Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative exploration. After describing conditions under which the consumption function converges, the paper shows that a 'target' buffer stock exists only under conditions strictly stronger than those that guarantee convergence of the consumption and value functions. It also shows that the average growth rate of consumption equals the average growth rate of permanent income (in a small open economy populated by buffer stock savers). Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis

JEL codes D81, D91, E21

PDF: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory.pdf`
Slides: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory-Slides.pdf`
Web: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory/`
Appendix: `http://llorracc.github.io/BufferStockTheory/BufferStockTheory#Appendices`
GitHub: `http://github.com/llorracc/BufferStockTheory`
(In GitHub repo, see `/Code` for tools for solving and simulating the model)

CLICK HERE for an interactive Jupyter Notebook that uses the Econ-ARK/HARK toolkit to produce all of the paper's figures (warning: the notebook may take several minutes to launch). Information about citing the toolkit can be found at Acknowleding Econ-ARK.

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All numerical results herein were produced using the Econ-ARK/HARK toolkit, which can be cited per our references

1 Introduction

In the presence of empirically realistic transitory and permanent shocks to income *a la* **?**, only one additional ingredient is required to define a testable model of optimal consumption: A description of preferences. Modelers usually specify preferences by geometric discounting of a constant relative risk aversion (CRRA) utility function, because, starting with Zeldes (**?**), a large literature has shown that models of this kind have quantitative predictions that can match microeconomic evidence reasonably well.

A companion theoretical literature has shown that standard numerical solution methods provide good approximations to limiting "true" mathematical solutions – but only for models more complex than the simple case with just shocks and utility. The extra complexity has been required because standard contraction mapping theorems (beginning with **?** and including those following Stokey et. al. (**?**)) cannot be applied when the utility function is unbounded (like CRRA - see section 2.1).[1]

This paper's first technical contribution is to articulate the (surprisingly loose) conditions under which the simple problem (without convenient shortcuts like a consumption floor or liquidity constraints) defines a contraction mapping with a nondegenerate consumption function (the main requirement is a 'Finite Value of Autarky' condition). Another contribution is to specify the conditions under which the resulting consumption function implies there is a 'target' wealth-to-permanent-income ratio. (This is the sense in which the paper studies the class of 'buffer stock' saving models.) The key requirement is that the model's parameters satisfy a "Growth Impatience Condition" (equation (32)) that relates preferences and uncertainty to the predictable growth rate of income.

Even without a formal proof, target saving of this kind has been intuitively understood to underlie central numerical results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the explanation by **?** of the fact that, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich - even if everyone has the same fundamental preferences. The theoretical logic articulated below explains this finding: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$) because they have little room to maneuver, and it does not change the behavior of the rich because the increase in uncertainty does not threaten their consumption much.

Conveniently, elements required for the convergence proof turn out to provide analytical foundations for many other results that have become familiar from the numerical literature. All theoretical conclusions are paired with numerically computed illustrations (using an open-source toolkit available from the Econ-ARK project). All of the insights of this paper are instantiated in the toolkit, which algorithmically flags parametric

[1]It is unclear whether newer methods such as those of **?** could overcome this problem, or how difficult it would be to do so; but in any case this particular problem does not seem to have been tackled by those methods or any others.

(**?**); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Metin Uyanik, Weifeng Wu, Xudong Zheng, and Jiaxiong Yao for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

choices under which a problem fails to define a contraction mapping, under which a target level of wealth does not exist, or under which the solution is otherwise degenerate.

Thus, the theoretical foundations provided here are valuable both because they provide intuition about the determinants of saving targets, and because they make it easier to develop reliable numerical solution methods (by providing new restrictions that valid solutions must satisfy).

The paper proceeds in three parts.

The first part articulates the conditions required for the problem to define a unique nondegenerate limiting consumption function, and discusses the relation of the paper's model to models previously considered in the literature. The required conditions are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives some limiting properties of the consumption function as cash approaches infinity and as it approaches its lower bound, and the theorem is proven explaining when the problem defines a contraction mapping. Finally, a related class of commonly-used models (exemplified by Deaton (?)) is shown to constitute a particular limit of this paper's more general model.

The next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, if the consumer is 'growth impatient,' a unique target cash-to-permanent-income ratio will exist. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent noncapital income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

Szeidl (?) has shown that such an economy will be characterized by stable invariant distributions for the consumption ratio, the wealth ratio, and other variables.[2] Using Szeidl's result, the final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, an economy populated by buffer stock consumers converges to a balanced growth equilibrium in which the growth rate of consumption tends toward the (exogenous) growth rate of permanent income.

[2]Szeidl's proof supplants the analysis in an earlier draft of this paper, which conjectured that such a result held and provided supportive simulation evidence.

2 The Problem

2.1 Setup

The consumer solves an optimization problem from period t until the end of life at T defined by the objective

$$\max\ \mathbb{E}_t\left[\sum_{n=0}^{T-t}\beta^n \mathrm{u}(\mathbf{c}_{t+n})\right] \tag{1}$$

where

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{2}$$

is a constant relative risk aversion utility function with $\rho > 1$.[3,4] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ (which **?** called 'cash-on-hand') and permanent noncapital income $\mathbf{p}_t$.

In the usual treatment, a dynamic budget constraint (DBC) simultaneously incorporates all of the elements that determine next period's $\mathbf{m}$ given this period's choices; but for the detailed analysis here, it will be useful to disarticulate the steps so that individual ingredients can be separately examined:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathbf{a}_t\mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t\underbrace{\Gamma\psi_{t+1}}_{\equiv\Gamma_{t+1}} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1}\xi_{t+1},
\end{aligned} \tag{3}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1+\mathsf{r})$ between periods,[5] so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[6] $\mathbf{m}_{t+1}$ ('market resources' or 'money') is the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; future transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1\ \forall\ n \geq 1$). Permanent noncapital income in period $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1\ \forall\ n \geq 1$ satisfying

[3] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[4] We will define the infinite horizon solution as the limit of the finite horizon problem as the horizon $T-t$ approaches infinity.

[5] See **?** for interesting new work that considers the case where capital returns are stochastic and liquidity constraints exist. **?** examines implications of capital income risk for the distribution of wealth.

[6] Allowing a stochastic interest factor is straightforward but adds little insight. The effects are more interesting for analysis of the invariant distribution (**?**).

$\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ where $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks.[7,8]

In future periods $t+n \; \forall \; n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{4}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \; \forall \; n > 0$) that has a distribution satisfying $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$ (degenerately $\underline{\theta} = \bar{\theta} = 1$). (See **?** and **?** for analyses of cases where the shock processes have unbounded support). Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (and $\mathcal{F}_\xi$ is derived trivially from (4) and $\mathcal{F}_\theta$). Permanent income and cash start out strictly positive, $\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty)$, and as usual the consumer cannot die in debt, so

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{5}$$

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable. However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,[9] analytically transforming that model back into the model analyzed here. Also, the assumption of a positive point mass (as opposed to positive density) for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.

This model differs from Bewley's (**?**) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that $\mathrm{u}(0)$ is well defined, or that $\mathrm{u}'(0)$ is well defined and finite, so neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (**?**) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds are quantitatively important in micro data[10] and which since **?** have been understood to be far more consequential for household welfare than are transitory fluctuations. It differs from Deaton (**?**) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* **?**); and the transitory shocks here can occasionally cause income to reach zero.[11] Finally, it

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[7] It is useful to emphasize that permanent noncapital income as defined here differs from what Deaton (**?**) calls permanent income (which is often adopted in the macro literature). Deaton defines permanent income as the amount that a perfect foresight consumer could spend while leaving total (human and nonhuman) wealth constant. Relatedly, we refer to $\mathbf{m}_t$ as 'cash-on-hand' or 'market resources' rather than as wealth to avoid any confusion for readers accustomed to thinking of the discounted value of future noncapital income as a part of wealth. The 'market resources' terminology is motivated by the model's assumption that human wealth cannot be capitalized, an implication of anti-slavery laws.

[8] Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

[9] So long as this PDV is a finite number and unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in section 2.11.

[10] MaCurdy (**?**); Abowd and Card (**?**); Carroll and Samwick (**?**); Jappelli and Pistaferri (**?**); Storesletten, Telmer, and Yaron (**?**); **?**

[11] Below it will become clear that the Deaton model is a particular limit of this paper's model.

differs from models found in Stokey et. al. (?) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[12] ? relaxed the bounds on the return function, but they address only the deterministic case.

The incorporation of permanent shocks rules out application of the tools of ?, who followed and corrected an error in the fundamental work on the local contraction mapping method developed in ?. ? provides a correction to ?, and provides conditions that are easier to verify than those of ? in many applications, but again only addresses the deterministic case.

2.2 The Problem Can Be Rewritten in Ratio Form

We establish a bit more notation by reviewing the standard result that in problems of this class (CRRA utility, permanent shocks) the number of relevant state variables can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with m just above), assume that value in the last period of life is $\mathrm{u}(\mathbf{m}_T)$, and consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} \ \mathrm{u}(\mathbf{c}_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1}c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho}\left\{\max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)]\right\}.
\end{aligned} \tag{6}$$

where the last line follows because for the CRRA utility function in (2), $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$.

Now, in a one-time deviation from the notational convention established in the last paragraph, define nonbold 'normalized value' not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to use the related problem

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ \mathrm{u}(\mathrm{c}_t) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1})a_t \ = \ \mathcal{R}_{t+1}a_t \\
m_{t+1} &= b_{t+1} + \xi_{t+1}
\end{aligned} \tag{7}$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the problem's first order condition is

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho}c_{t+1}^{-\rho}]. \tag{8}$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (7) for $t = T-1$, (6) reduces to

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

[12] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (?) and Clarida (?). See ? for an elegant analysis of a related but simpler continuous-time model.

This logic induces to all earlier periods, so that if we solve the normalized one-state-variable problem specified in (7) we will have solutions to the original problem for any $t < T$ from:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t),$$
$$\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t\mathrm{c}_t(m_t).$$

2.3 Definition of a Nondegenerate Solution

We say that this problem has a nondegenerate solution if, as the number of remaining periods of life gets arbitrarily large, it defines a unique limiting consumption function whose optimal c satisfies

$$0 < c < \infty \tag{9}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is either $\mathrm{c}(m) = 0$ or $\mathrm{c}(m) = \infty$.)

