0.1 Convergence of v_t in Euclidian Space

Boyd's theorem shows that \mathcal{T} defines a contraction mapping in a \mathcal{F} -bounded space. We now show that \mathcal{T} also defines a contraction mapping in Euclidian space.

Calling v* the unique fixed point of the operator \mathcal{T} , since v*(m) = \mathcal{T} v*(m),

$$\|\mathbf{v}_{T-n+1} - \mathbf{v}^*\|_F \le \alpha^{n-1} \|\mathbf{v}_T - \mathbf{v}^*\|_F.$$
 (1)

On the other hand, $\mathbf{v}_T - \mathbf{v}^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ and $\kappa = \|\mathbf{v}_T - \mathbf{v}^*\|_F < \infty$ because \mathbf{v}_T and \mathbf{v}^* are in $\mathcal{C}_{\mathcal{F}}(\mathcal{A},\mathcal{B})$. It follows that

$$|\mathbf{v}_{T-n+1}(m) - \mathbf{v}^*(m)| \le \kappa \alpha^{n-1} |F(m)|.$$
 (2)

Then we obtain

$$\lim_{n \to \infty} \mathbf{v}_{T-n+1}(m) = \mathbf{v}^*(m). \tag{3}$$

Since $\mathbf{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathbf{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathbf{v}_T(m)$. On the other hand, $\mathbf{v}_{T-1} \leq \mathbf{v}_T$ means $\Im \mathbf{v}_{T-1} \leq \Im \mathbf{v}_T$, in other words, $\mathbf{v}_{T-2}(m) \leq \mathbf{v}_{T-1}(m)$. Inductively one gets $\mathbf{v}_{T-n}(m) \geq \mathbf{v}_{T-n-1}(m)$. This means that $\{\mathbf{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^* .

0.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\{c_{T-n(i)}(m)\}$ of $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^* . By the definition of $c_{T-n}(m)$, we have

$$u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)] \ge u(c_{T-n(i)}) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)],$$
(4)

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting n(i) go to infinity, it follows that the left hand side converges to $u(c^*) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho}v(m)]$, and the right hand side converges to $u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho}v(m)]$. So the limit of the preceding inequality as n(i) approaches infinity implies

$$u(c^*) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}v(m)] \ge u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}v(m)].$$
 (5)

 $\underset{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]}{\arg\max} \left\{ \mathbf{u}(c_{T-n(i)}) + \beta \, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathbf{v}(m)] \right\}. \quad \text{By the uniqueness of } \mathbf{c}(m),$ $c^* = c(m)$.

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We start by showing that

$$c(m) = \underset{c_t \in [\underline{\kappa}m, \bar{\kappa}m]}{\arg\max} \left\{ u(c_t) + \beta \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} v(m_{t+1}) \right] \right\}$$
 (6)

is uniquely determined. We show this by contradiction. Suppose there exist c_1 and c_2 that both attain the supremum for some m, with mean $\tilde{c} = (c_1 + c_2)/2$. c_i satisfies

$$\Im \mathbf{v}(m) = \mathbf{u}(c_i) + \beta \underbrace{\mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathbf{v}(\mathbf{m}_{t+1}(m, c_i)) \right]}_{\equiv \mathbf{v}}$$
 (7)

where $m_{t+1}(m, c_i) = (m - c_i)\mathcal{R}_{t+1} + \xi_{t+1}$ and i = 1, 2. Tv is concave for concave \mathfrak{v} . Since the space of continuous and concave functions is closed, \mathfrak{v} is also concave and satisfies

$$\frac{1}{2} \sum_{i=1,2} \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathbf{v}(\mathbf{m}_{t+1}(m, c_i)) \right] \le \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathbf{v}(\mathbf{m}_{t+1}(m, \tilde{c})) \right]. \tag{8}$$

On the other hand, $\frac{1}{2} \{ \mathbf{u}(c_1) + \mathbf{u}(c_2) \} < \mathbf{u}(\tilde{c})$. Then one gets

$$\operatorname{Tv}(m) < \operatorname{u}(\tilde{c}) + \beta \operatorname{\mathbb{E}}_{t} \left[\Gamma_{t+1}^{1-\rho} \operatorname{v}(\operatorname{m}_{t+1}(m, \tilde{c})) \right]. \tag{9}$$

Since \tilde{c} is a feasible choice for c_i , the LHS of this equation cannot be a maximum, which contradicts the definition.

