Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative illustration. After describing conditions under which a consumption function exists, the paper shows that individual consumers subject to idiosyncratic shocks will engage in 'target' saving behavior when a particular 'growth impatience' condition is imposed. A similar (but looser) condition guarantees that a small open economy populated by such agents will exhibit a balanced-growth 'steady state' equilibrium. Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem

JEL codes D81, D91, E21

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All figures and numerical results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper's own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

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1 Introduction

In the presence of empirically realistic transitory and permanent income shocks *a la* Friedman (1957),[1] only one more ingredient is required to construct a testable model of optimal consumption: A description of preferences. Zeldes (1989) was the first to construct a quantitatively realistic version of such a model, spawning a subsequent literature showing that such models' predictions match microeconomic evidence reasonably well, whether or not liquidity constraints are imposed.[2]

A companion theoretical literature has derived key analytical properties of infinite-horizon solutions, but only for models more complex than the case with just shocks and preferences. The extra complexity has been required, in part, because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et. al. (1989)) cannot be applied when utility or marginal utility are unbounded. Many proof methods also rule out permanent shocks *a la* Friedman (1957), Muth (1960), and Zeldes (1989).[3]

This paper's first technical contribution is to articulate conditions under which the simple problem (without complications like a consumption floor or liquidity constraints) defines a contraction mapping whose limiting value and consumption functions are nondegenerate as as the horizon approaches infinity. A 'Finite Value of Autarky Condition' turns out to be sufficient (along with a 'Weak Return Impatience Condition' that is unlikely ever to bind). Conveniently, the resulting model has analytical properties, like continuous differentiability of the consumption function, that make it easier to analyze than the more complicated models. The paper's other main theoretical contribution is to identify conditions under which 'stable' values of the wealth-to-permanent-income ratio exist (the consumer, or the economy populated by such consumers, exhibits 'buffer stock' saving behavior). 'Target saving' arises when the model's parameters satisfy a "Normalized Growth Impatience Condition" that relates preferences and uncertainty to predictable income growth. A nonnormalized (therefore looser) "Growth Impatience Condition" turns out to guarantee existence of an 'individual steady-state' point at which the consumer expects 'balanced growth' (equal rates of growth in permanent labor income and market wealth).

Even without a formal proof of its existence, target saving has been intuitively understood to underlie central quantitative results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the recent claim by Krueger, Mitman, and Perri (2016) in the *Handbook of Macroeconomics* that such models explain why, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theory below provides the rigorous theoretical basis for this claim: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$ means they – optimally – have little room to maneuver). And, increased labor income uncertainty does not change the behavior

[1] As formalized by Muth (1960).

[2] See Carroll (1997) or Gourinchas and Parker (2002) for arguments that such models match a wide variety of facts; for a model with constraints that produces very similar results, see, e.g., Cagetti (2003).

[3] See the fuller discussion at the end of section 2.1.

of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending when uncertainty increases.

Analytical derivations required for the proofs also provide foundations for many other results familiar from the numerical literature. The main insights of the paper are instantiated in the Econ-ARK toolkit, whose buffer stock saving module flags parametric choices under which a problem is degenerate or a target level of wealth may not exist.

The paper proceeds in three parts.

The first part articulates sufficient conditions for the problem to define a nondegenerate limiting consumption function, and explains how the model relates to those previously considered in the literature. The conditions required for convergence are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; then the contraction mapping theorem is proven. The last result here is a proof that a corresponding model with an 'artificial' liquidity constraint (that is, a model that exogenously prohibits consumers from borrowing even if they could repay) is a particular limiting case of the model without constraints.

In the process of proving the remaining theorems, the next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Next, the central theorems articulate conditions under which different measures of 'growth impatience' imply useful conclusions about points of stability ('individual target' or 'individual steady-state' points). Fourth, at the 'individual target' ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent (noncapital) income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last holds if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

The final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, a small open economy populated by buffer stock consumers converges to a balanced growth equilibrium in which consumption, income, and wealth eventually match the exogenous growth rate of permanent income (equivalent, here, to productivity growth). In the terms of Schmitt-Grohé and Uribe (2003), buffer stock saving is a method of 'closing' a small open economy model, one that is attractive because it requires no ad-hoc assumptions. Not even liquidity constraints.

2 The Problem

2.1 Setup

The infinite horizon solution is the limiting first-period solution to a sequence of finite-horizon problems as the horizon (the last period of life) becomes arbitrarily distant.

That is, for the value function, fixing a terminal date T, we are interested in the final term $\mathbf{v}_{T-n}$ in the sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, ..., \mathbf{v}_{T-n}\}$. We will say that the problem has a 'nondegenerate' infinite horizon solution if, corresponding to that value function, there is a limiting consumption function $\mathring{c}(m) = \lim_{n\uparrow\infty} c_{T-n}$ which is neither zero everywhere nor infinity everywhere (this is fleshed out below).

Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max\ \mathbb{E}_t\left[\sum_{i=0}^{n} \beta^i \mathrm{u}(\mathbf{c}_{t+i})\right]$$

where the utility function

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{6}$$

exhibits relative risk aversion $\rho > 1$.[4] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ and permanent noncapital income $\mathbf{p}_t$, which both start out strictly positive,

$$\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty), \tag{7}$$

and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{8}$$

In the usual treatment, a dynamic budget constraint (DBC) incorporates several elements that jointly determine next period's $\mathbf{m}$ (given this period's choices); for the detailed analysis here, it will be useful to disarticulate the steps:

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R} \\ \mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1}, \end{aligned} \tag{9}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1 + \mathsf{r})$ between periods, so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[5] $\mathbf{m}_{t+1}$ ('market resources') is the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital

[4]The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[5]Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1\ \forall\ n \geq 1$). Permanent noncapital income in $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1\ \forall\ n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).[6]

Following Zeldes (1989), in future periods $t+n\ \forall\ n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{10}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1\ \forall\ n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.[7] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (10) and $\mathcal{F}_\theta$). For quick identification in tables and graphs, we will call this the Friedman/Muth model because it is a specific implementation of the Friedman (1957) model as interpreted by Muth (1960), needing only a calibration of the income process and a specification of preferences (here, geometric discounting and CRRA utility) to be solvable.

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable (though it has empirical support).[8] However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,[9] transforming that model back into this one. And no key results would change if the transitory shocks were persistent but mean-reverting, instead of IID. Also, the assumption of a positive point mass for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.[10]

This model differs from Bewley's (1977) classic formulation in several ways. The Constant Relative Risk Aversion (CRRA) utility function does not satisfy Bewley's assumption that $\mathrm{u}(0)$ is well defined, or that $\mathrm{u}'(0)$ is well defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds are quantitatively important in micro data[11] and which are far more consequential for household welfare than are transitory fluctuations. It differs from Deaton (1991) because

[6]Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

[7]See Rabault (2002) and Li and Stachurski (2014) for analyses of cases where the shock processes have unbounded support.

[8]We will calibrate this probability to 0.005 percent to match data from the Panel Study of Income Dynamics (Carroll (1992)).

[9]So long as unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in section 2.11.

[10]A strictly positive density over a strictly positive interval above the lower bound would work just as well, but would be cumbersome.

[11]MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004); Blundell, Low, and Preston (2008)

liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income to reach zero.[12] It differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[13,14] Li and Stachurski (2014) show how to allow unbounded returns by using policy function iteration, but also impose constraints.

The paper with perhaps the most in common with this one is Ma, Stachurski, and Toda (2020), henceforth MST, who establish the existence and uniqueness of a solution to a general income fluctuation problem in a Markovian setting. The most important differences are that MST impose liquidity constraints, assume that $\mathrm{u}'(0) = 0$, and assume that expected marginal utility of income is finite ($\mathbb{E}[\mathrm{u}'(Y)] < \infty$). These assumptions are not consistent with the combination of CRRA utility and income dynamics used here, whose combined properties are key to the derivation of the results.[15]

2.2 The Problem Can Be Normalized By Permanent Income

We establish a bit more notation by reviewing the result that in such problems (CRRA utility, permanent shocks) the number of states can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Value in the last period of life is $\mathrm{u}(\mathbf{m}_T)$; using (in the last line in (11)) the fact that for our CRRA utility function, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, and generically defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with $m = \mathbf{m}/\mathbf{p}$), consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} \ \mathrm{u}(\mathbf{c}_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1}c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho}\left\{\max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)]\right\}.
\end{aligned} \tag{11}$$

Now, in a one-time deviation from the notational convention established in the last sentence, define nonbold 'normalized value' not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ \mathrm{u}(\mathrm{c}_t) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1})a_t \ = \ \mathcal{R}_{t+1}a_t
\end{aligned} \tag{12}$$

[12] Below it will become clear that the Deaton model is a particular limit of this paper's model.

[13] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (1986), Clarida (1987), and Chamberlain and Wilson (2000). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

[14] Alvarez and Stokey (1998) relaxed the bounds on the return function, but they address only the deterministic case.