2.4 Perfect Foresight Benchmarks

Articulating the familiar analytical solution to the perfect foresight specialization of the model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology, and to define a convenient reference point.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (5) imply an exactly-holding intertemporal budget constraint (IBC)

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{10}$$

where $\mathbf{b}$ is nonhuman wealth and $\mathbf{h}_t$ is 'human wealth,' and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$,

$$\begin{aligned} \mathbf{h}_t &= \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + \ldots + \mathcal{R}^{t-T}\mathbf{p}_t \\ &= \underbrace{\left(\frac{1 - \mathcal{R}^{-(T-t+1)}}{1 - \mathcal{R}^{-1}}\right)}_{\equiv h_t}\mathbf{p}_t. \end{aligned} \tag{11}$$

Equation (11) makes plain that in order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC')

$$\underbrace{\Gamma/\mathsf{R}}_{\equiv \mathcal{R}^{-1}} < 1. \tag{12}$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 Unconstrained Solution

The consumption Euler equation holds in every period; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$, this says

$$\mathbf{c}_{t+1}/\mathbf{c}_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{13}$$

where the Old English letter 'thorn' represents what we will call the 'absolute patience factor' $(\mathsf{R}\beta)^{1/\rho}$.[13] The sense in which Þ captures patience is that if the 'absolute impatience condition' (AIC) holds,

$$\text{Þ} < 1, \tag{14}$$

the consumer will choose to spend an amount too large to sustain indefinitely (the level of consumption must fall over time). We call such a consumer 'absolutely impatient' (this is the key condition in **?**).

We next define a 'return patience factor' that relates absolute patience to the return factor:

$$\text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{15}$$

and note that since consumption is growing by Þ but discounted by R:

$$\begin{aligned} \mathrm{PDV}_t(\mathbf{c}) &= \left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \ldots + \text{Þ}_{\mathsf{R}}^{T-t}\right)\mathbf{c}_t \\ &= \left(\frac{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}{1-\text{Þ}_{\mathsf{R}}}\right)\mathbf{c}_t \end{aligned} \tag{16}$$

from which the IBC (10) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t}(\mathbf{b}_t + \mathbf{h}_t) \tag{17}$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{=b_{T-n}} + h_{T-n})\underline{\kappa}_{T-n} \tag{18}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) because it answers the question 'if the consumer had an extra unit of wealth, how much more would he spend.' (The overbar on c reflects the fact that this will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, the underbar for κ indicates that it is a lower bound). Equation (17) makes plain that for the limiting MPC to be strictly positive as $n = T - t$ goes to infinity we must impose the condition

$$\text{Þ}_{\mathsf{R}} < 1, \tag{19}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{20}$$

[13] Impatience conditions of one kind or another have figured in intertemporal optimization problems since such problems were first formalized in economics, most notably by **?**. Discussion of these issues was prominent in the literature of the 1960s and 1970s, and no brief citations here could do justice to the literature on the topic, so I refrain from the attempt.

Equation (19) thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth; that is, the condition rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$. Because the return patience factor $\text{\TH}_{\mathsf{R}}$ is the absolute patience factor divided by the return, we call equation (19) the 'return impatience condition' or RIC; we will say that a consumer who satisfies the condition is 'return impatient.'

Given that the RIC holds, and defining limiting objects by the absence of a time subscript (e.g., $\bar{\mathrm{c}}(m) = \lim_{n\uparrow\infty} \bar{\mathrm{c}}_{T-n}(m)$), the limiting consumption function will be

$$\bar{\mathrm{c}}(m) = (m + h - 1)\underline{\kappa}, \tag{21}$$

and we now see that in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m) = \infty$ we need h to be finite so we must impose the finite human wealth condition (12).

A final useful point is that since the perfect foresight growth factor for consumption is $\text{\TH}$, again using $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, yields an analytical expression for value:

$$\begin{aligned} v_t &= \mathrm{u}(c_t) + \beta \mathrm{u}(c_t \text{\TH}) + \beta^2 \mathrm{u}(c_t \text{\TH}^2) + ... \\ &= \mathrm{u}(c_t)\left(1 + \beta \text{\TH}^{1-\rho} + (\beta \text{\TH}^{1-\rho})^2 + ...\right) \\ &= \mathrm{u}(c_t)\left(\frac{1 - (\beta \text{\TH}^{1-\rho})^{T-t+1}}{1 - \beta \text{\TH}^{1-\rho}}\right) \end{aligned} \tag{22}$$

which asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if $\beta \text{\TH}^{1-\rho} < 1$ (related to a condition in **?**); with a bit of algebra, this requirement can be shown to be equivalent to the RIC.[14] Thus, the same conditions that guarantee a nondegenerate limiting consumption function also guarantee a nondegenerate limiting value function (which will *not* be true in the version of the model that incorporates uncertainty).

2.4.3 Constrained Solution

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, which obtains for a consumer who enters period t with $b_t = 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (8):

$$1^{-\rho} > \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{23}$$

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$$\begin{aligned} \beta((\mathsf{R}\beta)^{1/\rho})^{1-\rho} &< 1 \\ \beta(\mathsf{R}\beta)^{1/\rho}/\mathsf{R}\beta &< 1 \\ (\mathsf{R}\beta)^{1/\rho}/\mathsf{R} &< 1 \end{aligned}$$

.

By analogy to the return patience factor, we therefore define a 'perfect foresight growth patience factor' as

$$\mathbf{Þ}_{\Gamma} = \mathbf{Þ}/\Gamma, \tag{24}$$

and define a 'perfect foresight growth impatience condition' (PF-GIC)

$$\mathbf{Þ}_{\Gamma} < 1 \tag{25}$$

which is equivalent to (23) (exponentiate both sides by $1/\rho$).

If the RIC and the FHWC hold, appendix A shows that, for some $0 < m_{\#} < 1$, an unconstrained consumer behaving according to (21) would choose $c < m$ for all $m > m_{\#}$. The solution to the constrained consumer's problem in this case is simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future) and so the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[15] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would have to consume $c = m$. We use the $\circ$ accent to designate the limiting constrained consumption function:

$$\mathring{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases} \tag{26}$$

More useful is the case where the perfect foresight growth and return impatience conditions both hold. In this case appendix A shows that the limiting constrained consumption function is piecewise linear, with $\mathring{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m_{\#}^{1} > 1$, and with discrete declines in the MPC at successively increasing kink points $\{m_{\#}^{1}, m_{\#}^{2}, ...\}$. As $m \uparrow \infty$ the constrained consumption function $\mathring{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\mathring{\boldsymbol{\kappa}}(m) \equiv \mathring{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$. Similarly, the value function $\mathring{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer. Surprisingly, this logic holds even when the finite human wealth condition fails (denoted ~~FHWC~~). A solution exists because the constraint prevents the consumer from borrowing against infinite human wealth to finance infinite current consumption. Under these circumstances, the consumer who starts with any amount of resources $b_t > 1$ will, over time, run those resources down so that by some finite number of periods n in the future the consumer will reach $b_{t+n} = 0$, and thereafter will set $c = m = 1$ for eternity, a policy that will (using (22)) yield value of

$$\begin{aligned} \mathbf{v}_{t+n} &= \mathrm{u}(\mathbf{p}_{t+n})\left(1 + \beta\Gamma^{1-\rho} + (\beta\Gamma^{1-\rho})^2 + ...\right) \\ &= \Gamma^{n(1-\rho)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1 - (\beta\Gamma^{1-\rho})^{T-(t+n)+1}}{1 - \beta\Gamma^{1-\rho}}\right), \end{aligned}$$

which will be finite whenever

$$\begin{aligned} \overbrace{\beta\Gamma^{1-\rho}}^{\equiv \beth} &< 1 \\ \beta\mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma \end{aligned} \tag{27}$$

[15]"Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

$$\text{Þ}_\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}$$

which we call the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC, because it guarantees that a consumer who always spends all his permanent income will have finite value (the consumer has 'finite autarky value'). Note that the version of the PF-FVAC in (27) implies the PF-GIC $\text{Þ}_\Gamma < 1$ whenever $\cancel{\text{FHWC}}$ ($\mathsf{R} < \Gamma$) holds. So, if $\cancel{\text{FHWC}}$, value for any finite m will be the sum of two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+n} = 0$, and the finite component due to the value of consuming all income thereafter. The consumer's value function is therefore nondegenerate.

The most peculiar possibility occurs when the RIC fails. The appendix shows that under these circumstances the FHWC must also fail, and the constrained consumption function is nondegenerate. (See Figure 6 for a numerical example). While it is true that $\lim_{m\uparrow\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\mathring{\mathrm{c}}(m)$ is strictly positive and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where $\cancel{\text{RIC}}$ would suggest a dengenerate limit of $\mathring{\mathrm{c}}(m) = 0$ while $\cancel{\text{FHWC}}$ would suggest a degenerate limit of $\mathring{\mathrm{c}}(m) = \infty$.

Tables 3 and 4 (and appendix table 5) codify the key points to help the reader keep them straight (and to facilitate upcoming comparisons with the results in the presence of uncertainty but the absence of liquidity constraints (also tabulated for comparison)). The model without constraints but with uncertainty will turn out to be a close parallel to the model with constraints but without uncertainty.

2.5 Uncertainty-Modified Conditions

2.5.1 Impatience

When uncertainty is introduced, the expectation of b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[\mathcal{R}_{t+1}] = a_t\mathcal{R}\,\mathbb{E}_t[\psi_{t+1}^{-1}] \tag{28}$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define the object

$$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1}$$

because this permits us to write expressions like the RHS of (28) compactly as, e.g., $a_t\mathcal{R}/\underline{\psi}^{-1}$.[16] We refer to this as the 'compensated return,' because it compensates (in a risk-neutral way) for the effect of uncertainty on the expected growth-normalized return (in the sense implicitly defined in (28)).

We can now transparently generalize the PF-GIC (25) by defining a 'compensated growth factor'

$$\underline{\Gamma} = \Gamma\underline{\psi} \tag{29}$$

[16]One way to think of $\underline{\psi}$ is as a particular kind of a 'certainty equivalent' of the shock; this captures the intuition that mean-one shock renders a given mean level of income less valuable than if the shock did not exist, so that $\underline{\psi} < 1$.

and a compensated growth patience factor

$$\text{Þ}_{\underline{\Gamma}} = \text{Þ}/\underline{\Gamma} \tag{30}$$

$$= \mathbb{E}\left(\frac{\text{Þ}}{\Gamma\psi}\right) \tag{31}$$

and a straightforward derivation in (47) below yields the conclusion that

$$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{Þ}_{\underline{\Gamma}},$$

which implies that if we wish to prevent m from heading to infinity (that is, if we want m to be guaranteed to be expected to fall for some large enough value of m) we must impose a generalized version of the Perfect Foresight Growth Impatience Conditon (25) which we call simply the 'growth impatience condition' (GIC):[17]

$$\text{Þ}_{\underline{\Gamma}} < 1 \tag{32}$$

which is stronger than the perfect foresight version (25) because $\underline{\Gamma} < \Gamma$ (Jensen's inequality implies that $\underline{\psi} < 1$ for nondegenerate ψ).

