

1 Unique And Stable Target and Steady State Points

Theorem 1. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (36) holds, there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \quad (1)$$

Moreover, \check{m} is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}), \quad \mathbb{E}_t[m_{t+1}] &> m_t \\ \forall m_t \in (\check{m}, \infty), \quad \mathbb{E}_t[m_{t+1}] &< m_t. \end{aligned} \quad (2)$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.0.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that $c(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all t , $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and \mathcal{R}_t are strictly positive. With $m_t > 0$, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.0.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}a(m_t) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(\mathcal{R}/\Gamma_{t+1})\mathbf{P}_R] \\ &= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \quad (3)$$

where the last two lines are merely a restatement of the GIC-Nrm (36).

The Intermediate Value Theorem says that if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,\end{aligned}\tag{4}$$

so that $\zeta(\check{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\zeta'(m_t) &\equiv \left(\frac{d}{dm_t}\right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t}\right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}} (1 - c'(m_t)) - 1.\end{aligned}\tag{5}$$

Note that the statement of theorem 1 did not require the RIC to hold. Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (\mathbf{RIC}^*).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned}\bar{\mathcal{R}} (1 - c'(m_t)) - 1 &< \bar{\mathcal{R}} (1 - \underbrace{(1 - \mathbf{p}_R)}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}} \mathbf{p}_R - 1 \\ &= \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \frac{\mathbf{p}}{\mathbf{R}} \right] - 1 \\ &= \underbrace{\mathbb{E}_t \left[\frac{\mathbf{p}}{\Gamma \psi} \right]}_{=\mathbf{p}_\Gamma} - 1\end{aligned}$$

which is negative because the GIC-Nrm says $\mathbf{p}_\Gamma < 1$.

If RIC fails. Under \mathbf{RIC} , recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}} (1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that $\zeta'(m_t)$ from (11) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \right] < 1\tag{6}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\mathbf{p}}{\Gamma \psi} \right]}^{\mathbf{p}_\Gamma} < 1 < \overbrace{\frac{\mathbf{p}}{\mathbf{R}}}^{\mathbf{p}_R},$$

and multiplying all three elements by R/\mathbf{P} gives

$$\mathbb{E}_t \left[\frac{R}{\Gamma\psi} \right] < R/\mathbf{P} < 1$$

which satisfies our requirement in (6).

Theorem 2. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (36) holds, there exists a unique cash-on-hand-to-income ratio $\hat{m} > 0$ such that*

$$\hat{m}_{t+1}/m_t = 1 \text{ if } m_t = \hat{m}. \quad (7)$$

Moreover, \hat{m} satisfies

$$\begin{aligned} \forall m_t \in (0, \hat{m}), \hat{m}_{t+1} &> m_t \\ \forall m_t \in (\hat{m}, \infty), \hat{m}_{t+1} &< m_t. \end{aligned} \quad (8)$$

The elements of the proof are:

- Existence and continuity of \hat{m}_{t+1}/m_t
- Existence of a point where $\hat{m}_{t+1}/m_t = 1$
- $\hat{m}_{t+1} - m_t$ is monotonically decreasing

1.0.4 Existence and Continuity of The Ratio

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1).

Section 2.7 shows that for all t , $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and \mathcal{R}_t are strictly positive. With $m_t > 0$, the ratio \hat{m}_{t+1}/m_t inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.0.5 Existence of a stable point

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}\psi_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}\psi_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \hat{m}_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\Gamma_{t+1} ((R/\Gamma_{t+1})a(m_t) + \xi_{t+1})/\Gamma}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(R/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \left[\frac{(R/\Gamma)a(m_t) + 1}{m_t} \right] \\ &= (R/\Gamma)\mathbf{P}_R \\ &= \mathbf{P}_\Gamma \end{aligned} \quad (9)$$

$$< 1$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if \hat{m}_{t+1}/m_t is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.6 $\mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ is monotonically decreasing.

Now define $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ and note that

$$\begin{aligned}\zeta(m_t) &< 0 \Leftrightarrow \hat{m}_{t+1}/m_t < 1 \\ \zeta(m_t) &= 0 \Leftrightarrow \hat{m}_{t+1}/m_t = 1 \\ \zeta(m_t) &> 0 \Leftrightarrow \hat{m}_{t+1}/m_t > 1,\end{aligned}\tag{10}$$

so that $\zeta(\tilde{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\zeta'(m_t) &\equiv \left(\frac{d}{dm_t}\right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t}\right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\ &= (\mathcal{R}/\Gamma) (1 - c'(m_t)) - 1.\end{aligned}\tag{11}$$

Note that the statement of theorem 2 did not require the RIC to hold. Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (\mathbf{RIC}).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned}\mathcal{R} (1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= (\mathcal{R}/\Gamma)\mathbf{P}_R - 1\end{aligned}$$

which is negative because the GIC says $\mathbf{P}_R < 1$.

If RIC fails. Under \mathbf{RIC} , recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R} (1 - c'(m_t)) < \mathcal{R}$$

which means that $\zeta'(m_t)$ from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathcal{R}/\Gamma) < 1\tag{12}$$

But we showed in section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHC also fails (that is, (12) holds).

Figure 4 plots $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ rather than $\mathbb{E}_t[\mathbf{c}_{t+1}/c_t]$, and due to the model's nonlinearities the values of m at which the expected growth rate of \mathbf{c} matches Γ is very slightly different from the m at which the growth rate at which expected growth of \mathbf{m} is Γ . But to first order they are the same:

$$\mathbb{E}_t[c_{t+1}\psi_{t+1}] = c_t.$$

$$\begin{aligned}
\mathbb{E}_t[c(m_{t+1})\psi_{t+1}] &= c_t \\
\mathbb{E}_t[(c(m_t) + c'(m_t)(m_{t+1} - m_t))\psi_{t+1}] &= c(m_t) \\
\mathbb{E}_t[(c'(m_t)(m_{t+1} - m_t))\psi_{t+1}] &= 0 \\
\mathbb{E}_t[m_{t+1}\psi_{t+1}] &= m_t
\end{aligned}$$