[15] The incorporation of permanent shocks rules out application of the tools of Matkowski and Nowak (2011), who followed and corrected an error in the fundamental work on the local contraction mapping method developed in Rincón-Zapatero and Rodríguez-Palmero (2003). Martins-da Rocha and Vailakis (2010) provide a correction to Rincón-Zapatero and Rodríguez-Palmero (2003), that works under easier conditions to verify, but only addresses the deterministic case.

$$m_{t+1} = b_{t+1} + \xi_{t+1},$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the new problem's first order condition is[16]

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho}c_{t+1}^{-\rho}]. \tag{13}$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (12), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem (12), we will have solutions to the original problem for any $t < T$ from:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t),$$
$$\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t\mathrm{c}_t(m_t).$$

2.3 Definition of a Nondegenerate Solution

The problem has a nondegenerate solution if as the horizon n gets arbitrarily large the solution in the first period of life $\mathring{\mathrm{c}}_{T-n}(m)$ gets arbitrarily close to a limiting $\mathring{\mathrm{c}}(m)$:

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m) \tag{14}$$

that satisfies

$$0 < \mathrm{c}(m) < \infty \tag{15}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is $\mathring{\mathrm{c}}(m) = 0$ or $\mathring{\mathrm{c}}(m) = \infty$; below when we say a 'solution exists' we will always mean 'a nondegenerate solution.')

2.4 Perfect Foresight Benchmarks

The familiar analytical solution to the perfect foresight model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (8) imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{16}$$

[16]Leaving aside their assumptions about the marginal utility function and liquidity constraints, it is tempting to view this as a special case of the model of MST, with the $\mathcal{R}_{t+1} = \mathsf{R}/\Gamma_{t+1}$ (defined below equation (12)) corresponding to their stochastic rate of return on capital and the FVAF $\beta\Gamma_{t+1}^{1-\rho}$ defined below (45) corresponding to their stochastic discount factor. But a caveat is that, here, $\mathcal{R}_{t+1}$ and the modified discount factor are intimately related because Γ_{t+1} plays a role in each.

where $\mathbf{b}$ is nonhuman wealth and $\mathbf{h}_t$ is 'human wealth,' and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$,

$$
\begin{aligned}
\mathbf{h}_t &= \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + ... + \mathcal{R}^{t-T}\mathbf{p}_t \\
&= \underbrace{\left(\frac{1-\mathcal{R}^{-(T-t+1)}}{1-\mathcal{R}^{-1}}\right)}_{\equiv h_t}\mathbf{p}_t
\end{aligned}
\tag{17}
$$

In order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC'):

$$
\underbrace{\Gamma/\mathsf{R}}_{\equiv \mathcal{R}^{-1}} < 1. \tag{18}
$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 PF Unconstrained Solution Exists Under RIC and FHWC

Without constraints, the consumption Euler equation always holds; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$
\mathbf{c}_{t+1}/\mathbf{c}_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{19}
$$

where the archaic letter 'thorn' represents what we will call the 'Absolute Patience Factor,' or APF:

$$
\text{Þ} = (\mathsf{R}\beta)^{1/\rho}. \tag{20}
$$

The sense in which Þ captures patience is that if the 'absolute impatience condition' (AIC) holds,[17]

$$
\text{Þ} < 1, \tag{21}
$$

the consumer will choose to spend an amount too large to sustain indefinitely. We call such a consumer 'absolutely impatient.'

We next define a 'Return Patience Factor' (RPF) that relates absolute patience to the return factor:

$$
\text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{22}
$$

and since consumption is growing by Þ but discounted by R:

$$
\mathrm{PDV}_t(\mathbf{c}) = \left(\frac{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}{1-\text{Þ}_{\mathsf{R}}}\right)\mathbf{c}_t \tag{23}
$$

[17] Impatience conditions have figured in intertemporal optimization problems since the beginning, e.g. in Ramsey (1928). These issues are so central that it would be hopeless to attempt to cite conditions in every other paper that correspond to conditions named and briefly exposited here. I make no claim to novelty for any condition or implication except for the conditions implicated in my theorems, whose parallels *will* be articulated.

from which the IBC (16) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t}(\mathbf{b}_t + \mathbf{h}_t) \tag{24}$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{T-n} \tag{25}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) – it answers the question 'if the consumer had an extra unit of resources, how much more would be spent.' ($\bar{\mathrm{c}}$'s overbar signfies that $\bar{\mathrm{c}}$ will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is a lower bound).

Equation (24) makes plain that for the limiting MPC $\underline{\kappa}$ to be strictly positive as $n = T - t$ goes to infinity we must impose the Return Impatience Condition (RIC):

$$\text{Þ}_{\mathsf{R}} < 1, \tag{26}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{27}$$

The RIC thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{\mathrm{c}}(m) = 0$). A consumer who satisfies the RIC is 'return impatient.'

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting consumption function will be

$$\bar{\mathrm{c}}(m) = (m + h - 1)\underline{\kappa}, \tag{28}$$

and so in order to rule out the degenerate limiting solution $\bar{\mathrm{c}}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (18).

The fact that $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$ allows us to write a useful analytical expression for the value the consumer would achieve by spending permanent income $\mathbf{p}$ in every period:

$$\begin{aligned}\mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma) + \beta^2\mathrm{u}(\mathbf{p}_t\Gamma^2) + ... \\ &= \mathrm{u}(\mathbf{p}_t)\left(1 + \beta\Gamma^{1-\rho} + (\beta\Gamma^{1-\rho})^2 + ...\right) \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-t+1}}{1-\beta\Gamma^{1-\rho}}\right)\end{aligned} \tag{29}$$

which (for $\Gamma > 0$) asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if any of these equivalent conditions holds:

$$\begin{aligned}\overbrace{\beta\Gamma^{1-\rho}}^{\equiv \beth} &< 1 \\ \beta\mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma \\ \text{Þ}_{\mathsf{R}} &< (\Gamma/\mathsf{R})^{1-1/\rho},\end{aligned} \tag{30}$$

where we call $\beth$[18] the 'Perfect Foresight Value Of Autarky Factor' (PF-VAF), and the variants of (30) constitute alternative versions of the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC; they guarantee that a consumer who always spends all permanent income 'has finite autarky value.'[19]

If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied: Divide both sides of the third inequality in (30) by R:

$$\text{Þ}/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} \tag{31}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(\Gamma/\mathsf{R}) < 1$ (and the RHS is raised to a positive power (because $\rho > 1$)).

Likewise, if the FHWC and the GIC are both satisfied, PF-FVAC must hold:

$$\begin{aligned} \text{Þ} &< \Gamma < \mathsf{R} \\ \text{Þ}_{\mathsf{R}} &< \Gamma/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} < 1 \end{aligned} \tag{32}$$

where the last line holds because FHWC $\Rightarrow 0 \leq (\Gamma/\mathsf{R}) < 1$ and $\rho > 1 \Rightarrow 0 < 1-1/\rho < 1$.

The first panel of Table 4 summarizes: The PF-Unconstrained model has a non-degenerate limiting solution if we impose the RIC and FHWC (these conditions are necessary as well as sufficient). Imposing the PF-FVAC and the FHWC implies the RIC, so PF-FVAC and FHWC are sufficient. If we impose the GIC and the FHWC, both the PF-FVAC and the RIC follow, so GIC+FHWC are sufficient. But there are circumstances under which the RIC and FHWC can hold while the PF-FVAC fails (which we write ~~PF-FVAC~~). For example, if $\Gamma = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under the RIC.

Perhaps more useful than this prose or the table, the relations of the conditions for the unconstrained perfect foresight case are presented diagrammatically in Figure 1. Each node represents a quantity considered in the foregoing analysis. The arrow associated with each inequality reflects the imposition of that condition. For example, one way we wrote the PF-FVAC in equation (30) is $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, so imposition of the PF-FVAC is captured by the diagonal arrow connecting Þ and $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$. Traversing the boundary of the diagram clockwise starting at Þ involves imposing first the GIC then the FHWC, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply that the PF-FVAC holds. Reversal of a condition will reverse the arrow's direction; so, for example, the bottommost arrow going from R to $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ imposes ~~FHWC~~; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram in a clockwise direction from Þ to R, revealing that imposition of GIC and FHWC (and, redundantly, FHWC again) let us conclude that the RIC holds because the starting point is Þ and the endpoint is R. (Consult Appendix K for a detailed exposition of diagrams of this type).

[18]This is another kind of discount factor, so we use the Hebrew 'bet' which is a cognate of the Greek 'beta'.

[19]This is related to the key impatience condition in Alvarez and Stokey (1998).

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Þ} \arrow[r, "\mathrm{GIC}"] \arrow[d, "\mathrm{RIC}"'] \arrow[dr, "\mathrm{PF\text{-}FVAC}" description] \arrow[dr, phantom, shift left=6ex, "\neq"] \arrow[dr, phantom, shift right=6ex, "\neq"] & \Gamma \arrow[d, "\mathrm{FHWC}"] \\
\mathsf{R} \arrow[r, "\cancel{\mathrm{FHWC}}"'] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 1 Relation of GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is one way of writing the PF-FVAC, equation (30)

2.4.3 PF Constrained Solution Exists Under RIC or Under {~~RIC~~,GIC}

We next examine the perfect foresight constrained solution because it is a useful benchmark (and limit) for the unconstrained problem with uncertainty (examined next).

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound for entering the period, $b_t = 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (13):

$$1^{-\rho} > \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{33}$$

By analogy to the return patience factor, we therefore define a 'growth patience factor' (GPF) as

$$\text{Þ}_{\Gamma} = \text{Þ}/\Gamma, \tag{34}$$

and define a 'growth impatience condition' (GIC)

$$\text{Þ}_{\Gamma} < 1 \tag{35}$$

which is equivalent to (33) (exponentiate both sides by $1/\rho$).

