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Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative illustration. After describing conditions under which a consumption function exists, the paper shows that individual consumers subject to idiosyncratic shocks will engage in 'target' saving behavior when a particular 'growth impatience' condition is imposed. A similar (but looser) condition guarantees that a small open economy populated by such agents will exhibit a balanced-growth 'steady state' equilibrium. Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem

JEL codes D81, D91, E21

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All figures and numerical results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper's own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

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1 Introduction

In the presence of empirically realistic transitory and permanent income shocks *a la* Friedman (1957),[1] only one more ingredient is required to construct a testable model of optimal consumption: A description of preferences. Zeldes (1989) was the first to construct a quantitatively realistic version of such a model, spawning a subsequent literature showing that such models' predictions match microeconomic evidence reasonably well, whether or not liquidity constraints are imposed.[2]

A companion theoretical literature has derived key analytical properties of infinite-horizon solutions, but only for models more complex than the case with just shocks and preferences. The extra complexity has been required, in part, because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et. al. (1989)) cannot be applied when utility or marginal utility are unbounded. Many proof methods also rule out permanent shocks *a la* Friedman (1957), Muth (1960), and Zeldes (1989).[3]

This paper's first technical contribution is to articulate conditions under which the simple problem (without complications like a consumption floor or liquidity constraints) defines a contraction mapping whose limiting value and consumption functions are nondegenerate as as the horizon approaches infinity. A 'Finite Value of Autarky Condition' turns out to be sufficient (along with a 'Weak Return Impatience Condition' that is unlikely ever to bind). Conveniently, the resulting model has analytical properties, like continuous differentiability of the consumption function, that make it easier to analyze than the more complicated models. The paper's other main theoretical contribution is to identify conditions under which 'stable' values of the wealth-to-permanent-income ratio exist (the consumer, or the economy populated by such consumers, exhibits 'buffer stock' saving behavior). 'Target saving' arises when the model's parameters satisfy a "Normalized Growth Impatience Condition" that relates preferences and uncertainty to predictable income growth. A nonnormalized (therefore looser) "Growth Impatience Condition" turns out to guarantee existence of an 'individual steady-state' point at which the consumer expects 'balanced growth' (equal rates of growth in permanent labor income and market wealth).

Even without a formal proof of its existence, target saving has been intuitively understood to underlie central quantitative results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the recent claim by Krueger, Mitman, and Perri (2016) in the *Handbook of Macroeconomics* that such models explain why, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theory below provides the rigorous theoretical basis for this claim: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$ means they – optimally – have little room to maneuver). And, increased labor income uncertainty does not change the behavior

[1] As formalized by Muth (1960).

[2] See Carroll (1997) or Gourinchas and Parker (2002) for arguments that such models match a wide variety of facts; for a model with constraints that produces very similar results, see, e.g., Cagetti (2003).

[3] See the fuller discussion at the end of section 2.1.

of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending when uncertainty increases.

Analytical derivations required for the proofs also provide foundations for many other results familiar from the numerical literature. The main insights of the paper are instantiated in the Econ-ARK toolkit, whose buffer stock saving module flags parametric choices under which a problem is degenerate or a target level of wealth may not exist.

The paper proceeds in three parts.

The first part articulates sufficient conditions for the problem to define a nondegenerate limiting consumption function, and explains how the model relates to those previously considered in the literature. The conditions required for convergence are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; then the contraction mapping theorem is proven. The last result here is a proof that a corresponding model with an 'artificial' liquidity constraint (that is, a model that exogenously prohibits consumers from borrowing even if they could repay) is a particular limiting case of the model without constraints.

In the process of proving the remaining theorems, the next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Next, the central theorems articulate conditions under which different measures of 'growth impatience' imply useful conclusions about points of stability ('individual target' or 'individual steady-state' points). Fourth, at the 'individual target' ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent (noncapital) income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last holds if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

The final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, a small open economy populated by buffer stock consumers converges to a balanced growth equilibrium in which consumption, income, and wealth eventually match the exogenous growth rate of permanent income (equivalent, here, to productivity growth). In the terms of Schmitt-Grohé and Uribe (2003), buffer stock saving is a method of 'closing' a small open economy model, one that is attractive because it requires no ad-hoc assumptions. Not even liquidity constraints.

2 The Problem

2.1 Setup

The infinite horizon solution is the limiting first-period solution to a sequence of finite-horizon problems as the horizon (the last period of life) becomes arbitrarily distant.

That is, for the value function, fixing a terminal date T, we are interested in the final term $\mathbf{v}_{T-n}$ in the sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, ..., \mathbf{v}_{T-n}\}$. We will say that the problem has a 'nondegenerate' infinite horizon solution if, corresponding to that value function, there is a limiting consumption function $\mathring{\mathrm{c}}(m) = \lim_{n\uparrow\infty} \mathrm{c}_{T-n}$ which is neither zero everywhere nor infinity everywhere (this is fleshed out below).

Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max \ \mathbb{E}_t \left[\sum_{i=0}^{n} \beta^i \mathrm{u}(\mathbf{c}_{t+i}) \right]$$

where the utility function

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{1}$$

exhibits relative risk aversion $\rho > 1$.[4] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ and permanent noncapital income $\mathbf{p}_t$, which both start out strictly positive,

$$\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty), \tag{2}$$

and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{3}$$

In the usual treatment, a dynamic budget constraint (DBC) incorporates several elements that jointly determine next period's $\mathbf{m}$ (given this period's choices); for the detailed analysis here, it will be useful to disarticulate the steps:

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R} \\ \mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1}\xi_{t+1}, \end{aligned} \tag{4}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1 + \mathsf{r})$ between periods, so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[5] $\mathbf{m}_{t+1}$ ('market resources') is the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital

[4] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[5] Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n \geq 1$). Permanent noncapital income in $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1 \ \forall \ n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).[6]

Following Zeldes (1989), in future periods $t+n \ \forall \ n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{5}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \ \forall \ n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.[7] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (5) and $\mathcal{F}_\theta$). For quick identification in tables and graphs, we will call this the Friedman/Muth model because it is a specific implementation of the Friedman (1957) model as interpreted by Muth (1960), needing only a calibration of the income process and a specification of preferences (here, geometric discounting and CRRA utility) to be solvable.

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable (though it has empirical support).[8] However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,[9] transforming that model back into this one. And no key results would change if the transitory shocks were persistent but mean-reverting, instead of IID. Also, the assumption of a positive point mass for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.[10]

This model differs from Bewley's (1977) classic formulation in several ways. The Constant Relative Risk Aversion (CRRA) utility function does not satisfy Bewley's assumption that $\mathrm{u}(0)$ is well defined, or that $\mathrm{u}'(0)$ is well defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds are quantitatively important in micro data[11] and which are far more consequential for household welfare than are transitory fluctuations. It differs from Deaton (1991) because

[6] Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

[7] See Rabault (2002) and Li and Stachurski (2014) for analyses of cases where the shock processes have unbounded support.

[8] We will calibrate this probability to 0.005 percent to match data from the Panel Study of Income Dynamics (Carroll (1992)).

[9] So long as unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in section 2.11.

[10] A strictly positive density over a strictly positive interval above the lower bound would work just as well, but would be cumbersome.

[11] MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004); Blundell, Low, and Preston (2008)

liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income to reach zero.[12] It differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[13,14] Li and Stachurski (2014) show how to allow unbounded returns by using policy function iteration, but also impose constraints.

The paper with perhaps the most in common with this one is Ma, Stachurski, and Toda (2020), henceforth MST, who establish the existence and uniqueness of a solution to a general income fluctuation problem in a Markovian setting. The most important differences are that MST impose liquidity constraints, assume that $\mathrm{u}'(0) = 0$, and assume that expected marginal utility of income is finite ($\mathbb{E}[\mathrm{u}'(Y)] < \infty$). These assumptions are not consistent with the combination of CRRA utility and income dynamics used here, whose combined properties are key to the derivation of the results.[15]

2.2 The Problem Can Be Normalized By Permanent Income

We establish a bit more notation by reviewing the result that in such problems (CRRA utility, permanent shocks) the number of states can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Value in the last period of life is $\mathrm{u}(\mathbf{m}_T)$; using (in the last line in (6)) the fact that for our CRRA utility function, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, and generically defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with $m = \mathbf{m}/\mathbf{p}$), consider the problem in the second-to-last period,

$$
\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} \ \mathrm{u}(\mathbf{c}_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\mathbf{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1} c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho} \left\{ \max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)] \right\}. \qquad (6)
\end{aligned}
$$

Now, in a one-time deviation from the notational convention established in the last sentence, define nonbold 'normalized value' not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ \mathrm{u}(\mathrm{c}_t) + \beta\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \qquad (7) \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1}) a_t \ = \ \mathcal{R}_{t+1} a_t
\end{aligned}
$$

[12] Below it will become clear that the Deaton model is a particular limit of this paper's model.

[13] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (1986), Clarida (1987), and Chamberlain and Wilson (2000). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

[14] Alvarez and Stokey (1998) relaxed the bounds on the return function, but they address only the deterministic case.

[15] The incorporation of permanent shocks rules out application of the tools of Matkowski and Nowak (2011), who followed and corrected an error in the fundamental work on the local contraction mapping method developed in Rincón-Zapatero and Rodríguez-Palmero (2003). Martins-da Rocha and Vailakis (2010) provide a correction to Rincón-Zapatero and Rodríguez-Palmero (2003), that works under easier conditions to verify, but only addresses the deterministic case.

$$m_{t+1} = b_{t+1} + \xi_{t+1},$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the new problem's first order condition is[16]

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho} c_{t+1}^{-\rho}]. \tag{8}$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (7), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho} \mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem (7), we will have solutions to the original problem for any $t < T$ from:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t^{1-\rho} \mathrm{v}_t(m_t),$$
$$\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t \mathrm{c}_t(m_t).$$

2.3 Definition of a Nondegenerate Solution

The problem has a nondegenerate solution if as the horizon n gets arbitrarily large the solution in the first period of life $\mathring{\mathrm{c}}_{T-n}(m)$ gets arbitrarily close to a limiting $\mathring{\mathrm{c}}(m)$:

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m) \tag{9}$$

that satisfies

$$0 < \mathrm{c}(m) < \infty \tag{10}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is $\mathring{\mathrm{c}}(m) = 0$ or $\mathring{\mathrm{c}}(m) = \infty$; below when we say a 'solution exists' we will always mean 'a nondegenerate solution.')

2.4 Perfect Foresight Benchmarks

The familiar analytical solution to the perfect foresight model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (3) imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{11}$$

[16] Leaving aside their assumptions about the marginal utility function and liquidity constraints, it is tempting to view this as a special case of the model of MST, with the $\mathcal{R}_{t+1} = \mathsf{R}/\Gamma_{t+1}$ (defined below equation (7)) corresponding to their stochastic rate of return on capital and the FVAF $\beta\Gamma_{t+1}^{1-\rho}$ defined below (40) corresponding to their stochastic discount factor. But a caveat is that, here, $\mathcal{R}_{t+1}$ and the modified discount factor are intimately related because Γ_{t+1} plays a role in each.

where $\mathbf{b}$ is nonhuman wealth and $\mathbf{h}_t$ is 'human wealth,' and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$,

$$\begin{aligned}\mathbf{h}_t &= \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + ... + \mathcal{R}^{t-T}\mathbf{p}_t \\ &= \underbrace{\left(\frac{1-\mathcal{R}^{-(T-t+1)}}{1-\mathcal{R}^{-1}}\right)}_{\equiv h_t}\mathbf{p}_t \end{aligned} \tag{12}$$

In order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC'):

$$\underbrace{\Gamma/\mathsf{R}}_{\equiv \mathcal{R}^{-1}} < 1. \tag{13}$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 PF Unconstrained Solution Exists Under RIC and FHWC

Without constraints, the consumption Euler equation always holds; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$\mathbf{c}_{t+1}/\mathbf{c}_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{14}$$

where the archaic letter 'thorn' represents what we will call the 'Absolute Patience Factor,' or APF:

$$\text{Þ} = (\mathsf{R}\beta)^{1/\rho}. \tag{15}$$

The sense in which **Þ** captures patience is that if the 'absolute impatience condition' (AIC) holds,[17]

$$\text{Þ} < 1, \tag{16}$$

the consumer will choose to spend an amount too large to sustain indefinitely. We call such a consumer 'absolutely impatient.'

We next define a 'Return Patience Factor' (RPF) that relates absolute patience to the return factor:

$$\text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{17}$$

and since consumption is growing by **Þ** but discounted by R:

$$\mathrm{PDV}_t(\mathbf{c}) = \left(\frac{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}{1-\text{Þ}_{\mathsf{R}}}\right)\mathbf{c}_t \tag{18}$$

[17] Impatience conditions have figured in intertemporal optimization problems since the beginning, e.g. in Ramsey (1928). These issues are so central that it would be hopeless to attempt to cite conditions in every other paper that correspond to conditions named and briefly exposited here. I make no claim to novelty for any condition or implication except for the conditions implicated in my theorems, whose parallels *will* be articulated.

from which the IBC (11) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv\underline{\kappa}_t}(\mathbf{b}_t+\mathbf{h}_t) \tag{19}$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n}-1}^{\equiv b_{T-n}}+h_{T-n})\underline{\kappa}_{T-n} \tag{20}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) – it answers the question 'if the consumer had an extra unit of resources, how much more would be spent.' ($\bar{\mathrm{c}}$'s overbar signfies that $\bar{\mathrm{c}}$ will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is a lower bound).