Using uniqueness of c(m) we can now show

$$\lim_{n \to \infty} c_{T-n+1}(m) = c(m). \tag{10}$$

Suppose this does not hold for some $m=m^*$. In this case, $\{c_{T-n+1}(m^*)\}_{n=1}^{\infty}$ has a subsequence $\{c_{T-n(i)}(m^*)\}_{i=1}^{\infty}$ that satisfies $\lim_{i\to\infty} c_{T-n(i)}(m^*) = c^*$ and $c^*\neq c(m^*)$. Now define $c_{T-n+1}^* = c_{T-n+1}(m^*)$. $c^*>0$ because $\lim_{i\to\infty} v_{T-n(i)+1}(m^*) \leq \lim_{i\to\infty} u(c_{T-n(i)}^*)$. Because $a(m^*)>0$ and $\psi\in[\underline{\psi},\overline{\psi}]$ there exist $\{\underline{m}_+^*,\overline{m}_+^*\}$ satisfying $0<\underline{m}_+^*<\overline{m}_+^*$ and $m_{T-n+1}(m^*,c_{T-n+1}^*)\in[\underline{m}_+^*,\overline{m}_+^*]$. It follows that $\lim_{n\to\infty} v_{T-n+1}(m)=v(m)$ and the convergence is uniform on $m\in[\underline{m}_+^*,\overline{m}_+^*]$. (Uniform convergence is obtained from Dini's theorem.) Hence for any $\delta>0$, there exists an n_1 such that

$$\beta \mathbb{E}_{T-n} \left[\Gamma_{T-n+1}^{1-\rho} \left| \mathbf{v}_{T-n+1}(m_{T-n+1}(m^*, c_{T-n+1}^*)) - \mathbf{v}(m_{T-n+1}(m^*, c_{T-n+1}^*)) \right| \right] < \delta$$

for all $n \geq n_1$. It follows that if we define

$$w(m^*, z) = u(z) + \beta \mathbb{E}_{T-n} \left[\Gamma_{T-n+1}^{1-\rho} v(m_{T-n+1}(m^*, z)) \right]$$
(11)

then $v_{T-n}(m^*)$ satisfies

$$\lim_{n \to \infty} \left| \mathbf{v}_{T-n}(m^*) - \mathbf{w}(m^*, c_{T-n+1}^*) \right| = 0.$$
 (12)

On the other hand, there exists an $i_1 \in \mathbb{N}$ such that

$$\left| v(\mathbf{m}_{T-n(i)}(m^*, c_{T-n(i)}^*)) - v(\mathbf{m}_{T-n(i)}(m^*, c^*)) \right| \le \delta \text{ for all } i \ge i_1$$
 (13)

because v is uniformly continuous on $[\underline{m}_+^*, \bar{m}_+^*]$. $\lim_{i\to\infty} |c_{T-n(i)}(m^*) - c^*| = 0$ and

$$\left| \mathbf{m}_{T-n(i)}(m^*, c_{T-n(i)}^*) - \mathbf{m}_{T-n(i)}(m^*, c^*) \right| \le \frac{\mathsf{R}}{\Gamma \psi} \left| c_{T-n(i)}^* - c^* \right|.$$
 (14)

This implies

$$\lim_{i \to \infty} \left| \mathbf{w}(m^*, c_{T-n(i)+1}^*) - \mathbf{w}(m^*, c^*) \right| = 0.$$
 (15)

From (12) and (15), we obtain $\lim_{i\to\infty} \mathbf{v}_{T-n(i)}(m^*) = \mathbf{w}(m^*,c^*)$ and this implies

¹[**Dini's theorem**] For a monotone sequence of continuous functions $\{v_n(m)\}_{n=1}^{\infty}$ which is defined on a compact space and satisfies $\lim_{n\to\infty} v_n(m) = v(m)$ where v(m) is continuous, convergence is uniform.

 $\mathbf{w}(m^*,c^*)=\mathbf{v}(m^*).$ This implies that $\mathbf{c}(m)$ is not uniquely determined, which is a contradiction.

Thus, the consumption functions must converge.

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