EGM^N: The Sequential Endogenous Grid Method

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Alan Lujan

This paper describes a method for solving heterogeneous agent models with many decision choices. It extends the method proposed by (G2EGM) and complements recent work by (NEGM) on how to nest such models. I suggest first solving the pure consumption problem (inner loop) using a standard EGM step. Then, the outer loop is solved using an appropriately designed sequential EGM step. Finally, I used the method for curvilinear or "wraped grid" interpolation as in White (2015).

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Heterogeneous agent models with many decision choices are often solved using inefficient grid search methods that require a large number of points and are often very slow. This paper provides a method for solving such models using a novel extension of the endogenous grid method (EGM) that uses Gaussian Process Regression (GPR) to interpolate functions on unstructured grids. First, separating models into smaller, sequential problems allows for more tractable and digestable problems. Second, using an exogenous grid of post-decision states and solving for an endogenous grid of pre-decision states that obey a first order condition greatly speeds up the solution process. Third, as the resulting endogenous grid can often be curvilinear at best and unstructured at worst, GPR provides an efficient and accurate method for interpolating value, marginal value, and policy functions. Applied sequentially to each smaller problem, the method is able to solve heterogeneous agent models with many decision choices in a fraction of the time required by standard grid search methods. The method is applied to a number of increasingly complex models for demonstration purposes. Software is provided in the form of a Python module under the 'HARK' package.

html: https://llorracc.github.io/HAFiscal/
PDF: HAFiscal.pdf
GitHub: https://github.com/llorracc/HAFiscal

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[1] Department of Economics, The Ohio State University, alujan91@quantmacro.org .

1 Introduction

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1.1 Background

The endogenous grid method (EGM) developed by (Carroll, 1997) has allowed the solving of dynamic optimization problems to be done in a computationally efficient and fast manner. Many problems that before took hours to solve became much more easier to solve and allowed us to focus on estimation and simulation. However, the engodenous grid method is limited to a few class of problems. Recently, the class of problems to which EGM can be applied has been expanded by (cite a few papers), but with every new method comes a new set of limitations. This paper introduces a new approach to EGM in a multivariate setting. The method is called Sequential EGM (or EGM^N) and introduces a novel way of breaking down complex problems into a sequence of simpler, smaller, and more tractable problems, along with an exploration of new multidimensional interpolation methods that can be used to solve these problems.

1.2 Literature Review

The literature review should cite the following:
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1.3 Research Question

1.4 Methodology

1.5 Contributions

1.6 Outline

2 Model

The baseline problem which I will use to demonstrate the Sequential Endogenous Grid Method (EGM^N) is a discrete time version of Bodie, Merton, and Samuelson (1992) where a consumer has the ability to adjust their labor as well as their consumption in response to financial risk. Their objective consists of maximizing their present discounted lifetime utility of consumption and leisure.

$$\max \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(C_{t+n}, Z_{t+n}) \right] \tag{1}$$

In particular, this methodology makes use of a utility function that is based on Example 1 in the paper, which is that of additively separable utility of labor and leisure as

$$
\mathrm{u}(C, Z) = \frac{C^{1-\rho}}{1-\rho} + \nu^{1-\rho} \frac{Z^{1-\zeta}}{1-\zeta} \tag{2}
$$

where the term $\nu^{1-\rho}$ is introduced to allow for a balanced growth path as in Mertens and Ravn (2011).

This model represents a consumer who begins the period with a level of bank balances b_t and a given wage offer θ_t. Subsequently, they are able to choose consumption, labor intensity, and a risky portfolio share with the objective of maximizing their utility of consumption and leisure, as well as their future value of wealth.

The problem can be written in normalized recursive form as

$$
\begin{aligned}
\mathrm{v}_t(b_t, \theta_t) &= \max_{\{c_t, z_t, \varsigma_t\}} u(c_t) + v(z_t) + \beta \, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(b_{t+1}, \theta_{t+1}) \right] \\
&\text{s.t.} \\
\ell_t &= 1 - z_t \\
m_t &= b_t + \theta_t \ell_t \\
a_t &= m_t - c_t \\
\mathbb{R}_{t+1} &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R}) \varsigma_t \\
b_{t+1} &= a_t \mathbb{R}_{t+1} / \Gamma_{t+1}
\end{aligned} \tag{3}
$$

in which ℓ_t is the time supplied to labor net of leisure, m_t is the market resources totaling bank balances and labor income, a_t is the amount of saving assets held by the consumer, and ς_t is the risky share of assets, which induce a $\mathbb{R}_{t+1}$ return on portfolio that results in next period's bank balances b_{t+1} normalized by next period's permanent income Γ_{t+1}.

2.1 The Sequential Endogenous Grid Method

We can make a few choices to create a nested problem as follows. First, the agent decides their labor-leisure trade-off and receives a wage. Their wage plus their previous bank balance then becomes their market resources. Second, given market resources, the agent makes a pure consumption-saving decision. Finally, given an amount of savings, the consumer then decides their risky portfolio share.