~~GIC~~ and RIC. If the GIC fails but the RIC (26) holds, appendix A shows that, for some $0 < m_{\#} < 1$, an unconstrained consumer behaving according to (28) would choose $c < m$ for all $m > m_{\#}$. In this case the solution to the constrained consumer's problem is

simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[20] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using the $\grave{\bullet}$ accent to designate the version of a function $\bullet$ in the presence of constraints:

$$\grave{c}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{c}(m) & \text{if } m \geq m_{\#}. \end{cases} \tag{36}$$

GIC and RIC. More useful is the case where the return impatience and GIC conditions both hold. In this case appendix A shows that the limiting constrained consumption function is piecewise linear, with $\grave{c}(m) = m$ up to a first 'kink point' at $m^1_{\#} > 1$, and with discrete declines in the MPC at a set of kink points $\{m^1_{\#}, m^2_{\#}, ...\}$. As $m \uparrow \infty$ the constrained consumption function $\grave{c}(m)$ becomes arbitrarily close to the unconstrained $\bar{c}(m)$, and the marginal propensity to consume function $\grave{\boldsymbol{\kappa}}(m) \equiv \grave{c}'(m)$ limits to $\underline{\kappa}$. Similarly, the value function $\grave{v}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer.

This logic holds even when the finite human wealth condition fails ($\cancel{\text{FHWC}}$), because the constraint prevents the consumer from borrowing against infinite human wealth to finance infinite current consumption. Under these circumstances, the consumer who starts with any amount of resources $b_t > 1$ will, over time, run those resources down so that by some finite number of periods n in the future the consumer will reach $b_{t+n} = 0$, and thereafter will set $\mathbf{c} = \mathbf{p}$ for eternity (which the PF-FVAC says yields finite value). Using the same steps as for equation (29), value of the interim program is also finite:

$$\mathbf{v}_{t+n} = \Gamma^{n(1-\rho)} \mathrm{u}(\mathbf{p}_t) \left(\frac{1 - (\beta\Gamma^{1-\rho})^{T-(t+n)+1}}{1 - \beta\Gamma^{1-\rho}} \right).$$

So, if $\cancel{\text{FHWC}}$, value for any finite m will be the sum of two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+n} = 0$, and the finite component due to the value of consuming all $\mathbf{p}_{t+n}$ thereafter.

GIC and $\cancel{\textit{RIC}}$. The most peculiar possibility occurs when the RIC fails. Under these circumstances the FHWC must also fail (Appendix A), and the constrained consumption function is nondegenerate. (See appendix Figure 8 for a numerical example). While it is true that $\lim_{m \uparrow \infty} \grave{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\grave{c}(m)$ is strictly positive and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where $\cancel{\text{RIC}}$ would suggest a degenerate limit of $\grave{c}(m) = 0$ while $\cancel{\text{FHWC}}$ would suggest a degenerate limit of $\grave{c}(m) = \infty$.

Tables 3 and 4 (and appendix table 5) codify.

We now turn to the case with uncertainty but without constraints, which will turn out to be a close parallel to the model with constraints but without uncertainty.

[20]"Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

2.5 Uncertainty-Modified Conditions

2.5.1 Impatience

When uncertainty is introduced, the expectation of b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})] = a_t(\mathsf{R}/\Gamma)\,\mathbb{E}_t[\psi_{t+1}^{-1}] \tag{37}$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define

$$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1} \tag{38}$$

which satisfies $\underline{\psi} < 1$ (again thanks to Jensen's inequality), so we can define

$$\underline{\Gamma} \equiv \Gamma\underline{\psi} \tag{39}$$

which is useful because it allows us to write uncertainty-adjusted versions of equations and conditions in a manner exactly parallel to those for the perfect foresight case; for example, we define a normalized Growth Patience Pactor (GPF-Nrm):

$$\text{\th}_{\underline{\Gamma}} = \text{\th}/\underline{\Gamma} = \mathbb{E}[\text{\th}/\Gamma\psi] \tag{40}$$

and a normalized version of the Growth Impatience Condition, GIC-Nrm:

$$\text{\th}_{\underline{\Gamma}} < 1 \tag{41}$$

which is stronger than the perfect foresight version (35) because

$$\underline{\Gamma} < \Gamma. \tag{42}$$

2.5.2 Autarky Value

Analogously to (29), value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned}\mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + ... + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}...\Gamma_T)\right] \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1-\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right)\end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)} \tag{43}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ. Defining

$$\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}} \tag{44}$$

we can see that $\mathbf{v}_t$ will be finite as T approaches ∞ if

$$\overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}} < 1$$

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

$$\beta < \underline{\underline{\Gamma}}^{\rho-1} \tag{45}$$

which we call the 'finite value of autarky condition' (FVAC) because it guarantees that value is finite for a consumer who always consumes their (now stochastic) permanent income (and we will call $\underline{\underline{\sqsupset}}$ the 'Value of Autarky Factor,' VAF).[21] For nondegenerate ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (30) because $\underline{\underline{\Gamma}} < \Gamma$.[22]

2.6 The Baseline Numerical Solution

Figure 2, familiar from the literature, depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and earlier periods under baseline parameter values listed in Table 2. (The 45 degree line is $\mathring{\mathrm{c}}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge to a nondegenerate $\mathring{\mathrm{c}}(m)$. Our next purpose is to show that this appearance is not deceptive.

2.7 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (12) defines a sequence of continuously differentiable strictly increasing strictly concave[23] functions

[21] In a stationary environment – that is, with $\underline{\underline{\Gamma}} = 1$ – this corresponds to an impatience condition imposed by Ma, Stachurski, and Toda (2020); but their remaining conditions do not correspond to those here, because their problem differs and their method of proof differs.

[22] To see this, rewrite (45) as

$$\begin{aligned}
\beta\mathsf{R} &< \mathsf{R}\underline{\underline{\Gamma}}^{\rho-1} \\
(\beta\mathsf{R})^{1/\rho} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
\text{Þ}_\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho}
\end{aligned}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

[23] With one obvious exception: $\mathring{\mathrm{c}}_T(m)$ is linear (and so only weakly concave).

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Factor	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Factor	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	Þ_{R}	$\equiv$	$\text{Þ}/\mathsf{R}$	0.961
PF Growth Patience Factor	Þ_{Γ}	$\equiv$	$\text{Þ}/\Gamma$	0.970
Growth Patience Factor	$\text{Þ}_{\underline{\Gamma}}$	$\equiv$	$\text{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Factor	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941
Weak Impatience Factor	$\wp^{1/\rho}\text{Þ}$	$\equiv$	$(\wp\beta\mathsf{R})^{1/\rho}$	0.071

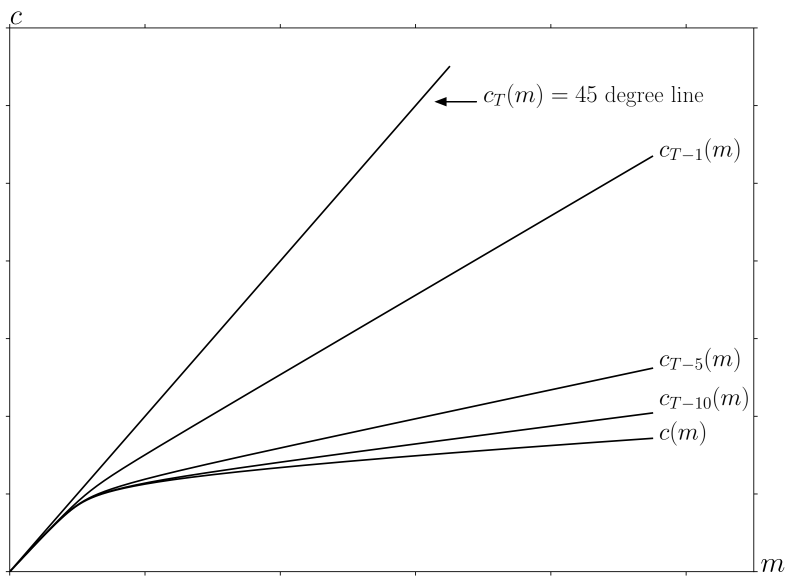


Figure 2 Convergence of the Consumption Rules

$\{\mathrm{c}_T, \mathrm{c}_{T-1}, ...\}$. The straightforward but tedious proof is relegated to appendix B. For present purposes, the most important point is that the income process induces what Aiyagari (1994) dubbed a 'natural borrowing constraint': $\mathring{\mathrm{c}}_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero income over the remaining horizon, requiring the consumer to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. Zeldes (1989) seems to have been the first to argue, based on his numerical results, that the natural borrowing constraint was a quantitatively plausible alternative to 'artificial' or 'ad hoc' borrowing constraints in a life cycle model.[24]

Strict concavity and continuous differentiability of the consumption function are key elements in many of the arguments below, but are not characteristics of models with 'artificial' borrowing constraints. The analytical convenience of these features is considerable, even if models with natural borrowing constraints in practice usually give similar results to those with artificial constraints.

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 2 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which will be needed in the contraction mapping proof.[25]

Assume that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathring{\mathrm{c}}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption that human wealth is finite, the proportion of consumption that will be financed out of human wealth approaches zero. In consequence, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero. In the course of proving this, appendix G provides a useful recursive expression (used below) for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+1}^{-1}. \tag{46}$$

2.8.1 Weak RIC Conditions

Appendix equation (92) presents a parallel expression for the limiting maximal MPC as $m_t \downarrow 0$:

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho} \text{Þ}_{\mathsf{R}} \bar{\kappa}_{t+1}^{-1} \tag{47}$$

[24] Carroll (1992) made the same (numerical) point for infinite horizon models (calibrated to actual empirical data on household income dynamics).