2.5.2 Autarky Value

Analogously to (22), a consumer who spent his permanent income every period would have value determined by the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned}
\mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + ... + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}...\Gamma_T)\right] \\
&= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}] + ... + \beta^{T-t}\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}]...\,\mathbb{E}_t[\Gamma_T^{1-\rho}]\right) \\
&= \mathrm{u}(\mathbf{p}_t)\left(\frac{1 - (\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1 - \beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right)
\end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ (and $\underline{\underline{\psi}} < \underline{\psi}$ for the preferred (though not required) case of $\rho > 2$); defining $\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}}$ we can see that $\mathbf{v}_t$ will be finite as T approaches ∞ if

$$\begin{aligned}
\overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}} &< 1 \\
\beta &< \underline{\underline{\Gamma}}^{\rho-1}
\end{aligned} \tag{33}$$

which we call the 'finite value of autarky' condition (FVAC) because it is the value obtained by always consuming (now stochastic) permanent income. For nondegenerate

[17] Equation (32) is a bit easier to satisfy than the similar condition imposed by Deaton (**?**): $\left(\mathbb{E}[\psi^{-\rho}]\right)^{1/\rho}\text{Þ}_{\Gamma} < 1$ to guarantee that his problem defined a contraction mapping.

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (27) because $\underline{\underline{\Gamma}} < \Gamma$.

To see this, rewrite (33) as

$$\beta\mathsf{R} < \underline{\underline{\Gamma}}^{\rho-1} \tag{34}$$

$$(\beta\mathsf{R})^{1/\rho} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \tag{35}$$

$$\text{Þ}_\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\Gamma}}^{1-1/\rho} \tag{36}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\Gamma}}^{1-1/\rho}$ which is strictly less than 1.

2.6 The Baseline Numerical Solution

Figure 1 depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and various earlier periods under the baseline parameter values listed in Table 2. (The 45 degree line is labelled as $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge as the horizon recedes (our purpose is to show that this appearance is not deceptive); we call the limiting infinite-horizon consumption rule

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m). \tag{37}$$

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Measure	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Measure	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}_t[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	Þ_{R}	$\equiv$	$\text{Þ}/\mathsf{R}$	0.961
PF Growth Patience Factor	Þ_{Γ}	$\equiv$	$\text{Þ}/\Gamma$	0.970
Growth Patience Factor	$\text{Þ}_{\underline{\Gamma}}$	$\equiv$	$\text{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Measure	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941

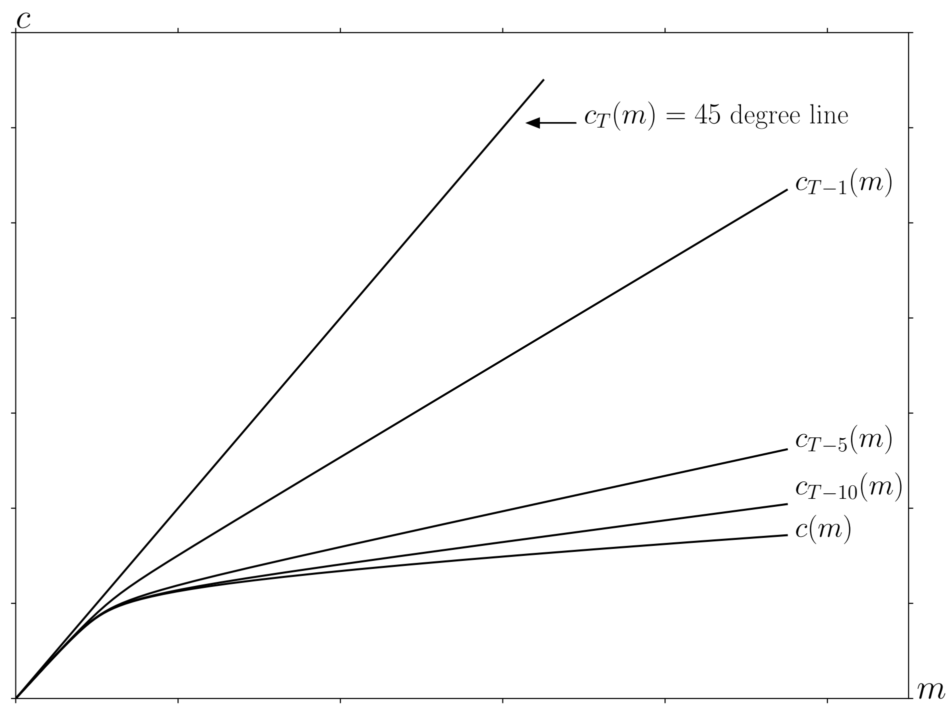


Figure 1 Convergence of the Consumption Rules

2.7 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (7) defines a sequence of continuously differentiable strictly increasing strictly concave[18] functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ...\}$.[19] The straightforward but tedious proof is relegated to appendix B. For present purposes, the most important point is the following intuition: $\mathrm{c}_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero noncapital income over the remaining horizon (an unbroken series of zero-income events is unlikely but possible). In such a case, the budget constraint and the can't-die-in-debt condition mean that the consumer would be forced to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. (This is an example of the 'natural borrowing constraint' induced by a precautionary motive (**?**).)[20]

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 1 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which also turn out to be useful in the contraction mapping proof. In a recent paper, **?** show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; it seems clear that their results would generalize to the limits derived here if capital income risk were added to the model. **?** establish the existence and uniqueness of a solution to a general income fluctuation problem with capital income risk in a Markovian setting and use such a model to study the tail behavior of wealth in the presence of risky returns to capital.

Assume (as discussed above) that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathrm{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption about finite human wealth, the proportion of consumption that will be financed out of human wealth approaches zero. The consequence is that the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero.

In the course of proving this point, appendix F provides a useful recursive expression for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+1}^{-1}. \tag{38}$$

[18] There is one obvious exception: $\mathrm{c}_T(m)$ is a linear (and so only weakly concave) function.

[19] **?** proved concavity but not the other desired properties.

[20] It would perhaps be better to call it the 'utility-induced borrowing constraint' as it follows from the assumptions on the utility function (in particular, $\lim_{c \downarrow 0} \mathrm{u}(c) = -\infty$); for example, no such constraint arises if utility is of the (implausible) Constant Absolute Risk Aversion form.

It turns out that there is a parallel expression for the limiting maximal MPC as $m \downarrow 0$: appendix equation (97) shows that, as $m_t \uparrow \infty$,

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}. \tag{39}$$

Then $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{40}$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero income events. (That consideration is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathrm{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{41}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave (?), and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is provided in appendix D).

These limits are useful at least in the sense that they can be hard-wired into a solution algorithm for the model, which has the potential to make the solution more efficient (cf. ?). Alternatively, they can provide a useful check on the accuracy of a solution algorithm that does not impose them directly.

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. This cannot be proven using the standard theorems in, say, Stokey et. al. (?), which require marginal utility to be bounded over the space of possible values of m, because the possibility (however unlikely) of an unbroken string of zero-income events for the remainder of life means that as m approaches zero c must approach zero (see the discussion in 2.7); thus, marginal utility is unbounded. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., ?), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that involves permanent shocks.[21]

Fortunately, Boyd (?) provided a weighted contraction mapping theorem that ? showed how to address the homogeneous case (of which CRRA formulation is an example) in a deterministic framework; ? showed how to extend ? approach to the stochastic case.

[21]See ? for a detailed discussion of the reasons the existing literature up through ? cannot handle the problem described here.

Definition 1. *Consider any function* $\bullet \in \mathcal{C}(\mathcal{A},\mathcal{B})$ *where* $\mathcal{C}(\mathcal{A},\mathcal{B})$ *is the space of continuous functions from* $\mathcal{A}$ *to* $\mathcal{B}$. *Suppose* $F \in \mathcal{C}(\mathcal{A},\mathcal{B})$ *with* $\mathcal{B} \subseteq \mathbb{R}$ *and* $F > 0$. *Then* $\bullet$ *is* F*-bounded if the* F*-norm of* $\bullet$,

$$\| \bullet \|_{F} = \sup_{m} \left[\frac{| \bullet (m)|}{F(m)} \right], \tag{42}$$

is finite.

For $\mathcal{C}_{F}(\mathcal{A},\mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A},\mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (**?**) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{F}(\mathcal{A},\mathcal{B}) \to \mathcal{C}(\mathcal{A},\mathcal{B})$ *such that*[22,23]

$$\begin{aligned}
&(1)\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \leq \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \leq \{\mathsf{T}\mathrm{y}\} \\
&(2)\{\mathsf{T}\mathbf{0}\} \in \ \mathcal{C}_{F}(\mathcal{A},\mathcal{B}) \\
&(3)\textit{There exists some real } 0 < \alpha < 1\textit{, such that} \\
&\quad \{\mathsf{T}(\mathrm{w} + \zeta F)\} \leq \{\mathsf{T}\mathrm{w}\} + \zeta \alpha F \quad \textit{holds for all real } \zeta > 0\ .
\end{aligned}$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{++}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t) = \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{z}(a_t \mathcal{R}_{t+1} + \xi_{t+1}) \right].$$

Using this, we introduce the mapping $\mathcal{T} : \mathcal{C}_{F}(\mathcal{A},\mathcal{B}) \to \mathcal{C}(\mathcal{A},\mathcal{B})$,[24]

$$\{\mathcal{T}\mathrm{z}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \mathrm{u}(c_t) + \beta \left(\{\mathsf{E}\mathrm{z}\}(m_t - c_t) \right). \tag{43}$$

We can show that our operator $\mathcal{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first restriction is the WRIC necessary for convergence of the maximal MPC, equation (40) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (33). (We discuss the interpretation of these restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. $\mathcal{T}$ *is a contraction mapping if the restrictions on parameter values* (40) *and* (33) *are true (that is, if the weak return impatience condition and the finite value of autarky condition hold).*

Intuitively, Boyd's theorem shows that if you can find a F that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with F, the normalized problem defines a contraction mapping. The intuition for the FVAC condition

[22] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[23] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \leq \mathrm{y}$ is short for $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)\ \forall \bullet \in \mathcal{A}$. In this notation, $\zeta \alpha F$ in (3) is a *function* which can be applied to any argument $\bullet$ (because F is a function).

[24] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous F-bounded functions z) and the compactness of $[\underline{\kappa} m_t, \bar{\kappa} m_t]$.

is just that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming all of your permanent income and nothing else every period (say, by consuming $(\mathsf{r}/RFree)b_0 - \epsilon$ for some small $\epsilon > 0$.

The details of the proof are cumbersome, and are therefore relegated to appendix D. Given that the value function converges, appendix D.3 shows that the consumption functions converge.

2.10 The Liquidity Constrained Solution as a Limit

This section shows that a related problem commonly considered in the literature (e.g., with a simpler income process, by Deaton (**?**)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The essence of the argument is easy to state. As noted above, there is a finite possibility of earning zero income over the remainder of the horizon, which prevents the consumer from ending the current period with zero assets because with some finite probability the consumer would be forced to consume zero, which would be infinitely painful.

But *extent* to which the consumer feels the need to make this precautionary provision depends on the probability that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the amount of precautionary saving approaches zero. But zero precautionary saving is the amount of saving that a liquidity constrained consumer with perfect foresight would choose.

Another way to think about this is just to think of the liquidity constraint as being imposed by specifying a component of the utility function that is zero whenever the consumer ends the period with (weakly) positive assets, but negative infinity if the consumer ended the period with (strictly) negative assets.

See appendix G for the formal proof justifying the foregoing intuitive discussion.