Equation (19) makes plain that for the limiting MPC $\underline{\kappa}$ to be strictly positive as $n = T-t$ goes to infinity we must impose the Return Impatience Condition (RIC):

$$\text{Þ}_{\mathsf{R}} < 1, \tag{21}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{22}$$

The RIC thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$). A consumer who satisfies the RIC is 'return impatient.'

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting consumption function will be

$$\bar{\mathrm{c}}(m) = (m + h - 1)\underline{\kappa}, \tag{23}$$

and so in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (13).

The fact that $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$ allows us to write a useful analytical expression for the value the consumer would achieve by spending permanent income $\mathbf{p}$ in every period:

$$\begin{aligned} \mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma) + \beta^2\mathrm{u}(\mathbf{p}_t\Gamma^2) + ... \\ &= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta\Gamma^{1-\rho} + (\beta\Gamma^{1-\rho})^2 + ...\right) \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-t+1}}{1-\beta\Gamma^{1-\rho}}\right) \end{aligned} \tag{24}$$

which (for $\Gamma > 0$) asymptotes to a finite number as $n = T-t$ approaches $+\infty$ if any of these equivalent conditions holds:

$$\begin{aligned} \overbrace{\beta\Gamma^{1-\rho}}^{\equiv\beth} &< 1 \\ \beta\mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma \\ \text{Þ}_{\mathsf{R}} &< (\Gamma/\mathsf{R})^{1-1/\rho}, \end{aligned} \tag{25}$$

where we call $\beth$[18] the 'Perfect Foresight Value Of Autarky Factor' (PF-VAF), and the variants of (25) constitute alternative versions of the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC; they guarantee that a consumer who always spends all permanent income 'has finite autarky value.'[19]

If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied: Divide both sides of the third inequality in (25) by R:

$$\text{Þ}/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} \tag{26}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(\Gamma/\mathsf{R}) < 1$ (and the RHS is raised to a positive power (because $\rho > 1$)).

Likewise, if the FHWC and the GIC are both satisfied, PF-FVAC must hold:

$$\begin{aligned} \text{Þ} &< \Gamma < \mathsf{R} \\ \text{Þ}_{\mathsf{R}} &< \Gamma/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} < 1 \end{aligned} \tag{27}$$

where the last line holds because FHWC $\Rightarrow 0 \leq (\Gamma/\mathsf{R}) < 1$ and $\rho > 1 \Rightarrow 0 < 1-1/\rho < 1$.

The first panel of Table 4 summarizes: The PF-Unconstrained model has a non-degenerate limiting solution if we impose the RIC and FHWC (these conditions are necessary as well as sufficient). Imposing the PF-FVAC and the FHWC implies the RIC, so PF-FVAC and FHWC are sufficient. If we impose the GIC and the FHWC, both the PF-FVAC and the RIC follow, so GIC+FHWC are sufficient. But there are circumstances under which the RIC and FHWC can hold while the PF-FVAC fails (which we write ~~PF-FVAC~~). For example, if $\Gamma = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under the RIC.

Perhaps more useful than this prose or the table, the relations of the conditions for the unconstrained perfect foresight case are presented diagrammatically in Figure 1. Each node represents a quantity considered in the foregoing analysis. The arrow associated with each inequality reflects the imposition of that condition. For example, one way we wrote the PF-FVAC in equation (25) is $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, so imposition of the PF-FVAC is captured by the diagonal arrow connecting Þ and $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$. Traversing the boundary of the diagram clockwise starting at Þ involves imposing first the GIC then the FHWC, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply that the PF-FVAC holds. Reversal of a condition will reverse the arrow's direction; so, for example, the bottommost arrow going from R to $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ imposes ~~FHWC~~; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram in a clockwise direction from Þ to R, revealing that imposition of GIC and FHWC (and, redundantly, FHWC again) let us conclude that the RIC holds because the starting point is Þ and the endpoint is R. (Consult Appendix K for a detailed exposition of diagrams of this type).

[18]This is another kind of discount factor, so we use the Hebrew 'bet' which is a cognate of the Greek 'beta'.
[19]This is related to the key impatience condition in Alvarez and Stokey (1998).

$$
\begin{tikzcd}
\mathbf{Þ} \arrow[r, "\text{GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}" description] \arrow[dr, phantom, shift left=4ex, "\neq"] \arrow[dr, phantom, shift right=4ex, "\neq"] & \Gamma \arrow[d, "\text{FHWC}"] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 1 Relation of GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is one way of writing the PF-FVAC, equation (25)

2.4.3 PF Constrained Solution Exists Under RIC or Under {~~RIC~~,GIC}

We next examine the perfect foresight constrained solution because it is a useful benchmark (and limit) for the unconstrained problem with uncertainty (examined next).

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound for entering the period, $b_t = 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (8):

$$1^{-\rho} > \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{28}$$

By analogy to the return patience factor, we therefore define a 'growth patience factor' (GPF) as

$$\mathbf{Þ}_\Gamma = \mathbf{Þ}/\Gamma, \tag{29}$$

and define a 'growth impatience condition' (GIC)

$$\mathbf{Þ}_\Gamma < 1 \tag{30}$$

which is equivalent to (28) (exponentiate both sides by $1/\rho$).

~~GIC~~ and RIC. If the GIC fails but the RIC (21) holds, appendix A shows that, for some $0 < m_\# < 1$, an unconstrained consumer behaving according to (23) would choose $c < m$ for all $m > m_\#$. In this case the solution to the constrained consumer's problem is

simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[20] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using the $\grave{\bullet}$ accent to designate the version of a function $\bullet$ in the presence of constraints:

$$\grave{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases} \tag{31}$$

GIC and RIC. More useful is the case where the return impatience and GIC conditions both hold. In this case appendix A shows that the limiting constrained consumption function is piecewise linear, with $\grave{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m_{\#}^{1} > 1$, and with discrete declines in the MPC at a set of kink points $\{m_{\#}^{1}, m_{\#}^{2}, ...\}$. As $m \uparrow \infty$ the constrained consumption function $\grave{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\grave{\boldsymbol{\kappa}}(m) \equiv \grave{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$. Similarly, the value function $\grave{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer.

This logic holds even when the finite human wealth condition fails (~~FHWC~~), because the constraint prevents the consumer from borrowing against infinite human wealth to finance infinite current consumption. Under these circumstances, the consumer who starts with any amount of resources $b_t > 1$ will, over time, run those resources down so that by some finite number of periods n in the future the consumer will reach $b_{t+n} = 0$, and thereafter will set $\mathbf{c} = \mathbf{p}$ for eternity (which the PF-FVAC says yields finite value). Using the same steps as for equation (24), value of the interim program is also finite:

$$\mathbf{v}_{t+n} = \Gamma^{n(1-\rho)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-(t+n)+1}}{1-\beta\Gamma^{1-\rho}}\right).$$

So, if ~~FHWC~~, value for any finite m will be the sum of two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+n} = 0$, and the finite component due to the value of consuming all $\mathbf{p}_{t+n}$ thereafter.

GIC and ~~RIC~~. The most peculiar possibility occurs when the RIC fails. Under these circumstances the FHWC must also fail (Appendix A), and the constrained consumption function is nondegenerate. (See appendix Figure 8 for a numerical example). While it is true that $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\grave{\mathrm{c}}(m)$ is strictly positive and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where ~~RIC~~ would suggest a degenerate limit of $\grave{\mathrm{c}}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\grave{\mathrm{c}}(m) = \infty$.

Tables 3 and 4 (and appendix table 5) codify.

We now turn to the case with uncertainty but without constraints, which will turn out to be a close parallel to the model with constraints but without uncertainty.

[20]"Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

2.5 Uncertainty-Modified Conditions

2.5.1 Impatience

When uncertainty is introduced, the expectation of b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})] = a_t(\mathsf{R}/\Gamma)\,\mathbb{E}_t[\psi_{t+1}^{-1}] \tag{32}$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define

$$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1} \tag{33}$$

which satisfies $\underline{\psi} < 1$ (again thanks to Jensen's inequality), so we can define

$$\underline{\Gamma} \equiv \Gamma\underline{\psi} \tag{34}$$

which is useful because it allows us to write uncertainty-adjusted versions of equations and conditions in a manner exactly parallel to those for the perfect foresight case; for example, we define a normalized Growth Patience Pactor (GPF-Nrm):

$$\text{\TH}_{\underline{\Gamma}} = \text{\TH}/\underline{\Gamma} = \mathbb{E}[\text{\TH}/\Gamma\psi] \tag{35}$$

and a normalized version of the Growth Impatience Condition, GIC-Nrm:

$$\text{\TH}_{\underline{\Gamma}} < 1 \tag{36}$$

which is stronger than the perfect foresight version (30) because

$$\underline{\Gamma} < \Gamma. \tag{37}$$

2.5.2 Autarky Value

Analogously to (24), value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned}\mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + ... + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}...\Gamma_T)\right] \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1-\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right)\end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)} \tag{38}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ. Defining

$$\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}} \tag{39}$$

we can see that $\mathbf{v}_t$ will be finite as T approaches ∞ if

$$\overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}} < 1$$

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

$$\beta < \underline{\underline{\Gamma}}^{\rho-1} \tag{40}$$

which we call the 'finite value of autarky condition' (FVAC) because it guarantees that value is finite for a consumer who always consumes their (now stochastic) permanent income (and we will call $\underline{\underline{\beth}}$ the 'Value of Autarky Factor,' VAF).[21] For nondegenerate ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (25) because $\underline{\underline{\Gamma}} < \Gamma$.[22]

2.6 The Baseline Numerical Solution

Figure 2, familiar from the literature, depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and earlier periods under baseline parameter values listed in Table 2. (The 45 degree line is $\mathring{\mathrm{c}}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge to a nondegenerate $\mathring{\mathrm{c}}(m)$. Our next purpose is to show that this appearance is not deceptive.

2.7 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (7) defines a sequence of continuously differentiable strictly increasing strictly concave[23] functions

[21] In a stationary environment – that is, with $\underline{\underline{\Gamma}} = 1$ – this corresponds to an impatience condition imposed by Ma, Stachurski, and Toda (2020); but their remaining conditions do not correspond to those here, because their problem differs and their method of proof differs.

[22] To see this, rewrite (40) as

$$\begin{aligned} \beta\mathsf{R} &< \mathsf{R}\underline{\underline{\Gamma}}^{\rho-1} \\ (\beta\mathsf{R})^{1/\rho} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\ \text{Þ}_\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \end{aligned}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

[23] With one obvious exception: $\mathring{\mathrm{c}}_T(m)$ is linear (and so only weakly concave).

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Factor	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Factor	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	Þ_{R}	$\equiv$	$\text{Þ}/\mathsf{R}$	0.961
PF Growth Patience Factor	Þ_{Γ}	$\equiv$	$\text{Þ}/\Gamma$	0.970
Growth Patience Factor	$\text{Þ}_{\underline{\Gamma}}$	$\equiv$	$\text{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Factor	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941
Weak Impatience Factor	$\wp^{1/\rho}\text{Þ}$	$\equiv$	$(\wp\beta\mathsf{R})^{1/\rho}$	0.071

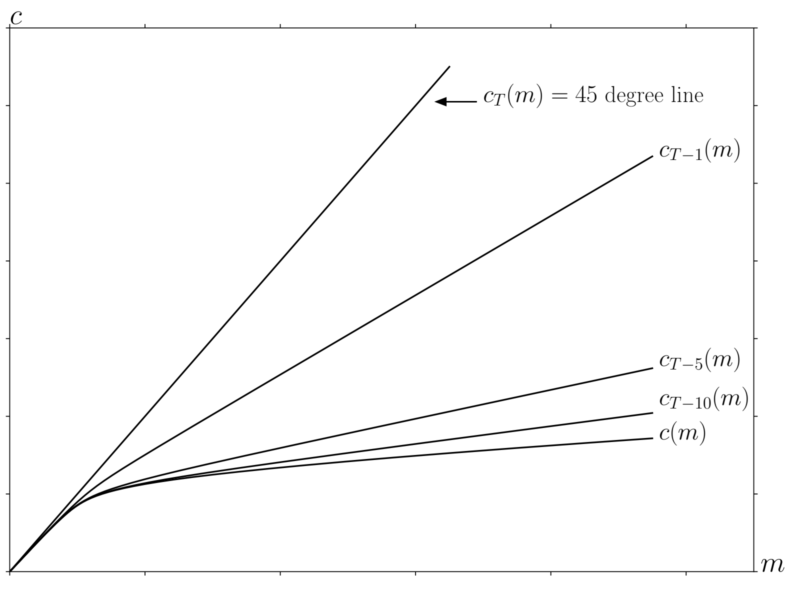


Figure 2 Convergence of the Consumption Rules

$\{c_T, c_{T-1}, ...\}$. The straightforward but tedious proof is relegated to appendix B. For present purposes, the most important point is that the income process induces what Aiyagari (1994) dubbed a 'natural borrowing constraint': $\mathring{c}_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero income over the remaining horizon, requiring the consumer to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. Zeldes (1989) seems to have been the first to argue, based on his numerical results, that the natural borrowing constraint was a quantitatively plausible alternative to 'artificial' or 'ad hoc' borrowing constraints in a life cycle model.[24]

Strict concavity and continuous differentiability of the consumption function are key elements in many of the arguments below, but are not characteristics of models with 'artificial' borrowing constraints. The analytical convenience of these features is considerable, even if models with natural borrowing constraints in practice usually give similar results to those with artificial constraints.

