

# 1 Unique And Stable Target and Steady State Points

**Theorem 1.** *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (36) holds, there exists a unique cash-on-hand-to-income ratio  $\check{m} > 0$  such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \quad (1)$$

*Moreover,  $\check{m}$  is a point of stability in the sense that*

$$\begin{aligned} \forall m_t \in (0, \check{m}), \quad \mathbb{E}_t[m_{t+1}] &> m_t \\ \forall m_t \in (\check{m}, \infty), \quad \mathbb{E}_t[m_{t+1}] &< m_t. \end{aligned} \quad (2)$$

The elements of the proof are:

- Existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$  is monotonically decreasing

## 1.0.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ .

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that  $c(m)$  is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all  $t$ ,  $a_{t-1} = m_{t-1} - c_{t-1} > 0$ . Since  $m_t = a_{t-1}\mathcal{R}_t + \xi_t$ , even if  $\xi_t$  takes on its minimum value of 0,  $a_{t-1}\mathcal{R}_t > 0$ , since both  $a_{t-1}$  and  $\mathcal{R}_t$  are strictly positive. With  $m_t > 0$ , the ratio  $\mathbb{E}_t[m_{t+1}/m_t]$  inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

## 1.0.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ .

Paralleling the logic for  $c$  in section 3.2: the ratio of  $\mathbb{E}_t[m_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$  because  $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$ .

The limit as  $m_t$  goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\mathcal{R}_{t+1}a(m_t) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(\mathcal{R}/\Gamma_{t+1})\mathbf{P}_R] \\ &= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \quad (3)$$

where the last two lines are merely a restatement of the GIC-Nrm (36).

The Intermediate Value Theorem says that if  $\mathbb{E}_t[m_{t+1}/m_t]$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.3  $\mathbb{E}_t[m_{t+1}] - m_t$  is monotonically decreasing.

Now define  $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$  and note that

$$\begin{aligned}\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,\end{aligned}\tag{4}$$

so that  $\zeta(\check{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned}\zeta'(m_t) &\equiv \left(\frac{d}{dm_t}\right) \zeta(m_t) = \mathbb{E}_t \left[ \left(\frac{d}{dm_t}\right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}}(1 - c'(m_t)) - 1.\end{aligned}\tag{5}$$

Note that the statement of theorem 1 did not require the RIC to hold. Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the RIC holds or fails ( $\mathbf{RIC}^*$ ).

**If RIC holds.** Equation (22) indicates that if the RIC holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.8.1 that if the RIC holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\begin{aligned}\bar{\mathcal{R}}(1 - c'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \mathbf{p}_R)}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}}\mathbf{p}_R - 1 \\ &= \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Gamma\psi} \frac{\mathbf{p}}{\mathbf{R}} \right] - 1 \\ &= \underbrace{\mathbb{E}_t \left[ \frac{\mathbf{p}}{\Gamma\psi} \right]}_{=\mathbf{p}_\Gamma} - 1\end{aligned}$$

which is negative because the GIC-Nrm says  $\mathbf{p}_\Gamma < 1$ .

**If RIC fails.** Under  $\mathbf{RIC}$ , recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\bar{\mathcal{R}}(1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that  $\zeta'(m_t)$  from (11) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Gamma\psi} \right] < 1\tag{6}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\underbrace{\mathbb{E}_t \left[ \frac{\mathbf{p}}{\Gamma\psi} \right]}_{\mathbf{p}_\Gamma} < 1 < \underbrace{\frac{\mathbf{p}}{\mathbf{R}}}_{\mathbf{p}_R},$$

and multiplying all three elements by  $R/\mathbf{P}$  gives

$$\mathbb{E}_t \left[ \frac{R}{\Gamma\psi} \right] < R/\mathbf{P} < 1$$

which satisfies our requirement in (6).

**Theorem 2.** *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (36) holds, there exists a unique cash-on-hand-to-income ratio  $\hat{m} > 0$  such that*

$$\hat{m}_{t+1}/m_t = 1 \text{ if } m_t = \hat{m}. \quad (7)$$

Moreover,  $\hat{m}$  satisfies

$$\begin{aligned} \forall m_t \in (0, \hat{m}), \hat{m}_{t+1} &> m_t \\ \forall m_t \in (\hat{m}, \infty), \hat{m}_{t+1} &< m_t. \end{aligned} \quad (8)$$

The elements of the proof are:

- Existence and continuity of  $\hat{m}_{t+1}/m_t$
- Existence of a point where  $\hat{m}_{t+1}/m_t = 1$
- $\hat{m}_{t+1} - m_t$  is monotonically decreasing

#### 1.0.4 Existence and Continuity of The Ratio

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1).