2.11 Discussion of Parametric Restrictions

2.11.1 The RIC

In the perfect foresight unconstrained problem (section 2.4.2), the RIC was required for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the RIC is neither necessary nor sufficient for a nondegenerate solution. We thus begin our discussion by asking what features the problem must exhibit (given the FVAC) if the RIC fails (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$\mathsf{R} < \overbrace{(\mathsf{R}\beta)^{1/\rho} \;<\; (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}}$$
$$\mathsf{R} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho}$$

$$\mathsf{R}/\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho}$$
$$(\mathsf{R}/\Gamma)^{1-1/\rho} < \underline{\underline{\psi}}^{1-1/\rho} \tag{44}$$

but since $\underline{\underline{\psi}} < 1$ and $0 < 1 - 1/\rho < 1$ (because we have assumed $\rho > 1$), this can hold only if $\mathsf{R}/\Gamma < 1$; that is, given the FVAC, the RIC can fail only if human wealth is unbounded. Unbounded human wealth is permitted here, as in the perfect foresight liquidity constrained problem. But, from equation (38), an implication of ~~RIC~~ is that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited wealth from resulting in infinite consumption. That is, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in consumption of zero) plus infinite human wealth (which the perfect foresight model rules out because it leads to infinite consumption) combine here to yield a unique finite limiting MPC for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {PF-GIC,~~RIC~~} case (for detailed analysis of this case see the appendix). There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.

2.11.2 The WRIC

The 'weakness' of the additional requirement for contraction, the weak RIC, can be seen by asking 'under what circumstances would the FVAC hold but the WRIC fail?' Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \ 1 \ < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{45}$$

If there were no conceivable parameter values that could satisfy both of these inequalities, the WRIC would have no force; it would be redundant. And if we require $\mathsf{R} \geq 1$, the WRIC is indeed redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that the RIC (and WRIC) must hold.

But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (45) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. Thus, the relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which section

2.10 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.11.3 When the GIC Fails

If both the GIC and the RIC hold, the arguments above establish that the limiting consumption function asymptotes to the consumption function for the perfect foresight unconstrained function. The more interesting case is where the GIC fails. A solution that satisfies the combination FVAC and ~~GIC~~ is depicted in Figure 2. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the code for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC does not, the consumption function's limiting slope $(1 - \text{\TH}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1 - \underline{\Gamma}/\mathsf{R})$,[25] so the gap between the two actually increases with m in the limit. That is, although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, he will always spend less than the amount that would keep m stable (in expectation).

For the reader's convenience, Tables 3 and 4 present a summary of the connections between the various conditions in the presence and the absence of uncertainty.

[25]This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\underline{\psi}^{-1} + 1$ for c and differentiate.

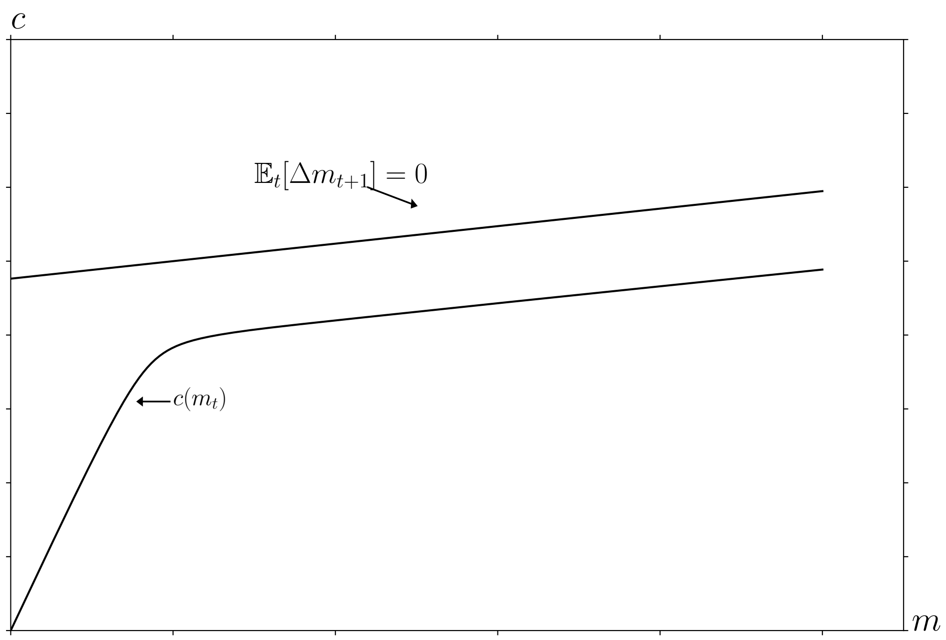


Figure 2 Example Solution when FVAC Holds but GIC Does Not

3 Analysis of the Converged Consumption Function

Figures 3 and 4a,b capture the main properties of the converged consumption rule when the RIC, GIC, and FHWC all hold.[26] Figure 3 shows the expected consumption growth factor $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ for a consumer behaving according to the converged consumption rule, while Figures 4a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to $\text{\textbf{Þ}}$, indicated by the lower bound in Figure 3, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{\textbf{Þ}}_{\mathsf{R}})$ (Figure 4), the same as the perfect foresight MPC.[27] Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 3) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{\textbf{Þ}}_{\mathsf{R}})$ (Figure 4). Third (Figure 3), there is a target cash-on-hand-to-income ratio $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ curve), the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then cash-on-hand will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 3), at the target m, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 3 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[28]

3.1 Limits as $m_t \uparrow \infty$

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathrm{c}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption.[29]

Assuming the FHWC holds, the infinite horizon perfect foresight solution (21) constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimball (**?**) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

[26] These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

[27] If the RIC fails, the limiting minimal MPC is 0; see appendix.

[28] Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathrm{c}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathrm{c}(m)$. It does not strictly follow that the limiting function $\mathrm{c}(m)$ is twice continuously differentiable, but I will assume that it is.

[29] We will assume the RIC holds here and subsequently so that $\underline{\kappa} > 0$; the situation is a bit more complex when the RIC does not hold. In that case the bound on consumption is given by the spending that would be undertaken by a consumer who faced binding liquidity constraints. Detailed analysis of this special case is not sufficiently interesting to warrant inclusion in the paper.

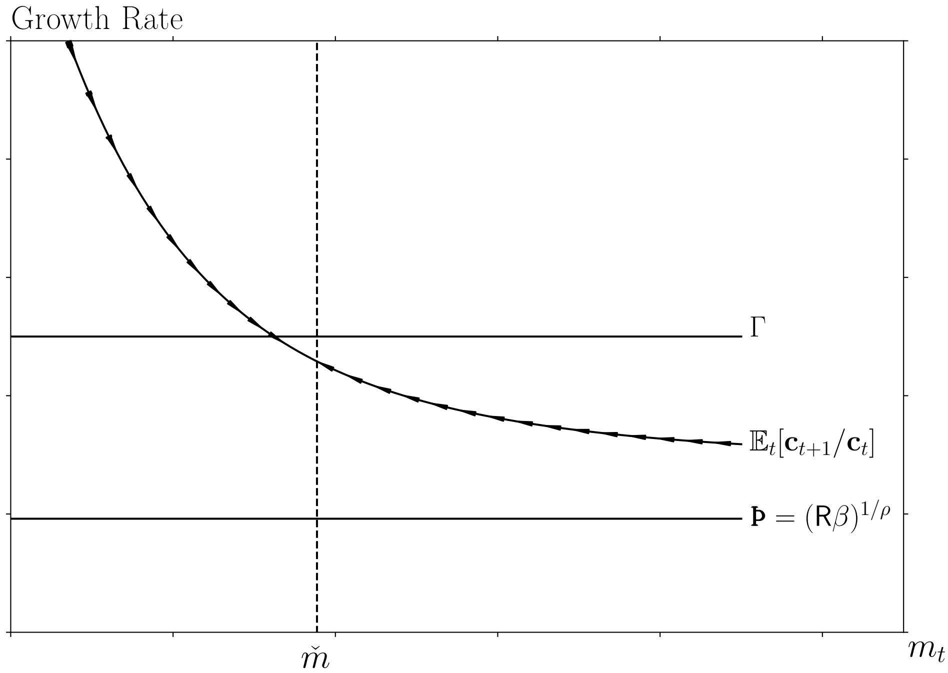


Figure 3 Target m, Expected Consumption Growth, and Permanent Income Growth

Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) <& \mathrm{c}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathrm{c}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned} \tag{46}$$

But

$$\begin{aligned} \lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m-1+h)/m \\ &= 1, \end{aligned}$$

so as $m \uparrow \infty, \mathrm{c}(m)/\underline{\mathrm{c}}(m) \rightarrow 1$, and the continuous differentiability and strict concavity of $\mathrm{c}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 4 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t\uparrow\infty} \mathbb{E}_t[\Gamma_{t+1} c_{t+1}/c_t].$$

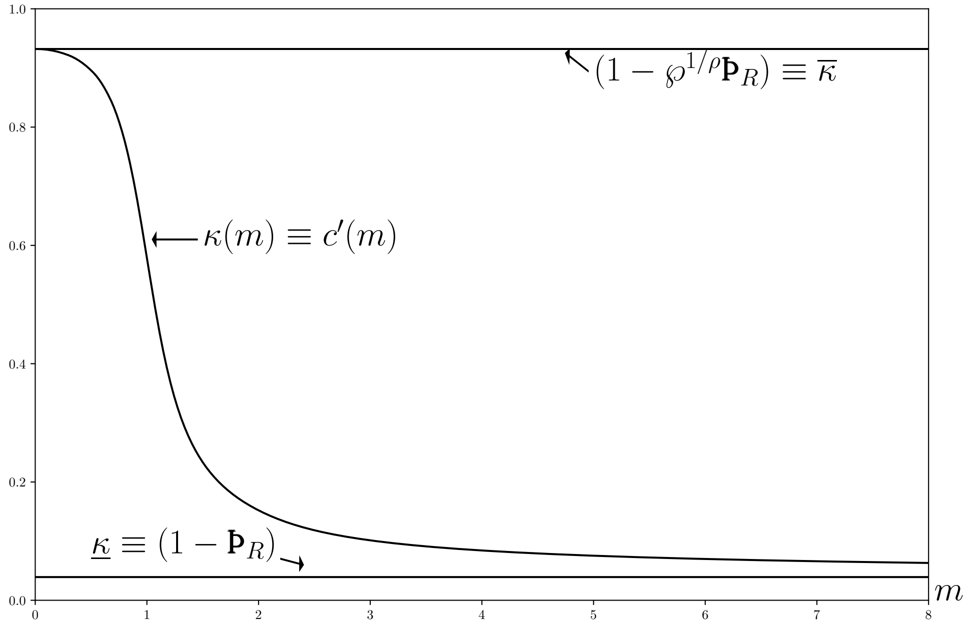


Figure 4 Limiting MPC's

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t\uparrow\infty} \Gamma_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while

$$\lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t = \lim_{m_t\uparrow\infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t}\right) \tag{47}$$