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 2 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which will be needed in the contraction mapping proof.[25]

Assume that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathring{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption that human wealth is finite, the proportion of consumption that will be financed out of human wealth approaches zero. In consequence, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero. In the course of proving this, appendix G provides a useful recursive expression (used below) for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}}\underline{\kappa}_{t+1}^{-1}. \tag{41}$$

2.8.1 Weak RIC Conditions

Appendix equation (92) presents a parallel expression for the limiting maximal MPC as $m_t \downarrow 0$:

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \tag{42}$$

[24] Carroll (1992) made the same (numerical) point for infinite horizon models (calibrated to actual empirical data on household income dynamics).

[25] Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; Ma and Toda (2020) show that these results generalize to the limits derived here if capital income is added to the model.

where $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ satisfies:

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{43}$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathring{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{44}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave,[26] and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is in appendix D).

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

We can now articulate conditions under which the problem defines a contraction mapping. As mentioned above, standard theorems in the literature following Stokey et. al. (1989) require utility or marginal utility to be bounded over the space of possible values of m, which does not hold here because the possibility (however unlikely) of an unbroken string of zero-income events through the end of the horizon means that utility (and marginal utility) are unbounded as $m \downarrow 0$. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., Matkowski and Nowak (2011)), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that incorporates permanent shocks.[27]

Fortunately, Boyd (1990) provided a weighted contraction mapping theorem that Alvarez and Stokey (1998) showed could be used to address the homogeneous case (of which CRRA is an example) in a deterministic framework; later, Durán (2003) showed how to extend the Boyd (1990) approach to the stochastic case.

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ where $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then $\bullet$ is F-bounded if the F-norm of $\bullet$,*

$$\| \bullet \|_F = \sup_m \left[\frac{| \bullet (m)|}{F(m)} \right], \tag{45}$$

is finite.

[26] Carroll and Kimball (1996)

[27] See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

For $\mathcal{C}_{F}(\mathcal{A},\mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A},\mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m)=0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T}:\mathcal{C}_{F}(\mathcal{A},\mathcal{B})\to\mathcal{C}(\mathcal{A},\mathcal{B})$ *such that*[28,29]

$$
\begin{aligned}
&(1)\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x}\le\mathrm{y}\Rightarrow\{\mathsf{T}\mathrm{x}\}\le\{\mathsf{T}\mathrm{y}\}\\
&(2)\{\mathsf{T}\mathbf{0}\}\in\ \mathcal{C}_{F}(\mathcal{A},\mathcal{B})\\
&(3)\textit{There exists some real } 0<\alpha<1 \textit{ such that}\\
&\quad\{\mathsf{T}(\mathrm{w}+\zeta F)\}\le\{\mathsf{T}\mathrm{w}\}+\zeta\alpha F\quad \textit{holds for all real } \zeta>0\quad.
\end{aligned}
$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{>0}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t)=\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1}+\xi_{t+1})\right].$$

Using this, we introduce the mapping $\mathcal{T}:\mathcal{C}_{F}(\mathcal{A},\mathcal{B})\to\mathcal{C}(\mathcal{A},\mathcal{B})$,[30]

$$\{\mathcal{T}\mathrm{z}\}(m_t)=\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\mathrm{u}(c_t)+\beta\left(\{\mathsf{E}\mathrm{z}\}(m_t-c_t)\right). \tag{46}$$

We can show that our operator $\mathcal{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first is the WRIC necessary for convergence of the maximal MPC, equation (43) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (40). (We discuss the interpretation of these restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. *$\mathcal{T}$ is a contraction mapping if the restrictions on parameter values* (43) *and* (40) *are true (that is, if the weak return impatience condition and the finite value of autarky condition hold).*

Intuitively, Boyd's theorem shows that if you can find a F that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with F, the normalized problem defines a contraction mapping. The intuition for the FVAC condition is just that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly the sustainable amount (say, by consuming $(\mathsf{r}/\mathsf{R})b_0-\epsilon$ for some small $\epsilon>0$).

The details of the proof are cumbersome, and are therefore relegated to appendix D. Given that the value function converges, appendix E.2 shows that the consumption functions converge.[31]

[28] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[29] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x}\le\mathrm{y}$ is short for $\mathrm{x}(\bullet)\le\mathrm{y}(\bullet)\ \forall\bullet\in\mathcal{A}$. In this notation, $\zeta\alpha F$ in (3) is a *function* which can be applied to any argument $\bullet$ (because F is a function).

[30] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous F-bounded functions z) and the compactness of $[\underline{\kappa}m_t,\bar{\kappa}m_t]$.

[31] MST's proof is for convergence of the consumption policy function directly, rather than of the value function, which is why their conditions are on u', which governs behavior.

2.10 The Liquidity Constrained Solution as a Limit

This section explains why a related problem commonly considered in the literature (e.g., by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The 'related' problem makes two changes to the problem defined above:

1. An 'artificial' liquidity constraint is imposed: $a_t \geq 0$
2. The probability of zero-income events is zero: $\wp = 0$

The essence of the argument is simple. Imposing the artificial constraint without changing $\wp$ would not change behavior at all: The possibility of earning zero income over the remaining horizon already prevents the consumer from ending the period with zero assets. So, for precautionary reasons, the consumer will save something.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the *probability* that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the *amount* of precautionary saving induced by the zero-income events approaches zero as $\wp \downarrow 0$. But "zero" is the amount of precautionary saving that would be induced by a zero-probability event for the impatient liquidity constrained consumer.

Another way to understand this is just to think of the liquidity constraint reflecting a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ends the period with (weakly) negative assets.

See appendix H for the formal proof justifying the foregoing intuitive discussion.[32]

The conditions required for convergence and nondegeneracy are thus strikingly similar between the liquidity constrained perfect foresight model and the model with uncertainty but no explicit constraints: The liquidity constrained perfect foresight model is just the limiting case of the model with uncertainty as the degree of all three kinds of uncertainty (zero-income events, other transitory shocks, and permanent shocks) approaches zero.

2.11 Discussion of Parametric Restrictions

The full relationship among all the conditions is represented in Figure 3. Though the diagram looks complex, it is merely a modified version of the earlier diagram with further (mostly intermediate) inequalities inserted. (Arrows with a "because" are a new element to label relations that always hold under the model's assumptions.) Again readers unfamiliar with such diagrams should see Appendix K) for a more detailed explanation.

[32]It seems likely that a similar argument would apply even in the context of a model like that of MST, perhaps with some weak restrictions on returns.

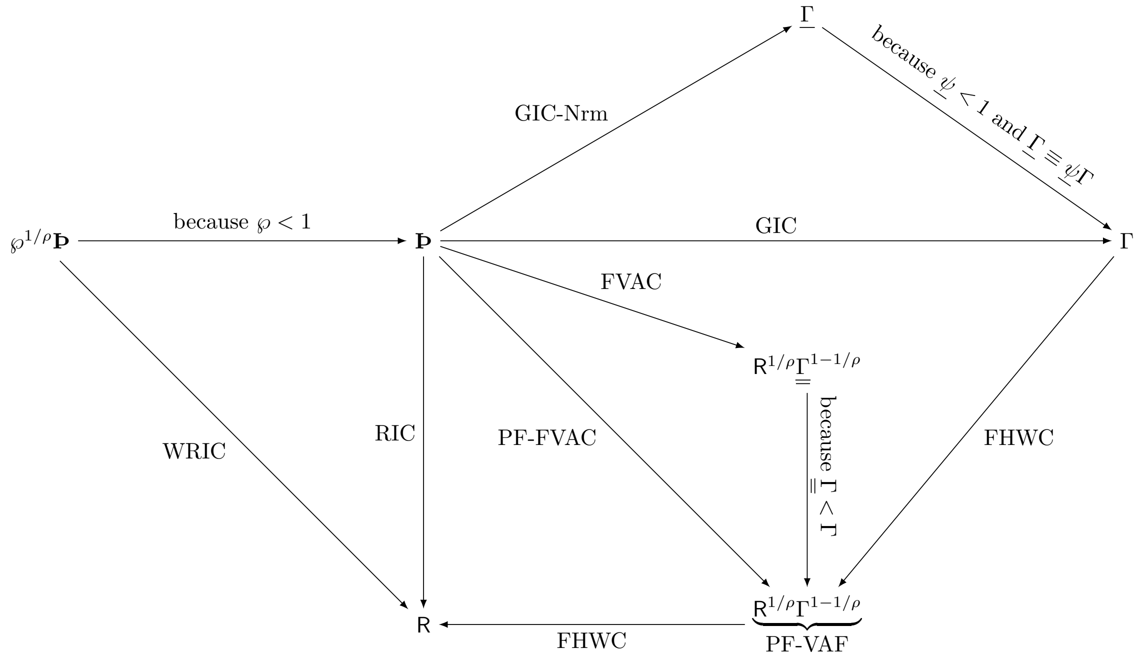


Figure 3 Relation of All Inequality Conditions
See Table 2 for Numerical Values of Nodes Under Baseline Parameters

2.11.1 The WRIC

The 'weakness' of the additional condition sufficient for contraction beyond the FVAC, the WRIC, can be seen by asking 'under what circumstances would the FVAC hold but the WRIC fail?' Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \ 1 \ < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{47}$$

If we require $\mathsf{R} \geq 1$, the WRIC is redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that (with $\rho > 1$ and $\beta < 1$) the RIC (and WRIC) must hold. But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (47) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. The relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which section

2.10 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.11.2 When the RIC Fails

In the perfect foresight problem (section 2.4.2), the RIC was necessary for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the much weaker WRIC is sufficient for nondegeneracy (assuming that the FVAC holds). We can directly derive the features the problem must exhibit (given the FVAC) under $\cancel{\text{RIC}}$ (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$
\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} \;<\; (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< \underline{\underline{\psi}}
\end{aligned}
\tag{48}
$$

but since $\underline{\underline{\psi}} < 1$ (cf. the argument below (38)), this reduces to $\mathsf{R}/\Gamma < 1$; so, given the FVAC, the RIC can fail only if human wealth is unbounded. As an illustration of the usefulness of our diagrams, note that this algebraically complicated conclusion could be easily reached diagrammatically in figure 3 by starting at the R node and imposing $\cancel{\text{RIC}}$ (reversing the RIC arrow) and then traversing the diagram along any clockwise path to the PF-VAF node at which point we realize that we *cannot* impose the FHWC because that would let us conclude $\mathsf{R} > \mathsf{R}$.

As in the perfect foresight constrained problem, unbounded limiting human wealth ($\cancel{\text{FHWC}}$) here does not lead to a degenerate limiting consumption function (finite human wealth is not a condition required for the convergence theorem). But, from equation (41) and the discussion surrounding it, an implication of $\cancel{\text{RIC}}$ is that $\lim_{m\uparrow\infty} \mathring{c}'(m) = 0$. Thus, interestingly, in the special $\{\cancel{\text{RIC}},\cancel{\text{FHWC}}\}$ case (unavailable in the perfect foresight model) the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited human wealth from resulting in infinite consumption at any finite m. Intutively, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in a limiting consumption function of $\mathring{c}(m) = 0$) plus unbounded human wealth (which the perfect foresight model prohibits (by assumption FHWC) because it leads to a limiting consumption function $\mathring{c}(m) = \infty$) combine to yield a unique finite level of consumption and the MPC for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the $\{\text{GIC},\cancel{\text{RIC}}\}$ case. There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.[33]

2.11.3 When the RIC Holds

FHWC. If the RIC and FHWC both hold, a perfect foresight solution exists (see 2.4.2 above). As $m \uparrow \infty$ the limiting consumption function and value function become arbitrarily close to those in the perfect foresight model, because human wealth pays for a vanishingly small portion of spending. This will be the main case analyzed in detail below.

~~FHWC~~. The more exotic case is where FHWC does not hold; in the perfect foresight model, $\{\text{RIC}, \cancel{\text{FHWC}}\}$ is the degenerate case with limiting $\bar{\mathrm{c}}(m) = \infty$. Here, since the FVAC implies that the PF-FVAC holds (traverse Figure 3 clockwise from Þ by imposing FVAC and continue to the PF-VAF node), reversing the arrow connecting the R and PF-VAF nodes implies that under ~~FHWC~~:

$$\begin{aligned} &\overbrace{\text{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma}^{\text{PF-FVAC}} \\ &\text{Þ} < \Gamma \end{aligned}$$

where the transition from the first to the second lines is justified because $\cancel{\text{FHWC}} \Rightarrow (\mathsf{R}/\Gamma)^{1/\rho} < 1$. So, $\{\text{RIC}, \cancel{\text{FHWC}}\}$ implies the GIC holds. However, we are not entitled to conclude that the GIC-Nrm holds: $\text{Þ} < \Gamma$ does not imply $\text{Þ} < \underline{\psi}\Gamma$ where $\underline{\psi} < 1$. See further discussion of this illuminating case in section 3.3.3.

