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0.1 The EGM^n in Higher Dimensions

The problem in Section name demonstrates the simplicity of solving problems sequentially. However, as constructed, the problem has only one state variable and one post-decision state variable per stage. Can EGM^n be used to solve higher dimensional problems? In short, yes, but it requires additional thought on interpolation.

0.1.1 A more complex problem

For a demonstration, we now turn to the problem of a worker saving up for retirement. This worker must consume, save, and deposit resources into a tax-advantaged account that can not be liquidated until retirement. In the recursive problem, the worker begins a new period with a liquid account of market resources m_t and an illiquid account of retirement savings n_t. The worker maximizes their utility by choosing consumption c_t and pension deposit d_t. The pension deposit is set aside on a retirement account that is exposed to a risky return, while their post-consumption liquid assets accrue risk-free interest every period. The worker additionally receives an income that faces a permanent (Γ_{t+1}) and a transitory (θ_{t+1}) shock every period. At the age of 65, the worker is retired and their assets are liquidated, at which point the state reduces to one liquid account of market resources. The worker's recursive problem is:

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t) &= \max_{c_t, d_t} u(c_t) + \beta \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1}, n_{t+1}) \right] \\
\text{s.t.} &\quad c_t \geq 0, \quad d_t \geq 0 \\
a_t &= m_t - c_t - d_t \\
b_t &= n_t + d_t + g(d_t) \\
m_{t+1} &= a_t \mathsf{R} / \Gamma_{t+1} + \theta_{t+1} \\
n_{t+1} &= b_t \mathbf{R}_{t+1} // \Gamma_{t+1}
\end{aligned}
\tag{1}
$$

where

$$
\mathrm{g}(d) = \chi \log(1 + d). \tag{2}
$$

This problem can subsequently be broken down into 3 stages: a pension deposit stage, a consumption stage, and an income shock stage.

0.1.2 Breaking down the problem

In the deposit stage, the worker begins with market resources and a retirement savings account. The worker must maximize their value of liquid wealth l_t and retirement balance b_t by choosing a pension deposit d_t, which must be positive. The retirement balance b is the cash value of their retirement account plus their pension deposit and an additional amount $g(d_t)$ that provides an incentive to save for retirement. As we'll see, this additional term will allow us to use the Endogenous Grid Method to solve this subproblem.

$$\begin{aligned} \mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(l_t, b_t) \\ \text{s.t.} \quad d_t \geq 0 \\ l_t = m_t - d_t \\ b_t = n_t + d_t + g(d_t) \end{aligned} \tag{3}$$

After making their pension decision, the worker begins their consumption stage with liquid wealth l_t and retirement balance b_t. From their liquid wealth, the worker must choose a level of consumption to maximize utility and continuation value w_t. After consumption, the worker is left with post-decision states that represent liquid assets a_t and retirement balance b_t, which passes through this problem unaffected because it can't be liquidated until retirement.

$$\begin{aligned} \tilde{\mathfrak{v}}_t(l_t, b_t) = \max_{c_t} u(c_t) + \beta \mathrm{w}_t(a_t, b_t) \\ \text{s.t.} \quad c_t \geq 0 \\ a_t = l_t - c_t \end{aligned} \tag{4}$$

Finally, the post-decision value function w_t represents the value of both liquid and illiquid account balances before the realization of uncertainty regarding the risky return and income shocks. Since we are dealing with a normalized problem, this stage handles the normalization of state variables and value functions into the next period.

$$\begin{aligned} \mathrm{w}_t(a_t, b_t) = \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1}, m_{t+1}) \right] \\ \text{s.t.} \quad a_t \geq 0, \quad b_t \geq 0 \\ m_{t+1} = a_t \mathsf{R} / \Gamma_{t+1} + \theta_{t+1} \\ n_{t+1} = b_t \mathbf{R}_{t+1} / \Gamma_{t+1} \end{aligned} \tag{5}$$

The advantage of conceptualizing this subproblem as a separate stage is that we can construct a function w_t and use it in the prior optimization problems without having to worry about stochastic optimization and taking expectations repeatedly.

0.1.3 The consumption-saving problem

As seen in the consumption stage above, the retirement balance b_t passes through the problem unaffected because it can't be liquidated until retirement. In this sense, it is already a post-decision state variable. To solve this problem, we can use a fixed grid of b_t and for each obtain endogenous consumption and ex-ante market resources using the simple Endogenous Grid Method for the consumption problem.

0.1.4 The pension deposit problem

In the deposit stage, both the state variables and post-decision variables are different since both are affected by the pension deposit decision.

First, we can rewrite the pension deposit problem more compactly:

$$\mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(m_t - d_t, n_t + d_t + \mathrm{g}(d_t)) \tag{6}$$

The first-order condition is

$$\tilde{\mathfrak{v}}_t^l(l_t, b_t)(-1) + \tilde{\mathfrak{v}}_t^b(l_t, b_t)(1 + \mathrm{g}'(d_t)) = 0. \tag{7}$$

Rearranging this equation gives

$$\mathrm{g}'(d_t) = \frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1 \tag{8}$$

where

$$\mathrm{g}'(d) = \frac{\chi}{1+d} \qquad \mathrm{g}'^{-1}(y) = \chi/y - 1 \tag{9}$$

Given that $\mathrm{g}'(d)$ exists and is invertible, we can find

$$\mathfrak{d}_t(l_t, b_t) = \mathrm{g}'^{-1}\left(\frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1\right) \tag{10}$$

Using this, we can back out n_t as

$$\mathfrak{n}_t(l_t, b_t) = b_t - \mathfrak{d}_t(l_t, b_t) - \mathrm{g}(\mathfrak{d}_t(l_t, b_t)) \tag{11}$$

and m_t as

$$\mathfrak{m}_t(l_t, b_t) = l_t + \mathfrak{d}_t(l_t, b_t) \tag{12}$$

In sum, given an exogenous grid (l_t, b_t) we obtain the triple

$(\mathfrak{m}_t(l_t, b_t), \mathfrak{n}_t(l_t, b_t), \mathfrak{d}_t(l_t, b_t))$, which

we can use to

create an interpolator for the decision rule d_t.

To close the solution method, the envelope conditions are

$$\begin{aligned} \mathrm{v}_t^m(m_t, n_t) &= \tilde{\mathfrak{v}}_t^l(l_t, b_t) \\ \mathrm{v}_t^n(m_t, n_t) &= \tilde{\mathfrak{v}}_t^b(l_t, b_t) \end{aligned} \tag{13}$$

0.1.5 Unstructured Grid Interpolation

As in Section name, the resulting endogenous grid is not rectilinear, and in this more complex problem it is not even a regular grid. We can see in Figure Figure 1 that starting from a regular and rectilinear exogenous grid of liquid assets post-consumption l_t and pension balances post-deposit b_t, we obtain Figure Figure 2 which shows an irregular and unstructured endogenous grid of market resources m_t and pension balances pre-deposit n_t.

Figure 1: A regular, rectilinear exogenous grid of pension balances after deposit b_t and liquid assets after consumption l_t.

To interpolate a function defined on an unstructured grid, we use Gaussian Process Regression as in Scheidegger and Bilionis (2019).

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Figure 2: An irregular, unstructured endogenous grid of market resources m_t and pension balances before deposit n_t.

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References

S. Scheidegger and I. Bilionis. Machine learning for high-dimensional dynamic stochastic economies. *Journal of computational science*, 33:68–82, Apr. 2019. ISSN 1877-7503. doi:10.1016/j.jocs.2019.03.004.