EGMn

The Sequential Endogenous Grid Method

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Motivation

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- Structural Economics for modeling **decision-making under uncertainty**
  - household: consumption, savings, labor, portfolio, retirement
  - firms: production, investment, hiring, entry/exit
  - governments: fiscal and monetary policy, taxation, redistribution
  - interdisciplinary: climate change, public health, education, etc.

. . .

- Structural modeling is **hard**
  - modern economics requires solving complex problems
  - with many state variables, many decisions, and non-convexities
  - computationally challenging and time-consuming

Outline

- Functional Approximation
  - Interpolation on different spaces/dimensions
  - Conventional techniques are **insufficient** for complex problems
- Dynamic Programming
  - The Endogenous Grid Method
  - The **Sequential** Endogenous Grid Method

- Machine Learning in Economics
  - Neural Nets as **function approximators**
  - The Deep Learning Revolution
  - **Uncertainty Quantification**
- Conclusion
  - Computational Economics solving increasingly complex problems
  - **Econ-ARK** provides **open source** tools for researchers

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Functional Approximation

Linear Interpolation on a Uniform Grid

videos/LinearInterpolationUniform.mp4

Linear Interpolation on a Non-linear Grid

videos/LinearInterpolationGeometric.mp4

Bilinear Interpolation

videos/BilinearInterpolation.mp4

Curvilinear (Warped) Grid Interpolation

videos/CurvilinearInterpolation.mp4

See: White (2015)

What about Unstructured Grids?

videos/UnstructuredGrid.mp4

See: Ludwig and Schön (2018)

Dynamic Programming

A simple consumption-savings problem

Agent maximizes present discounted value (PDV) of lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \mathrm{u}(c_t) \tag{1}$$

. . .

Recursive Bellman equation

$$\begin{aligned} v_t(m) &= \max_{c_t} \mathrm{u}(c_t) + \beta \mathbb{E}_t \left[v_{t+1}(m_{t+1}) \right] \\ \text{s.t.} &\quad 0 \leq c_t \leq m_t \\ a_t &= m_t - c_t \\ m_{t+1} &= \mathsf{R} a_t + \theta_{t+1} \end{aligned} \tag{2}$$

A simple consumption-savings problem

Recursive Bellman equation

$$
\begin{aligned}
v_t(m) &= \max_{c_t} \mathrm{u}(c_t) + \beta \mathbb{E}_t \left[v_{t+1}(m_{t+1}) \right] \\
\text{s.t.} \quad & 0 \leq c_t \leq m_t \\
a_t &= m_t - c_t \\
m_{t+1} &= \mathsf{R} a_t + \theta_{t+1}
\end{aligned}
\tag{3}
$$

How do we solve this problem?

- **Value Function Iteration** (VFI)
  - Discretize state space (interpolation)
  - Grid search optimization (brute force, iterative)

The Endogenous Grid Method by Carroll (2006)

. . .

$$
c_t = \mathrm{u}'^{-1} \left(\beta \mathbb{E}_t \left[v'_{t+1}(\mathsf{R} a_t + \theta_{t+1}) \right] \right) \tag{4}
$$

. . .

- Simple
  - Inverted Euler equation
- Fast
  - No root-finding or grid search optimization required
- Efficient
  - Finds exact solution at each gridpoint

Limitations of EGM

- **One-dimensional** problems/subproblems (nested)
  - (GEGM) Barillas and Fernández-Villaverde (2007)
  - (NEGM) Druedahl (2021)
- Can result in **non-rectangular grids**
  - (Curvilinear) White (2015)
  - (Triangular) Ludwig and Schön (2018)
- **Non-convexities** (discrete choices) can be problematic
  - (DCEGM) Iskhakov et al. (2017)
  - (G2EGM) Druedahl and Jørgensen (2017)

EGMn: The Sequential Endogenous Grid Method

. . .