We have now established the principal points of comparison between the perfect foresight solutions and the solutions under uncertainty; these are codified in the remaining parts of Tables 3 and 4.

3 Analysis of the Converged Consumption Function

Figures 4 and 5a,b capture the main properties of the converged consumption rule when the RIC, GIC-Nrm, and FHWC all hold.[34] Figure 4 shows the expected consumption growth factor $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ for a consumer behaving according to the converged consumption rule, while Figures 5a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to Þ, indicated by the lower bound in Figure 4, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{Þ}_{\mathsf{R}})$ (Figure 5), the same as the perfect foresight MPC. Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 4) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}})$ (Figure 5). Third (Figure 4), there are two special values of m, which we will call the 'individual steady

[33]Ma and Toda (2020) derive conditions under which the limiting MPC is zero in an even more general case where there is also capital income risk.

[34]These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

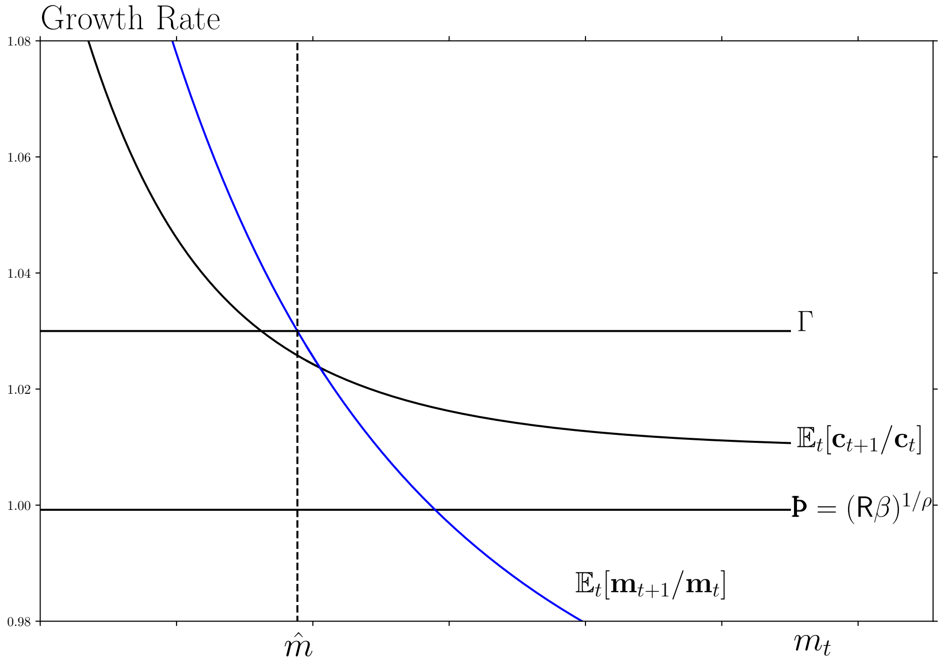


Figure 4 Target m, Expected Consumption Growth, and Permanent Income Growth

state' point $\hat{m}$ because it is the point where consumption growth and income growth are balanced, and the 'individual target' $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$. As indicated by the arrows of motion on the $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ curve, the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then m will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 4), at the market resources target $\check{m}$, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 4 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[35]

3.1 Limits as m approaches Infinity

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

[35]Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathring{\mathrm{c}}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathring{\mathrm{c}}(m)$. It does not strictly follow that the limiting function $\mathring{\mathrm{c}}(m)$ is twice continuously differentiable, but I will assume that it is.

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathring{\mathrm{c}}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption. Our imposition of the RIC guarantees that $\underline{\kappa} > 0$, so this solution satisfies our definition of nondegeneracy, and because this solution is always available it defines a lower bound on both the consumption and value functions.

Assuming the FHWC holds, the infinite horizon perfect foresight solution (23) constitutes an upper bound on consumption in the presence of uncertainty, since the introduction of uncertainty strictly decreases the level of consumption at any m (Carroll and Kimball (1996)). Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) <& \mathring{\mathrm{c}}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathring{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned} \tag{49}$$

But

$$\begin{aligned} \lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m - 1 + h)/m \\ &= 1, \end{aligned}$$

so as $m \uparrow \infty, \mathring{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) \rightarrow 1$, and the continuous differentiability and strict concavity of $\mathring{\mathrm{c}}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathring{\mathrm{c}}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 5 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t\uparrow\infty} \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t\uparrow\infty} \Gamma_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathring{\mathrm{c}}(m_t)$),

$$\begin{aligned} \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t &= \lim_{m_t\uparrow\infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t} \right) \\ &= (\mathsf{R}\beta)^{1/\rho} = \text{\textbf{Þ}} \end{aligned} \tag{50}$$

because $\lim_{m_t\uparrow\infty} \mathrm{a}'(m) = \text{\textbf{Þ}}_{\mathsf{R}}$[36] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

[36]This is because $\lim_{m_t\uparrow\infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathring{\mathrm{c}}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathring{\mathrm{c}}'(m_t) = \text{\textbf{Þ}}_{\mathsf{R}}$.

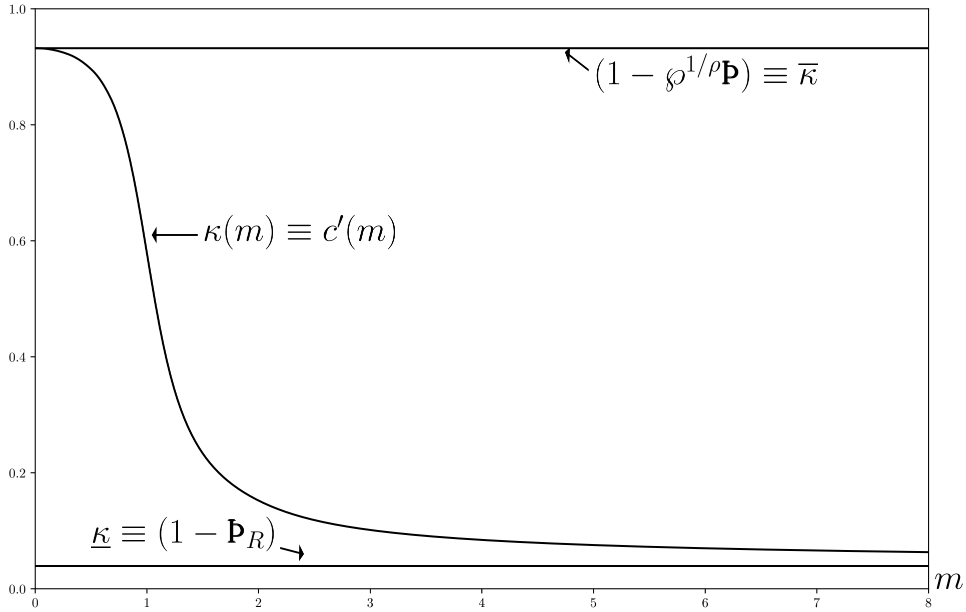


Figure 5 Limiting MPC's

Hence we have

$$\text{Þ} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model.

3.2 Limits as m Approaches Zero

Equation (42) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho}.$$

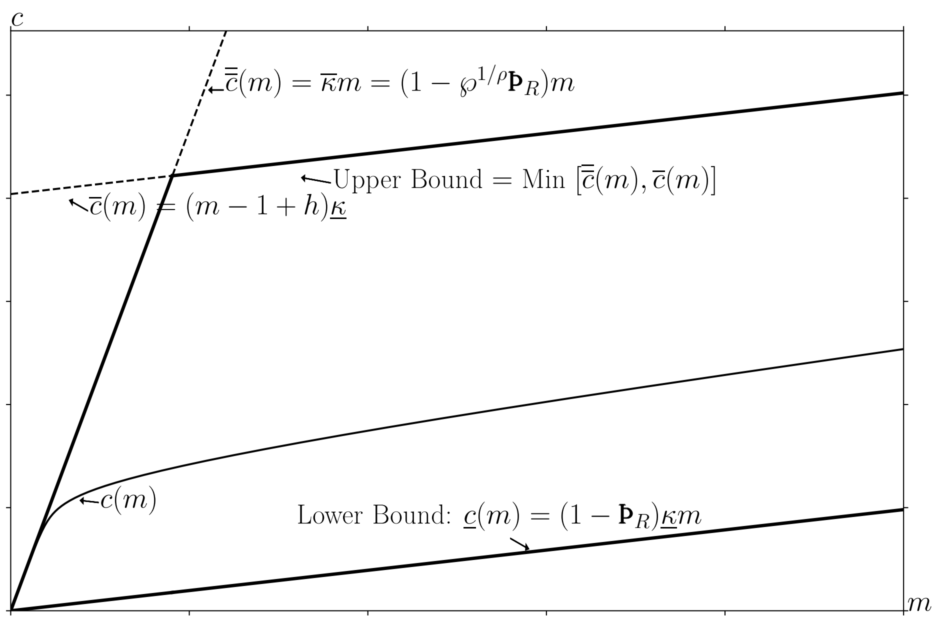
Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

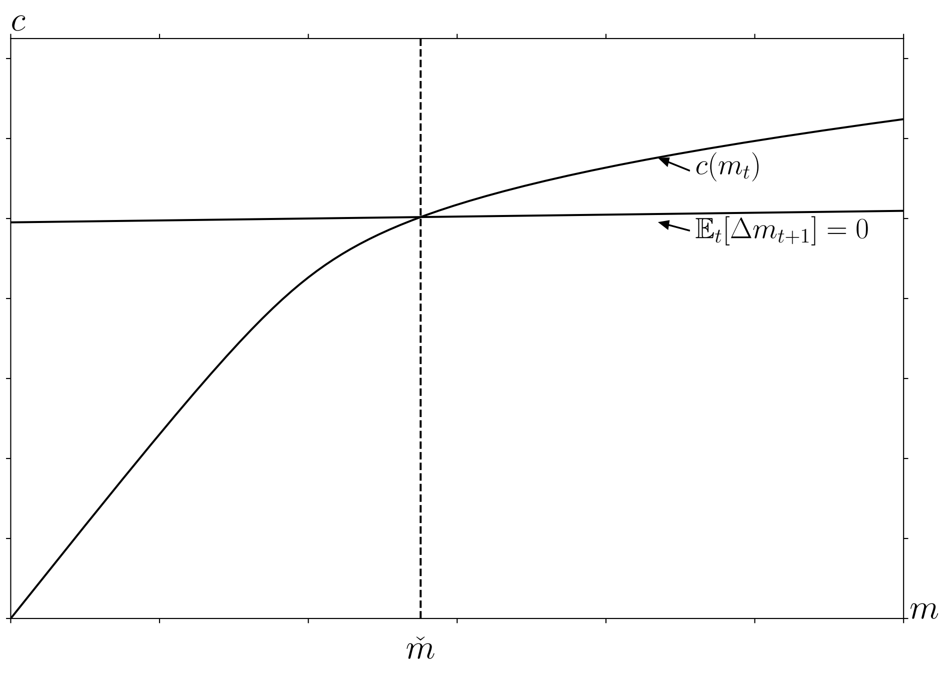
Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathring{\mathrm{c}}'(m) = \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 5 confirms that the numerical solution obtains this limit for the MPC as m approaches zero.



(a) Bounds



(b) Target m

Figure 6 The Consumption Function

For consumption growth, as $m \downarrow 0$ we have

$$
\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}
$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[37]

3.3 Unique 'Stable' Points

3.3.1 The 'Individual Target'

The most obvious definition of a 'stable' point is a value $\check{m}$ such that if $m_t = \check{m}$, then $\mathbb{E}_t[m_{t+1}] = m_t$. Existence of such a target turns out to require the GIC-Nrm condition, equation (36).

Theorem 2. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (36) holds, there exists a unique 'individual target' $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{51}$$

Moreover, $\check{m}$ is a point of stablity in the sense that

$$
\begin{aligned}
\forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] &> m_t \\
\forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] &< m_t.
\end{aligned}
\tag{52}
$$

The full proof is in Appendix M, but the key points can be made informally here. Defining $\bar{\mathcal{R}} = (\mathsf{R}/\underline{\Gamma})$, since $\mathbb{E}_t[m_{t+1}] = \bar{\mathcal{R}}(m_t - c_t) + 1$, the implicit equation for $\check{m}$ is

$$\bar{\mathcal{R}}(\check{m} - \mathring{\mathrm{c}}(\check{m})) + 1 = \check{m} \tag{53}$$

which can be differentiated and rearranged to yield

$$1 - \bar{\mathcal{R}}^{-1} = \mathring{\mathrm{c}}'(\check{m}). \tag{54}$$

[37] None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

The fact (cf. (22)) that the minimum value of $\mathring{c}'$ is $1-\text{\textthorn}_{\mathsf{R}}$ converts (54) to the inequality $1-\bar{\mathcal{R}}^{-1} > 1-\text{\textthorn}_{\mathsf{R}}$ from which we have

$$\begin{aligned}\underline{\Gamma}/\mathsf{R} &> \text{\textthorn}/\mathsf{R} \\ 1 &> \text{\textthorn}/\underline{\Gamma} \end{aligned} \tag{55}$$

which is the GIC-Nrm; thus, if a stable point exists, it must satisfy the GIC-Nrm. (The appendix proves that if the GIC-Nrm is satisfied, such a point must exist, and be globally stable).