Starting from the end of the period, we can define the risky share decision problem as

$$
\begin{aligned}
\mathfrak{v}_t(a_t) &= \max_{\varsigma_t} \beta \, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(b_{t+1}, \theta_{t+1}) \right] \\
&\text{s.t.} \\
\mathbb{R}_{t+1} &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R}) \varsigma_t \\
b_{t+1} &= a_t \mathbb{R}_{t+1} / \Gamma_{t+1}
\end{aligned} \tag{4}
$$

The pure consumption problem is then

$$\begin{aligned}\tilde{\mathfrak{v}}_t(m_t) &= \max_{c_t} u(c_t) + \mathfrak{v}_t(a_t) \\ &\text{s.t.} \\ a_t &= m_t - c_t\end{aligned} \tag{5}$$

And finally, the labor-leisure problem is:

$$\begin{aligned}\mathrm{v}_t(b_t, \theta_t) &= \max_{z_t} v(z_t) + \tilde{\mathfrak{v}}_t(m_t) \\ &\text{s.t.} \\ \ell_t &= 1 - z_t \\ m_t &= b_t + \theta_t \ell_t\end{aligned} \tag{6}$$

This sequential approach is explicitly modeled after the nested approach explored in (NEGM paper). However, I will offer additional mechanisms that expand on NEGM. An important observation is that now, every single choice is self contained to a sub-problem, and although the structure is specifically chosen to minimize the number of state variables at every stage, the problem does not change by this structural imposition. This is because there is no additional information or realization of uncertainty that happens between decisions, as can be seen by the expectation operator being in the last sub-problem. From the perspective of the consumer, these decisions are essentially simultaneous, but a careful organization into sub-period problems enables us to solve the model more efficiently and can provide key economic insights. In this problem, as we will see, a key insight will be the ability to explicitly calculate the marginal value of wealth and the Frisch elasticity of labor.

2.2 Solving the problem

The consumption-saving EGM step should be familiar but I will cover it for exposition.

2.3 Consumption-Savings

We can begin the solution process by restating the consumption-savings subproblem in a more compact form, substituting the market resources constraint and ignoring the no-borrowing constraint for now. The problem is:

$$\tilde{\mathfrak{v}}_t(m_t) = \max_{c_t} u(c_t) + \mathfrak{v}_t(m_t - c_t) \tag{7}$$

To solve, we derive the first order condition with respect to c_t which gives the familiar Euler equation:

$$u'(c_t) = \mathfrak{v}_t'(m_t - c_t) = \mathfrak{v}_t'(a_t) \tag{8}$$

Inverting the above equation is the (first) EGM step.

$$\mathfrak{c}_t(a_t) = u'^{-1}\left(\mathfrak{v}'_t(a_t)\right) \tag{9}$$

Carroll (cite year) demonstrates that by using an exogenous grid of $[a]$ points we can find the unique $\mathfrak{c}_t([a])$ that optimizes the consumption-saving problem, since the first order condition is necessary and sufficient. Further, using the market resources constraint, we can recover the exact amount of market resources that is consistent with this consumption-saving decision as

$$\mathfrak{m}_t([a]) = \mathfrak{c}_t([a]) + [a] \tag{10}$$

This $\mathfrak{m}_t([a])$ is the "endogenous" grid that is consistent with the exogenous decision grid a_t. Now that we have a $(\mathfrak{m}_t([a]), \mathfrak{c}_t([a]))$ pair for each $a \in [a]$, we can construct an interpolating consumption function for market resources points that are off-the-grid.

The envelope condition will be useful in the next section, but for completeness is defined here.

$$\tilde{\mathfrak{v}}'_t(m_t) = \mathfrak{v}'_t(a_t) = u'(c_t) \tag{11}$$

2.4 Labor-Leisure

The labor-leisure sub-problem can also be restated more compactly as:

$$\mathrm{v}_t(b_t, \theta_t) = \max_{z_t} v(z_t) + \tilde{\mathfrak{v}}_t(b_t + \theta_t(1 - z_t)) \tag{12}$$

The first order condition with respect to leisure implies the labor-leisure Euler equation

$$v'(z_t) = \tilde{\mathfrak{v}}'_t(m_t)\theta_t \tag{13}$$

For now, let's assume that $v'(z_t)$ exists and invertible. Using an exogenous grid of $[m_t]$ and $[\theta_t]$, we can find leisure as

$$\mathfrak{z}_t([m_t], [\theta_t]) = v'^{-1}\left(\tilde{\mathfrak{v}}'_t([m_t])[\theta_t]\right) \tag{14}$$

In this case, it's important to note that there are conditions on leisure itself. An agent with a small level of market resources m might want to work more than their available time endowment, especially at higher levels of income θ, if the utility of leisure is not enough to compensate for their low wealth. In these situations, the optimal unconstrained leisure might be negative, so we must impose a constraint on the optimal leisure function. This is similar to the treatment of an artificial borrowing constraint in the pure consumption sub-problem. From now on, let's call this constrained optimal function $\hat{\mathfrak{z}}_t([m_t], [\theta_t])$.

