1 Unique And Stable Target and Steady State Points

Theorem 1. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC-Nrm (36) holds, there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{1}$$

Moreover, $\check{m}$ is a point of stablity in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{2}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.0.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that $\mathrm{c}(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With $m_t > 0$, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.0.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{Þ}_{\mathsf{R}}] \\ &= \mathbb{E}_t[\text{Þ}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \tag{3}$$

where the last two lines are merely a restatement of the GIC-Nrm (36).

The Intermediate Value Theorem says that if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{4}$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{5}$$

Note that the statement of theorem 1 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails ($\cancel{\text{RIC}}$).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 < \bar{\mathcal{R}}(1 - &\underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under $\cancel{\text{RIC}}$, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (11) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1 \tag{6}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{\th}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{\th} < 1$$

which satisfies our requirement in (6).

Theorem 2. *For a nondegenerate solution to the problem defined in section 2.1, if the GIC (36) holds, there exists a unique cash-on-hand-to-income ratio $\hat{m} > 0$ such that*

$$\hat{m}_{t+1}/m_t = 1 \text{ if } m_t = \check{m}. \tag{7}$$

Moreover, $\hat{m}$ satisfies

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \hat{m}_{t+1} > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \hat{m}_{t+1} < m_t. \end{aligned} \tag{8}$$

The elements of the proof are:

- Existence and continuity of $\hat{m}_{t+1}/m_t$
- Existence of a point where $\hat{m}_{t+1}/m_t = 1$
- $\hat{m}_{t+1} - m_t$ is monotonically decreasing

1.0.4 Existence and Continuity of The Ratio

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1).

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With $m_t > 0$, the ratio $\hat{m}_{t+1}/m_t$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.0.5 Existence of a stable point

Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}\psi_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}\psi_{t+1}] > 0$.

The limit as m_t goes to infinity is

$$\begin{aligned} \lim_{m_t\uparrow\infty} \hat{m}_{t+1}/m_t &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t}\right] \\ &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t}\right] \\ &= \lim_{m_t\uparrow\infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t}\right] \\ &= (\mathsf{R}/\Gamma)\text{\th}_{\mathsf{R}} \\ &= \text{\th}_{\Gamma} \end{aligned} \tag{9}$$

$$< 1$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if $\hat{m}_{t+1}/m_t$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.0.6 $\mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}\psi_{t+1}] - m_t$ and note that

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \hat{m}_{t+1}/m_t < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \hat{m}_{t+1}/m_t = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \hat{m}_{t+1}/m_t > 1, \end{aligned} \tag{10}$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t\right)\right] \\ &= (\mathsf{R}/\Gamma)\left(1 - \mathrm{c}'(m_t)\right) - 1. \end{aligned} \tag{11}$$

Note that the statement of theorem 2 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} - 1 \end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\Gamma} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1 \tag{12}$$

But we showed in section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (12) holds).

Figure 4 plots $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ rather than $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$, and due to the model's nonlinearities the values of m at which the expected growth rate of $\mathbf{c}$ matches Γ is very slightly different from the m at which the growth rate at which expected growth of $\mathbf{m}$ is Γ. But to first order they are the same:

$$\mathbb{E}_t[c_{t+1}\psi_{t+1}] = c_t.$$

$$
\begin{aligned}
\mathbb{E}_t[\mathrm{c}(m_{t+1})\psi_{t+1}] &= c_t \\
\mathbb{E}_t[(\mathrm{c}(m_t) + \mathrm{c}'(m_t)(m_{t+1} - m_t))\, \psi_{t+1}] &= \mathrm{c}(m_t) \\
\mathbb{E}_t[(\mathrm{c}'(m_t)(m_{t+1} - m_t))\, \psi_{t+1}] &= 0 \\
\mathbb{E}_t[m_{t+1}\psi_{t+1}] &= m_t
\end{aligned}
$$