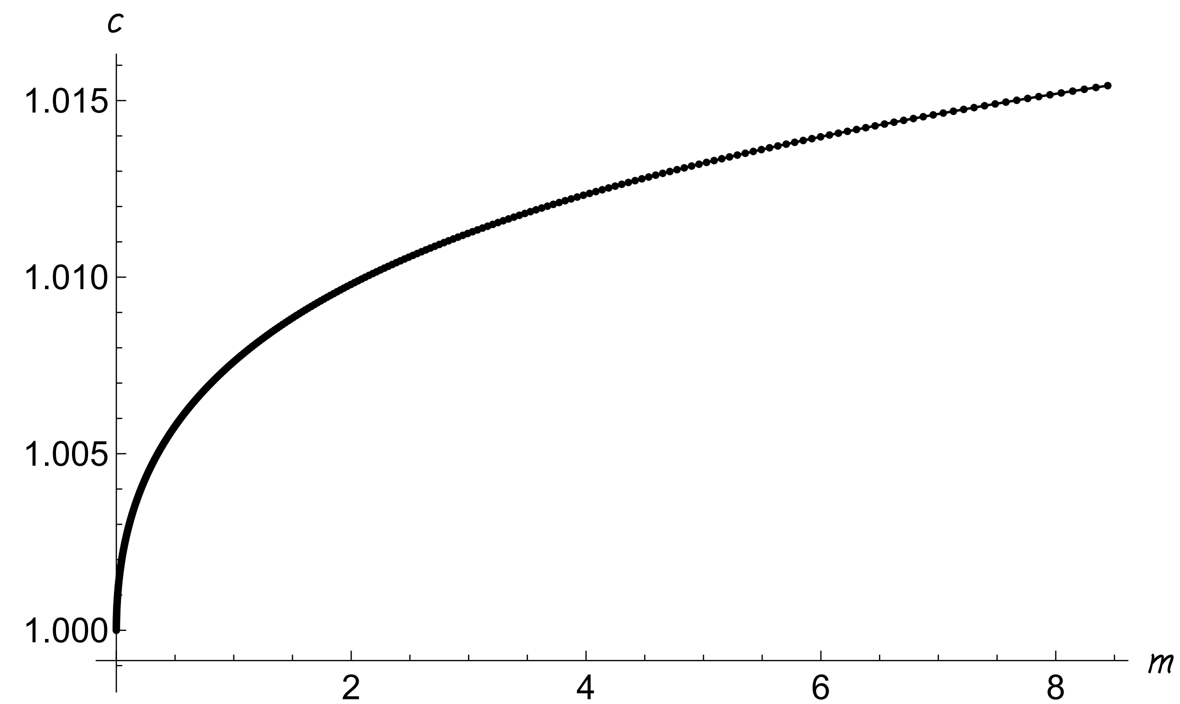

Figure 1 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

CRRA utility as well as capital and labor income risks in a Markovian setting, and find that in that much more general framework the limiting MPC is also zero.

B Existence of a Concave Consumption Function

To show that (7) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$
6. $\lim_{z \downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})\right]. \tag{22}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0}\mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0}\mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (?).)

Next define $\underline{\mathrm{v}}_t(m,c)$ as

$$\underline{\mathrm{v}}_t(m,c) = \mathrm{u}(c) + \mathfrak{v}_t(m-c) \tag{23}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (7) can be written as

$$\mathrm{v}_t(m) = \max_{c}\ \underline{\mathrm{v}}_t(m,c). \tag{24}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m,c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m,c) = -\infty$, $\frac{\partial^2\underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \arg\max_{0<c<m}\ \underline{\mathrm{v}}_t(m,c) \tag{25}$$

exists and is unique, and (7) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{26}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (26) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{27}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm\to+0}\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm\to+0}\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{28}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

D Proof that $\mathfrak{T}$ Is a Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (**?**).

Consider condition (1). For this problem,

$$\begin{aligned} \{\mathfrak{T}\mathrm{x}\}(m_t) \text{ is } &\max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\} \\ \{\mathfrak{T}\mathrm{y}\}(m_t) \text{ is } &\max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\}, \end{aligned}$$

so $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)$ implies $\{\mathfrak{T}\mathrm{x}\}(m_t) \le \{\mathfrak{T}\mathrm{y}\}(m_t)$ by inspection.[7]

Condition (2) requires that $\{\mathfrak{T}\mathbf{0}\} \in \mathcal{C}_F\left(\mathcal{A}, \mathcal{B}\right)$. By definition,

$$\{\mathfrak{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right) + \beta 0\right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is F-bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \tag{29}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of F, $\{\mathfrak{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly F-bounded.