3.3.2 The 'Individual Steady-State'

Heterogeneous agents in a small open economy are often represented by models of this kind. A traditional question in such models is whether there is a 'balanced growth' equilibrium in which aggregate variables (income, consumption, market resources) all grow at the same rate. As an input to our more focused small-open-economy analysis (in section 4.2) it will be useful to derive here the conditions under which an m will exist at which an individual consumer expects 'balanced growth' in their own individual market resources and permanent income:

$$\begin{aligned} \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &= \mathbb{E}_t[\mathbf{p}_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t[m_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/(m_t\mathbf{p}_t) &= \mathbb{E}_t[\mathbf{p}_t\Gamma\psi_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t\left[\psi_{t+1}\underbrace{((m_t-\mathring{c}(m_t))\mathsf{R}/(\Gamma\psi_{t+1}))+\xi_{t+1})}_{m_{t+1}}\right] &= m_t. \end{aligned} \tag{56}$$

If some value $m_t = \hat{m}$ exists for which equation (56) holds, we will call that point the 'individual steady-state' $\hat{m}$:

$$\begin{aligned} \mathbb{E}\left[(\hat{m}-\mathring{c}(\hat{m}))\overbrace{\mathsf{R}/\Gamma}^{\mathcal{R}}+\psi\xi\right] &= \hat{m} \\ (\hat{m}-\mathring{c}(\hat{m}))\mathcal{R}+1 &= \hat{m}, \end{aligned} \tag{57}$$

and derivations parallel to those after (53) yield the conclusion that existence of an individual steady-state implies the GIC, $1 > \text{\textthorn}/\Gamma$. Note that since $\underline{\Gamma} < \Gamma$, $\mathring{c}'(\check{m}) > \mathring{c}'(\hat{m})$ which implies that $\check{m} < \hat{m}$.

If $\hat{m}$ exists, a weak sense in which it is a point of stability is that for an economy that started in period t with all consumers at $m_t = \hat{m}$, the ratio of the aggregate value $\mathbf{M}_{t+1} = \int \mathbf{m}_{t+1}$ to aggregate $\mathbf{P}_{t+1} = \int \mathbf{p}_{t+1}$ would be $\mathbf{M}_{t+1}/\mathbf{P}_{t+1} = \hat{m}$ so that, at least between those two periods, the aggregate economy would exhibit balanced growth and stability of the $\mathbf{M}/\mathbf{P}$ ratio.[38] After that, however, the nondegenerate distribution across values of m_{t+1} and $\mathbf{p}_{t+1}$ would almost certainly lead to some drift in the aggregate ratio.

[38]This does not require all consumers to have the same $\mathbf{p}$. The unlimited size of the population means that the expectation in (56) holds for the set of consumers at each value of $\mathbf{p}$ represented in the population, and so holds for the entire population.

There is one circumstance in which $\hat{m}$ would constitute a perpetually stable steady state, at both the micro and aggregate levels: if after the date t at which all consumers had $m_t = \hat{m}$, all consumers always drew exactly the expected values of the idiosyncratic shocks ($\psi_{i,t+n} = \xi_{j,t+n} = 1 \ \forall \ i, j$ and for all $n > 0$). Such an economy would exhibit perpetual 'balanced growth': $\mathbf{M}_{t+n}/\mathbf{P}_{t+n} = \hat{m}$ for all $n > 0$.

Theorem 5 formally states the relevant propositions.

Theorem 3. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (30) holds, there exists a unique 'individual steady state' $\hat{m} > 0$ such that*

$$\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{58}$$

Moreover, $\hat{m}$ is a point of stablity in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] > \Gamma = \mathbb{E}_t[\mathbf{p}_{t+1}/\mathbf{p}_t] \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] < \Gamma = \mathbb{E}_t[\mathbf{p}_{t+1}/\mathbf{p}_t]. \end{aligned} \tag{59}$$

The proofs of the two theorems are almost completely parallel; to save space, they are relegated to Appendix M. In sum, they involve three steps:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1}/m_t]$
   - This follows from existence and continuity of the constitutents
2. Existence of the equilibrium point
   - This follows from the upper and lower bound limiting MPC's, existence and continuity, and the Intermediate Value Theorem
3. Monotonicity of $\mathbb{E}_t[m_{t+1} - m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1} - m_t]$
   - This follows from concavity of the consumption function

3.3.3 Example Where There Is A Solution Without A Target

To build intuition, it is useful to exhibit an example in which a nondegenerate solution exists but a target $\check{m}$ does not. An example that satisfies the combination FVAC and ~~GIC-Nrm~~ is depicted in Figure 7. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the code for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC-Nrm does not, the consumption function's limiting slope $(1 - \text{\textbf{Þ}}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1 - \underline{\Gamma}/\mathsf{R})$,[39] so the gap between the two *increases* with m in the limit. Although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, the consumer will always spend less than the amount that would keep m stable (in expectation).

[39] This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\underline{\psi}^{-1} + 1$ for c and differentiate.

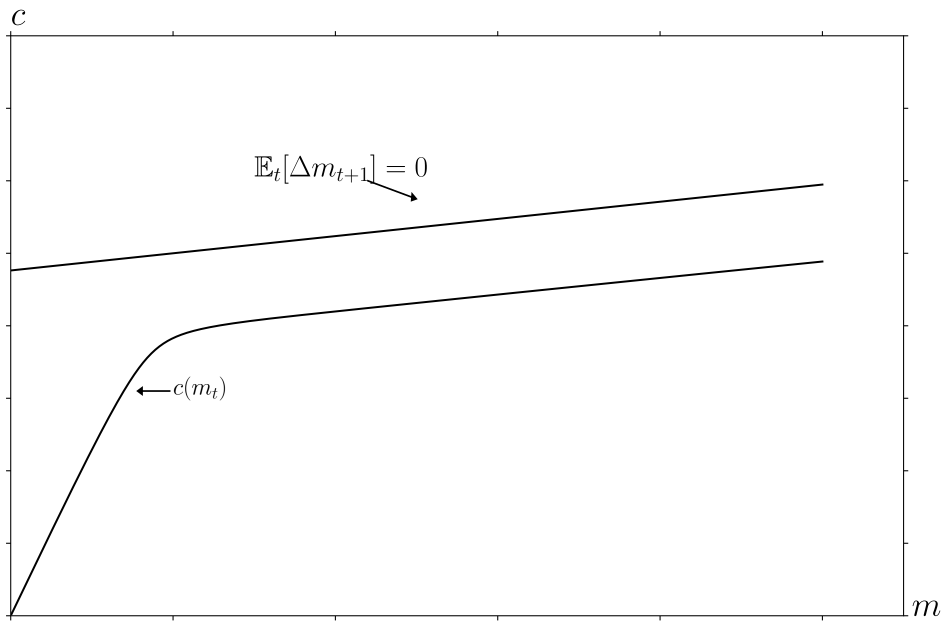


Figure 7 Example Solution under {FVAC,~~GIC-Nrm~~}

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 4 the intersection of the individual target m ratio $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the expected growth rate of permanent income. We now prove this.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\Gamma_{t+1}\mathring{c}(m_{t+1})}{\mathring{c}(m_t)}\right] &< \mathbb{E}_t\left[\left(\frac{\Gamma_{t+1}(\mathring{c}(\check{m}) + \mathring{c}'(\check{m})(m_{t+1} - \check{m}))}{\mathring{c}(\check{m})}\right)\right] \\
&= \mathbb{E}_t\left[\Gamma_{t+1}\left(1 + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)(m_{t+1} - \check{m})\right)\right] \\
&= \Gamma + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)\mathbb{E}_t\left[\Gamma_{t+1}\left(m_{t+1} - \check{m}\right)\right] \\
&= \Gamma + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)\left[\mathbb{E}_t[\Gamma_{t+1}]\underbrace{\mathbb{E}_t[m_{t+1} - \check{m}]}_{=0} + \mathrm{cov}_t(\Gamma_{t+1}, m_{t+1})\right] \qquad (60)
\end{aligned}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m}) + \xi_{t+1}$ and $\mathrm{a}(\check{m}) > 0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (60) is negative, as required.

3.5 Is Expected Consumption Growth a Declining Function of m_t?

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathring{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathring{\mathrm{c}}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv\boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathring{\mathrm{c}}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathring{\mathrm{c}}(m_t) - \mathring{\mathrm{c}}(m_{t+1})\mathring{\mathrm{c}}'(m_t)}{\mathring{\mathrm{c}}(m_t)^2}\right)\right] < 0. \tag{61}$$

Appendix L shows that the proposition holds true if there are only transitory (and no permanent) shocks. The software archive associated with this paper contains an example in which exotic interactions between permanent shocks and extreme curvature that occurs with very small $\wp$ generate a (small) region where the proposition does not hold. In practice, for plausible parametric choices (and in models without an artificial liquidity constraint), $\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

A large (infinite) collection of small (infinitesimal) buffer-stock consumers with identical parameter values can be thought of as a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy with an exogenous (constant) interest rate.[40]

Until now for convenience we have assumed infinite horizons, with the implicit understanding that Poisson mortality should be handled by adjusting the effective discount factor for mortality. On that basis, section 4.1 continues to omit mortality. But a reason for explicitly introducing mortality will appear at the end of section 4.2, so implications of alternative assumptions about mortality are briefly examined in Section 4.3.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above. Szeidl (2013) proves that whenever the GIC holds such a population will be characterized by invariant distributions of m, c, and a;[41] designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.

[40] It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

[41] Szeidl (2013)'s equation (9), in our notation, is:

$$\begin{aligned} \mathbb{E}\log \mathsf{R}(1-\kappa) &< \mathbb{E}\log \Gamma\psi \\ \mathbb{E}\log \mathsf{R}\text{Þ}_{\mathsf{R}} &< \mathbb{E}\log \Gamma\psi \\ \log \text{Þ}_{\Gamma} &< \mathbb{E}\log \psi \end{aligned}$$

4.1 Consumption and Income Growth at the Household Level

The operator $\mathbb{M}[\bullet]$ yields the mean of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ used above, which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}
\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2)
\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ (Szeidl (2013)) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$. Thus, the same GIC that guaranteed the existence of an 'individual steady state' value of m at the microeconomic level guarantees both that there will be an invariant distribution of the population across values of the model variables and that in that invariant distribution the mean growth rates of all idiosyncratic variables are the same (see Szeidl (2013) for details).

4.2 Balanced Growth of Aggregate Income, Consumption, and Wealth

Using boldface capital letters for aggregates, the growth factor for aggregate income is:

$$\begin{aligned}
\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/\mathbb{M}[\mathbf{p}_t\xi_t] \\
&= \Gamma
\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

From the perspective of period t,

$$\begin{aligned}
\mathbf{A}_{t+1} &= \mathbb{M}[a_{t+1}\mathbf{p}_{t+1}] \\
&= \Gamma\mathbb{M}[(a_t + (a_{t+1} - a_t))\mathbf{p}_t\psi_{t+1}] \\
&= \Gamma\left(\underbrace{\mathbb{M}[a_t\mathbf{p}_t\psi_{t+1}]}_{=\mathbf{A}_t} + \underbrace{\mathbb{M}[(a_{t+1} - a_t)]}_{=0 \text{ (Szeidl (2013))}}\mathbb{M}[\mathbf{p}_t\psi_{t+1}] + \text{cov}_t(a_{t+1} - a_t, \mathbf{p}_t\psi_{t+1})\right) \\
\mathbf{A}_{t+1}/\mathbf{A}_t &= \Gamma\left(1 + \frac{\text{cov}(a_{t+1}, \mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right)
\end{aligned}$$

Unfortunately, the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1}. Matters are simpler if there are no permanent shocks; see Appendix F for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

---

and under our assumption that $\log \psi \sim \mathcal{N}(-\sigma_\psi^2/2, \sigma_\psi^2)$ we can exponentiate both sides to obtain the GIC, $\text{Þ}_\Gamma < 1$. If the permanent income shocks are not lognormally distributed the expression must be tested in Szeidl's original form.

One way of thinking about the problem here is that it may reflect the fact that, under our assumptions, the $\mathbf{p}$ variable does not have an ergodic distribution; the distribution of permanent income becomes forever wider over time, because our consumers never die and each immortal person is perpetually subject to symmetric shocks to their $\log \mathbf{p}$.

This is why we need to introduce mortality.

4.3 Mortality and Redistribution

Most heterogeneous agent models incorporate a constant positive probability of death, following Blanchard (1985). In a model that mostly follows Blanchard (1985), for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, Carroll, Slacalek, Tokuoka, and White (2017) show that the limiting distribution of permanent income has a finite variance, which is a useful step in the direction of taming the problems caused by an unbounded distribution of $\boldsymbol{p}$. Numerical results in that paper confirm the intuition that, under appropriate impatience conditions, balanced growth arises (though a formal proof remains elusive).

