

0.1 Convergence of v_t in Euclidian Space

Boyd's theorem shows that \mathcal{T} defines a contraction mapping in a F -bounded space. We now show that \mathcal{T} also defines a contraction mapping in Euclidian space.

Since $v^*(m) = \mathcal{T}v^*(m)$,

$$\|v_{T-n+1} - v^*\|_F \leq \alpha^{n-1} \|v_T - v^*\|_F. \quad (58)$$

On the other hand, $v_T - v^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ and $\kappa = \|v_T - v^*\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$. It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa \alpha^{n-1} |F(m)|. \quad (59)$$

Then we obtain

$$\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v^*(m). \quad (60)$$

Since $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $v_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < v_T(m)$. On the other hand, $v_{T-1} \leq v_T$ means $\mathcal{T}v_{T-1} \leq \mathcal{T}v_T$, in other words, $v_{T-2}(m) \leq v_{T-1}(m)$. Inductively one gets $v_{T-n}(m) \geq v_{T-n-1}(m)$. This means that $\{v_{T-n+1}(m)\}_{n=1}^\infty$ is a decreasing sequence, bounded below by v^* .

0.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{c_{T-n+1}(m)\}_{n=1}^\infty$.

We start by showing that

$$c(m) = \arg \max_{c_t \in [\underline{\kappa}m, \bar{\kappa}m]} \{u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1})]\} \quad (61)$$

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m , with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

$$\mathcal{T}v(m) = u(c_i) + \beta \underbrace{\mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, c_i))]}_{\equiv \mathbf{v}} \quad (62)$$

where $m_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$ and $i = 1, 2$. $\mathcal{T}v$ is concave for concave v . Since the space of continuous and concave functions is closed, \mathbf{v} is also concave and satisfies

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, c_i))] \leq \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, \tilde{c}))]. \quad (63)$$

On the other hand, $\frac{1}{2} \{u(c_1) + u(c_2)\} < u(\tilde{c})$. Then one gets

$$\mathcal{T}v(m) < u(\tilde{c}) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v(m_{t+1}(m, \tilde{c}))]. \quad (64)$$

Since \tilde{c} is a feasible choice for c_i , the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of $c(m)$ we can now show

$$\lim_{n \rightarrow \infty} c_{T-n+1}(m) = c(m). \quad (65)$$

Suppose this does not hold for some $m = m^*$. In this case, $\{c_{T-n+1}(m^*)\}_{n=1}^\infty$ has a subsequence $\{c_{T-n(i)}(m^*)\}_{i=1}^\infty$ that satisfies $\lim_{i \rightarrow \infty} c_{T-n(i)}(m^*) = c^*$ and $c^* \neq c(m^*)$. Now define $c_{T-n+1}^* = c_{T-n+1}(m^*)$. $c^* > 0$ because $\lim_{i \rightarrow \infty} v_{T-n(i)+1}(m^*) \leq \lim_{i \rightarrow \infty} u(c_{T-n(i)}^*)$. Because $a(m^*) > 0$ and $\psi \in [\underline{\psi}, \bar{\psi}]$ there exist $\{\underline{m}_+^*, \bar{m}_+^*\}$ satisfying $0 < \underline{m}_+^* < \bar{m}_+^*$ and $m_{T-n+1}(m^*, c_{T-n+1}^*) \in [\underline{m}_+^*, \bar{m}_+^*]$. It follows that $\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v(m)$ and the convergence is uniform on $m \in [\underline{m}_+^*, \bar{m}_+^*]$. (Uniform convergence is obtained from Dini's theorem.¹) Hence for any $\delta > 0$, there exists an n_1 such that

$$\beta \mathbb{E}_{T-n} [\Gamma_{T-n+1}^{1-\rho} |v_{T-n+1}(m_{T-n+1}(m^*, c_{T-n+1}^*)) - v(m_{T-n+1}(m^*, c_{T-n+1}^*))|] < \delta$$

for all $n \geq n_1$. It follows that if we define

$$w(m^*, z) = u(z) + \beta \mathbb{E}_{T-n} [\Gamma_{T-n+1}^{1-\rho} v(m_{T-n+1}(m^*, z))] \quad (66)$$

then $v_{T-n}(m^*)$ satisfies

$$\lim_{n \rightarrow \infty} |v_{T-n}(m^*) - w(m^*, c_{T-n+1}^*)| = 0. \quad (67)$$

On the other hand, there exists an $i_1 \in \mathbb{N}$ such that

$$|v(m_{T-n(i)}(m^*, c_{T-n(i)}^*)) - v(m_{T-n(i)}(m^*, c^*))| \leq \delta \text{ for all } i \geq i_1 \quad (68)$$

because v is uniformly continuous on $[\underline{m}_+^*, \bar{m}_+^*]$. $\lim_{i \rightarrow \infty} |c_{T-n(i)}(m^*) - c^*| = 0$ and

$$|m_{T-n(i)}(m^*, c_{T-n(i)}^*) - m_{T-n(i)}(m^*, c^*)| \leq \frac{R}{\Gamma \underline{\psi}} |c_{T-n(i)}^* - c^*|. \quad (69)$$

This implies

$$\lim_{i \rightarrow \infty} |w(m^*, c_{T-n(i)+1}^*) - w(m^*, c^*)| = 0. \quad (70)$$

From (67) and (70), we obtain $\lim_{i \rightarrow \infty} v_{T-n(i)}(m^*) = w(m^*, c^*)$ and this implies $w(m^*, c^*) = v(m^*)$. This implies that $c(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

¹[Dini's theorem] For a monotone sequence of continuous functions $\{v_n(m)\}_{n=1}^\infty$ which is defined on a compact space and satisfies $\lim_{n \rightarrow \infty} v_n(m) = v(m)$ where $v(m)$ is continuous, convergence is uniform.