Section 2.7 shows that for all  $t$ ,  $a_{t-1} = m_{t-1} - c_{t-1} > 0$ . Since  $m_t = a_{t-1}\mathcal{R}_t + \xi_t$ , even if  $\xi_t$  takes on its minimum value of 0,  $a_{t-1}\mathcal{R}_t > 0$ , since both  $a_{t-1}$  and  $\mathcal{R}_t$  are strictly positive. With  $m_t > 0$ , the ratio  $\hat{m}_{t+1}/m_t$  inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

#### 1.0.5 Existence of a stable point

Paralleling the logic for  $c$  in section 3.2: the ratio of  $\mathbb{E}_t[m_{t+1}\psi_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$  because  $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}\psi_{t+1}] > 0$ .

The limit as  $m_t$  goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \hat{m}_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\Gamma_{t+1} ((R/\Gamma_{t+1})a(m_t) + \xi_{t+1})/\Gamma}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{(R/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \left[ \frac{(R/\Gamma)a(m_t) + 1}{m_t} \right] \\ &= (R/\Gamma)\mathbf{P}_R \\ &= \mathbf{P}_\Gamma \end{aligned} \quad (9)$$

$$< 1$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if  $\hat{m}_{t+1}/m_t$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.6  $\mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$  is monotonically decreasing.

Now define  $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$  and note that

$$\begin{aligned}\zeta(m_t) &< 0 \Leftrightarrow \hat{m}_{t+1}/m_t < 1 \\ \zeta(m_t) &= 0 \Leftrightarrow \hat{m}_{t+1}/m_t = 1 \\ \zeta(m_t) &> 0 \Leftrightarrow \hat{m}_{t+1}/m_t > 1,\end{aligned}\tag{10}$$

so that  $\zeta(\tilde{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned}\zeta'(m_t) &\equiv \left(\frac{d}{dm_t}\right) \zeta(m_t) = \mathbb{E}_t \left[ \left(\frac{d}{dm_t}\right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\ &= (\mathcal{R}/\Gamma) (1 - c'(m_t)) - 1.\end{aligned}\tag{11}$$

Note that the statement of theorem 2 did not require the RIC to hold. Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the RIC holds or fails ( $\mathbf{RIC}^*$ ).

**If RIC holds.** Equation (22) indicates that if the RIC holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.8.1 that if the RIC holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\begin{aligned}\mathcal{R} (1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= (\mathcal{R}/\Gamma)\mathbf{P}_R - 1\end{aligned}$$

which is negative because the GIC says  $\mathbf{P}_\Gamma < 1$ .

**If RIC fails.** Under  $\mathbf{RIC}^*$ , recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\mathcal{R} (1 - c'(m_t)) < \mathcal{R}$$

which means that  $\zeta'(m_t)$  from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathcal{R}/\Gamma) < 1\tag{12}$$

But we showed in section ?? that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHCW also fails (that is, (12) holds).

Figure 4 plots  $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$  rather than  $\mathbb{E}_t[\mathbf{c}_{t+1}/c_t]$ , and due to the model's nonlinearities the values of  $m$  at which the expected growth rate of  $\mathbf{c}$  matches  $\Gamma$  is very slightly different from the  $m$  at which the growth rate at which expected growth of  $\mathbf{m}$  is  $\Gamma$ . But to first order they are the same:

$$\mathbb{E}_t[c_{t+1}\psi_{t+1}] = c_t.$$

$$\begin{aligned}
\mathbb{E}_t[c(m_{t+1})\psi_{t+1}] &= c_t \\
\mathbb{E}_t[(c(m_t) + c'(m_t)(m_{t+1} - m_t))\psi_{t+1}] &= c(m_t) \\
\mathbb{E}_t[(c'(m_t)(m_{t+1} - m_t))\psi_{t+1}] &= 0 \\
\mathbb{E}_t[m_{t+1}\psi_{t+1}] &= m_t
\end{aligned}$$