- **Insight**: Problems in which agent makes several **simultaneous choices** can be decomposed into **sequence of problems**
- **Problem**: Rectilinear exogenous grid results in **unstructured** endogenous grid
- **Solution**: Using machine learning to **interpolate** on unstructured grids

. . .

Contribution:

- **Simple, Fast, Efficient**
  - Inherits properties of EGM
- **Multi-dimensional**
  - Can be used for problems with multiple state variables and decisions
- **Cutting-edge**
  - Functional approximation and uncertainty quantification approach using Gaussian Process Regression

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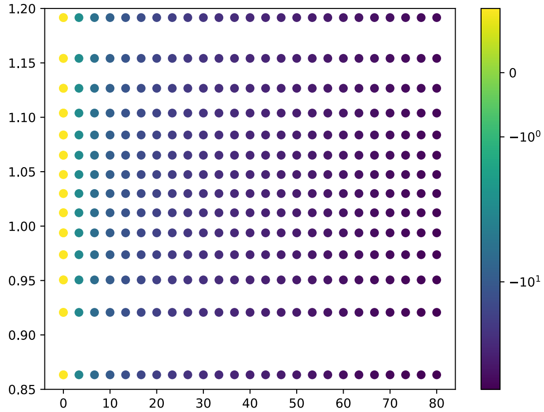
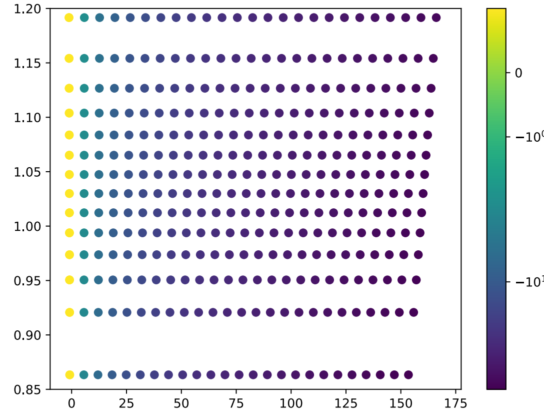
Exogenous Rectangular Grid



Endogenous Curvilinear Grid



A more complex problem

Consumption - Pension Deposit Problem as in Druedahl and Jørgensen (2017)

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t) = \max_{c_t, d_t} u(c_t) + \beta \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1}, n_{t+1}) \right] \\
\text{s.t.} \quad c_t > 0, \quad d_t \geq 0 \\
a_t &= m_t - c_t - d_t \\
b_t &= n_t + d_t + g(d_t) \\
m_{t+1} &= a_t \mathsf{R} / \Gamma_{t+1} + \theta_{t+1} \\
n_{t+1} &= b_t \mathbf{R}_{t+1} / \Gamma_{t+1}
\end{aligned} \tag{5}
$$

where

$$
\mathrm{u}(c) = \frac{c^{1-\rho}}{1-\rho} \qquad \text{and} \qquad \mathrm{g}(d) = \chi \log(1+d). \tag{6}
$$

is a tax-advantaged premium on pension contributions.

Breaking up the problem makes it easier to solve it

Consider the problem of a consumer who chooses how much to put into a pension account:

$$\begin{aligned} \mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(l_t, b_t) \\ \text{s.t.} \quad d_t \geq 0 \\ l_t = m_t - d_t \\ b_t = n_t + d_t + g(d_t) \end{aligned} \tag{7}$$

. . .

After, the consumer chooses how much to consume out of liquid savings:

$$\begin{aligned} \tilde{\mathfrak{v}}_t(l_t, b_t) = \max_{c_t} u(c_t) + \beta \mathrm{w}_t(a_t, b_t) \\ \text{s.t.} \quad c_t \geq 0 \\ a_t = l_t - c_t \end{aligned} \tag{8}$$

Solving the pension problem

The pension problem, more compactly

$$\mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(m_t - d_t, n_t + d_t + \mathrm{g}(d_t)) \tag{9}$$

. . .