Then, we derive labor as $\mathfrak{l}_t(m_t,\theta_t) = 1 - \hat{\mathfrak{z}}_t(m_t,\theta_t)$. Finally, for each θ_t and m_t as an exogenous grid, we can find the endogenous grid of bank balances as $\mathfrak{b}_t(m_t,\theta_t) = m_t - \theta_t\mathfrak{l}_t(m_t,\theta_t)$.

The envelope condition is simply

$$\mathrm{v}_t^b(b_t,\theta_t) = \tilde{\mathfrak{v}}_t'(m_t) = v'(z_t)/\theta_t \tag{15}$$

2.5 The portfolio decision problem

As useful as it is to be able to use the EGM step more than once, there are clear problems where the EGM step is not applicable. This basic labor-portfolio choice problem demonstrates where we can use an additional EGM step, and where we can not. Now, we go over a sub-problem where we can not use the EGM step.

In reorganizing the labor-portfolio problem into subproblems, we assigned the utility of leisure to the leisure-labor sub-problem and the utility of consumption to the consumption-savings sub-problem. There are no more separable convex utility functions to assign to this problem, and even if we re-organized the problem in a way that moved one of the utility functions into this subproblem, they would not be useful in solving this sub-problem via EGM as there is no direct relation between the risky share of portfolio and consumption or leisure. Therefore, the only way to solve this sub-problem is through standard convex optimization and root-finding techniques.

Restating the problem in compact form:

$$\mathfrak{v}_t(a_t) = \max_{\varsigma_t} \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}\left(a_t(\mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma_t), \theta_{t+1}\right)\right] \tag{16}$$

The first order condition with respect to the risky portfolio share is:

$$\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{-\rho}\mathrm{v}_{t+1}^b\left(b_{t+1},\theta_{t+1}\right)a_t(\mathbf{R}_{t+1} - \mathsf{R})\right] = 0 \tag{17}$$

Finding the optimal risky share requires numerical optimization and root-solving.

To close out the problem, we can calculate the envelope condition as:

$$\mathfrak{v}_t'(a_t) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{-\rho}\mathrm{v}_{t+1}^b\left(b_{t+1},\theta_{t+1}\right)\mathbb{R}_{t+1}\right] \tag{18}$$

2.6 A note on avoiding taking expectations more than once.

We could instead define the portfolio choice sub-problem as:

$$\mathfrak{v}_t(a_t) = \max_{\varsigma_t} \grave{\tilde{\mathfrak{v}}}(a_t,\varsigma_t) \tag{19}$$

where

$$
\begin{aligned}
\grave{\tilde{\mathfrak{v}}}_t(a_t, \varsigma_t) &= \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}\left(b_{t+1}, \theta_{t+1}\right)\right] \\
\mathbb{R}_{t+1} &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma_t \\
b_{t+1} &= a_t \mathbb{R}_{t+1} / \Gamma_{t+1}
\end{aligned}
\tag{20}
$$

In this case, the process is similar. The only difference is that we don't have to take expectations more than once. Given the next period's solution, we can calculate the marginal value functions as:

$$
\begin{aligned}
\grave{\tilde{\mathfrak{v}}}_t^a(a_t, \varsigma_t) &= \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{-\rho} \mathrm{v}_{t+1}'\left(b_{t+1}, \theta_{t+1}\right) \mathbb{R}_{t+1}\right] \\
\grave{\tilde{\mathfrak{v}}}_t^\varsigma(a_t, \varsigma_t) &= \beta\, \mathbb{E}_t \left[\Gamma_{t+1}^{-\rho} \mathrm{v}_{t+1}'\left(b_{t+1}, \theta_{t+1}\right) a_t(\mathbf{R}_{t+1} - \mathsf{R})\right]
\end{aligned}
\tag{21}
$$

If we are clever, we can calculate both of these in one step. Now, The optimal risky share can be found by the first order condition:

$$
\grave{\tilde{\mathfrak{v}}}_t^\varsigma(a_t, \varsigma_t^*) = 0 \tag{22}
$$

and the envelope condition is

$$
\mathfrak{v}_t^a(a_t) = \grave{\tilde{\mathfrak{v}}}_t^a(a_t, \varsigma_t^*) \tag{23}
$$

evaluated at the optimal risky share.

2.7 The Utility functions

Assume

$$
u(c) = \frac{c^{1-\rho}}{1-\rho} \tag{24}
$$

and

$$
v(z) = \zeta \frac{z^{1-\nu}}{1-\nu} \tag{25}
$$

The marginal utility of consumption and its inverse are

$$
u'(c) = c^{-\rho} \qquad u'^{-1}(x) = x^{-1/\rho} \tag{26}
$$

The marginal utility of leisure and it's inverse are

$$
v'(z) = \zeta z^{-\nu} \qquad v'^{-1}(x) = (x/\zeta)^{-1/\nu} \tag{27}
$$

Evidently, both utility functions are invertible and we can proceed with the sequential endogenous grid method.