Finally, we turn to condition (3), $\{\mathfrak{T}(\mathrm{z} + \zeta F)\}(m_t) \le \{\mathfrak{T}\mathrm{z}\}(m_t) + \zeta \alpha F(m_t)$. The proof

[7]For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

will be more compact if we define č and ǎ as the consumption and assets functions[8] associated with $\mathcal{T}$z and ĉ and â as the functions associated with $\mathcal{T}(\mathrm{z}+\zeta \mathsf{F})$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta \mathsf{F})\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}})+\zeta\alpha \mathsf{F}.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta \mathsf{F})\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha \mathsf{F} \\
\beta\{\mathsf{E}(\mathrm{z}+\zeta \mathsf{F})\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha \mathsf{F} \\
\beta\zeta\{\mathsf{E}\mathsf{F}\}(\hat{\mathrm{a}}) &\leq \zeta\alpha \mathsf{F} \\
\beta\{\mathsf{E}\mathsf{F}\}(\hat{\mathrm{a}}) &\leq \alpha \mathsf{F} \\
\beta\{\mathsf{E}\mathsf{F}\}(\hat{\mathrm{a}}) &< \mathsf{F}.
\end{aligned}$$

where the last line follows because $0<\alpha<1$ by assumption.[9]

Using $\mathsf{F}(m)=\eta+m^{1-\rho}$ and defining $\hat{a}_t=\hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\xi_{t+1})^{1-\rho}]-m_t^{1-\rho}<\eta(1-\underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (27), which says $\beth<1$) can be rewritten as:

$$\eta>\frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\xi_{t+1})^{1-\rho}\right]-m_t^{1-\rho}}{1-\beth}. \tag{30}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (30) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho} \\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho} \\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right) \\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)
\end{aligned} \tag{31}$$

$$<(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.$$

[8]Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[9]The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

We can thus conclude that equation (30) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{32}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (40) and (33) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (41)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (7) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (31). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned} \wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1 \\ (\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &> 1 \\ (\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}) &> 1 \end{aligned}$$

where we have used (39) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned} (\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &> 1 \\ (\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\ \beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1. \end{aligned}$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty}\underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (7) converge, they converge to the same unique v defined by $\mathcal{T}$.[10]

D.2 Convergence of v_t in Euclidian Space

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathcal{T}$, since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_{\digamma} \leq \alpha^{n-1} \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} . \tag{33}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1} |\digamma(m)| . \tag{34}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{35}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathcal{T}\mathrm{v}_{T-1} \leq \mathcal{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D.3 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

We start by showing that

$$\mathrm{c}(m) = \underset{c_t\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m_{t+1})\right]\right\} \tag{36}$$

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

$$\mathcal{T}\mathrm{v}(m) = \mathrm{u}(c_i) + \beta\,\underbrace{\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, c_i))\right]}_{\equiv\mathfrak{v}} \tag{37}$$

where $\mathrm{m}_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$ and $i = 1, 2$. $\mathcal{T}\mathrm{v}$ is concave for concave $\mathfrak{v}$. Since the space of continuous and concave functions is closed, $\mathfrak{v}$ is also concave and satisfies

$$\frac{1}{2}\sum_{i=1,2} \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, c_i))\right] \leq \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m, \tilde{c}))\right] . \tag{38}$$

[10] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

On the other hand, $\frac{1}{2}\{\mathrm{u}(c_1)+\mathrm{u}(c_2)\}<\mathrm{u}(\tilde{c})$. Then one gets

$$\mathcal{T}\mathrm{v}(m)<\mathrm{u}(\tilde{c})+\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}(\mathrm{m}_{t+1}(m,\tilde{c}))\right]. \tag{39}$$

Since $\tilde{c}$ is a feasible choice for c_i, the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of $\mathrm{c}(m)$ we can now show

$$\lim_{n\to\infty}\mathrm{c}_{T-n+1}(m)=\mathrm{c}(m). \tag{40}$$

Suppose this does not hold for some $m=m^*$. In this case, $\{\mathrm{c}_{T-n+1}(m^*)\}_{n=1}^{\infty}$ has a subsequence $\left\{c_{T-n(i)}(m^*)\right\}_{i=1}^{\infty}$ that satisfies $\lim_{i\to\infty}\mathrm{c}_{T-n(i)}(m^*)=c^*$ and $c^*\neq\mathrm{c}(m^*)$. Now define $c^*_{T-n+1}=\mathrm{c}_{T-n+1}(m^*)$. $c^*>0$ because $\lim_{i\to\infty}\mathrm{v}_{T-n(i)+1}(m^*)\leq\lim_{i\to\infty}\mathrm{u}(c^*_{T-n(i)})$. Because $\mathrm{a}(m^*)>0$ and $\psi\in[\underline{\psi},\bar{\psi}]$ there exist $\{\underline{m}^*_+,\bar{m}^*_+\}$ satisfying $0<\underline{m}^*_+<\bar{m}^*_+$ and $m_{T-n+1}(m^*,c^*_{T-n+1})\in\left[\underline{m}^*_+,\bar{m}^*_+\right]$. It follows that $\lim_{n\to\infty}\mathrm{v}_{T-n+1}(m)=\mathrm{v}(m)$ and the convergence is uniform on $m\in\left[\underline{m}^*_+,\bar{m}^*_+\right]$. (Uniform convergence is obtained from Dini's theorem.[11]) Hence for any $\delta>0$, there exists an n_1 such that