Interior solution must satisfy the first-order condition:

$$\mathrm{g}'(d_t) = \frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1 \tag{10}$$

Solving the pension problem

Inverting, we can obtain the optimal choice of d_t:

$$\mathfrak{d}_t(l_t, b_t) = \mathrm{g}'^{-1}\left(\frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1\right) \tag{11}$$

. . .

Using resource constraints we obtain endogenous grids:

$$\mathfrak{n}_t(l_t, b_t) = b_t - \mathfrak{d}_t(l_t, b_t) - \mathrm{g}(\mathfrak{d}_t(l_t, b_t))\mathfrak{m}_t(l_t, b_t) = l_t + \mathfrak{d}_t(l_t, b_t) \tag{12}$$

Unstructured Grids

Problem: **Rectilinear** exogenous grid results in **unstructured** endogenous grid

Exogenous Rectangular Grid

Endogenous Unstructured Grid

How do we **interpolate** on this grid?

Machine Learning in Economics

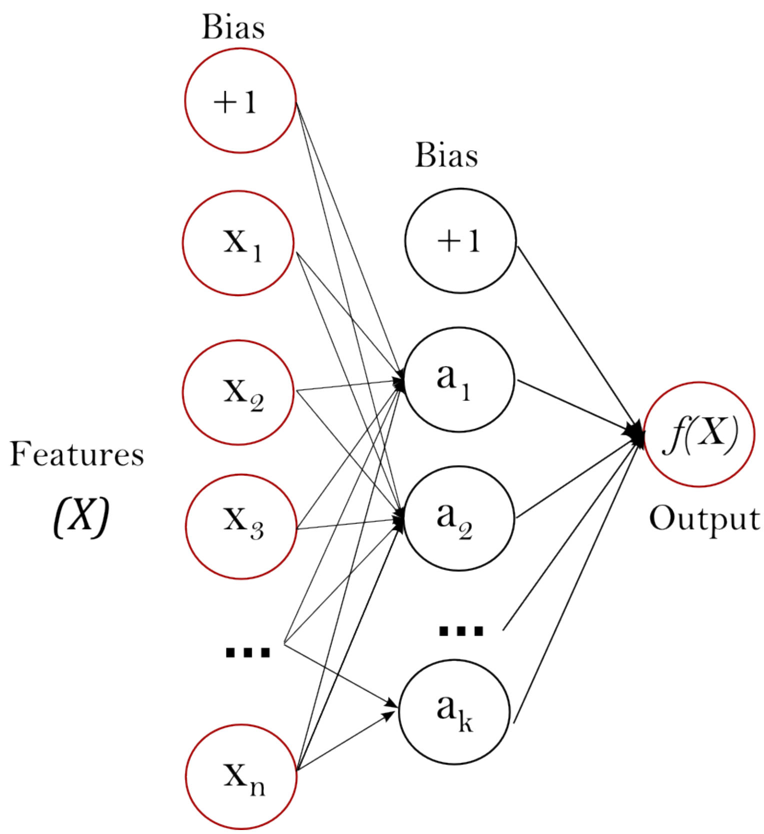


Figure 1: Figure 1: ANN (Source: scikit-learn.org)