2.7.1 Alternative

An alternative formulation for the utility of leisure is to state it in terms of the disutility of labor as in (references)

$$v(\ell) = -\zeta \frac{\ell^{1+\nu}}{1+\nu} \tag{28}$$

In this case, we can restate the problem as

$$v(z) = -\zeta \frac{(1-z)^{1+\nu}}{1+\nu} \tag{29}$$

The marginal utility of leisure and it's inverse are

$$v'(z) = \zeta(1-z)^{\nu} \qquad v'^{-1}(x) = 1 - (x/\zeta)^{1/\nu} \tag{30}$$

3 Parameterizing the model

4 The EGM^N in Higher Dimmensions

4.1 A more complex problem

The problem above demonstrates the simplicity of solving problems sequentially. However, as constructed, the problem has only 1 state variable and 1 post-decision state variable per stage. Can EGM^N be used to solve higher dimmensional problems? In short, yes, but it requires additional thought on interpolation.

For a demonstration, we now turn to the problem of a worker saving up for retirement. This worker must consume, save, and deposit resources into a tax-advantaged account that can not be liquidated until retirement. In the recursive problem, the worker begins a new period with a liquid account of market resources m and an illiquid account of retirement savings n. The worker maximizes their utility by choosing consumption c and pension deposit d. The pension deposit is set aside on a retirement account that is exposed to a risky return, while their post-consumption liquid assets accrue risk-free interest every period. The worker additionally receives an income which faces a permanent (Γ) and a transitory (θ) shock every period. At the age of 65, the worker is retired and their assets are liquidated, at which point the state reduces to one liquid account of market resources. The worker's recursive problem is:

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t) = \max_{c_t, d_t} u(c_t) + \beta\,\mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1}, n_{t+1})\right] \\
\text{s.t.} \quad c_t \geq 0, \quad d_t \geq 0 \\
a_t = m_t - c_t - d_t \\
b_t = n_t + d_t + g(d_t) \\
m_{t+1} = a_t \mathsf{R}/\Gamma_{t+1} + \theta_{t+1} \\
n_{t+1} = b_t \mathbf{R}_{t+1}//\Gamma_{t+1}
\end{aligned} \tag{31}
$$

This problem can subsequently be broken down into 3 stages: a pension deposit stage, a consumption stage, and an expectation stage.

In the deposit stage, the worker begins with market resources and a retirement savings account. The worker must maximize their value of liquid wealth l and retirement balance b by choosing a pension deposit d, which must be positive. The retirement balance b is the cash-value of their retirement account plus their pension deposit and an additional amount $g(d_t)$ that provides an incentive to save for retirement. As we'll see, this additional term will allow us to use the endogenous grid method to solve this sub-problem.

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(l_t, b_t) \\
\text{s.t.} \quad d_t \geq 0 \\
l_t = m_t - d_t \\
b_t = n_t + d_t + g(d_t)
\end{aligned} \tag{32}
$$

After making their pension decision, the worker begins their consumption stage with liquid wealth l and retirement balance b. From their liquid wealth, the worker must choose a level of consumption to maximize utility and continuation value w. After consumption, the worker is left with post-decision states that represent liquid assets a and retirement balance b, which passes through this problem unaffected because it can't be liquidated until retirement.

$$
\begin{aligned}
\tilde{\mathfrak{v}}_t(l_t, b_t) = \max_{c_t} u(c_t) + \beta \mathrm{w}_t(a_t, b_t) \\
\text{s.t.} \quad c_t \geq 0 \\
a_t = m_t - c_t - d_t \\
b_t = n_t + d_t + g(d_t)
\end{aligned} \tag{33}
$$