$$= (\mathsf{R}\beta)^{1/\rho} = \text{Þ} \tag{48}$$

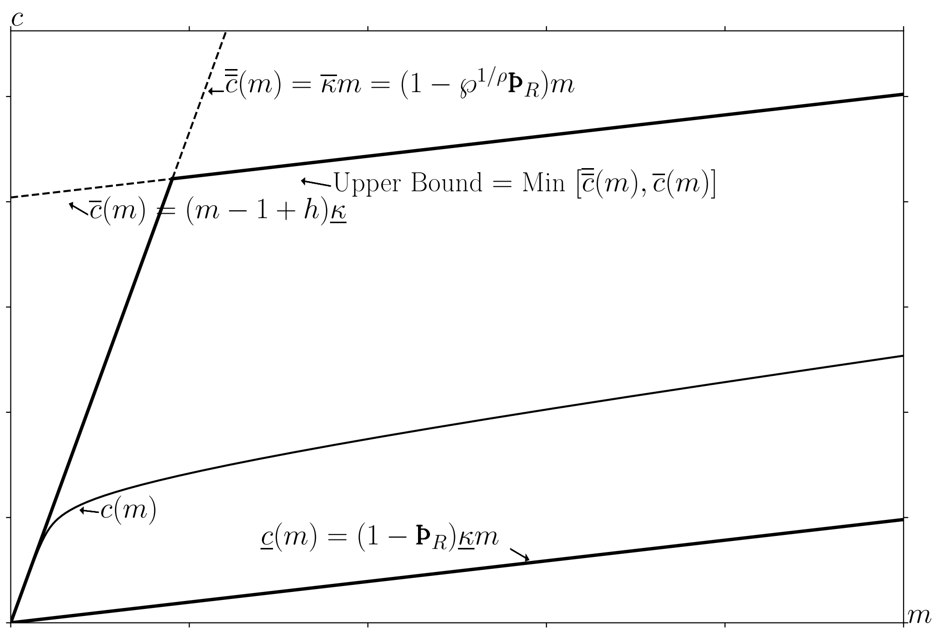
because $\lim_{m_t\uparrow\infty} \mathrm{a}'(m) = \text{Þ}_{\mathsf{R}}$[30] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

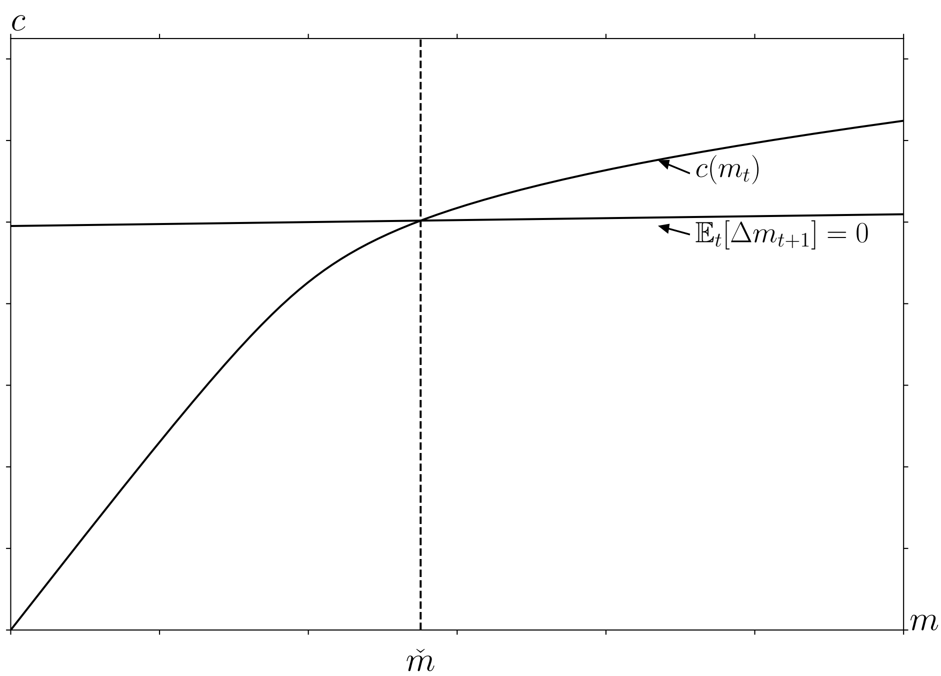
$$\text{Þ} \leq \lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model.

[30]This is because $\lim_{m_t\uparrow\infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathrm{c}'(m_t) = \text{Þ}_{\mathsf{R}}$.



(a) Bounds



(b) Target m

Figure 5 The Consumption Function

This argument applies equally well to the problem of the restrained consumer, because as m approaches infinity the constraint becomes irrelevant (assuming the FHWC holds).

3.2 Limits as $m_t \downarrow 0$

Now consider the limits of behavior as m_t gets arbitrarily small.

Equation (39) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho} \text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathrm{c}'(m) = \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 4 confirms that the numerical solution method obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1} \mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[31]

3.3 There Exists Exactly One Target Cash-on-Hand Ratio, which is Stable

Define the target cash-on-hand-to-income ratio $\check{m}$ as the value of m such that

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{49}$$

where the $\vee$ accent is meant to invoke the fact that this is the value that other m's 'point to.'

[31] The same arguments establish $\lim_{m \downarrow 0} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \infty$ for the problem of the restrained consumer.

We prove existence by arguing that $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous on $m_t > 0$, and takes on values both above and below 1, so that it must equal 1 somewhere by the intermediate value theorem.

Specifically, the same logic used in section 3.2 shows that $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}/m_t] = \infty$.

The limit as m_t goes to infinity is

$$
\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\textbf{Þ}}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}] \\
&< 1
\end{aligned}
$$

where the last line is guaranteed by our imposition of the GIC (32).

Stability means that in a local neighborhood of $\check{m}$, values of m_t above $\check{m}$ will result in a smaller ratio of $\mathbb{E}_t[m_{t+1}/m_t]$ than at $\check{m}$. That is, if $m_t > \check{m}$ then $\mathbb{E}_t[m_{t+1}/m_t] < 1$. This will be true if

$$
\left(\frac{d}{dm_t}\right) \mathbb{E}_t[m_{t+1}/m_t] < 0
$$

at $m_t = \check{m}$. But

$$
\begin{aligned}
\left(\frac{d}{dm_t}\right) \mathbb{E}_t[m_{t+1}/m_t] &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)[\mathcal{R}_{t+1}(1 - \mathrm{c}(m_t)/m_t) + \xi_{t+1}/m_t]\right] \\
&= \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(\mathrm{c}(m_t) - \mathrm{c}'(m_t)m_t) - \xi_{t+1}}{m_t^2}\right]
\end{aligned}
$$

which will be negative if its numerator is negative. Define $\boldsymbol{\zeta}(m_t)$ as the expectation of the numerator,

$$
\boldsymbol{\zeta}(m_t) = \underbrace{\mathbb{E}_t[\mathcal{R}_{t+1}]}_{\equiv \bar{\mathcal{R}}}(\mathrm{c}(m_t) - \mathrm{c}'(m_t)m_t) - 1. \tag{50}
$$

The target level of market resources is the m such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = \check{m}$.

$$
\begin{aligned}
\mathbb{E}_t[m_{t+1}] &= \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t) + \xi_{t+1}] \\
\check{m} &= \bar{\mathcal{R}}(\check{m} - \mathrm{c}(\check{m})) + 1 \\
\bar{\mathcal{R}}\mathrm{c}(\check{m}) &= 1 + (\bar{\mathcal{R}} - 1)\check{m}. \qquad (51)
\end{aligned}
$$

At the target, equation (50) is

$$
\boldsymbol{\zeta}(\check{m}) = \bar{\mathcal{R}}\mathrm{c}(\check{m}) - \bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m} - 1.
$$

Substituting for the first term in this expression using (51) gives

$$
\begin{aligned}
\boldsymbol{\zeta}(\check{m}) &= 1 + (\bar{\mathcal{R}} - 1)\check{m} - \bar{\mathcal{R}}\mathrm{c}'(\check{m})\check{m} - 1 \\
&= \check{m}\left(\bar{\mathcal{R}} - 1 - \bar{\mathcal{R}}\mathrm{c}'(\check{m})\right) \\
&= \check{m}\left(\bar{\mathcal{R}}(1 - \mathrm{c}'(\check{m})) - 1\right) \\
&< \check{m}\left(\bar{\mathcal{R}}(1 - (1 - \mathsf{R}^{-1}(\mathsf{R}\beta)^{1/\rho})) - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \check{m}\left(\bar{\mathcal{R}}\text{\textbf{Þ}}_{\mathsf{R}} - 1\right) \\
&= \check{m}\left(\underbrace{\mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}]}_{<1 \text{ from } (32)} - 1\right) \\
&< 0
\end{aligned}
$$

where the step introducing the inequality imposes the fact that $\mathrm{c}' > \text{\textbf{Þ}}_{\mathsf{R}}$ which is an implication of the concavity of the consumption function.

We have now proven that some target $\check{m}$ must exist, and that at any such $\check{m}$ the solution is stable. Nothing so far, however, rules out the possibility that there will be multiple values of m that satisfy the definition (49) of a target.

Multiple targets can be ruled out as follows. Suppose there exist multiple targets; these can be arranged in ascending order and indexed by an integer superscript, so that the target with the smallest value is, e.g., $\check{m}^1$. The argument just completed implies that since $\mathbb{E}_t[m_{t+1}/m_t]$ is continuously differentiable there must exist some small ϵ such that $\mathbb{E}_t[m_{t+1}/m_t] < 1$ for $m_t = \check{m}^1 + \epsilon$. (Continuous differentiability of $\mathbb{E}_t[m_{t+1}/m_t]$ follows from the continuous differentiability of $\mathrm{c}(m_t)$.)

Now assume there exists a second value of m satisfying the definition of a target, $\check{m}^2$. Since $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, it must be approaching 1 from below as $m_t \to \check{m}^2$, since by the intermediate value theorem it could not have gone above 1 between $\check{m}^1 + \epsilon$ and $\check{m}^2$ without passing through 1, and by the definition of $\check{m}^2$ it cannot have passed through 1 before reaching $\check{m}^2$. But saying that $\mathbb{E}_t[m_{t+1}/m_t]$ is approaching 1 from below as $m_t \to \check{m}^2$ implies that

$$
\left(\frac{d}{dm_t}\right)\mathbb{E}_t[m_{t+1}/m_t] > 0 \tag{52}
$$

at $m_t = \check{m}^2$. However, we just showed above that, under our assumption that the GIC holds, precisely the opposite of equation (52) must hold for any m that satisfies the definition of a target. Thus, assuming the existence of more than one target implies a contradiction.

The foregoing arguments rely on the continuous differentiability of $\mathrm{c}(m)$, so the arguments do not directly go through for the restrained consumer's problem in which the existence of liquidity constraints can lead to discrete changes in the slope $\mathrm{c}'(m)$ at particular values of m. But we can use the fact that the restrained model is the limit of the baseline model as $\wp \downarrow 0$ to conclude that there is likely a unique target cash level even in the restrained model.