[25] Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; Ma and Toda (2020) show that these results generalize to the limits derived here if capital income is added to the model.

where $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ satisfies:

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{48}$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathring{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{49}$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave,[26] and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is in appendix D).

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

We can now articulate conditions under which the problem defines a contraction mapping. As mentioned above, standard theorems in the literature following Stokey et. al. (1989) require utility or marginal utility to be bounded over the space of possible values of m, which does not hold here because the possibility (however unlikely) of an unbroken string of zero-income events through the end of the horizon means that utility (and marginal utility) are unbounded as $m \downarrow 0$. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., Matkowski and Nowak (2011)), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that incorporates permanent shocks.[27]

Fortunately, Boyd (1990) provided a weighted contraction mapping theorem that Alvarez and Stokey (1998) showed could be used to address the homogeneous case (of which CRRA is an example) in a deterministic framework; later, Durán (2003) showed how to extend the Boyd (1990) approach to the stochastic case.

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A},\mathcal{B})$ where $\mathcal{C}(\mathcal{A},\mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $\digamma \in \mathcal{C}(\mathcal{A},\mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $\digamma > 0$. Then $\bullet$ is $\digamma$-bounded if the $\digamma$-norm of $\bullet$,*

$$\| \bullet \|_{\digamma} = \sup_m \left[\frac{| \bullet (m)|}{\digamma(m)} \right], \tag{50}$$

is finite.

[26] Carroll and Kimball (1996)

[27] See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

For $\mathcal{C}_{F}(\mathcal{A}, \mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A}, \mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$ *such that*[28,29]

$$
\begin{aligned}
&(1)\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \leq \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \leq \{\mathsf{T}\mathrm{y}\} \\
&(2)\{\mathsf{T}\mathbf{0}\} \in \ \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \\
&(3) \textit{There exists some real } 0 < \alpha < 1 \textit{ such that} \\
&\quad \{\mathsf{T}(\mathrm{w} + \zeta F)\} \leq \{\mathsf{T}\mathrm{w}\} + \zeta \alpha F \quad \textit{holds for all real } \zeta > 0 \quad .
\end{aligned}
$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{>0}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$
\{\mathsf{E}\mathrm{z}\}(a_t) = \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1} + \xi_{t+1})\right].
$$

Using this, we introduce the mapping $\mathcal{T} : \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$,[30]

$$
\{\mathcal{T}\mathrm{z}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \mathrm{u}(c_t) + \beta\left(\{\mathsf{E}\mathrm{z}\}(m_t - c_t)\right). \tag{51}
$$

We can show that our operator $\mathcal{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first is the WRIC necessary for convergence of the maximal MPC, equation (48) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (45). (We discuss the interpretation of these restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. $\mathcal{T}$ *is a contraction mapping if the restrictions on parameter values* (48) *and* (45) *are true (that is, if the weak return impatience condition and the finite value of autarky condition hold).*

Intuitively, Boyd's theorem shows that if you can find a F that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with F, the normalized problem defines a contraction mapping. The intuition for the FVAC condition is just that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly the sustainable amount (say, by consuming $(\mathsf{r}/\mathsf{R})b_0 - \epsilon$ for some small $\epsilon > 0$).

The details of the proof are cumbersome, and are therefore relegated to appendix D. Given that the value function converges, appendix E.2 shows that the consumption functions converge.[31]

---

[28] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[29] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \leq \mathrm{y}$ is short for $\mathrm{x}(\bullet) \leq \mathrm{y}(\bullet)\ \forall \bullet \in \mathcal{A}$. In this notation, $\zeta\alpha F$ in (3) is a *function* which can be applied to any argument $\bullet$ (because F is a function).

[30] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous F-bounded functions z) and the compactness of $[\underline{\kappa} m_t, \bar{\kappa} m_t]$.

[31] MST's proof is for convergence of the consumption policy function directly, rather than of the value function, which is why their conditions are on u', which governs behavior.

2.10 The Liquidity Constrained Solution as a Limit

This section explains why a related problem commonly considered in the literature (e.g., by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The 'related' problem makes two changes to the problem defined above:

1. An 'artificial' liquidity constraint is imposed: $a_t \geq 0$
2. The probability of zero-income events is zero: $\wp = 0$

The essence of the argument is simple. Imposing the artificial constraint without changing $\wp$ would not change behavior at all: The possibility of earning zero income over the remaining horizon already prevents the consumer from ending the period with zero assets. So, for precautionary reasons, the consumer will save something.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the *probability* that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the *amount* of precautionary saving induced by the zero-income events approaches zero as $\wp \downarrow 0$. But "zero" is the amount of precautionary saving that would be induced by a zero-probability event for the impatient liquidity constrained consumer.

Another way to understand this is just to think of the liquidity constraint reflecting a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ends the period with (weakly) negative assets.

See appendix H for the formal proof justifying the foregoing intuitive discussion.[32]

The conditions required for convergence and nondegeneracy are thus strikingly similar between the liquidity constrained perfect foresight model and the model with uncertainty but no explicit constraints: The liquidity constrained perfect foresight model is just the limiting case of the model with uncertainty as the degree of all three kinds of uncertainty (zero-income events, other transitory shocks, and permanent shocks) approaches zero.

2.11 Discussion of Parametric Restrictions

The full relationship among all the conditions is represented in Figure 3. Though the diagram looks complex, it is merely a modified version of the earlier diagram with further (mostly intermediate) inequalities inserted. (Arrows with a "because" are a new element to label relations that always hold under the model's assumptions.) Again readers unfamiliar with such diagrams should see Appendix K) for a more detailed explanation.

[32] It seems likely that a similar argument would apply even in the context of a model like that of MST, perhaps with some weak restrictions on returns.

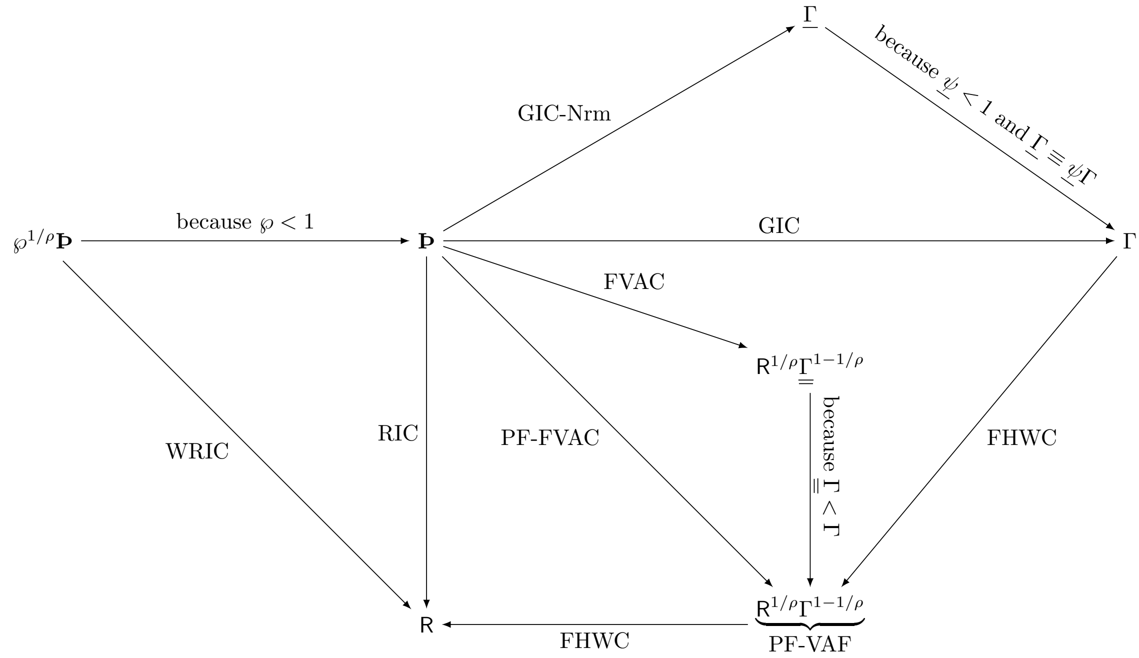


Figure 3 Relation of All Inequality Conditions
See Table 2 for Numerical Values of Nodes Under Baseline Parameters

2.11.1 The WRIC

The 'weakness' of the additional condition sufficient for contraction beyond the FVAC, the WRIC, can be seen by asking 'under what circumstances would the FVAC hold but the WRIC fail?' Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \ 1 \ < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{52}$$

If we require $\mathsf{R} \geq 1$, the WRIC is redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that (with $\rho > 1$ and $\beta < 1$) the RIC (and WRIC) must hold. But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (52) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. The relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which section

2.10 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.11.2 When the RIC Fails

In the perfect foresight problem (section 2.4.2), the RIC was necessary for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the much weaker WRIC is sufficient for nondegeneracy (assuming that the FVAC holds). We can directly derive the features the problem must exhibit (given the FVAC) under ~~RIC~~ (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} \ < \ (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< \underline{\underline{\psi}} \qquad (53)
\end{aligned}$$

but since $\underline{\underline{\psi}} < 1$ (cf. the argument below (43)), this reduces to $\mathsf{R}/\Gamma < 1$; so, given the FVAC, the RIC can fail only if human wealth is unbounded. As an illustration of the usefulness of our diagrams, note that this algebraically complicated conclusion could be easily reached diagrammatically in figure 3 by starting at the R node and imposing ~~RIC~~ (reversing the RIC arrow) and then traversing the diagram along any clockwise path to the PF-VAF node at which point we realize that we *cannot* impose the FHWC because that would let us conclude $\mathsf{R} > \mathsf{R}$.