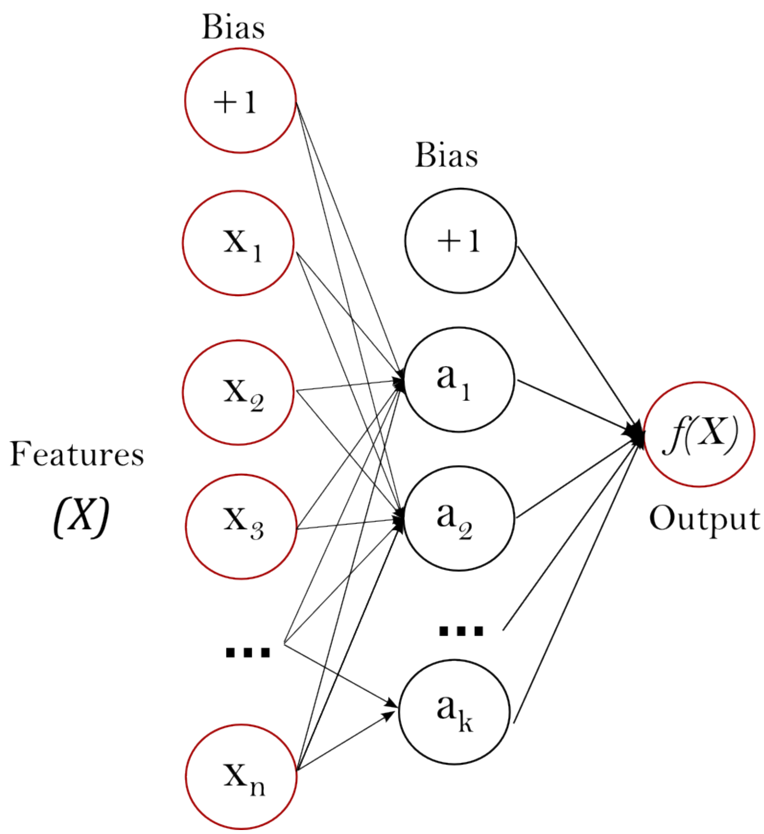


Figure 2: Figure 1: ANN (Source: scikit-learn.org)

Artificial Neural Networks

Artificial Neural Networks

- Based on biological neural pathways (neurons in a brain)
- Learns function $f(X) : R^n \to R^m$
- Consists of
  - input (features) X
  - hidden layers $g(\cdots)$
  - output (target) $y = f(X)$
- Hidden layers can have many nodes
- Neural nets can have many hidden layers (deep learning)

A single neuron, and a bit of math

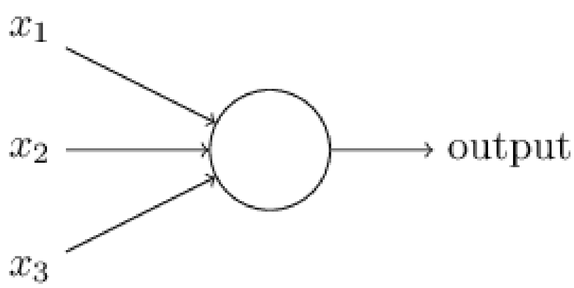


Figure 3: Figure 2: Perceptron

$$y = g(w_0 + \sum_{i=1}^{n} w_i x_i) = g(w_0 + \mathbf{x}'\mathbf{w}) \tag{13}$$

A single neuron, and a bit of math

$$y = g(w_0 + \sum_{i=1}^{n} w_i x_i) = g(w_0 + \mathbf{x}'\mathbf{w}) \tag{14}$$

. . .

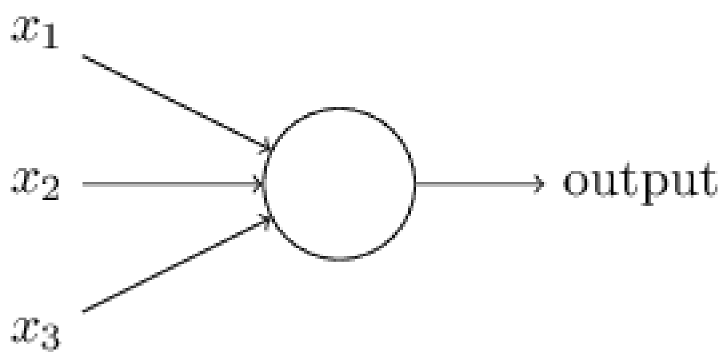


Figure 4: Figure 2: Perceptron

- y is the output or target
- x_i are the inputs or features
- w_0 is the bias
- w_i are the weights

- $g(\cdot)$ is the activation function (non-linear)

$$g(z) = \frac{1}{1 + e^{-z}} \tag{15}$$

- usually a sigmoid, but there are many others

The Deep Learning Revolution

. . .