If consumers are sufficiently impatient, the limiting target level in the restrained model will be $\check{m} = \mathbb{E}_t[\xi_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $\mathrm{a}(1) = 0$, then the target level of m in the restrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of cash-on-hand will be greater than the expected level of income.

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 3 the intersection of the target cash-on-hand ratio locus at $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the growth rate of expected permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\Gamma_{t+1}\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)}\right] &< \mathbb{E}_t\left[\left(\frac{\Gamma_{t+1}(\mathrm{c}(\check{m}) + \mathrm{c}'(\check{m})(m_{t+1} - \check{m}))}{\mathrm{c}(\check{m})}\right)\right] \\
&= \mathbb{E}_t\left[\Gamma_{t+1}\left(1 + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)(m_{t+1} - \check{m})\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\mathbb{E}_t\left[\Gamma_{t+1}\left(m_{t+1} - \check{m}\right)\right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})}\right)\left[\mathbb{E}_t[\Gamma_{t+1}]\underbrace{\mathbb{E}_t[m_{t+1} - \check{m}]}_{=0} + \mathrm{cov}_t(\Gamma_{t+1}, m_{t+1})\right] \qquad (53)
\end{aligned}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m}) + \xi_{t+1}$ and $\mathrm{a}(\check{m}) > 0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (53) is negative, as required.

3.5 Expected Consumption Growth Is a Declining Function of m_t (or Is It?)

Figure 3 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv\boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \qquad (54)$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (54) in brackets can be manipulated to yield

$$\begin{aligned}
\mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\
&= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}. \qquad (55)
\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$
\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho}] \\
0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}\boldsymbol{\Upsilon}'_{t+1}] \\
&= \mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) \\
\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}] \qquad (56)
\end{aligned}
$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (56) that (54) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (55), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\mathrm{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

The software archive associated with this paper presents an example in which this perverse effect dominates. However, extreme assumptions were required (in particular, a very small probability of the zero-income shock) and the region in which $\boldsymbol{\Upsilon}'_{t+1} > 0$ was tiny. In practice, for plausible parametric choices, $\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy.[32]

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0,1]$, all behaving according to the model specified above.[33]

Szeidl (?) proves that such a population will be characterized by an invariant distribution of m that induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.[34]

4.1 Consumption and Income Growth at the Household Level

It is useful to define the operator $\mathbb{M}[\bullet]$ which yields the mean value of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}
\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2)
\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ (see ?) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$.[35]

Thus, in a population that has reached its invariant distribution, the growth rate of idiosyncratic log consumption matches the growth rate of idiosyncratic log permanent income.

4.2 Growth Rates of Aggregate Income and Consumption

Attanasio and Weber (?) point out that concavity of the consumption function (or other nonlinearities) can imply that it is quantitatively important to distinguish between the growth rate of average consumption and the average growth rate of consumption.[36] We have just examined the average growth rate; we now examine the growth rate of the average.

[32] We will continue to take the aggregate interest rate as exogenous and constant. It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

[33] One inconvenient aspect of the model as specified is that it does not exhibit a stationary distribution of idiosyncratic permanent noncapital income; the longer the economy lasts, the wider is the distribution. This problem can be remedied by assuming a constant probability of death, and replacing deceased households with newborns whose initial idiosyncratic permanent income matches the mean idiosyncratic permanent income of the population. For a fully worked-out general equilibrium version of such a model, see ?.

[34] Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper.

[35] Papers in the simulation literature have observed an approximate equivalence between the average growth rates of idiosyncratic consumption and permanent income, but formal proof was not possible until Szeidl's proof of ergodicity.

[36] Since we assume number of the households are normalized to 1, aggregate and average variables are identical.

Using boldface capital letters for aggregate variables, the growth factor for aggregate income is given by:

$$
\begin{aligned}
\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}\left[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t\right]/\mathbb{M}\left[\mathbf{p}_t\xi_t\right] \\
&= \Gamma
\end{aligned}
$$

because of the independence assumptions we have made about ξ and ψ.

The growth factor for aggregate assets is:

$$
\begin{aligned}
\left(\frac{\mathbf{A}_{t+1}}{\mathbf{A}_t}\right) &= \frac{\mathbb{M}[a_{t+1}\mathbf{p}_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]} \\
&= \Gamma\left[\frac{\mathbb{M}[a_{t+1}\mathbf{p}_t\psi_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]}\right] \\
&= \Gamma\left[\frac{\mathbb{M}[(a_t+(a_{t+1}-a_t))\mathbf{p}_t\psi_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]}\right] \\
&= \Gamma\left[1+\frac{\mathbb{M}[(a_{t+1}-a_t)\mathbf{p}_t\psi_{t+1}]}{\mathbb{M}[a_t\mathbf{p}_t]}\right] \\
&= \Gamma\left[1+\frac{\mathbb{M}[a_{t+1}-a_t]\mathbb{M}[\mathbf{p}_t\psi_{t+1}]+\mathrm{cov}((a_{t+1}-a_t),\mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right] \\
&= \Gamma\left[1+\frac{\mathrm{cov}(a_{t+1},\mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right]
\end{aligned}
$$

where the second-to-last line follows from **?**'s proof the ergodicity of the distributions of normalized variables for this problem, which implies that $\mathbb{M}[a_{t+1}-a_t]=0$.

Unfortunately, it is clear that the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1}.

One way of thinking about this is that it reflects the fact that, under our assumptions, the $\boldsymbol{p}$ variable does not have an ergodic distribution; the distribution of permanent income becomes forever wider and wider as time progress in this model.

There is a simple solution to that problem, however. In practice most modelers incorporate a constant positive probability of death in their models, following **?**. **?** show that for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, the distribution of permanent income has a finite variance. In such cases, numerical results confirm the intuition that the growth rate of aggregate assets ends up matching the growth rate of permanent income.

Matters are simpler if there are no permanent shocks; E for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

5 Conclusions

This paper provides theoretical foundations for many characteristics of buffer stock saving models that have heretofore been observed in numerical solutions but not proven. Perhaps the most important such proposition is the existence of a target cash-to-permanent-income ratio toward which actual resources will move. The intuition provided by the existence of such a target can be a powerful aid to understanding a host of numerical results.

Another contribution is integration of the paper's results with an the open-source Econ-ARK toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Table 3 Definitions and Comparisons of Conditions

<table>
<tr><th>Perfect Foresight Versions</th><th>Uncertainty Versions</th></tr>
<tr><td colspan="2">Finite Human Wealth Condition (FHWC)</td></tr>
<tr><td>$\Gamma/\mathsf{R} < 1$
The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.</td><td>$\Gamma/\mathsf{R} < 1$
The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.</td></tr>
<tr><td colspan="2">Absolute Impatience Condition (AIC)</td></tr>
<tr><td>$\text{Þ} < 1$
The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time:
$c_{t+1} < c_t$</td><td>$\text{Þ} < 1$
If wealth is large enough, the expectation of consumption next period will be smaller than this period's consumption:
$\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}] < c_t$</td></tr>
<tr><td colspan="2">Return Impatience Conditions</td></tr>
<tr><td>Return Impatience Condition (RIC)</td><td>Weak RIC (WRIC)</td></tr>
<tr><td>$\text{Þ}/\mathsf{R} < 1$
The growth factor for consumption Þ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite:
$\mathrm{c}'(m) = 1 - \text{Þ}/\mathsf{R} < 1$</td><td>$\wp^{1/\rho}\text{Þ}/\mathsf{R} < 1$
If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker.
$\mathrm{c}'(m) < 1 - \wp^{1/\rho}\text{Þ}/\mathsf{R} < 1$</td></tr>
<tr><td colspan="2">Growth Impatience Conditions</td></tr>
<tr><td>PF-GIC</td><td>GIC</td></tr>
<tr><td>$\text{Þ}/\Gamma < 1$
Guarantees that for an unconstrained consumer, the ratio of consumption to permanent income will fall over time. For a constrained consumer, guarantees the constraint will eventually be binding.</td><td>$\text{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$
By Jensen's inequality, stronger than the PF-GIC.
Ensures consumers will not expect to accumulate m unboundedly.
$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{Þ}_{\underline{\Gamma}}$</td></tr>
<tr><td colspan="2">Finite Value of Autarky Conditions</td></tr>
<tr><td>PF-FVAC</td><td>FVAC</td></tr>
<tr><td>$\beta\Gamma^{1-\rho} < 1$
equivalently $\text{Þ}/\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}$
The discounted utility of constrained consumers who spend their permanent income each period should be finite.</td><td>$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$
By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.</td></tr>
</table>

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Model	Conditions	Comments
PF Unconstrained	RIC, FHWC[°]	RIC $\Rightarrow \lvert \mathrm{v}(m) \rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m) \rvert$
		RIC prevents $\bar{\mathrm{c}}(m) = 0$
		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
PF Constrained	PF-GIC*	If RIC, $\lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		If $\cancel{\text{RIC}}$, $\lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
Buffer Stock Model	FVAC, WRIC	FHWC $\Rightarrow \lim_{m\to\infty} \mathring{c}(m) = \bar{c}(m), \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+RIC $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		$\cancel{\text{FHWC}}$+$\cancel{\text{RIC}}$ $\Rightarrow \lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
		GIC guarantees finite target wealth ratio
		FVAC is stronger than PF-FVAC
		WRIC is weaker than RIC

‡For feasible m, the limiting consumption function defines the unique value of c satisfying $0 < c < \infty$. °RIC, FHWC are necessary as well as sufficient. *Solution also exists for ~~PF-GIC~~ and RIC, but is identical to the unconstrained model's solution for feasible $m \geq 1$.

Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions. Results are summarized in table 5.

A.1 If PF-GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{PF-GIC}}$, $1 < \text{Þ}/\Gamma$). Under $\cancel{\text{PF-GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[37]

$$1 < (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \tag{57}$$
$$1 < \mathsf{R}\beta\Gamma^{-\rho} \tag{58}$$
$$\mathrm{u}'(1) < \mathsf{R}\beta\mathrm{u}'(\Gamma). \tag{59}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{60}$$

(that is, consumption is zero for any level of human or nonhuman wealth).