Even with those (numerical) results in hand, the centrality of the mortality assumptions to the existence and nature of steady states requires a brief discussion here.

4.3.1 Blanchard Lives

Blanchard (1985)'s model assumes the existence of an annuitization scheme in which estates of dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, the most notable of which is that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor: If the probability of remaining alive is $\aleph$, then assuming that no utility accrues after death makes the effective discount factor $\hat{\beta} = \beta\aleph$, while the enhancement to the rate of return from the annuity scheme yields an effective interest rate of $\hat{\mathsf{R}}/\aleph$ (recall that because of Poisson mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\hat{\text{Þ}}$ is unchanged from its value in the infinite horizon model:

$$\hat{\text{Þ}} \equiv (\beta\aleph\mathsf{R}/\aleph)^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ}. \tag{62}$$

The only adjustments this requires to the analysis from prior parts of this paper are therefore to the few elements that involve a role for R distinct from its contribution to Þ (principally, the RIC). These would need to be adjusted to incorporate in interest factor with a higher rate of return.

The numerical finding that the covariance term above is approximately zero allows us to conclude again that the key requirement for aggregate balanced growth is presumably the GIC.

4.3.2 Modigliani Lives

Blanchard (1985)'s innovation was useful not only for the insight it provided but also because the principal alternative, the Life Cycle model of Modigliani (1966), was computationally challenging given the then-available technologies. Aside from its (considerable) conceptual value, there is no need for Blanchard's analytical solution today, when serious modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard's mortality is to follow Modigliani in assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2001, 2016)).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it again represents something of a polar alternative to Blanchard: Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean; the estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium. If $\aleph$ is the probability of remaining alive, the condition changes from the plain GIC to a looser mortality-adjusted GIC:

$$\aleph \text{Þ}_{\Gamma} < 1. \tag{63}$$

With no income growth, the condition required to prohibit unbounded growth in aggregate wealth would be the condition that prevents the per-capita wealth of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite horizon (usually, 'life cycle') versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to Friedman (1957) and the standard specification of preferences – and without complications like liquidity constraints – have attractive analytical properties (like continuous differentiability of the consumption function, and analytical limiting MPC's as resources approach their minimum and maximum possible values), and that (more widely used) models with liquidity constraints can be viewed as a particular limiting case of this simpler model.

The main focus of the paper, though, is on the limiting solution of the finite horizon model as the horizon extends to infinity. The paper shows that the simple model has additional attractive properties: A ‘Finite Value of Autarky’ condition guarantees convergence of the consumption function, under the mild requirement of a ‘Weak Return Impatience Condition’ that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model’s relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite horizon versions of such models is in heterogeneous agent macroeconomics. The paper articulates intuitive ‘Growth Impatience Conditions’ under which populations of such agents, with Blancharidan (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for a number of results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

The paper’s results are all easily reproducible interactively on the web or on any standard computer system. Such reproducibility reflects the paper’s use of the open-source Econ-ARK toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$\Gamma/\mathsf{R} < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.	$\Gamma/\mathsf{R} < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\mathbf{Þ} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $\mathbf{c}_{t+1} < \mathbf{c}_t$	$\mathbf{Þ} < 1$ *If wealth is large enough,* the *expectation* of consumption next period will be smaller than this period's consumption: $\lim_{m_t \to \infty} \mathbb{E}_t[\mathbf{c}_{t+1}] < \mathbf{c}_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\mathbf{Þ}/\mathsf{R} < 1$ The growth factor for consumption $\mathbf{Þ}$ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite: $\mathrm{c}'(m) = 1 - \mathbf{Þ}/\mathsf{R} < 1$	$\wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $\mathrm{c}'(m) < 1 - \wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$
Growth Impatience Conditions	
GIC	GIC-Nrm
$\mathbf{Þ}/\Gamma < 1$ For an unconstrained PF consumer, the ratio of $\mathbf{c}$ to $\mathbf{p}$ will fall over time. For constrained, guarantees the constraint eventually binds. Guarantees $\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = \mathbf{Þ}_\Gamma$	$\mathbf{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality stronger than GIC Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \mathbf{Þ}_{\underline{\Gamma}}$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta\Gamma^{1-\rho} < 1$ equivalently $\mathbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate‡ Solution

$\mathrm{c}(m)$: Model Reference	Conditions	Comments or or Logic
$\bar{\mathrm{c}}(m)$: PF Unconstrained	RIC, FHWC°	RIC $\Rightarrow \lvert \mathrm{v}(m) \rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m) \rvert$
$\underline{\mathrm{c}}(m) = \underline{\kappa} m$: PF $h = 0$		PF model with no human wealth
Section 2.4.2		RIC prevents $\bar{\mathrm{c}}(m) = \underline{\mathrm{c}}(m) = 0$
Section 2.4.2		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
Eq (26)		PF-FVAC+FHWC $\Rightarrow$ RIC
Eq (27)		GIC+FHWC $\Rightarrow$ PF-FVAC
$\grave{\mathrm{c}}(m)$: PF Constrained	$\cancel{\text{GIC}}$, RIC	FHWC holds ($\Gamma < \text{Þ} < \mathsf{R} \Rightarrow \Gamma < \mathsf{R}$)
Section 2.4.3		$\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m > m_{\#} < 1$
		($\cancel{\text{RIC}}$ would yield $m_{\#} = 0$ so $\grave{\mathrm{c}}(m) = 0$)
Appendix A	GIC,RIC	$\lim_{m \to \infty} \grave{c}(m) = \bar{c}(m), \lim_{m \to \infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		kinks at pts where horizon to $b = 0$ changes*
Appendix A	GIC,$\cancel{\text{RIC}}$	$\lim_{m \to \infty} \grave{\boldsymbol{\kappa}}(m) = 0$
		kinks at pts where horizon to $b = 0$ changes*
$\mathring{\mathrm{c}}(m)$: Friedman/Muth	Section 3.1,	$\underline{\mathrm{c}}(m) < \mathring{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)$
	Section 3.2	$\underline{\mathrm{v}}(m) < \mathring{\mathrm{v}}(m) < \bar{\mathrm{v}}(m)$
Section 2.9	FVAC, WRIC	Sufficient for Contraction
Section 2.11.1		WRIC is weaker than RIC
Figure 3		FVAC is stronger than PF-FVAC
Section 2.11.3		$\cancel{\text{FHWC}}$+RIC $\Rightarrow$GIC, $\lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
Section 2.11.2		$\cancel{\text{RIC}}$ $\Rightarrow \cancel{\text{FHWC}}$, $\lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
Section 3.3		"Buffer Stock Saving" Conditions
Section 3.3.2		GIC $\Rightarrow \exists \quad 0 < \hat{m} < \infty$: Steady-State
Section 3.3.1		GIC-Nrm $\Rightarrow \exists \quad 0 < \check{m} < \infty$: Target

‡For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines a unique optimal value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < \mathrm{v}(m) < 0$. °RIC, FHWC are necessary as well as sufficient for the perfect foresight case. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc. **In the Friedman/Muth model, the RIC+FHWC are sufficient, but *not* necessary for nondegeneracy

Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{c}(m)$ that arise under various parametric conditions. Results are summarized in table 5.

A.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{GIC}}$, $1 < \text{Þ}/\Gamma$). Under $\cancel{\text{GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[42]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \\
1 &< \mathsf{R}\beta\Gamma^{-\rho} \\
\mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\Gamma).
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$
\grave{c}_{T-n}(m) = 0(b_t + h). \tag{64}
$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. $\cancel{\text{FHWC}}$ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{c}_{T-n}(m) = 0$ or $\grave{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[43]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying $\cancel{\text{GIC}}$ we must impose the RIC (and the FHWC can be shown to be a consequence of $\cancel{\text{GIC}}$ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer

[42] The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_\#) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_\# = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{c}(m) = \min[m, c_\# + (m - c_\#)\underline{\kappa}]$.

[43] The knife-edge case is where $\text{Þ} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\grave{c}(m) = \min[m, 1]$.

would choose $c = m$ from equation (23):

$$\begin{aligned} m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\ m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\ m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \end{aligned} \tag{65}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[44] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[45] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

A.2 If GIC Holds

Imposition of the GIC reverses the inequality in (64), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^{n}$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^{n}$ would behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b_{\#}^{0}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^{1}$ was on the 'cusp' of being constrained in period $t - 1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^{0} = 1 + b_{\#}^{0} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^{0} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\mathbf{Þ}_{\Gamma}$, so consumption n periods in the past must have been

$$c_{\#}^{n} = \mathbf{Þ}_{\Gamma}^{-n} c_t = \mathbf{Þ}_{\Gamma}^{-n}. \tag{66}$$

[44] Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[45] As an illustration, consider a consumer for whom $\mathbf{Þ} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}_{t-n}^{t} &= c_{t-n}(1+\text{Þ}/\mathsf{R}+\ldots+(\text{Þ}/\mathsf{R})^{n}) \\
&= c_{\#}^{n}(1+\text{Þ}_{\mathsf{R}}+\ldots+\text{Þ}_{\mathsf{R}}^{n}) \\
&= \text{Þ}_{\Gamma}^{-n}\left(\frac{1-\text{Þ}_{\mathsf{R}}^{n+1}}{1-\text{Þ}_{\mathsf{R}}}\right) \\
&= \left(\frac{\text{Þ}_{\Gamma}^{-n}-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1+\ldots+\mathcal{R}^{-n} \tag{67}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} = b_{\#}^{n}+h_{\#}^{n}$$

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n}=b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t=0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h_{\#}^{n}}. \tag{68}$$

Defining $m_{\#}^{n}=b_{\#}^{n}+1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m)=m$ for $m<1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m=m_{\#}^{n}-\epsilon$ is discretely higher than for a consumer with assets $m=m_{\#}^{n}+\epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (68) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{69}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1}<1$, in which case the second term in (68) limits to a constant as $n\uparrow\infty$, and (69) reduces to a requirement that

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}-(\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) = \infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}-\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) = \infty$$
$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) = \infty.$$

Given the GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{70}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (69) requires

$$\lim_{n\to\infty}\left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}-\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)+\left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) = \infty$$
$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1}-\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty$$

If RIC Holds. When the RIC holds, rearranging (71) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned}\text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1}\\ \Gamma/\text{Þ} &> \Gamma/\mathsf{R}\\ 1 &> \text{Þ}/\mathsf{R}\end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n} = \lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{71}$$

which with a bit of algebra[46] can be shown to asymptote to the MPC in the perfect

[46] Calculate the limit of

$$\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\Gamma}^{-n}/(1-\text{Þ}_{\mathsf{R}})-(1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right) = \left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}})+\mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{72}$$

foresight model:[47]

$$\lim_{m \to \infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \text{\bf Þ}_{\mathsf{R}}. \tag{73}$$

If RIC Fails. Consider now the $\cancel{\text{RIC}}$ case, $\text{\bf Þ}_{\mathsf{R}} > 1$. We can rearrange (71)as

$$\lim_{n \to \infty} \left(\frac{\text{\bf Þ}_{\mathsf{R}}(\mathcal{R}^{-1} - 1)}{(\mathcal{R}^{-1} - 1)(\text{\bf Þ}_{\mathsf{R}} - 1)} - \frac{\mathcal{R}^{-1}(\text{\bf Þ}_{\mathsf{R}} - 1)}{(\mathcal{R}^{-1} - 1)(\text{\bf Þ}_{\mathsf{R}} - 1)} \right) \mathcal{R}^{-n} - \left(\frac{\text{\bf Þ}_{\Gamma}^{-n}}{\text{\bf Þ}_{\mathsf{R}} - 1} \right) = \infty. \tag{74}$$

which makes clear that with $\cancel{\text{FHWC}} \Rightarrow \mathcal{R}^{-1} > 1$ and $\cancel{\text{RIC}} \Rightarrow \text{\bf Þ}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (71) will be positive if

$$\begin{aligned} \text{\bf Þ}_{\mathsf{R}}\mathcal{R}^{-1} - \text{\bf Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{\bf Þ}_{\mathsf{R}} - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \text{\bf Þ}_{\mathsf{R}} \\ \Gamma &> \text{\bf Þ} \end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{\bf Þ}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \text{\bf Þ}_{\Gamma}^{-1} \\ \Gamma/\mathsf{R} &> \Gamma/\text{\bf Þ} \\ \text{\bf Þ}/\mathsf{R} &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\Gamma/\text{\bf Þ}$ while the $b_{\#}$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/\mathsf{R} > \Gamma/\text{\bf Þ} \tag{75}$$

because $\cancel{\text{RIC}} \Rightarrow \text{\bf Þ} > \mathsf{R}$.