- Most of these ideas are not new
  - Perceptron (1957)
  - Deep Learning (1965)
  - Stochastic Gradient Descent (1967)

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- What changed?
  - **Big data** (more data)
  - More computing power (**GPUs**, TPUs, etc.)
  - **Algorithmic** innovations (ReLU, Adam, regularization, etc.)

– Better and **open source** software (scikit-learn, TensorFlow, PyTorch, etc.)

Gaussian Process Regression

A Gaussian Process is a probability distribution over functions

$$\begin{aligned} \mathbf{X} \sim \mathcal{N}(\ ,\ ) \quad \text{s.t.} \quad x_i \sim \mathcal{N}(\mu_i, \sigma_{ii}) \\ \text{and} \quad \sigma_{ij} = \mathbb{E}[(x_i - \mu_i)(x_j - \mu_j)] \quad \forall i, j \in \{1, \ldots, n\}. \end{aligned} \tag{16}$$

where

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \qquad = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}. \tag{17}$$

A Gaussian Process Regression is used to find the function that best fits a set of data points

$$\mathbb{P}(\mathbf{f}|\mathbf{X}) = \mathcal{N}(\mathbf{f}|\mathbf{m}, \mathbf{K}) \tag{18}$$

I use standard covariance function, exploring alternatives is an active area of research

$$k(\mathbf{x}_i, \mathbf{x}_j) = \sigma_f^2 \exp\left(-\frac{1}{2l^2}(\mathbf{x}_i - \mathbf{x}_j)'(\mathbf{x}_i - \mathbf{x}_j)\right). \tag{19}$$

An example

Consider the true function $f(x) = x\cos(1.5x)$ sampled at random points

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An example

A random sample of the GP posterior distribution of functions

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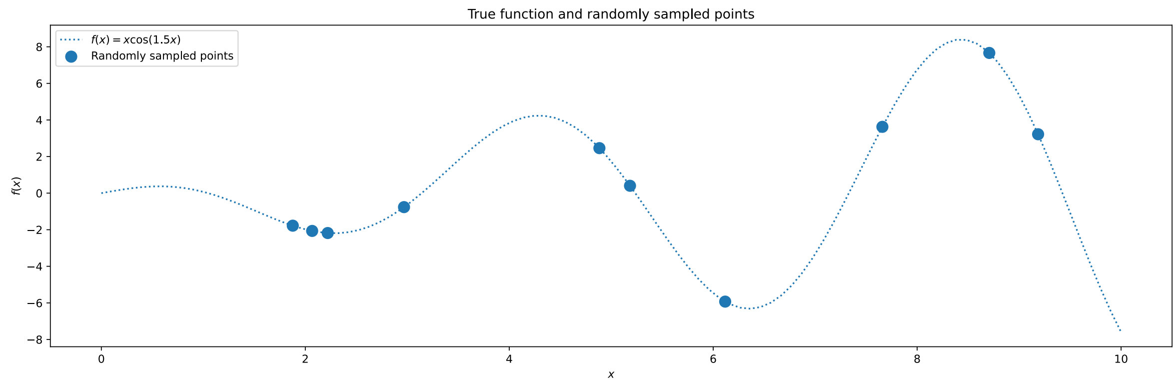