If the RIC fails and the FHWC fails, human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[38]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying $\cancel{\text{PF-GIC}}$ we must impose the RIC (and the FHWC can be shown to be a consequence of $\cancel{\text{PF-GIC}}$ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from equation (21):

$$m_{\#} = (m_{\#} - 1 + h)\underline{\kappa} \tag{61}$$

[37]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[38]The knife-edge case is where $\text{Þ} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

$$m_{\#}(1-\underline{\kappa}) = (h-1)\underline{\kappa} \tag{62}$$

$$m_{\#} = (h-1)\left(\frac{\underline{\kappa}}{1-\underline{\kappa}}\right) \tag{63}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[39] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{c}(m) = m$.[40] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

(**?** obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

A.2 If PF-GIC Holds

Imposition of the PF-GIC reverses the inequality in (59), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^n$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^n$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b_{\#}^0$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^1$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the PF-GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^0 = 1 + b_{\#}^0 = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^0 = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{\TH}_\Gamma$, so consumption n periods in the past must have been

$$c_{\#}^n = \text{\TH}_\Gamma^{-n} c_t = \text{\TH}_\Gamma^{-n}. \tag{64}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}_{t-n}^t &= c_{t-n}(1 + \text{\TH}/\mathsf{R} + \ldots + (\text{\TH}/\mathsf{R})^n) \\
&= c_{\#}^n(1 + \text{\TH}_\mathsf{R} + \ldots + \text{\TH}_\mathsf{R}^n) \\
&= \text{\TH}_\Gamma^{-n}\left(\frac{1-\text{\TH}_\mathsf{R}^{n+1}}{1-\text{\TH}_\mathsf{R}}\right) &\quad (65) \\
&= \left(\frac{\text{\TH}_\Gamma^{-n} - \text{\TH}_\mathsf{R}}{1-\text{\TH}_\mathsf{R}}\right) &\quad (66)
\end{aligned}$$

[39]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~PF-GIC~~.

[40]As an illustration, consider a consumer for whom $\text{\TH} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1 + ... + \mathcal{R}^{-n} \tag{67}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} \quad = b_{\#}^{n} + h_{\#}^{n}$$

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n} = b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h_{\#}^{n}}. \tag{68}$$

Defining $m_{\#}^{n} = b_{\#}^{n} + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (68) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{69}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (68) limits to a constant as $n \uparrow \infty$, and (69) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{\Thorn}_{\Gamma}^{-n} - (\text{\Thorn}_{\mathsf{R}}/\text{\Thorn}_{\Gamma})^{n}\text{\Thorn}_{\mathsf{R}}}{1-\text{\Thorn}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{\Thorn}_{\Gamma}^{-n} - \mathcal{R}^{-n}\text{\Thorn}_{\mathsf{R}}}{1-\text{\Thorn}_{\mathsf{R}}}\right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{\Thorn}_{\Gamma}^{-n}}{1-\text{\Thorn}_{\mathsf{R}}}\right) &= \infty.
\end{aligned}$$

Given the PF-GIC $\text{\Thorn}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{\Thorn}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the PF-GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the PF-GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{70}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (69) requires

$$\begin{aligned}\lim_{n\to\infty} \left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}} - \text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}} - 1}\right) + \left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) &= \infty \\ \lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}} - 1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}} - 1}\right) &= \infty\end{aligned} \tag{71}$$

If RIC Holds. When the RIC holds, rearranging (71) gives

$$\lim_{n\to\infty} \left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) - \mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}} + \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1} \\ \Gamma/\text{Þ} &> \Gamma/\mathsf{R} \\ 1 &> \text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty} \kappa_{\#}^{n} = \lim_{n\to\infty} \left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{72}$$

which with a bit of algebra[41] can be shown to asymptote to the MPC in the perfect foresight model:[42]

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \text{Þ}_{\mathsf{R}}. \tag{74}$$

If RIC Fails. Consider now the ~~RIC~~ case, $\text{Þ}_{\mathsf{R}} > 1$. We can rearrange (71)as

$$\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)} - \frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty. \tag{75}$$

---

[41]Calculate the limit of

$$\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\Gamma}^{-n}/(1-\text{Þ}_{\mathsf{R}}) - (1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right) = \left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}}) + \mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{73}$$

[42]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

which makes clear that with $\cancel{\text{FHWC}} \Rightarrow \mathcal{R}^{-1} > 1$ and $\cancel{\text{RIC}} \Rightarrow \text{Þ}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (71) will be positive if

$$\begin{aligned} \text{Þ}_{\mathsf{R}}\mathcal{R}^{-1} - \text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}} - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}} \\ \Gamma &> \text{Þ} \end{aligned}$$

which is merely the PF-GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \text{Þ}_{\Gamma}^{-1} \\ \Gamma/\mathsf{R} &> \Gamma/\text{Þ} \\ \text{Þ}/\mathsf{R} &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\Gamma/\text{Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/\mathsf{R} > \Gamma/\text{Þ} \tag{76}$$

because $\cancel{\text{RIC}} \Rightarrow \text{Þ} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (72) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that $\cancel{\text{RIC}}$ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{77}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 6 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting – $c = m$ – so not worth plotting).

We can summarize as follows. Given that the PF-GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

? characterize the limits of MPC in a more general framework that allows for non-

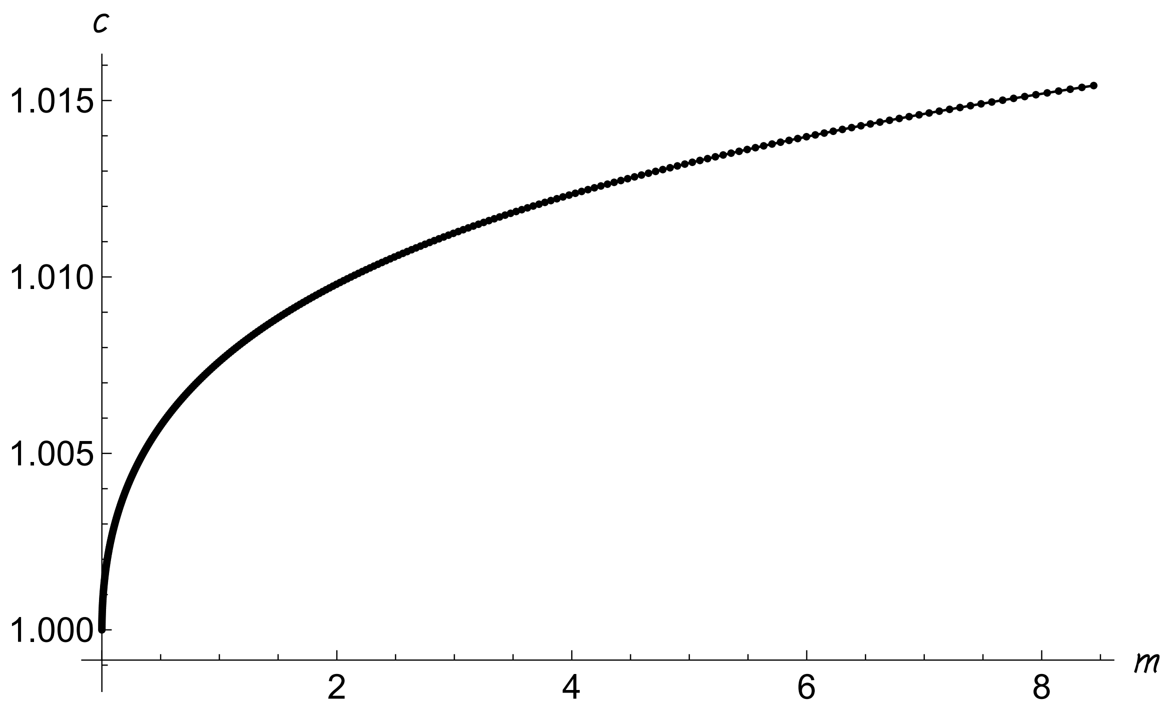


Figure 6 Nondegenerate Consumption Function with ~~EHWC~~ and ~~RIC~~

CRRA utility as well as capital and labor income risks in a Markovian setting, and find that in that much more general framework the limiting MPC is also zero.

B Existence of a Concave Consumption Function

To show that (7) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$
6. $\lim_{z\downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})\right]. \tag{78}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0}\mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0}\mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (**?**).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c) \tag{79}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (7) can be written as

$$\mathrm{v}_t(m) = \max_{c}\ \underline{\mathrm{v}}_t(m, c). \tag{80}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m, c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2\underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \underset{0<c<m}{\arg\max}\ \underline{\mathrm{v}}_t(m, c) \tag{81}$$

exists and is unique, and (7) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{82}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (82) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{83}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim\limits_{dm\to+0}\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim\limits_{dm\to+0}\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}=\frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)}+\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m)=\frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m))+\mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m)=\mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m))+\mathfrak{v}_t''(\mathrm{a}_t(m))<0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m)=\frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t)+\mathfrak{v}_t''(a_t)\right]-\mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t)+a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t)+\mathfrak{v}_t''(a_t)\right]^2}. \tag{84}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

D Proof that $\mathfrak{T}$ Is a Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_{F}(\mathcal{A},\mathcal{B})$ to $\mathcal{C}(\mathcal{A},\mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\}\in\ \mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (**?**).

Consider condition (1). For this problem,

$$\begin{aligned}\{\mathfrak{T}\mathrm{x}\}(m_t)\text{ is }&\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\}\\ \{\mathfrak{T}\mathrm{y}\}(m_t)\text{ is }&\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\},\end{aligned}$$

so $\mathrm{x}(\bullet)\le\mathrm{y}(\bullet)$ implies $\{\mathfrak{T}\mathrm{x}\}(m_t)\le\{\mathfrak{T}\mathrm{y}\}(m_t)$ by inspection.[43]

Condition (2) requires that $\{\mathfrak{T}\mathbf{0}\}\in\mathcal{C}_{F}\left(\mathcal{A},\mathcal{B}\right)$. By definition,

$$\{\mathfrak{T}\mathbf{0}\}(m_t)=\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right)+\beta 0\right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa}m_t)$. Thus, condition (2) will hold if $(\bar{\kappa}m_t)^{1-\rho}$ is F-bounded. We use the bounding function

$$F(m)=\eta+m^{1-\rho}, \tag{85}$$

for some real scalar $\eta>0$ whose value will be determined in the course of the proof. Under this definition of F, $\{\mathfrak{T}\mathbf{0}\}(m_t)=\mathrm{u}(\bar{\kappa}m_t)$ is clearly F-bounded.