As in the perfect foresight constrained problem, unbounded limiting human wealth (~~FHWC~~) here does not lead to a degenerate limiting consumption function (finite human wealth is not a condition required for the convergence theorem). But, from equation (46) and the discussion surrounding it, an implication of ~~RIC~~ is that $\lim_{m\uparrow\infty} \mathring{\mathrm{c}}'(m) = 0$. Thus, interestingly, in the special {~~RIC~~,~~FHWC~~} case (unavailable in the perfect foresight model) the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited human wealth from resulting in infinite consumption at any finite m. Intutively, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in a limiting consumption function of $\mathring{\mathrm{c}}(m) = 0$) plus unbounded human wealth (which the perfect foresight model prohibits (by assumption FHWC) because it leads to a limiting consumption function $\mathring{\mathrm{c}}(m) = \infty$) combine to yield a unique finite level of consumption and the MPC for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {GIC,~~RIC~~} case. There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.[33]

2.11.3 When the RIC Holds

FHWC. If the RIC and FHWC both hold, a perfect foresight solution exists (see 2.4.2 above). As $m \uparrow \infty$ the limiting consumption function and value function become arbitrarily close to those in the perfect foresight model, because human wealth pays for a vanishingly small portion of spending. This will be the main case analyzed in detail below.

~~FHWC~~. The more exotic case is where FHWC does not hold; in the perfect foresight model, {RIC,~~FHWC~~} is the degenerate case with limiting $\bar{\mathrm{c}}(m) = \infty$. Here, since the FVAC implies that the PF-FVAC holds (traverse Figure 3 clockwise from **Þ** by imposing FVAC and continue to the PF-VAF node), reversing the arrow connecting the R and PF-VAF nodes implies that under ~~FHWC~~:

$$\overbrace{\text{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma}^{\text{PF-FVAC}}$$
$$\text{Þ} < \Gamma$$

where the transition from the first to the second lines is justified because ~~FHWC~~ $\Rightarrow (\mathsf{R}/\Gamma)^{1/\rho} < 1$. So, {RIC, ~~FHWC~~} implies the GIC holds. However, we are not entitled to conclude that the GIC-Nrm holds: **Þ** $< \Gamma$ does not imply **Þ** $< \underline{\psi}\Gamma$ where $\underline{\psi} < 1$. See further discussion of this illuminating case in section 3.3.3.

We have now established the principal points of comparison between the perfect foresight solutions and the solutions under uncertainty; these are codified in the remaining parts of Tables 3 and 4.

3 Analysis of the Converged Consumption Function

Figures 4 and 5a,b capture the main properties of the converged consumption rule when the RIC, GIC-Nrm, and FHWC all hold.[34] Figure 4 shows the expected consumption growth factor $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ for a consumer behaving according to the converged consumption rule, while Figures 5a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to **Þ**, indicated by the lower bound in Figure 4, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{Þ}_{\mathsf{R}})$ (Figure 5), the same as the perfect foresight MPC. Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 4) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}})$ (Figure 5). Third (Figure 4), there are two special values of m, which we will call the 'individual steady

[33] Ma and Toda (2020) derive conditions under which the limiting MPC is zero in an even more general case where there is also capital income risk.

[34] These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

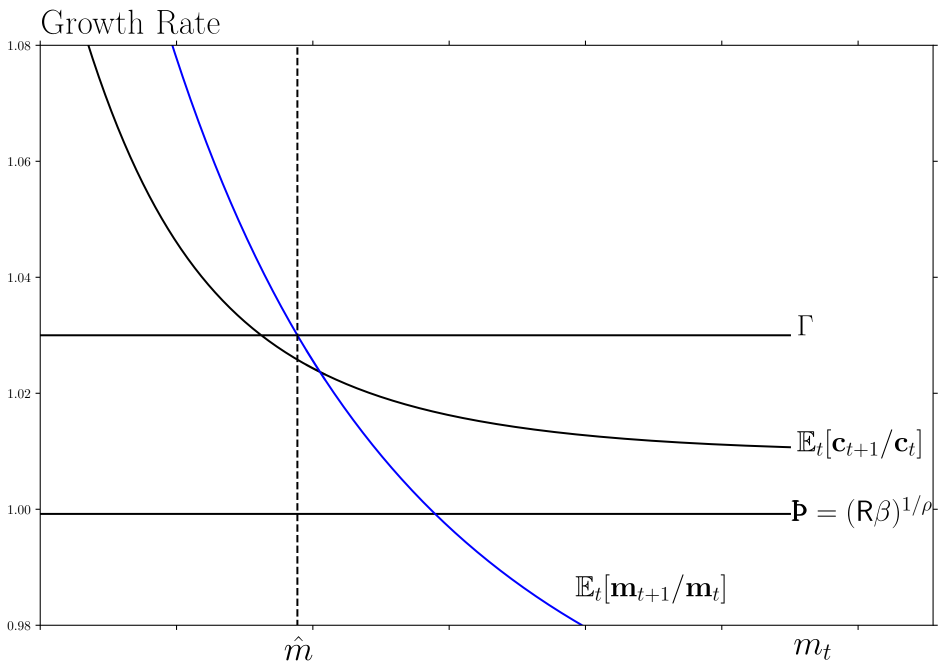


Figure 4 Target m, Expected Consumption Growth, and Permanent Income Growth

state' point $\hat{m}$ because it is the point where consumption growth and income growth are balanced, and the 'individual target' $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$. As indicated by the arrows of motion on the $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ curve, the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then m will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 4), at the market resources target $\check{m}$, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 4 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[35]

3.1 Limits as m approaches Infinity

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

[35]Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathring{\mathrm{c}}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathring{\mathrm{c}}(m)$. It does not strictly follow that the limiting function $\mathring{\mathrm{c}}(m)$ is twice continuously differentiable, but I will assume that it is.

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathring{\mathrm{c}}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption. Our imposition of the RIC guarantees that $\underline{\kappa} > 0$, so this solution satisfies our definition of nondegeneracy, and because this solution is always available it defines a lower bound on both the consumption and value functions.

Assuming the FHWC holds, the infinite horizon perfect foresight solution (28) constitutes an upper bound on consumption in the presence of uncertainty, since the introduction of uncertainty strictly decreases the level of consumption at any m (Carroll and Kimball (1996)). Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) <& \mathring{\mathrm{c}}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathring{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned} \tag{54}$$

But

$$\begin{aligned} \lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m - 1 + h)/m \\ &= 1, \end{aligned}$$

so as $m \uparrow \infty, \mathring{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) \to 1$, and the continuous differentiability and strict concavity of $\mathring{\mathrm{c}}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathring{\mathrm{c}}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 5 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t\uparrow\infty} \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t\uparrow\infty} \Gamma_{t+1}\underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}\bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathring{\mathrm{c}}(m_t)$),

$$\begin{aligned} \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t &= \lim_{m_t\uparrow\infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t}\right) \\ &= (\mathsf{R}\beta)^{1/\rho} = \text{Þ} \end{aligned} \tag{55}$$

because $\lim_{m_t\uparrow\infty} \mathrm{a}'(m) = \text{Þ}_{\mathsf{R}}$[36] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

[36]This is because $\lim_{m_t\uparrow\infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathring{\mathrm{c}}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathring{\mathrm{c}}'(m_t) = \text{Þ}_{\mathsf{R}}$.

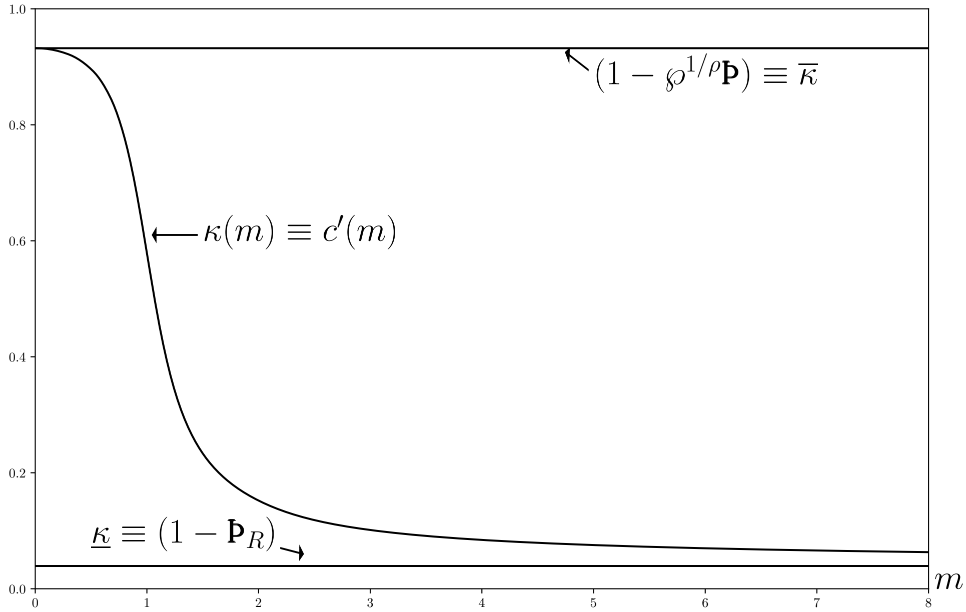


Figure 5 Limiting MPC's

Hence we have

$$\text{Þ} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value **Þ** in the perfect foresight model.

