

1 Unique And Stable Target and Steady State Points

The two theorems and lemma to be proven in this appendix are:

Theorem 2. *For the nondegenerate solution to the problem defined in section 1.1 when FVAC, WRIC, and GIC-Nrm all hold, there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \quad (1)$$

Moreover, \check{m} is a point of ‘wealth stability’ in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}), \quad \mathbb{E}_t[m_{t+1}] &> m_t \\ \forall m_t \in (\check{m}, \infty), \quad \mathbb{E}_t[m_{t+1}] &< m_t. \end{aligned} \quad (2)$$

Theorem 3. *For the nondegenerate solution to the problem defined in section 1.1 when FVAC, WRIC, and GIC all hold, there exists a unique ‘m-balanced-growth’ cash-on-hand-to-income ratio $\hat{m} > 0$ such that*

$$\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \quad (3)$$

Moreover, \hat{m} is a point of ‘ \mathbf{m} -growth stability’ in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &> \Gamma \\ \forall m_t \in (\hat{m}, \infty), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &< \Gamma. \end{aligned} \quad (4)$$

Lemma 1. *If both \check{m} and \hat{m} exist, then $\hat{m} < \check{m}$.*

1.1 Proof of Theorem 2

The elements of the proof of theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.1.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that $c(m)$ is not just continuous, but twice continuously differentiable.)

Section 1.7 shows that for all t , $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and \mathcal{R}_t are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.1.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$.

If RIC holds. Logic exactly parallel to that of section 2.1 leading to equation (49), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(\mathcal{R}/\Gamma_{t+1})\mathbf{P}_R] \\ &= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \tag{5}$$

where the inequality reflects imposition of the GIC-Nrm (36).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} c'(m) = 0$ (see equation (40)) means that the limit of the RHS of (5) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$.

Paralleling the logic for c in section 2.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.1.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned} \zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1, \end{aligned} \tag{6}$$

so that $\zeta(\check{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \zeta'(m_t) &\equiv \left(\frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}} (1 - c'(m_t)) - 1. \end{aligned} \tag{7}$$

Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 1.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned} \bar{\mathcal{R}}(1 - c'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}}\mathbf{P}_R - 1 \\ &= \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma\psi} \frac{\mathbf{P}}{\mathbf{R}} \right] - 1 \\ &= \underbrace{\mathbb{E}_t \left[\frac{\mathbf{P}}{\Gamma\psi} \right]}_{=\mathbf{P}_\Gamma} - 1 \end{aligned}$$

which is negative because the GIC-Nrm says $\mathbf{P}_\Gamma < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}(1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that $\zeta'(m_t)$ from (11) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma\psi} \right] < 1. \quad (8)$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\underbrace{\mathbb{E}_t \left[\frac{\mathbf{P}}{\Gamma\psi} \right]}_{\mathbf{P}_\Gamma} < 1 < \underbrace{\frac{\mathbf{P}_R}{\mathbf{R}}}_{\mathbf{P}_\Gamma},$$

and multiplying all three elements by \mathbf{R}/\mathbf{P} gives

$$\mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma\psi} \right] < \mathbf{R}/\mathbf{P} < 1$$

which satisfies our requirement in (8).

1.2 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

1.2.1 Existence and Continuity of The Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in 1.1.1 that demonstrated existence and continuity of $\mathbb{E}_t[\bar{m}_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

1.2.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in subsection 1.1.1 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\Gamma_{t+1} ((R/\Gamma_{t+1})a(m_t) + \xi_{t+1}) / \Gamma}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(R/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(R/\Gamma)a(m_t) + 1}{m_t} \right] \\
&= (R/\Gamma)\mathbf{P}_R \\
&= \mathbf{P}_\Gamma \\
&< 1
\end{aligned} \tag{9}$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.2.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\zeta(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{10}$$

so that $\zeta(\check{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\zeta'(m_t) &\equiv \left(\frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\
&= (R/\Gamma) (1 - c'(m_t)) - 1.
\end{aligned} \tag{11}$$

Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (\mathbf{RIC}).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 1.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned} \mathcal{R}(1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= (\mathbf{R}/\Gamma)\mathbf{P}_R - 1 \end{aligned}$$

which is negative because the GIC says $\mathbf{P}_R < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}(1 - c'(m_t)) < \mathcal{R}$$

which means that $\zeta'(m_t)$ from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathbf{R}/\Gamma) < 1. \tag{12}$$

But we showed in section 1.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHC also fails (that is, (12) holds).

1.3 Proof of Lemma