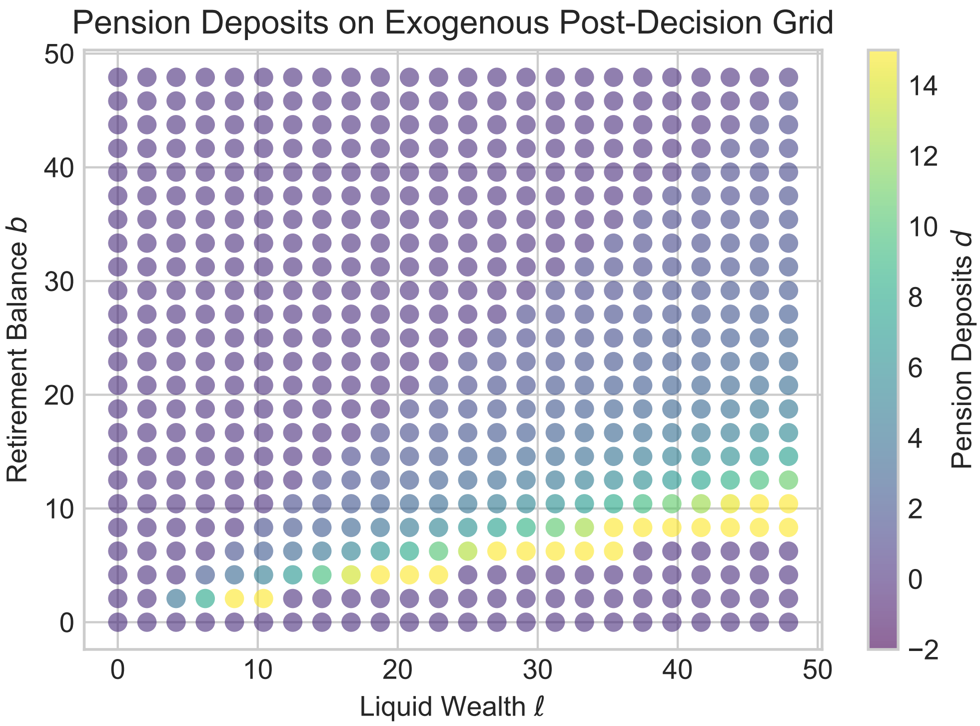
Finally, the post-decision value function w represents the value of both liquid and illiquid account balances before the realization of uncertainty regarding the risky return and income shocks. Since we are dealing with a normalized problem, this stage handles the normalization of state variables and value functions into the next period. The advantage of conceptualizing this sub-problem as a separate stage is that we can construct a function w and use it in the prior optimization problems without having to worry about stochastic optimization.

$$
\begin{aligned}
\mathrm{w}_t(a_t, b_t) &= \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1}, m_{t+1})\right] \\
\text{s.t.} \quad a_t &\geq 0, \quad b_t \geq 0 \\
m_{t+1} &= a_t\mathsf{R}/\Gamma_{t+1} + \theta_{t+1} \\
n_{t+1} &= b_t\mathbf{R}_{t+1}//\Gamma_{t+1}
\end{aligned}
\tag{34}
$$

4.2 The pass-through case

As seen in the consumption stage above, the retirement balance b passes through the problem unaffected because it can't be liquidated until retirement. In this sense, it is already a post-decision state variable. To solve this problem, we can use a fixed grid of b and for each obtain endogenous consumption and ex-ante market resources using the simple endogenous grid method for the consumption problem.

4.3 Non-rectilinear grids



In the deposit stage, both the state variables and post-decision variables are different since both are affected by the pension deposit decision.

First, we can rewrite the pension deposit problem more compactly as:

$$\mathrm{v}_t(m_t, n_t) = \max_{d_t} \tilde{\mathfrak{v}}_t(m_t - d_t, n_t + d_t + \mathrm{g}(d_t)) \tag{35}$$

The first order condition is

$$\tilde{\mathfrak{v}}_t^l(l_t, b_t)(-1) + \tilde{\mathfrak{v}}_t^b(l_t, b_t)(1 + \mathrm{g}'(d_t)) = 0 \tag{36}$$

Rearranging this equation gives

$$\mathrm{g}'(d_t) = \frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1 \tag{37}$$

Assuming that $\mathrm{g}'(d)$ exists and is invertible, we can find

$$\mathfrak{d}_t(l_t, b_t) = \mathrm{g}'^{-1}\left(\frac{\tilde{\mathfrak{v}}_t^l(l_t, b_t)}{\tilde{\mathfrak{v}}_t^b(l_t, b_t)} - 1\right) \tag{38}$$

Using this, we can back out n_t as

$$\mathfrak{n}_t(l_t, b_t) = b_t - \mathfrak{d}_t(l_t, b_t) - \mathrm{g}(\mathfrak{d}_t(l_t, b_t)) \tag{39}$$