$$\beta\,\mathbb{E}_{T-n}\left[\Gamma_{T-n+1}^{1-\rho}\left|\mathrm{v}_{T-n+1}(m_{T-n+1}(m^*,c^*_{T-n+1}))-\mathrm{v}(m_{T-n+1}(m^*,c^*_{T-n+1}))\right|\right]<\quad\delta$$

for all $n\geq n_1$. It follows that if we define

$$\mathrm{w}(m^*,z)=\mathrm{u}(z)+\beta\,\mathbb{E}_{T-n}\left[\Gamma_{T-n+1}^{1-\rho}\mathrm{v}(m_{T-n+1}(m^*,z))\right] \tag{41}$$

then $\mathrm{v}_{T-n}(m^*)$ satisfies

$$\lim_{n\to\infty}\left|\mathrm{v}_{T-n}(m^*)-\mathrm{w}(m^*,c^*_{T-n+1})\right|=0. \tag{42}$$

On the other hand, there exists an $i_1\in\mathbb{N}$ such that

$$\left|\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*,c^*_{T-n(i)}))-\mathrm{v}(\mathrm{m}_{T-n(i)}(m^*,c^*))\right|\leq\delta\text{ for all }i\geq i_1 \tag{43}$$

because v is uniformly continuous on $[\underline{m}^*_+,\bar{m}^*_+]$. $\lim_{i\to\infty}\left|\mathrm{c}_{T-n(i)}(m^*)-c^*\right|=0$ and

$$\left|\mathrm{m}_{T-n(i)}(m^*,c^*_{T-n(i)})-\mathrm{m}_{T-n(i)}(m^*,c^*)\right|\leq\frac{\mathsf{R}}{\Gamma\underline{\psi}}\left|c^*_{T-n(i)}-c^*\right|. \tag{44}$$

This implies

$$\lim_{i\to\infty}\left|\mathrm{w}(m^*,c^*_{T-n(i)+1})-\mathrm{w}(m^*,c^*)\right|=0. \tag{45}$$

From (42) and (45), we obtain $\lim_{i\to\infty}\mathrm{v}_{T-n(i)}(m^*)=\mathrm{w}(m^*,c^*)$ and this implies $\mathrm{w}(m^*,c^*)=\mathrm{v}(m^*)$. This implies that $\mathrm{c}(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

[11][**Dini's theorem**] For a monotone sequence of continuous functions $\{\mathrm{v}_n(m)\}_{n=1}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty}\mathrm{v}_n(m)=\mathrm{v}(m)$ where $\mathrm{v}(m)$ is continuous, convergence is uniform.

E The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (8) can be rewritten

$$\mathrm{e}_t(m_t)^{-\rho} = \beta\mathsf{R}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t)+\Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t}\right)\right)^{-\rho}\right] \tag{46}$$

$$\begin{aligned} =&\quad (1-\wp)\beta\mathsf{R}m_t^{\rho}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1}>0\right] \\ &+\wp\beta\mathsf{R}^{1-\rho}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t-\mathrm{c}_t(m_t)}{m_t}\right)^{-\rho} \mid \xi_{t+1}=0\right]. \end{aligned} \tag{47}$$

Consider the first conditional expectation in (46), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m\downarrow 0}\mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \tag{48}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}. \end{aligned} \tag{49}$$

As noted in the main text, we need the WRIC (40) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{50}$$

Since $\bar{\kappa}_T = 1$, iterating (49) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{51}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (46) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} \quad = \beta \mathsf{R}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) \left(\mathsf{R}\mathrm{a}_t(m_t)\right))^{-\rho} \right]$$

and we know from L'Hôpital's rule that $\lim_{m_t \to \infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned}
(m_t \underline{\kappa}_t)^{-\rho} &= \beta \mathsf{R} \left(\underline{\kappa}_{t+1} \mathsf{R}(1 - \underline{\kappa}_t) m_t \right)^{-\rho} \\
\underbrace{\mathsf{R}^{-1} \text{Þ}}_{\equiv \text{Þ}_{\mathsf{R}} = (1-\underline{\kappa})} \ \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1}
\end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \text{Þ}_{\mathsf{R}} \tag{52}$$

so that $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \kappa_{T-n}^{-1} \tag{53}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n \to \infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \ldots\right)}_{= 1 + \text{Þ}_{\mathsf{R}}(1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+2}^{-1}) \ldots} \quad = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \tag{54}$$