3.2 Limits as m Approaches Zero

Equation (47) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp \mathsf{R}\beta)^{1/\rho}.$$

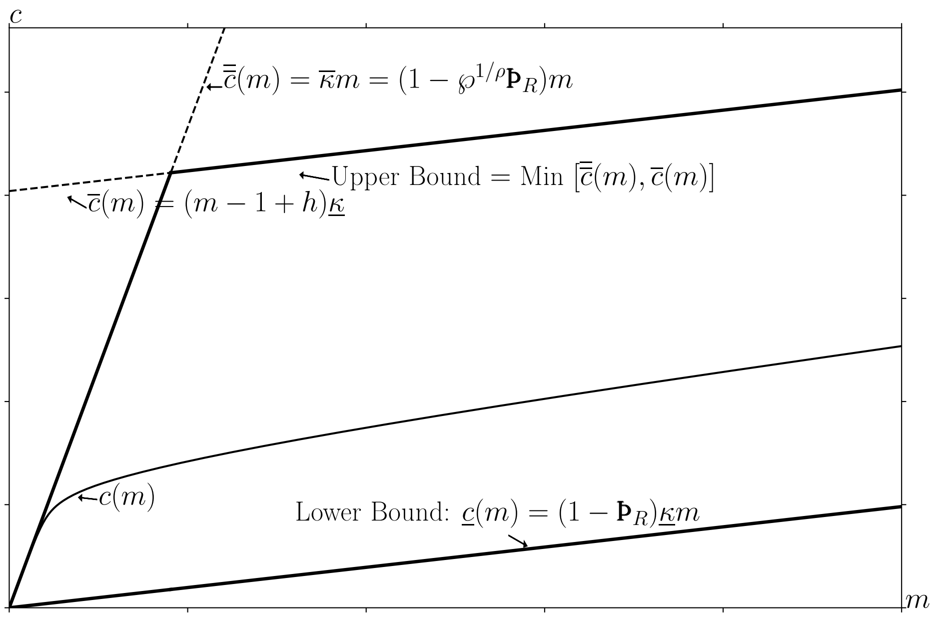
Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

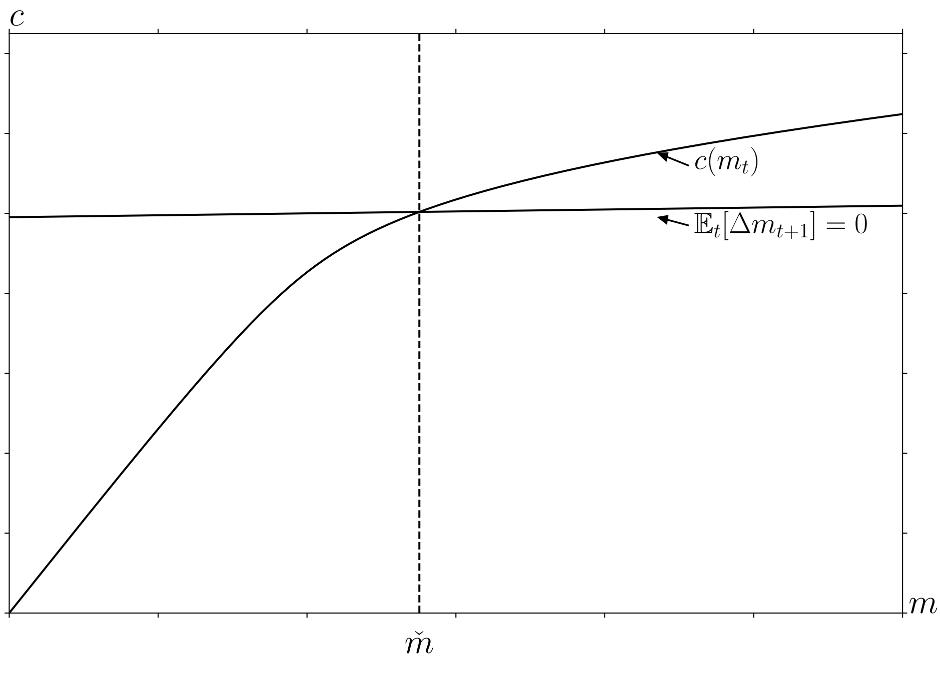
Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathring{\mathrm{c}}'(m) = \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 5 confirms that the numerical solution obtains this limit for the MPC as m approaches zero.



(a) Bounds



(b) Target m

Figure 6 The Consumption Function

For consumption growth, as $m \downarrow 0$ we have

$$
\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}
$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[37]

3.3 Unique ‘Stable’ Points

3.3.1 The ‘Individual Target’

The most obvious definition of a ‘stable’ point is a value $\check{m}$ such that if $m_t = \check{m}$, then $\mathbb{E}_t[m_{t+1}] = m_t$. Existence of such a target turns out to require the GIC-Nrm condition, equation (41).

Theorem 2. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (41) holds, there exists a unique ‘individual target’ $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{56}$$

Moreover, $\check{m}$ is a point of stablity in the sense that

$$
\begin{aligned}
\forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\
\forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t.
\end{aligned} \tag{57}
$$

The full proof is in Appendix M, but the key points can be made informally here. Defining $\bar{\mathcal{R}} = (\mathsf{R}/\underline{\Gamma})$, since $\mathbb{E}_t[m_{t+1}] = \bar{\mathcal{R}}(m_t - c_t) + 1$, the implicit equation for $\check{m}$ is

$$\bar{\mathcal{R}}(\check{m} - \mathring{\mathrm{c}}(\check{m})) + 1 = \check{m} \tag{58}$$

which can be differentiated and rearranged to yield

$$1 - \bar{\mathcal{R}}^{-1} = \mathring{\mathrm{c}}'(\check{m}). \tag{59}$$

[37] None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

The fact (cf. (27)) that the minimum value of $\mathring{c}'$ is $1-\text{\textthorn}_{\mathsf{R}}$ converts (59) to the inequality $1-\bar{\mathcal{R}}^{-1} > 1-\text{\textthorn}_{\mathsf{R}}$ from which we have

$$\begin{aligned} \underline{\Gamma}/\mathsf{R} &> \text{\textthorn}/\mathsf{R} \\ 1 &> \text{\textthorn}/\underline{\Gamma} \end{aligned} \tag{60}$$

which is the GIC-Nrm; thus, if a stable point exists, it must satisfy the GIC-Nrm. (The appendix proves that if the GIC-Nrm is satisfied, such a point must exist, and be globally stable).

3.3.2 The 'Individual Steady-State'

Heterogeneous agents in a small open economy are often represented by models of this kind. A traditional question in such models is whether there is a 'balanced growth' equilibrium in which aggregate variables (income, consumption, market resources) all grow at the same rate. As an input to our more focused small-open-economy analysis (in section 4.2) it will be useful to derive here the conditions under which an m will exist at which an individual consumer expects 'balanced growth' in their own individual market resources and permanent income:

$$\begin{aligned} \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &= \mathbb{E}_t[\mathbf{p}_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t[m_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/(m_t\mathbf{p}_t) &= \mathbb{E}_t[\mathbf{p}_t\Gamma\psi_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t\left[\psi_{t+1}\underbrace{((m_t-\mathring{c}(m_t))\mathsf{R}/(\Gamma\psi_{t+1}))+\xi_{t+1})}_{m_{t+1}}\right] &= m_t. \end{aligned} \tag{61}$$

If some value $m_t = \hat{m}$ exists for which equation (61) holds, we will call that point the 'individual steady-state' $\hat{m}$:

$$\begin{aligned} \mathbb{E}\left[(\hat{m}-\mathring{c}(\hat{m}))\overbrace{\mathsf{R}/\Gamma}^{\mathcal{R}}+\psi\xi\right] &= \hat{m} \\ (\hat{m}-\mathring{c}(\hat{m}))\mathcal{R}+1 &= \hat{m}, \end{aligned} \tag{62}$$

and derivations parallel to those after (58) yield the conclusion that existence of an individual steady-state implies the GIC, $1 > \text{\textthorn}/\Gamma$. Note that since $\underline{\Gamma} < \Gamma$, $\mathring{c}'(\check{m}) > \mathring{c}'(\hat{m})$ which implies that $\check{m} < \hat{m}$.

If $\hat{m}$ exists, a weak sense in which it is a point of stability is that for an economy that started in period t with all consumers at $m_t = \hat{m}$, the ratio of the aggregate value $\mathbf{M}_{t+1} = \int \mathbf{m}_{t+1}$ to aggregate $\mathbf{P}_{t+1} = \int \mathbf{p}_{t+1}$ would be $\mathbf{M}_{t+1}/\mathbf{P}_{t+1} = \hat{m}$ so that, at least between those two periods, the aggregate economy would exhibit balanced growth and stability of the $\mathbf{M}/\mathbf{P}$ ratio.[38] After that, however, the nondegenerate distribution across values of m_{t+1} and $\mathbf{p}_{t+1}$ would almost certainly lead to some drift in the aggregate ratio.