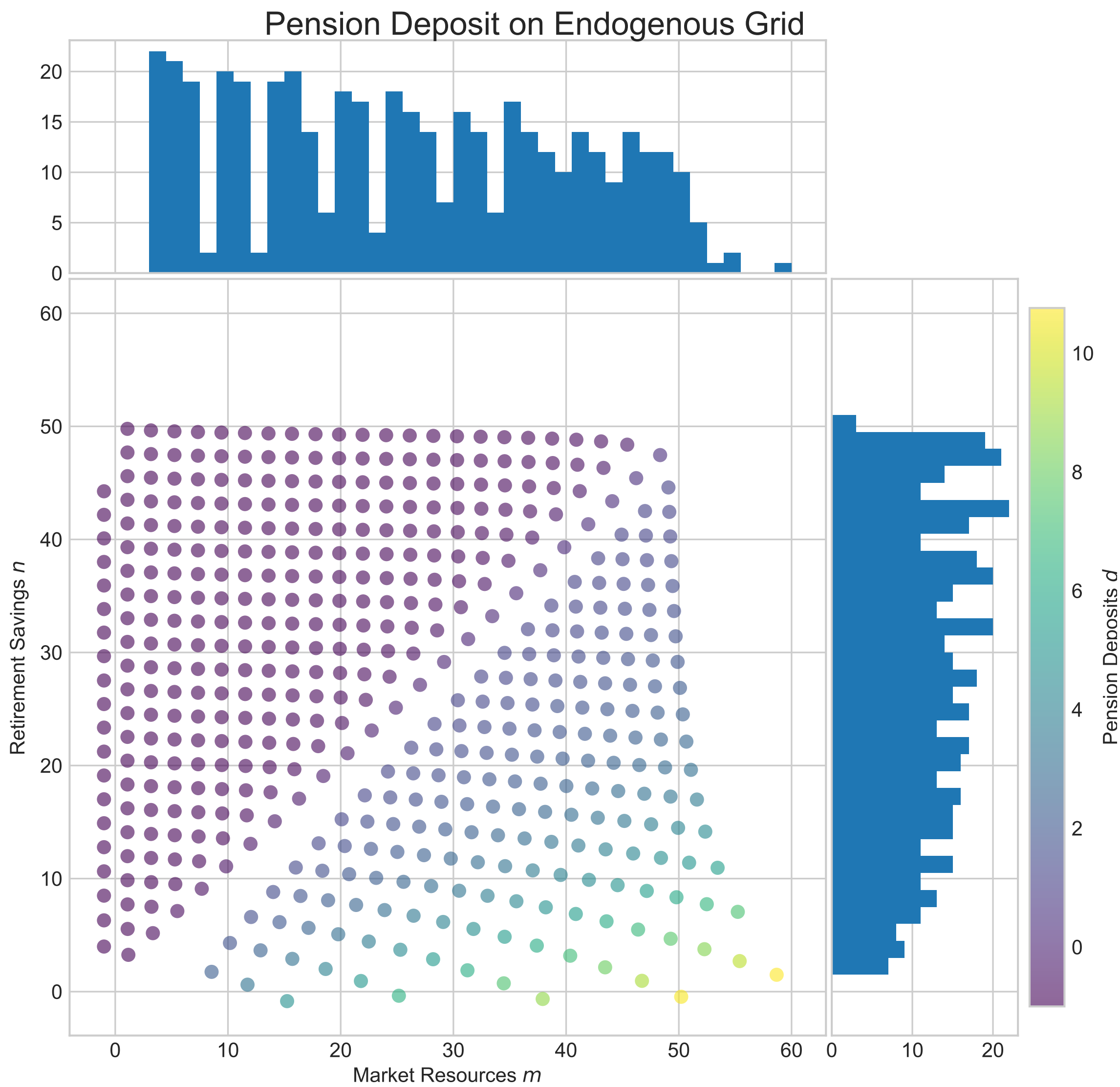
and m_t as

$${}_t(l_t, b_t) = l_t + \mathfrak{d}_t(l_t, b_t) \tag{40}$$

In sum, given an exogenous grid (l_t, b_t) we obtain the triple $({}_t(l_t, b_t), \mathfrak{n}_t(l_t, b_t), \mathfrak{d}_t(l_t, b_t))$, which we can use to create an interpolator for the decision rule d_t.

To close the solution method, the envelope conditions are

$$\begin{aligned}
\mathrm{v}_t^m(m_t, n_t) &= \tilde{\mathfrak{v}}_t^l(l_t, b_t) \\
\mathrm{v}_t^n(m_t, n_t) &= \tilde{\mathfrak{v}}_t^b(l_t, b_t)
\end{aligned} \tag{41}$$



4.4 Unstructured Grid Interpolation

5 Comparing fiscal stimulus policies

6 Solving the illustrative G2EGM model with EGM^N

6.1 The problem for a reitired household

I designate as $\mathrm{w}_t(m_t)$ the problem of a retired household at time t with total resources m. The retired household solves a simple consumption-savings problem with no income uncertainty and a certain next period pension of $\underline{\theta}$.

$$
\begin{aligned}
\mathrm{w}_t(m_t) &= \max_{c_t} u(c_t) + \beta \mathrm{w}_{t+1}(m_t) \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \mathsf{R}_a a_t + \underline{\theta}
\end{aligned} \tag{42}
$$

Notice that there is no uncertainty and the household receives a retirement income $\underline{\theta}$ every period until death.

6.2 The problem for a worker household

The value function of a worker household is

$$
\mathrm{V}_t(m_t, n_t) = \mathbb{E}_\epsilon \max \left\{ \mathrm{v}_t(m_t, n_t, \mathbb{W}) + \sigma_\epsilon \epsilon_{\mathbb{W}}, \mathrm{v}_t(m_t, n_t, \mathbb{R}) + \sigma_\epsilon \epsilon_{\mathbb{R}} \right\} \tag{43}
$$

where the choice specific problem for a working household that decides to continue working is