F The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned}
\wp \quad &= 0 \\
c_t \quad &\leq m_t,
\end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp\downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{55}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{56}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (56) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{57}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (?)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (57), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp) \int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/(1-\wp))\right)^{-\rho} d\mathcal{F}_\theta \right], \tag{58}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{59}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (57) and (59) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (58) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (55). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (51) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp\downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

G Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (?): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[12]

[12] The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

$$\mathrm{u}'(\mathfrak{c}_t(\vec{a})) = \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))] \tag{60}$$
$$\vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) = \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}.$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (60) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\begin{aligned}
\mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}_{t+1}^m\mathcal{R}_{t+1}] \\
&= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m] \\
\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m]/\mathrm{u}''(\mathfrak{c}_t) \qquad (61)
\end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$\begin{aligned}
c &= \mathfrak{m} - a \\
\mathfrak{c}^a + 1 &= \mathfrak{m}^a
\end{aligned}$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$\begin{aligned}
\mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1+\mathfrak{c}^a)
\end{aligned}$$

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

H The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, ...\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[13] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight

[13] Unless $\beta \approx +0$.

liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_{\#}, m^2_{\#}, ...\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathring{\mathrm{c}}(m)$ for integer values of n (e.g., $c^n_{\#} = \mathbf{Þ}_{\Gamma}^{-n}$) are continuous functions of n; the conclusion that $\mathring{\mathrm{c}}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\breve{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[14]

This strategy generates a smooth limiting consumption function – except at the remaining kink point defined by $\{m^0_{\#}, c^0_{\#}\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_{\#}$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_{\#}} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_{\#}} \mathrm{c}'(m) = \kappa^0_{\#} < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\breve{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\breve{\mathrm{c}}(m)$ between $m^0_{\#}$ and $m^{\underline{n}}_{\#}$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

1. $\hat{\mathrm{c}}(m^0_{\#}) = c^0_{\#}$
2. $\hat{\mathrm{c}}'(m^0_{\#}) = 1$
3. $\hat{\mathrm{c}}'(m^{\underline{n}}_{\#}) = (d\mathrm{c}^n_{\#}/dn)(d\mathrm{m}^n_{\#}/dn)^{-1}|_{n=\underline{n}}$
4. $\hat{\mathrm{c}}''(m^{\underline{n}}_{\#}) = (d^2\mathrm{c}^n_{\#}/dn^2)(d^2\mathrm{m}^n_{\#}/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\breve{\mathrm{c}}(m)$ and fidelity of that function to the $\mathring{\mathrm{c}}(m)$ (see the actual code for details).

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m^0_{\#} & m \\ m^0_{\#} < m < m^{\underline{n}}_{\#} & \breve{\mathrm{c}}(m) \\ m^{\underline{n}}_{\#} < m & \mathring{\mathrm{c}}(m) \end{cases} \tag{62}$$

[14] In practice, we calculate the first and second derivatives of $\mathring{\mathrm{c}}$ and use piecewise polynomial approximation methods that match the function at these points.

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\check{c}(m) \geq \mathring{c}(m)$, implicitly defining $m = e^{\mu}$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^{\mu})/\mathrm{c}_T(e^{\mu})) \tag{63}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \check{c}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi_t'(\mu) = \left(\frac{-\left(\frac{\mathrm{c}_t'\mathrm{c}_T - \mathrm{c}_t\mathrm{c}_T'}{\mathrm{c}_T^2} e^{\mu}\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{64}$$

which can be solved for

$$\mathrm{c}_t' = (\mathrm{c}_t\mathrm{c}_T'/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi_t'. \tag{65}$$

Similarly, we can solve (63) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right) \mathrm{c}_T(m). \tag{66}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

Table 1 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{\mathrm{c}}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~PF-GIC~~	$1 < \mathbf{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \Gamma$); $\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \mathbf{Þ}/\mathsf{R}$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
PF-GIC	$\mathbf{Þ}/\Gamma < 1$	Constraint binds in finite time for any m
and RIC	$\mathbf{Þ}/\mathsf{R} < 1$	FHWC may or may not hold
		$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \mathbf{Þ}/\mathsf{R}$	~~FHWC~~
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~PF-GIC~~ and RIC both hold, while the third row indicates that when the PF-GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the PF-GIC holds, the constraint will bind in finite time.