Consequently, while $\lim_{n \uparrow \infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (71) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that $\cancel{\text{RIC}}$ implies a limiting MPC of zero,

$$\lim_{m \to \infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{76}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 8 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m - 1$; the part of the consumption function below the depicted points is uninteresting – $c = m$ – so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into

[47]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

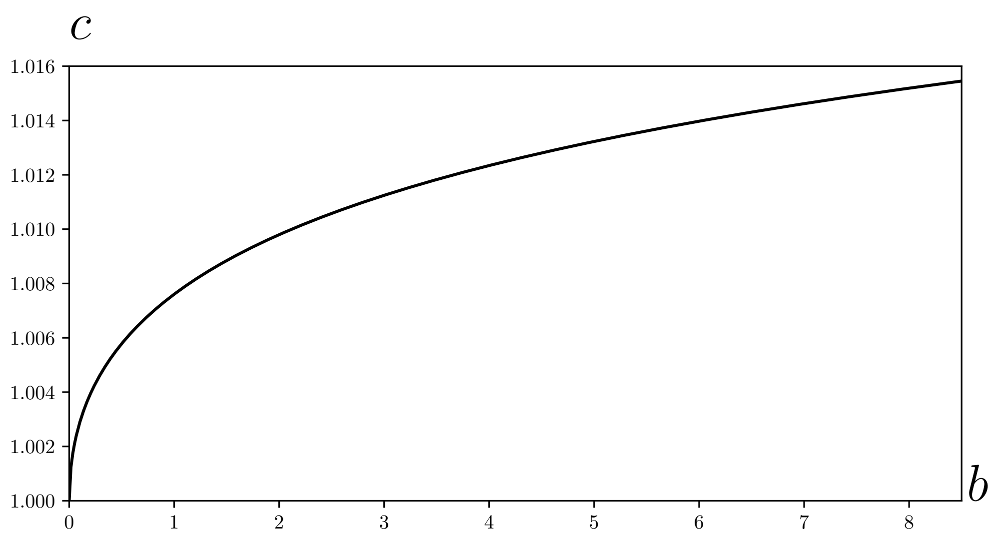


Figure 8 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

B Existence of a Concave Consumption Function

To show that (7) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ..., \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$

5. $\mathrm{n}(z) < 0$
6. $\lim_{z\downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})\right]. \tag{77}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0}\mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0}\mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m,c)$ as

$$\underline{\mathrm{v}}_t(m,c) = \mathrm{u}(c) + \mathfrak{v}_t(m-c) \tag{78}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (7) can be written as

$$\mathrm{v}_t(m) = \max_c\ \underline{\mathrm{v}}_t(m,c). \tag{79}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m,c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m,c) = -\infty$, $\frac{\partial^2\underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \arg\max_{0<c<m}\ \underline{\mathrm{v}}_t(m,c) \tag{80}$$

exists and is unique, and (7) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{81}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (81) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{82}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}=\left(\frac{\mathrm{u}'\left(\mathrm{c}_t(y)\right)-\mathrm{u}'\left(\mathrm{c}_t(m)\right)}{\mathrm{c}_t(y)-\mathrm{c}_t(m)}+\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm\to+0}\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)}<0$ and $\lim_{dm\to+0}\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}<0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)}+\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}<0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}=\frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)}+\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m)=\frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m))+\mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m)=\mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m))+\mathfrak{v}_t''\left(\mathrm{a}_t(m)\right)<0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m)=\frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t)+\mathfrak{v}_t''(a_t)\right]-\mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t)+a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t)+\mathfrak{v}_t''(a_t)\right]^2}. \tag{83}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

D Proof that $\mathcal{T}$ Is a Contraction Mapping

We must show that our operator $\mathcal{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A},\mathcal{B})$ to $\mathcal{C}(\mathcal{A},\mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathcal{T}\mathrm{z}\}\in\mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\begin{aligned}\{\mathcal{T}\mathrm{x}\}(m_t)\text{ is }&\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\}\\ \{\mathcal{T}\mathrm{y}\}(m_t)\text{ is }&\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\},\end{aligned}$$

so $\mathrm{x}(\bullet)\le\mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t)\le\{\mathcal{T}\mathrm{y}\}(m_t)$ by inspection.[48]

Condition (2) requires that $\{\mathcal{T}\mathbf{0}\}\in\mathcal{C}_F\left(\mathcal{A},\mathcal{B}\right)$. By definition,

$$\{\mathcal{T}\mathbf{0}\}(m_t)=\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right)+\beta 0\right\}$$

[48] For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is $\digamma$-bounded. We use the bounding function

$$\digamma(m) = \eta + m^{1-\rho}, \tag{84}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of $\digamma$, $\{\mathfrak{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly $\digamma$-bounded.

Finally, we turn to condition (3), $\{\mathfrak{T}(\mathrm{z}+\zeta\digamma)\}(m_t) \leq \{\mathfrak{T}\mathrm{z}\}(m_t) + \zeta\alpha\digamma(m_t)$. The proof will be more compact if we define $\check{\mathrm{c}}$ and $\check{\mathrm{a}}$ as the consumption and assets functions[49] associated with $\mathfrak{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathfrak{T}(\mathrm{z}+\zeta\digamma)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}) + \zeta\alpha\digamma.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\zeta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \zeta\alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &< \digamma.
\end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.[50]

Using $\digamma(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (25), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1-\beth}. \tag{85}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

[49]Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[50]The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (85) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (85) will certainly hold for any:

$$\eta>\underline{\eta}=\frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{86}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (43) and (40) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (44)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (7) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (86). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho}&<1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1})&>1\\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}})^{-1})&>1
\end{aligned}$$

where we have used (42) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho>1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}})^{-1}\approx 1-\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}$ so that it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}>1$$

$$
\begin{aligned}
(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1\\
\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.
\end{aligned}
$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (7) converge, they converge to the same unique v defined by $\mathfrak{T}$.[51]

E Convergence in Euclidian Space

E.1 Convergence of v_t

Boyd's theorem shows that $\mathfrak{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathfrak{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathfrak{T}$, since $\mathrm{v}^*(m) = \mathfrak{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_{\digamma} \leq \alpha^{n-1} \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} . \tag{87}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1} |\digamma(m)| . \tag{88}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{89}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

E.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

[51] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m))+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)})+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \tag{90}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{91}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

F Equality of Aggregate Consumption Growth and Income Growth with Transitory Shocks

Section 4.2 asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\check{m}$ is the invariant mean level of m we can define a 'mean MPS away from $\check{m}$' function :

$$\bar{\acute{\mathrm{a}}}(\Delta) = \Delta^{-1}\int_{\check{m}}^{\check{m}+\Delta} \mathrm{a}'(z)dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$, if we define a as the value of a corresponding to $m = \check{m}$, we can write

$$a_{t+1,i} = a + (m_{t+1,i} - \check{m})\bar{\acute{\mathrm{a}}}(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \check{m})$$

so

$$\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \mathrm{cov}_t\left(\bar{\acute{\mathrm{a}}}(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \check{m}), \Gamma\mathbf{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho} < \bar{\acute{\mathrm{a}}}(m) < \text{\textbf{Þ}}_{\mathsf{R}}$,

$$|\mathrm{cov}_t((\wp\mathsf{R}\beta)^{1/\rho}a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(\text{\textbf{Þ}}a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC-Nrm says that $\text{\textbf{Þ}} < \Gamma$, while the FHWC says that $\Gamma < \mathsf{R}$

$$|\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma|\mathrm{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathbb{M}\left[\mathcal{R}_{t+1,i}\right]$ and consider a first order Taylor expansion of $\bar{\mathrm{a}}(m_{t+1,i})$ around $\breve{m}_{t+1,i} = \breve{\mathcal{R}}a_{t,i} + 1$,

$$\bar{\mathrm{a}}_{t+1,i} \approx \bar{\mathrm{a}}(\breve{m}_{t+1,i}) + \bar{\mathrm{a}}'(\breve{m}_{t+1,i})\left(m_{t+1,i} - \breve{m}_{t+1,i}\right).$$

The problem comes from the $\bar{\mathrm{a}}'$ term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\mathrm{a}}$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\mathrm{a}}'$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

G The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (8) can be rewritten

$$\begin{aligned}
\mathrm{e}_t(m_t)^{-\rho} &= \beta\mathsf{R}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t}\right)\right)^{-\rho}\right] \\
&= \quad (1-\wp)\beta\mathsf{R}m_t^{\rho}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0\right] \\
&\quad + \wp\beta\mathsf{R}^{1-\rho}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t - \mathrm{c}_t(m_t)}{m_t}\right)^{-\rho} \mid \xi_{t+1} = 0\right].
\end{aligned}$$

Consider the first conditional expectation in (8), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m\downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$
\begin{aligned}
(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}.
\end{aligned}
$$

As noted in the main text, we need the WRIC (43) for this to be a convergent sequence:

$$
0 \le \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{92}
$$

Since $\bar{\kappa}_T = 1$, iterating (92) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$
\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{93}
$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (92) can be neglected, leading to a revised limiting Euler equation

$$
(m_t \mathrm{e}_t(m_t))^{-\rho} \quad = \beta\mathsf{R}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\left(\mathsf{R}\mathrm{a}_t(m_t)\right)\right)^{-\rho}\right]
$$

and we know from L'Hôpital's rule that $\lim_{m_t\to\infty}\mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty}\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$
\begin{aligned}
(m_t\underline{\kappa}_t)^{-\rho} &= \beta\mathsf{R}\left(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\right)^{-\rho} \\
\underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\text{Þ}_{\mathsf{R}}=(1-\underline{\kappa})}\ \underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1}
\end{aligned}
$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \le \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$
\underline{\kappa}_t^{-1} = 1+\underline{\kappa}_{t+1}^{-1}\text{Þ}_{\mathsf{R}} \tag{94}
$$

so that $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$
\underline{\kappa}^{-1} \equiv \lim_{n\uparrow\infty} \kappa_{T-n}^{-1} \tag{95}
$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty}\underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$
c_t\underbrace{\left(1+\text{Þ}_{\mathsf{R}}+\text{Þ}_{\mathsf{R}}^2+\ldots\right)}_{=1+\text{Þ}_{\mathsf{R}}(1+\text{Þ}_{\mathsf{R}}\underline{\kappa}_{t+2}^{-1})\ldots} \quad = c_t\underline{\kappa}_{T-n}^{-1}.
$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{96}$$

H The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m; \wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m; \wp) = \grave{\mathrm{c}}_t(m). \tag{97}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{98}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (98) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{99}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (99), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\theta}}^{\bar{\theta}} \left(\mathrm{c}_T(a+\theta/(1-\wp))\right)^{-\rho} d\mathcal{F}_\theta\right], \tag{100}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{101}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp \downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (99) and (101) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (100) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (97). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (93) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

I Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (Carroll (2006)): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[52]

$$\begin{aligned} \mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))] \\ \vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}. \end{aligned}$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (102) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\begin{aligned} \mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}_{t+1}^m\mathcal{R}_{t+1}] \\ &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m] \\ \mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m]/\mathrm{u}''(\mathfrak{c}_t) \end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$\begin{aligned} c &= \mathfrak{m} - a \\ \mathfrak{c}^a + 1 &= \mathfrak{m}^a \end{aligned}$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$\begin{aligned} \mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\ \mathrm{c}^m &= \mathfrak{c}^a/(1 + \mathfrak{c}^a) \end{aligned}$$

[52] The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

J The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, ...\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[53] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_{\#}, m^2_{\#}, ...\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathring{\mathrm{c}}(m)$ for integer values of n (e.g., $c^n_{\#} = \mathbf{Þ}_{\Gamma}^{-n}$) are continuous functions of n; the conclusion that $\mathring{\mathrm{c}}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\breve{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[54]

This strategy generates a smooth limiting consumption function – except at the remaining kink point defined by $\{m^0_{\#}, c^0_{\#}\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_{\#}$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_{\#}} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_{\#}} \mathrm{c}'(m) = \kappa^0_{\#} < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\breve{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\breve{\mathrm{c}}(m)$ between $m^0_{\#}$ and $m^{\underline{n}}_{\#}$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

---

[53]Unless $\beta \approx +0$.

[54]In practice, we calculate the first and second derivatives of $\mathring{\mathrm{c}}$ and use piecewise polynomial approximation methods that match the function at these points.

1. $\hat{\mathrm{c}}(m^0_\#) = c^0_\#$
2. $\hat{\mathrm{c}}'(m^0_\#) = 1$
3. $\hat{\mathrm{c}}'(m^{\underline{n}}_\#) = (dc^n_\#/dn)(\mathrm{dm}^n_\#/dn)^{-1}|_{n=\underline{n}}$
4. $\hat{\mathrm{c}}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\#/dn^2)(d^2\mathrm{m}^n_\#/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\breve{\mathrm{c}}(m)$ and fidelity of that function to the $\mathring{\mathrm{c}}(m)$ (see the actual code for details).