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t, \mathbb{W}) &= \max_{c_t, d_t} u(c_t) - \alpha + \beta\, \mathbb{E}_t \left[\mathrm{V}_{t+1}(m_{t+1}, n_{t+1}) \right] \\
&\text{s.t.} \\
a_t &= m_t - c_t - d_t \\
b_t &= n_t + d_t + (d_t) \\
m_{t+1} &= \mathsf{R}_a a_t + \theta_{t+1} \\
n_{t+1} &= \mathsf{R}_b b_t
\end{aligned} \tag{44}
$$

and the choice specific problem for a working household that decides to retire is

$$
\mathrm{v}_t(m_t, n_t, \mathbb{R}) = \mathrm{w}_t(m_t + n_t) \tag{45}
$$

6.3 Applying the Sequential EGM

The first step is to define a post-decision value function. Once the household decides their level of consumption and pension deposits, they are left with liquid assets they are saving for the future and illiquid assets in their pension account which they can't access again until retirement. The post-decision value function can be defined as

$$
\begin{aligned}
\mathfrak{v}_t(a_t, b_t) &= \beta\, \mathbb{E}_t \left[\mathrm{V}_{t+1}(m_{t+1}, n_{t+1}) \right] \\
&\text{s.t.} \\
m_{t+1} &= \mathsf{R}_a a_t + \theta_{t+1} \\
n_{t+1} &= \mathsf{R}_b b_t
\end{aligned} \tag{46}
$$

Then redefine the working agent's problem as

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t, \mathbb{W}) &= \max_{c_t, d_t} u(c_t) - \alpha + \mathfrak{v}_t(a_t, b_t) \\
a_t &= m_t - c_t - d_t \\
b_t &= n_t + d_t + (d_t)
\end{aligned}
\tag{47}
$$

Clearly, the structure of the problem remains the same, and this is the problem that G2EGM solves. We've only moved some of the stochastic mechanics out of the problem. Now, we can apply the sequential EGM^N method. Let the agent first decide d_t, the deposit amount into their retirement; we will call this the deposit problem, or outer loop. Thereafter, the agent will have net liquid assets of $_t$ and pension assets of b_t.

$$
\begin{aligned}
\mathrm{v}_t(m_t, n_t, \mathbb{W}) &= \max_{d_t} \tilde{\mathfrak{v}}_t(_t, b_t) \\
&\text{s.t.} \\
_t &= m_t - d_t \\
b_t &= n_t + d_t + (d_t)
\end{aligned}
\tag{48}
$$

Now, the agent can move on to picking their consumption and savings; we can call this the pure consumption problem or inner loop.

$$
\begin{aligned}
\tilde{\mathfrak{v}}_t(_t, b_t) &= \max_{c_t} u(c_t) - \alpha + \mathfrak{v}_t(a_t, b_t) \\
&\text{s.t.} \\
a_t &=_t - c_t
\end{aligned}
\tag{49}
$$

Because we've already made the pension decision, the amount of pension assets does not change in this loop and it just passes through to the post-decision value function.

6.4 Solving the problem

6.4.1 Solving the Inner Consumption Saving Problem

Let's start with the pure consumption-saving problem, which we can summarize by substitution as

$$
\tilde{\mathfrak{v}}_t(_t, b_t) = \max_{c_t} u(c_t) - \alpha + \mathfrak{v}_t(_t - c_t, b_t) \tag{50}
$$

The first order condition is

$$
u'(c_t) = \mathfrak{v}_t^a(_t - c_t, b_t) = \mathfrak{v}_t^a(a_t, b_t) \tag{51}
$$

We can invert this Euler equation as in standard EGM to obtain the consumption function.

$$\mathfrak{c}_t(a_t, b_t) = u'^{-1}\left(\mathfrak{v}_t^a(a_t, b_t)\right) \tag{52}$$

Again as before, $\mathfrak{l}_t(a_t, b_t) = \mathfrak{c}_t(a_t, b_t) + a_t$. To sum up, using an exogenous grid of (a_t, b_t) we obtain the trio $(\mathfrak{c}_t(a_t, b_t), \mathfrak{l}_t(a_t, b_t), b_t)$ which provides an interpolating function for our optimal consumption decision rule over the $(, b)$ grid. Without loss of generality, assume $\mathfrak{l}_t = \mathfrak{l}_t(a_t, b_t)$ and define the interpolating function as

$$\check{c}_t(\mathfrak{l}_t, b_t) \equiv \mathfrak{c}_t(a_t, b_t) \tag{53}$$

For completeness, we derive the envelope conditions as well, and as we will see, these will be useful when solving the next section.

$$\begin{aligned} \tilde{\mathfrak{v}}_t(_t, b_t) &= \mathfrak{v}_t^a(a_t, b_t) = u'(c_t) \\ \tilde{\mathfrak{v}}_t^b(_t, b_t) &= \mathfrak{v}_t^b(a_t, b_t) \end{aligned} \tag{54}$$

6.4.2 Solving the Outer Pension Deposit Problem

Now, we can move on to solving the deposit problem, which we can also summarize as

$$\mathrm{v}_t(m_t, n_t, \mathbb{W}) = \max_{d_t} \tilde{\mathfrak{v}}_t(m_t - d_t, n_t + d_t + (d_t)) \tag{55}$$

The first order condition is

$$\tilde{\mathfrak{v}}_t(_t, b_t)(-1) + \tilde{\mathfrak{v}}_t^b(_t, b_t)(1 +' (d_t)) = 0 \tag{56}$$