Figure 5: True Function

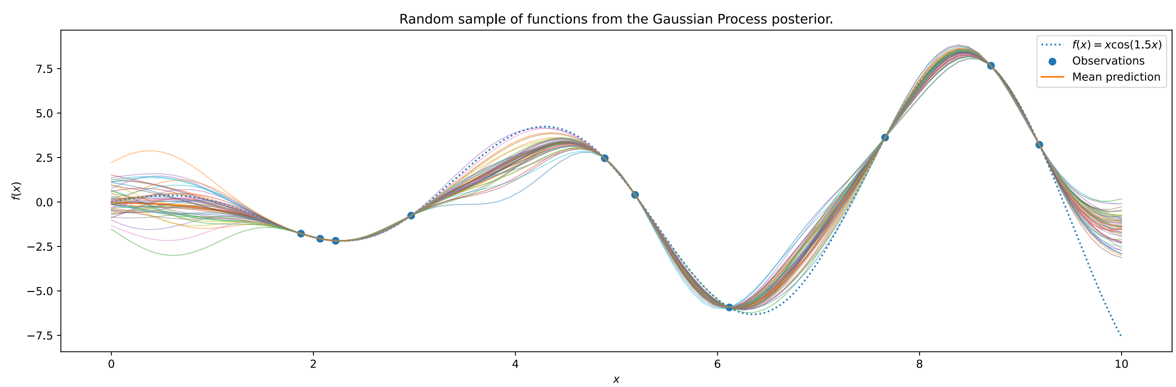


Figure 6: Posterior Sample

An example

Gaussian Process Regression finds the function that best fits the data

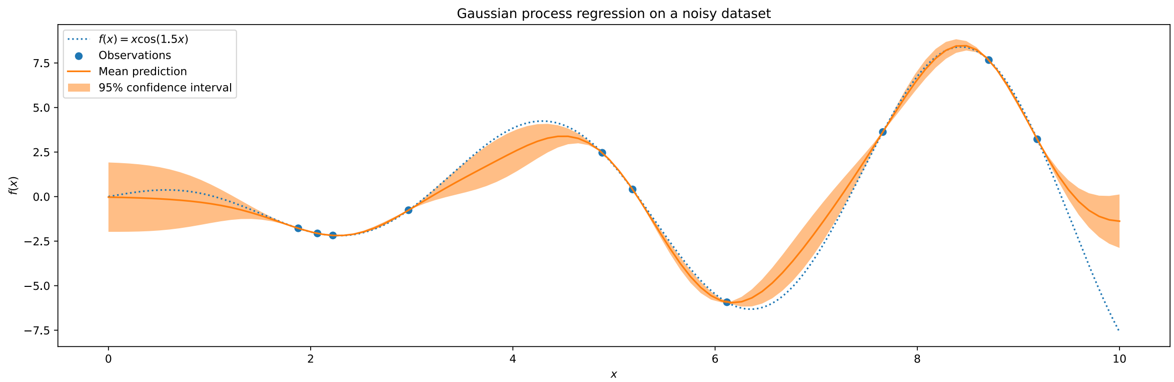
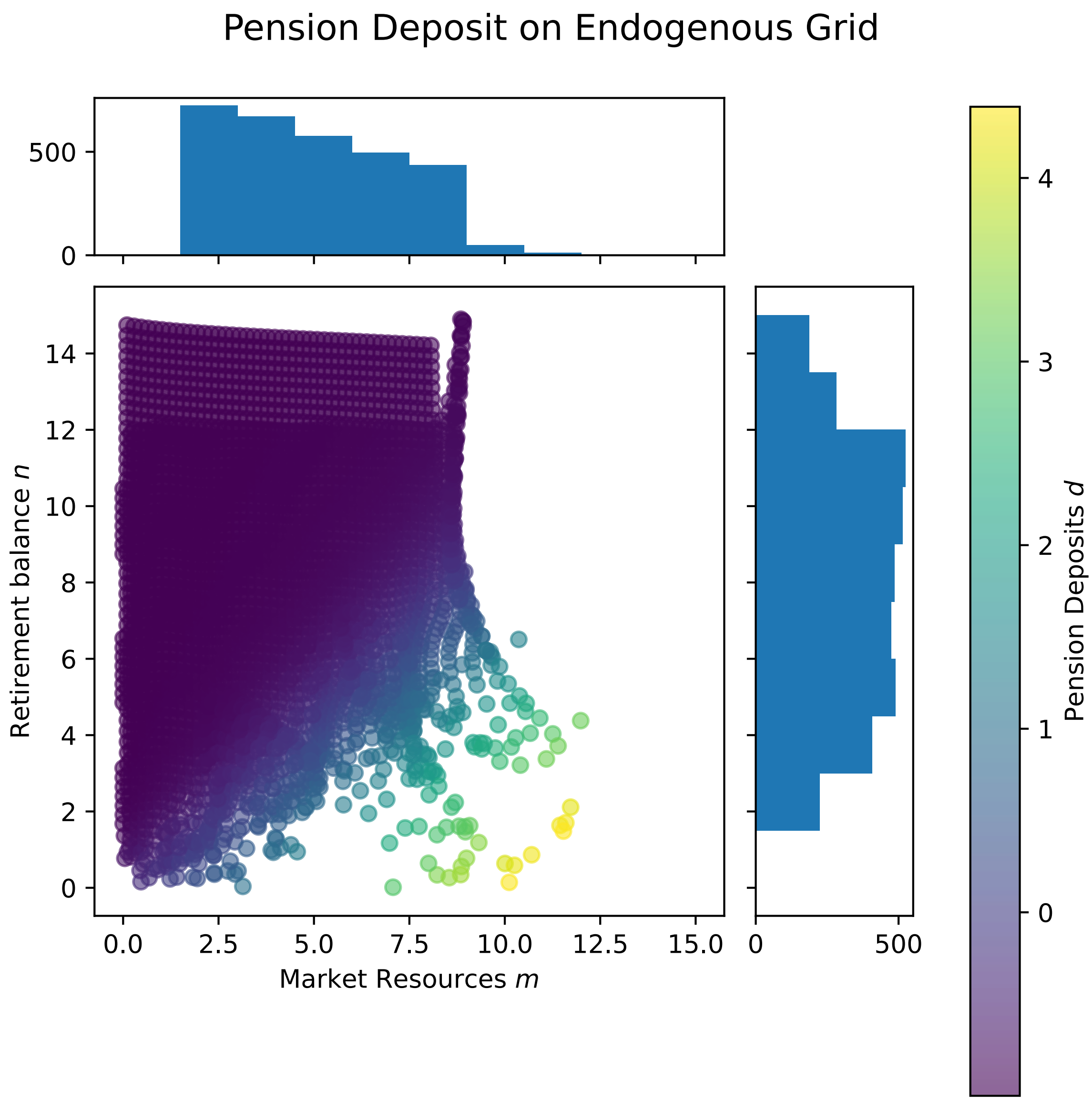


Figure 7: Alt text

- **Gaussian Process Regression** gives us
  - **Mean** function of the posterior distribution
  - **Uncertainty quantification** of the mean function
  - Can be useful to predict ex-post where we might need **more points**

Back to the model

Second Stage Pension Endogenous Grid



Some Results

Consumption Function

Deposit Function

Conclusion

Conditions for using Sequential EGM

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- Model must be
  - concave
  - differentiable
  - continuous
  - separable

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Need an **additional** function to exploit **invertibility**

. . .

Examples in this paper:

- Separable utility function
  - $\mathrm{u}(c, z) = \mathrm{u}(c) + h(z)$
- Continuous and differentiable transition
  - $b_t = n_t + d_t + g(d_t)$

Resources

- An Introduction to Statistical Learning `statlearning.com`
- Neural Networks and Deep Learning `neuralnetworksanddeeplearning.com`
- Deep Learning `deeplearningbook.org`
- Probabilistic machine learning `probml.github.io/pml-book`
- A Neural Network Playground `playground.tensorflow.org`

Thank you!

engine: github.com/econ-ark/HARK

code: github.com/alanlujan91/SequentialEGM

website: alanlujan91.github.io/SequentialEGM/egmn

References

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