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m^0_\# & m \\ m^0_\# < m < m^{\underline{n}}_\# & \breve{\mathrm{c}}(m) \\ m^{\underline{n}}_\# < m & \mathring{\mathrm{c}}(m) \end{cases} \tag{102}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\breve{\mathrm{c}}(m) \geq \mathring{\mathrm{c}}(m)$, implicitly defining $m = e^\mu$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^\mu)/\mathrm{c}_T(e^\mu)) \tag{103}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \breve{\mathrm{c}}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi'_t(\mu) = \left(\frac{-\left(\frac{\mathrm{c}'_t\mathrm{c}_T - \mathrm{c}_t\mathrm{c}'_T}{\mathrm{c}^2_T} e^\mu\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{104}$$

which can be solved for

$$\mathrm{c}'_t = (\mathrm{c}_t\mathrm{c}'_T/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi'_t. \tag{105}$$

Similarly, we can solve (103) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right)\mathrm{c}_T(m). \tag{106}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

K Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures 1,3).

$$
\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \text{\textbf{Þ}} \arrow[rr, "\text{GIC}(\equiv \text{\textbf{Þ}} < \Gamma)"] \arrow[dr, "(\text{\textbf{Þ}} < \mathsf{R} \equiv)\ \text{RIC}"'] & \arrow[d, phantom, "\neq"] & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}
$$

Figure 9 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

K.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 9, whose three nodes represent values of the absolute patience factor **Þ**, the permanent-income growth factor Γ, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from **Þ** to Γ, reflects imposition of the PF-GICNrm condition (clicking PF-GICNrm should take you to its definition; definitions of other conditions are also linked below).[55] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the **Þ** node and impose the PF-GICNrm and then the FHWC, and see that imposition of these conditions allows us to conclude that **Þ** $< \mathsf{R}$.

One could also impose **Þ** $< \mathsf{R}$ directly (without imposing PF-GICNrm and FHWC) by following the downward-sloping diagonal arrow exiting **Þ**. Although alternate routes from one node to another all justify the same core conclusion (**Þ** $< \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[56] to indicate that the diagram is not commutative.[57]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, ~~RIC~~ $\equiv$ **Þ** $> \mathsf{R}$, would be represented by moving the

[55] For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[56] For a popular introduction to category theory, see Riehl (2017).

[57] But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

arrowhead from the bottom right to the top left of the line segment connecting **Þ** and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and Γ, but the Γ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed PF-GICNrm (that is, if we imposed ~~PF-GICNrm~~), that would take us to the **Þ** node, and we could deduce $\mathsf{R} > \textbf{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the **Þ** node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~PF-GICNrm~~) requires us also to reverse the final condition, giving us ~~RIC~~.[58]

Under these conventions, Figure 1 in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

$$\begin{tikzcd}
\textbf{Þ} \arrow[rr, "\text{GIC}"] \arrow[dd, "\text{RIC}"'] \arrow[ddrr, "\text{PF-FVAC}"] \arrow[drr, phantom, "\neq"] \arrow[ddr, phantom, "\neq"] & & \Gamma \arrow[dd, "\text{FHWC}"] \\
& & \\
\mathsf{R} \arrow[rr, "\cancel{\text{FHWC}}"'] & & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 10 Relation of PF-GICNrm, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\textbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (25)

This diagram can be interpreted, for example, as saying that, starting at the **Þ** node, it is possible to derive the PF-FVAC[59] by imposing both the PF-GICNrm and the FHWC;

[58]The corresponding algebra is

$$\begin{aligned}
\cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\
\cancel{\text{PF-GICNrm}}: &\quad \Gamma < \textbf{Þ} \\
\Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \textbf{Þ},
\end{aligned}$$

.

[59]in the form $\textbf{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition of the FHWC (twice - reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\text{Þ} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FHWC}: \quad & \Gamma < \mathsf{R} \\ \cancel{\text{RIC}}: \quad & \mathsf{R} < \text{Þ} \\ & \Gamma < \text{Þ} \end{aligned} \tag{107}$$

which leads to the negation of both of the conditions leading into Þ. ~~PF-GICNrm~~ is obtained directly as the last line in (107) and ~~PF-FVAC~~ follows if we start by multipling the Return Patience Factor (RPF=Þ/R) by the FHWF(=Γ/R) raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$\begin{aligned} 1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \text{ from FHWC}} \\ 1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right) \\ \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma &< \text{Þ} \end{aligned}$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

L When Is Consumption Growth Declining in m?

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (61) in brackets can be manipulated to yield

$$\begin{aligned} \mathrm{c}_t \boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\ &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}. \end{aligned}$$

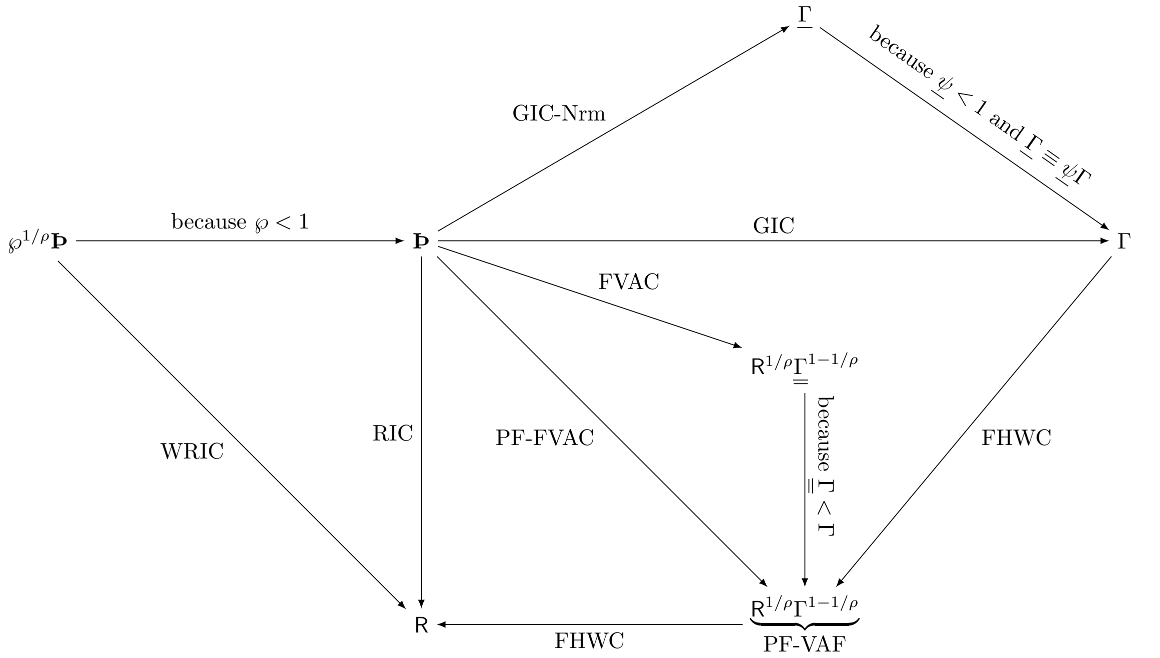


Figure 11 Relation of All Inequality Conditions

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho}] \\
0 &= \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1}\mathbf{\Upsilon}'_{t+1}] \\
&= \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1}]\,\mathbb{E}_t[\mathbf{\Upsilon}'_{t+1}] + \text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) \\
\mathbb{E}_t[\mathbf{\Upsilon}'_{t+1}] &= -\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) / \,\mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1}]
\end{aligned}$$

but since $\mathbf{\Upsilon}_{t+1} > 0$ we can see from (108) that (61) is equivalent to

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) > 0$$

which, using (108), will be true if

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\mathbf{\Upsilon}_{t+1}) > 0$$

which in turn will be true if both

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\text{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy

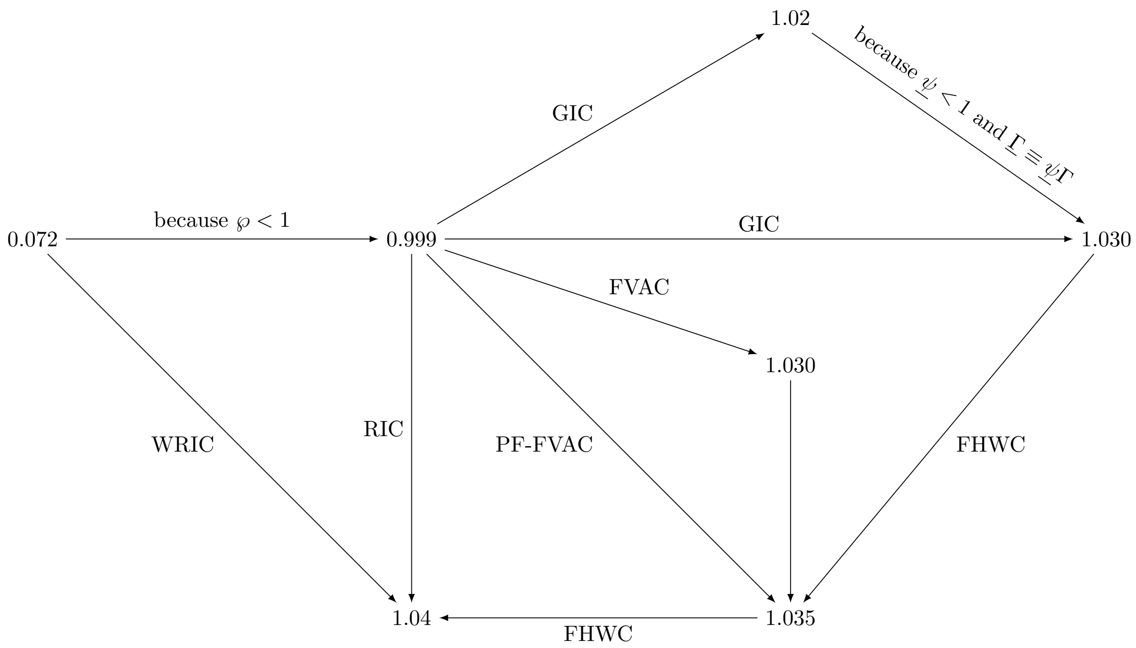


Figure 12 Numerical Relation of All Inequality Conditions

the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

M Unique And Stable Target and Steady State Points

Theorem 4. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (36) holds, there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{108}$$

Moreover, $\check{m}$ is a point of stablity in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{109}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$

- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

M.0.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that $\mathrm{c}(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With $m_t > 0$, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

M.0.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t} \right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\textbf{Þ}}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}] \\
&< 1
\end{aligned} \tag{110}$$

where the last two lines are merely a restatement of the GIC-Nrm (36).

The Intermediate Value Theorem says that if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.0.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{111}$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{112}$$

Note that the statement of theorem 4 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under $\cancel{\text{RIC}}$, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (118) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1 \tag{113}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (113).

Theorem 5. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (36) holds, there exists a unique cash-on-hand-to-income ratio $\hat{m} > 0$ such that*

$$\hat{m}_{t+1}/m_t = 1 \text{ if } m_t = \check{m}. \tag{114}$$

Moreover, $\hat{m}$ satisfies

$$\begin{aligned}
\forall m_t \in (0, \hat{m}),\ \hat{m}_{t+1} > m_t \\
\forall m_t \in (\hat{m}, \infty),\ \hat{m}_{t+1} < m_t.
\end{aligned} \tag{115}$$

The elements of the proof are:

- Existence and continuity of $\hat{m}_{t+1}/m_t$
- Existence of a point where $\hat{m}_{t+1}/m_t = 1$
- $\hat{m}_{t+1} - m_t$ is monotonically decreasing

M.0.4 Existence and Continuity of The Ratio

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1).

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With $m_t > 0$, the ratio $\hat{m}_{t+1}/m_t$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

M.0.5 Existence of a stable point

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}\psi_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}\psi_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \hat{m}_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\Gamma)\text{\TH}_{\mathsf{R}} \qquad (116) \\
&= \text{\TH}_{\Gamma} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if $\hat{m}_{t+1}/m_t$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.0.6 $\mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \hat{m}_{t+1}/m_t < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \hat{m}_{t+1}/m_t = 1 \qquad (117) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \hat{m}_{t+1}/m_t > 1,
\end{aligned}$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t\right)\right] \qquad (118) \\
&= (\mathsf{R}/\Gamma)\left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned}$$

Note that the statement of theorem 5 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails ($\cancel{\text{RIC}}$).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}\mathcal{R}\,(1-\mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1-\underbrace{(1-\text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} - 1\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\Gamma} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\,(1-\mathrm{c}'(m_t)) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (118) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1 \tag{119}$$

But we showed in section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (119) holds).

Figure 4 plots $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ rather than $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$, and due to the model's nonlinearities the values of m at which the expected growth rate of $\mathbf{c}$ matches Γ is very slightly different from the m at which the growth rate at which expected growth of $\mathbf{m}$ is Γ. But to first order they are the same:

$$\begin{aligned}\mathbb{E}_t[c_{t+1}\psi_{t+1}] &= c_t. \\ \mathbb{E}_t[\mathrm{c}(m_{t+1})\psi_{t+1}] &= c_t \\ \mathbb{E}_t[(\mathrm{c}(m_t)+\mathrm{c}'(m_t)(m_{t+1}-m_t))\,\psi_{t+1}] &= \mathrm{c}(m_t) \\ \mathbb{E}_t[(\mathrm{c}'(m_t)(m_{t+1}-m_t))\,\psi_{t+1}] &= 0 \\ \mathbb{E}_t[m_{t+1}\psi_{t+1}] &= m_t\end{aligned}$$

Table 5 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~GIC~~	$1 < \text{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \Gamma$); $\grave{c}(m) = \bar{c}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	$\grave{c}(m)$ is degenerate: $\grave{c}(m) = 0$
GIC	$\text{Þ}/\Gamma < 1$	Constraint binds in finite time for any m
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC may or may not hold $\lim_{m\uparrow\infty} \bar{c}(m) - \grave{c}(m) = 0$ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	~~FHWC~~ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GIC holds, the constraint will bind in finite time.

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