Rearranging this equation gives

$$'(d_t) = \frac{\tilde{\mathfrak{v}}_t(_t, b_t)}{\tilde{\mathfrak{v}}_t^b(_t, b_t)} - 1 \tag{57}$$

Assuming that $'(d)$ exists and is invertible, we can find

$$\mathfrak{d}_t(_t, b_t) =^{'-1} \left(\frac{\tilde{\mathfrak{v}}_t(_t, b_t)}{\tilde{\mathfrak{v}}_t^b(_t, b_t)} - 1 \right) \tag{58}$$

Using this, we can back out n_t as

$$\mathfrak{n}_t(_t, b_t) = b_t - \mathfrak{d}_t(_t, b_t) - (\mathfrak{d}_t(_t, b_t)) \tag{59}$$

and m_t as

$$_t(_t, b_t) =_t + \mathfrak{d}_t(_t, b_t) \tag{60}$$

In sum, given an exogenous grid $({}_t, b_t)$ we obtain the triple $({}_t({}_t, b_t), \mathfrak{n}_t({}_t, b_t), \mathfrak{d}_t({}_t, b_t))$, which we can use to create an interpolator for the decision rule d_t.

To close the solution method, the envelope conditions are

$$\begin{aligned} \mathrm{v}_t^m(m_t, n_t, \mathbb{W}) &= \tilde{\mathfrak{v}}_t({}_t, b_t) \\ \mathrm{v}_t^n(m_t, n_t, \mathbb{W}) &= \tilde{\mathfrak{v}}_t^b({}_t, b_t) \end{aligned} \tag{61}$$

6.5 Is g invertible?

We've already seen that $u'(\cdot)$ is invertible, but is ?

$$(d) = \chi \log(1+d) \qquad {}'(d) = \frac{\chi}{1+d} \qquad {}'^{-1}(y) = \chi/y - 1 \tag{62}$$

6.6 The Post-Decision Value and Marginal Value Functions

$$\begin{aligned} \mathfrak{v}_t(a, b) &= \beta\, \mathbb{E}_t \left[\mathrm{V}(m_{t+1}, n_{t+1})\right] \\ &\text{s.t.} \\ m_{t+1} &= \mathsf{R}_a a_t + \theta_{t+1} \\ n_{t+1} &= \mathsf{R}_b b_t \end{aligned} \tag{63}$$

and

$$\begin{aligned} \mathfrak{v}_t^a(a_t, b_t) &= \beta \mathsf{R}_a\, \mathbb{E}_t \left[\mathrm{V}_{t+1}^m(m_{t+1}, n_{t+1})\right] \\ &\text{s.t.} \\ m_{t+1} &= \mathsf{R}_a a_t + \theta_{t+1} \\ n_{t+1} &= \mathsf{R}_b b_t \end{aligned} \tag{64}$$

and

$$\begin{aligned} \mathfrak{v}_t^b(a_t, b_t) &= \beta \mathsf{R}_b\, \mathbb{E}_t \left[\mathrm{V}_{t+1}^n(m_{t+1}, n_{t+1})\right] \\ &\text{s.t.} \\ m_{t+1} &= \mathsf{R}_a a_t + \theta_{t+1} \\ n_{t+1} &= \mathsf{R}_b b_t \end{aligned} \tag{65}$$

6.7 Taste Shocks

From decision choice theory and from DCEGM paper, we know that

$$\mathbb{E}_t \left[\mathrm{V}_{t+1}(m_{t+1}, n_{t+1}, \epsilon_{t+1})\right] = \sigma \log \left[\sum_{\mathbb{D} \in \{\mathbb{W}, \mathbb{R}\}} \exp\left(\frac{\mathrm{v}_{t+1}(m_{t+1}, n_{t+1}, \mathbb{D})}{\sigma_\epsilon} \right) \right] \tag{66}$$

and

$$\mathbb{P}_t(\mathbb{D} \mid m_{t+1}, n_{t+1}) = \frac{\exp\left(\mathrm{v}_{t+1}(m_{t+1}, n_{t+1}, \mathbb{D})/\sigma_\epsilon\right)}{\sum\limits_{\mathbb{D}\in\{\mathbb{W},\mathbb{R}\}} \exp\left(\frac{\mathrm{v}_{t+1}(m_{t+1},n_{t+1},\mathbb{D})}{\sigma_\epsilon}\right)} \tag{67}$$

the first order conditions are therefore

$$\grave{\tilde{\mathfrak{v}}}_t^m(m_{t+1}, n_{t+1}) = \sum_{\mathbb{D}\in\{\mathbb{W},\mathbb{R}\}} \mathbb{P}_t(\mathbb{D} \mid m_{t+1}, n_{t+1}) \mathrm{v}_{t+1}^m(m_{t+1}, n_{t+1}, \mathbb{D}) \tag{68}$$

7 When can EGM^N be used?

8 Robustness

9 Conclusion

Appendices

A Estimating discount factor distributions for different interest rates