Finally, we turn to condition (3), $\{\mathfrak{T}(\mathrm{z}+\zeta F)\}(m_t)\le\{\mathfrak{T}\mathrm{z}\}(m_t)+\zeta\alpha F(m_t)$. The proof

[43]For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

will be more compact if we define $\check{\mathrm{c}}$ and $\check{\mathrm{a}}$ as the consumption and assets functions[44] associated with $\mathfrak{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathfrak{T}(\mathrm{z}+\zeta F)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}})+\zeta\alpha F.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha F\\
\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha F\\
\beta\zeta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \zeta\alpha F\\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \alpha F\\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &< F.
\end{aligned}$$

where the last line follows because $0<\alpha<1$ by assumption.[45]

Using $F(m)=\eta+m^{1-\rho}$ and defining $\hat{a}_t=\hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\xi_{t+1})^{1-\rho}]-m_t^{1-\rho}<\eta(1-\underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (27), which says $\beth<1$) can be rewritten as:

$$\eta>\frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\xi_{t+1})^{1-\rho}\right]-m_t^{1-\rho}}{1-\beth}. \tag{86}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (86) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned} \tag{87}$$

[44] Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[45] The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

We can thus conclude that equation (86) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{88}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (40) and (33) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa} m_t \leq c_t \leq \bar{\kappa} m_t$. However, in the discussion of the consumption function bounds, we showed only (in (41)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (7) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (87). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} &< 1 \\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) &> 1 \\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}) &> 1
\end{aligned}$$

where we have used (39) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned}
(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &> 1 \\
(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.
\end{aligned}$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty}\underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (7) converge, they converge to the same unique v defined by $\mathcal{T}$.[46]

D.2 Convergence of v_t in Euclidian Space

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathcal{T}$, since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\left\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\right\|_{\digamma} \leq \alpha^{n-1} \left\|\mathrm{v}_T - \mathrm{v}^*\right\|_{\digamma}. \tag{89}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ and $\kappa = \left\|\mathrm{v}_T - \mathrm{v}^*\right\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. It follows that

$$\left|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right| \leq \kappa\alpha^{n-1} \left|\digamma(m)\right|. \tag{90}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{91}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathcal{T}\mathrm{v}_{T-1} \leq \mathcal{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D.3 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m))+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)})+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \tag{92}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{93}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

[46] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

E Equality of Aggregate Consumption Growth and Income Growth with Transitory Shocks

Section 4.2 asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\check{m}$ is the invariant mean level of m we can define a 'mean MPS away from $\check{m}$' function :

$$\bar{\mathrm{a}}(\Delta) = \Delta^{-1} \int_{\check{m}}^{\check{m}+\Delta} \mathrm{a}'(z) dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$, if we define a as the value of a corresponding to $m = \check{m}$, we can write

$$a_{t+1,i} = a + (m_{t+1,i} - \check{m})\bar{\mathrm{a}}(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \check{m})$$

so

$$\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \mathrm{cov}_t\left(\bar{\mathrm{a}}(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \check{m}), \Gamma \mathbf{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho} < \bar{\mathrm{a}}(m) < \text{\textbf{Þ}}_{\mathsf{R}}$,

$$|\mathrm{cov}_t((\wp\mathsf{R}\beta)^{1/\rho} a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(\text{\textbf{Þ}} a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC says that $\text{\textbf{Þ}} < \Gamma$, while the FHWC says that $\Gamma < \mathsf{R}$

$$|\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma |\mathrm{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathbb{M}\left[\mathcal{R}_{t+1,i}\right]$ and consider a first order Taylor expansion of $\bar{\mathrm{a}}(m_{t+1,i})$ around $\check{m}_{t+1,i} = \breve{\mathcal{R}}a_{t,i} + 1$,

$$\bar{\mathrm{a}}_{t+1,i} \approx \bar{\mathrm{a}}(\check{m}_{t+1,i}) + \bar{\mathrm{a}}'(\check{m}_{t+1,i})\left(m_{t+1,i} - \check{m}_{t+1,i}\right).$$

The problem comes from the $\bar{\mathrm{a}}'$ term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\mathrm{a}}$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\bar{a}}'$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

F The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (8) can be rewritten

$$\mathrm{e}_t(m_t)^{-\rho} = \beta \mathsf{R}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t}\right)\right)^{-\rho}\right] \tag{94}$$

$$\begin{aligned} &= (1-\wp)\beta \mathsf{R} m_t^{\rho}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0\right] \\ &+ \wp\beta\mathsf{R}^{1-\rho}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t - \mathrm{c}_t(m_t)}{m_t}\right)^{-\rho} \mid \xi_{t+1} = 0\right]. \end{aligned} \tag{95}$$

Consider the first conditional expectation in (94), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m\downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\mathbf{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1} \tag{96}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}. \end{aligned} \tag{97}$$

As noted in the main text, we need the WRIC (40) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}} < 1, \tag{98}$$

Since $\bar{\kappa}_T = 1$, iterating (97) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}} \tag{99}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (94) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} = \beta \mathsf{R}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) \left(\mathsf{R}\mathrm{a}_t(m_t)\right))^{-\rho} \right]$$

and we know from L'Hôpital's rule that $\lim_{m_t \to \infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\rho} &= \beta \mathsf{R} \left(\underline{\kappa}_{t+1} \mathsf{R} (1 - \underline{\kappa}_t) m_t \right)^{-\rho} \\ \underbrace{\mathsf{R}^{-1} \text{\TH}}_{\equiv \text{\TH}_{\mathsf{R}} = (1 - \underline{\kappa})} \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \text{\TH}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \text{\TH}_{\mathsf{R}} \tag{100}$$

so that $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \kappa_{T-n}^{-1} \tag{101}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n \to \infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1 + \text{\TH}_{\mathsf{R}} + \text{\TH}_{\mathsf{R}}^2 + \ldots\right)}_{=1 + \text{\TH}_{\mathsf{R}}(1 + \text{\TH}_{\mathsf{R}} \underline{\kappa}_{t+2}^{-1}) \ldots} = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \tag{102}$$

G The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp\downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{103}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta)d\mathcal{F}_\theta \right\}. \tag{104}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta))d\mathcal{F}_\theta,$$

and the solution to (104) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{105}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \le m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (**?**)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (105), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\theta}}^{\bar{\theta}} (\mathrm{c}_T(a+\theta/(1-\wp)))^{-\rho}\, d\mathcal{F}_\theta\right], \tag{106}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{107}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a>0$, $\lim_{\wp\downarrow 0}\mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (105) and (107) are identical, this means that $\lim_{\wp\downarrow 0}\mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (106) is $\lim_{a\downarrow 0}\wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m>0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0}\mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0}\mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0}\mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (103). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (99) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp\downarrow 0}\bar{\kappa} \; = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

H Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (**?**): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[47]

$$\begin{aligned} \mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))] \qquad (108)\\ \vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}. \end{aligned}$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (108) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\begin{aligned} \mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}_{t+1}^m\mathcal{R}_{t+1}]\\ &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m]\\ \mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m]/\mathrm{u}''(\mathfrak{c}_t) \qquad (109) \end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$\begin{aligned} c &= \mathfrak{m} - a\\ \mathfrak{c}^a + 1 &= \mathfrak{m}^a \end{aligned}$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$\begin{aligned} \mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a\\ \mathrm{c}^m &= \mathfrak{c}^a/(1+\mathfrak{c}^a) \end{aligned}$$

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

I The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, ...\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged

[47]The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

consumption rule $\mathrm{c}(m)$,[48] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_\#, m^2_\#, ...\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathring{\mathrm{c}}(m)$ for integer values of n (e.g., $c^n_\# = \mathbf{Þ}_\Gamma^{-n}$) are continuous functions of n; the conclusion that $\mathring{\mathrm{c}}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\breve{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[49]

This strategy generates a smooth limiting consumption function – except at the remaining kink point defined by $\{m^0_\#, c^0_\#\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_\#$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_\#} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_\#} \mathrm{c}'(m) = \kappa^0_\# < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\breve{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\breve{\mathrm{c}}(m)$ between $m^0_\#$ and $m^{\underline{n}}_\#$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

1. $\hat{\mathrm{c}}(m^0_\#) = c^0_\#$
2. $\hat{\mathrm{c}}'(m^0_\#) = 1$
3. $\hat{\mathrm{c}}'(m^{\underline{n}}_\#) = (d\mathrm{c}^n_\#/dn)(d\mathrm{m}^n_\#/dn)^{-1}|_{n=\underline{n}}$
4. $\hat{\mathrm{c}}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\#/dn^2)(d^2\mathrm{m}^n_\#/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\breve{\mathrm{c}}(m)$ and fidelity of that function to the $\mathring{\mathrm{c}}(m)$ (see the actual code for details).

[48]Unless $\beta \approx +0$.

[49]In practice, we calculate the first and second derivatives of $\mathring{\mathrm{c}}$ and use piecewise polynomial approximation methods that match the function at these points.

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m_{\#}^{0} & m \\ m_{\#}^{0} < m < m_{\#}^{n} & \breve{\mathrm{c}}(m) \\ m_{\#}^{n} < m & \mathring{\mathrm{c}}(m) \end{cases} \tag{110}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\breve{\mathrm{c}}(m) \geq \mathring{\mathrm{c}}(m)$, implicitly defining $m = e^{\mu}$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^{\mu})/\mathrm{c}_T(e^{\mu})) \tag{111}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \breve{\mathrm{c}}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi_t'(\mu) = \left(\frac{-\left(\frac{\mathrm{c}_t'\mathrm{c}_T - \mathrm{c}_t\mathrm{c}_T'}{\mathrm{c}_T^2} e^{\mu} \right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{112}$$

which can be solved for

$$\mathrm{c}_t' = (\mathrm{c}_t\mathrm{c}_T'/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi_t'. \tag{113}$$

Similarly, we can solve (111) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right) \mathrm{c}_T(m). \tag{114}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

I.1 Commutative Diagrams for the Perfect Foresight Model

The diagrams below illustrate the order of the several conditions in the text:

$$\begin{tikzcd} \text{Þ} \arrow[rr, "\text{PF-GIC}"] \arrow[dr, "\text{RIC}"'] & & \Gamma \arrow[dl, "\text{FHWC}"] \\ & \mathsf{R} & \end{tikzcd}$$

and to further incorporate the Perfect Foresight Finite Value of Autarky Condition:

$$\begin{tikzcd} \text{Þ} \arrow[r, "\text{PF-GIC}"] \arrow[d, "\text{RIC}"] \arrow[dr, "\text{PF-FVAC}"] & \Gamma \arrow[d, "\text{FHWC}"] \\ \mathsf{R} & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} \arrow[l, "\text{FHWC}"] \end{tikzcd}$$

In both diagrams, an arrow means “$<$”, which indicates the annotated condition holds, so if a condition is violated, the corresponding arrow is to be reversed.

These diagrams also keep track of the hierarchy among the conditions. For example, if the right vertical arrow in the second diagram is reversed, then the top right triangle says PF-FVAC+ ~~EHWC~~ implies PF-GIC. If the left vertical arrow is reversed, then ~~RIC~~ + PF-GIC implies ~~EHWC~~.

Table 5 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math			Outcome, Comments or Results
$\cancel{\text{PF-GIC}}$		$1 <$	$\text{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R}$	< 1		FHWC holds $(\mathsf{R} > \Gamma)$; $\grave{c}(m) = \bar{c}(m)$ for $m \geq 1$
and $\cancel{\text{RIC}}$		$1 <$	$\text{Þ}/\mathsf{R}$	$\grave{c}(m)$ is degenerate: $\grave{c}(m) = 0$
PF-GIC	$\text{Þ}/\Gamma$	< 1		Constraint binds in finite time for any m
and RIC	$\text{Þ}/\mathsf{R}$	< 1		FHWC may or may not hold $\lim_{m \uparrow \infty} \bar{c}(m) - \grave{c}(m) = 0$ $\lim_{m \uparrow \infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and $\cancel{\text{RIC}}$		$1 <$	$\text{Þ}/\mathsf{R}$	$\cancel{\text{FHWC}}$ $\lim_{m \uparrow \infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~PF-GIC~~ and RIC both hold, while the third row indicates that when the PF-GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the PF-GIC holds, the constraint will bind in finite time.