[38]This does not require all consumers to have the same $\mathbf{p}$. The unlimited size of the population means that the expectation in (61) holds for the set of consumers at each value of $\mathbf{p}$ represented in the population, and so holds for the entire population.

There is one circumstance in which $\hat{m}$ would constitute a perpetually stable steady state, at both the micro and aggregate levels: if after the date t at which all consumers had $m_t = \hat{m}$, all consumers always drew exactly the expected values of the idiosyncratic shocks ($\psi_{i,t+n} = \xi_{j,t+n} = 1 \ \forall \ i, j$ and for all $n > 0$). Such an economy would exhibit perpetual 'balanced growth': $\mathbf{M}_{t+n}/\mathbf{P}_{t+n} = \hat{m}$ for all $n > 0$.

Theorem 3 formally states the relevant propositions.

Theorem 3. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (35) holds, there exists a unique 'individual steady state' $\hat{m} > 0$ such that*

$$\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{63}$$

Moreover, $\hat{m}$ is a point of stablity in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] > \Gamma = \mathbb{E}_t[\mathbf{p}_{t+1}/\mathbf{p}_t] \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] < \Gamma = \mathbb{E}_t[\mathbf{p}_{t+1}/\mathbf{p}_t]. \end{aligned} \tag{64}$$

The proofs of the two theorems are almost completely parallel; to save space, they are relegated to Appendix M. In sum, they involve three steps:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1}/m_t]$
   - This follows from existence and continuity of the constitutents
2. Existence of the equilibrium point
   - This follows from the upper and lower bound limiting MPC's, existence and continuity, and the Intermediate Value Theorem
3. Monotonicity of $\mathbb{E}_t[m_{t+1} - m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1} - m_t]$
   - This follows from concavity of the consumption function

3.3.3 Example Where There Is A Solution Without A Target

To build intuition, it is useful to exhibit an example in which a nondegenerate solution exists but a target $\check{m}$ does not. An example that satisfies the combination FVAC and ~~GIC-Nrm~~ is depicted in Figure 7. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the code for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC-Nrm does not, the consumption function's limiting slope $(1 - \text{\textbf{Þ}}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1 - \underline{\Gamma}/\mathsf{R})$,[39] so the gap between the two *increases* with m in the limit. Although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, the consumer will always spend less than the amount that would keep m stable (in expectation).

[39]This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\underline{\psi}^{-1} + 1$ for c and differentiate.

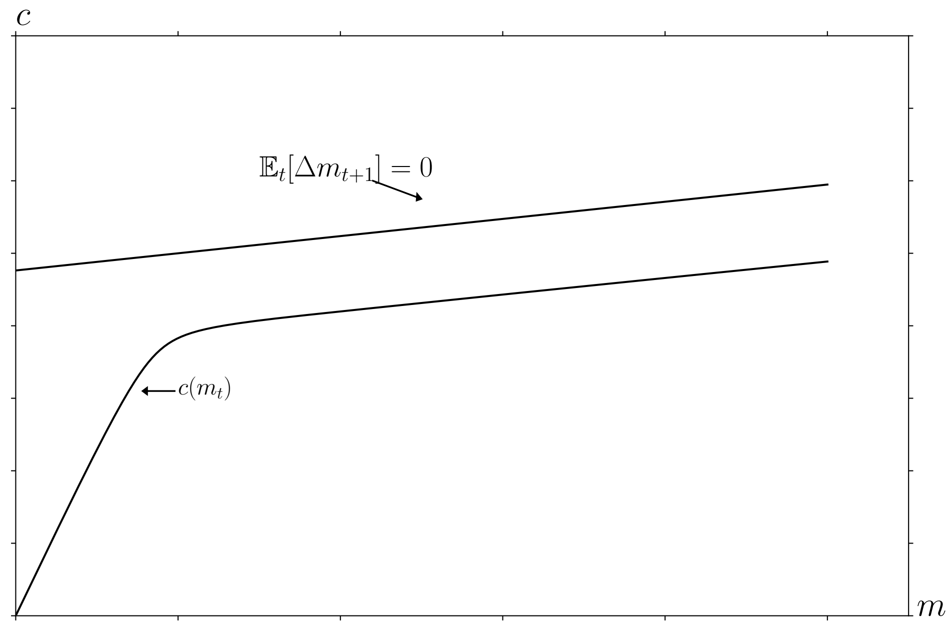


Figure 7 Example Solution under {FVAC,~~GIC-Nrm~~}

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 4 the intersection of the individual target m ratio $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the expected growth rate of permanent income. We now prove this.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t\left[\frac{\Gamma_{t+1}\mathring{c}(m_{t+1})}{\mathring{c}(m_t)}\right] &< \mathbb{E}_t\left[\left(\frac{\Gamma_{t+1}(\mathring{c}(\check{m}) + \mathring{c}'(\check{m})(m_{t+1} - \check{m}))}{\mathring{c}(\check{m})}\right)\right] \\
&= \mathbb{E}_t\left[\Gamma_{t+1}\left(1 + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)(m_{t+1} - \check{m})\right)\right] \\
&= \Gamma + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)\mathbb{E}_t\left[\Gamma_{t+1}\left(m_{t+1} - \check{m}\right)\right] \\
&= \Gamma + \left(\frac{\mathring{c}'(\check{m})}{\mathring{c}(\check{m})}\right)\left[\mathbb{E}_t[\Gamma_{t+1}]\underbrace{\mathbb{E}_t[m_{t+1} - \check{m}]}_{=0} + \text{cov}_t(\Gamma_{t+1}, m_{t+1})\right] \qquad (65)
\end{aligned}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\text{a}(\check{m}) + \xi_{t+1}$ and $\text{a}(\check{m}) > 0$ it is clear that $\text{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (65) is negative, as required.

3.5 Is Expected Consumption Growth a Declining Function of m_t?

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathring{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathring{\mathrm{c}}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv\boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathring{\mathrm{c}}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathring{\mathrm{c}}(m_t) - \mathring{\mathrm{c}}(m_{t+1})\mathring{\mathrm{c}}'(m_t)}{\mathring{\mathrm{c}}(m_t)^2}\right)\right] < 0. \tag{66}$$

Appendix L shows that the proposition holds true if there are only transitory (and no permanent) shocks. The software archive associated with this paper contains an example in which exotic interactions between permanent shocks and extreme curvature that occurs with very small $\wp$ generate a (small) region where the proposition does not hold. In practice, for plausible parametric choices (and in models without an artificial liquidity constraint), $\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

A large (infinite) collection of small (infinitesimal) buffer-stock consumers with identical parameter values can be thought of as a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy with an exogenous (constant) interest rate.[40]

Until now for convenience we have assumed infinite horizons, with the implicit understanding that Poisson mortality should be handled by adjusting the effective discount factor for mortality. On that basis, section 4.1 continues to omit mortality. But a reason for explicitly introducing mortality will appear at the end of section 4.2, so implications of alternative assumptions about mortality are briefly examined in Section 4.3.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above. Szeidl (2013) proves that whenever the GIC holds such a population will be characterized by invariant distributions of m, c, and a;[41] designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.

[40] It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

[41] Szeidl (2013)'s equation (9), in our notation, is:

$$\begin{aligned} \mathbb{E}\log \mathsf{R}(1-\kappa) &< \mathbb{E}\log \Gamma\psi \\ \mathbb{E}\log \mathsf{R}\text{Þ}_{\mathsf{R}} &< \mathbb{E}\log\Gamma\psi \\ \log \text{Þ}_{\Gamma} &< \mathbb{E}\log\psi \end{aligned}$$

4.1 Consumption and Income Growth at the Household Level

The operator $\mathbb{M}[\bullet]$ yields the mean of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ used above, which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma^2_\psi/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma^2_\psi/2)\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ (Szeidl (2013)) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$. Thus, the same GIC that guaranteed the existence of an 'individual steady state' value of m at the microeconomic level guarantees both that there will be an invariant distribution of the population across values of the model variables and that in that invariant distribution the mean growth rates of all idiosyncratic variables are the same (see Szeidl (2013) for details).

4.2 Balanced Growth of Aggregate Income, Consumption, and Wealth

Using boldface capital letters for aggregates, the growth factor for aggregate income is:

$$\begin{aligned}\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/\mathbb{M}[\mathbf{p}_t\xi_t] \\ &= \Gamma\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

From the perspective of period t,

$$\begin{aligned}\mathbf{A}_{t+1} &= \mathbb{M}[a_{t+1}\mathbf{p}_{t+1}] \\ &= \Gamma\mathbb{M}[(a_t + (a_{t+1} - a_t))\mathbf{p}_t\psi_{t+1}] \\ &= \Gamma\left(\underbrace{\mathbb{M}[a_t\mathbf{p}_t\psi_{t+1}]}_{=\mathbf{A}_t} + \underbrace{\mathbb{M}[(a_{t+1} - a_t)]}_{=0\ \text{(Szeidl (2013))}}\mathbb{M}[\mathbf{p}_t\psi_{t+1}] + \mathrm{cov}_t(a_{t+1} - a_t, \mathbf{p}_t\psi_{t+1})\right) \\ \mathbf{A}_{t+1}/\mathbf{A}_t &= \Gamma\left(1 + \frac{\mathrm{cov}(a_{t+1}, \mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right)\end{aligned}$$

Unfortunately, the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1}. Matters are simpler if there are no permanent shocks; see Appendix F for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

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and under our assumption that $\log \psi \sim \mathcal{N}(-\sigma^2_\psi/2, \sigma^2_\psi)$ we can exponentiate both sides to obtain the GIC, $\text{Þ}_\Gamma < 1$. If the permanent income shocks are not lognormally distributed the expression must be tested in Szeidl's original form.

One way of thinking about the problem here is that it may reflect the fact that, under our assumptions, the $\mathbf{p}$ variable does not have an ergodic distribution; the distribution of permanent income becomes forever wider over time, because our consumers never die and each immortal person is perpetually subject to symmetric shocks to their $\log \mathbf{p}$.

This is why we need to introduce mortality.

4.3 Mortality and Redistribution

Most heterogeneous agent models incorporate a constant positive probability of death, following Blanchard (1985). In a model that mostly follows Blanchard (1985), for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, Carroll, Slacalek, Tokuoka, and White (2017) show that the limiting distribution of permanent income has a finite variance, which is a useful step in the direction of taming the problems caused by an unbounded distribution of $\boldsymbol{p}$. Numerical results in that paper confirm the intuition that, under appropriate impatience conditions, balanced growth arises (though a formal proof remains elusive).

Even with those (numerical) results in hand, the centrality of the mortality assumptions to the existence and nature of steady states requires a brief discussion here.

4.3.1 Blanchard Lives

Blanchard (1985)'s model assumes the existence of an annuitization scheme in which estates of dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, the most notable of which is that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor: If the probability of remaining alive is $\aleph$, then assuming that no utility accrues after death makes the effective discount factor $\hat{\beta} = \beta\aleph$, while the enhancement to the rate of return from the annuity scheme yields an effective interest rate of $\hat{\mathsf{R}}/\aleph$ (recall that because of Poisson mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\hat{\textbf{Þ}}$ is unchanged from its value in the infinite horizon model:

$$\hat{\textbf{Þ}} \equiv (\beta\aleph\mathsf{R}/\aleph)^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} \equiv \textbf{Þ}. \tag{67}$$

The only adjustments this requires to the analysis from prior parts of this paper are therefore to the few elements that involve a role for R distinct from its contribution to $\textbf{Þ}$ (principally, the RIC). These would need to be adjusted to incorporate in interest factor with a higher rate of return.

The numerical finding that the covariance term above is approximately zero allows us to conclude again that the key requirement for aggregate balanced growth is presumably the GIC.

4.3.2 Modigliani Lives

Blanchard (1985)'s innovation was useful not only for the insight it provided but also because the principal alternative, the Life Cycle model of Modigliani (1966), was computationally challenging given the then-available technologies. Aside from its (considerable) conceptual value, there is no need for Blanchard's analytical solution today, when serious modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard's mortality is to follow Modigliani in assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2001, 2016)).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it again represents something of a polar alternative to Blanchard: Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean; the estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium. If $\aleph$ is the probability of remaining alive, the condition changes from the plain GIC to a looser mortality-adjusted GIC:

$$\aleph \text{Þ}_{\Gamma} < 1. \tag{68}$$

With no income growth, the condition required to prohibit unbounded growth in aggregate wealth would be the condition that prevents the per-capita wealth of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite horizon (usually, 'life cycle') versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to Friedman (1957) and the standard specification of preferences – and without complications like liquidity constraints – have attractive analytical properties (like continuous differentiability of the consumption function, and analytical limiting MPC's as resources approach their minimum and maximum possible values), and that (more widely used) models with liquidity constraints can be viewed as a particular limiting case of this simpler model.

The main focus of the paper, though, is on the limiting solution of the finite horizon model as the horizon extends to infinity. The paper shows that the simple model has additional attractive properties: A 'Finite Value of Autarky' condition guarantees convergence of the consumption function, under the mild requirement of a 'Weak Return Impatience Condition' that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model's relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite horizon versions of such models is in heterogeneous agent macroeconomics. The paper articulates intuitive 'Growth Impatience Conditions' under which populations of such agents, with Blancharidan (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for a number of results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

The paper's results are all easily reproducible interactively on the web or on any standard computer system. Such reproducibility reflects the paper's use of the open-source Econ-ARK toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Table 3 Definitions and Comparisons of Conditions

<table>
<tr><td>Perfect Foresight Versions</td><td>Uncertainty Versions</td></tr>
<tr><td colspan="2">Finite Human Wealth Condition (FHWC)</td></tr>
<tr><td>$\Gamma/\mathsf{R} < 1$

The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.</td><td>$\Gamma/\mathsf{R} < 1$

The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.</td></tr>
<tr><td colspan="2">Absolute Impatience Condition (AIC)</td></tr>
<tr><td>$\text{Þ} < 1$

The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time:

$\mathbf{c}_{t+1} < \mathbf{c}_t$</td><td>$\text{Þ} < 1$

If wealth is large enough, the *expectation* of consumption next period will be smaller than this period's consumption:

$\lim_{m_t \rightarrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}] < \mathbf{c}_t$</td></tr>
<tr><td colspan="2">Return Impatience Conditions</td></tr>
<tr><td>Return Impatience Condition (RIC)</td><td>Weak RIC (WRIC)</td></tr>
<tr><td>$\text{Þ}/\mathsf{R} < 1$

The growth factor for consumption Þ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite:

$\mathrm{c}'(m) = 1 - \text{Þ}/\mathsf{R} < 1$</td><td>$\wp^{1/\rho}\text{Þ}/\mathsf{R} < 1$

If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker.

$\mathrm{c}'(m) < 1 - \wp^{1/\rho}\text{Þ}/\mathsf{R} < 1$</td></tr>
<tr><td colspan="2">Growth Impatience Conditions</td></tr>
<tr><td>GIC</td><td>GIC-Nrm</td></tr>
<tr><td>$\text{Þ}/\Gamma < 1$

For an unconstrained PF consumer, the ratio of $\mathbf{c}$ to $\mathbf{p}$ will fall over time. For constrained, guarantees the constraint eventually binds. Guarantees $\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = \text{Þ}_\Gamma$</td><td>$\text{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$

By Jensen's inequality stronger than GIC Ensures consumers will not expect to accumulate m unboundedly.

$\lim_{m_t \rightarrow \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{Þ}_{\underline{\Gamma}}$</td></tr>
<tr><td colspan="2">Finite Value of Autarky Conditions</td></tr>
<tr><td>PF-FVAC</td><td>FVAC</td></tr>
<tr><td>$\beta\Gamma^{1-\rho} < 1$
equivalently $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$

The discounted utility of constrained consumers who spend their permanent income each period should be finite.</td><td>$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$

By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.</td></tr>
</table>

Table 4 Sufficient Conditions for Nondegenerate‡ Solution

$\mathrm{c}(m)$: Model Reference	Conditions	Comments or or Logic
$\bar{\mathrm{c}}(m)$: PF Unconstrained	RIC, FHWC°	RIC $\Rightarrow \lvert \mathrm{v}(m)\rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m)\rvert$
$\underline{\mathrm{c}}(m) = \underline{\kappa} m$: PF $h = 0$		PF model with no human wealth
Section 2.4.2		RIC prevents $\bar{\mathrm{c}}(m) = \underline{\mathrm{c}}(m) = 0$
Section 2.4.2		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
Eq (31)		PF-FVAC+FHWC $\Rightarrow$ RIC
Eq (32)		GIC+FHWC $\Rightarrow$ PF-FVAC
$\grave{\mathrm{c}}(m)$: PF Constrained	$\cancel{\text{GIC}}$, RIC	FHWC holds ($\Gamma < \text{Þ} < \mathsf{R} \Rightarrow \Gamma < \mathsf{R}$)
Section 2.4.3		$\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m > m_{\#} < 1$
		($\cancel{\text{RIC}}$ would yield $m_{\#} = 0$ so $\grave{\mathrm{c}}(m) = 0$)
Appendix A	GIC,RIC	$\lim_{m\to\infty} \grave{c}(m) = \bar{c}(m), \lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		kinks at pts where horizon to $b = 0$ changes*
Appendix A	GIC,$\cancel{\text{RIC}}$	$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0$
		kinks at pts where horizon to $b = 0$ changes*
$\mathring{\mathrm{c}}(m)$: Friedman/Muth	Section 3.1,	$\underline{\mathrm{c}}(m) < \mathring{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)$
	Section 3.2	$\underline{\mathrm{v}}(m) < \mathring{\mathrm{v}}(m) < \bar{\mathrm{v}}(m)$
Section 2.9	FVAC, WRIC	Sufficient for Contraction
Section 2.11.1		WRIC is weaker than RIC
Figure 3		FVAC is stronger than PF-FVAC
Section 2.11.3		$\cancel{\text{FHWC}}$+RIC $\Rightarrow$GIC, $\lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
Section 2.11.2		$\cancel{\text{RIC}} \Rightarrow \cancel{\text{FHWC}}$, $\lim_{m\to\infty} \mathring{\boldsymbol{\kappa}}(m) = 0$
Section 3.3		"Buffer Stock Saving" Conditions
Section 3.3.2		GIC $\Rightarrow \exists \quad 0 < \hat{m} < \infty$: Steady-State
Section 3.3.1		GIC-Nrm $\Rightarrow \exists \quad 0 < \check{m} < \infty$: Target

‡For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines a unique optimal value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < \mathrm{v}(m) < 0$. °RIC, FHWC are necessary as well as sufficient for the perfect foresight case. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc. **In the Friedman/Muth model, the RIC+FHWC are sufficient, but *not* necessary for nondegeneracy

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