Key Continuous Assessment Test -1: September 2022

Programme	:	B.Tech.	Semester	:	Fall 2022-23
Course Title	:	Complex Variables and Linear Algebra	Code	:	BMAT201L
Class No.	:	CH2022231001185, 1186, 1187, 1188, 1189, 1190, 1191, 1192, 1193, 1194, 1195, 1196, 1197	Slot	:	A2+TA2+TAA2

(1)

(a) $v(x,y) = \log\left((x-1)^2+y^2\right)$

$$\frac{\partial v}{\partial x} = \frac{2(x-1)}{(x-1)^2+y^2} \qquad \frac{\partial v}{\partial y} = \frac{2y}{(x-1)^2+y^2}$$

$$\frac{\partial^2 v}{\partial x^2} = 2\left\{\frac{(x-1)^2+y^2-(x-1)\,2(x-1)}{\left((x-1)^2+y^2\right)^2}\right\} = 2\cdot\frac{y^2-(x-1)^2}{\left((x-1)^2+y^2\right)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = 2\left\{\frac{(x-1)^2+y^2-y\cdot 2y}{\left((x-1)^2+y^2\right)^2}\right\} = \frac{2\left((x-1)^2-y^2\right)}{\left((x-1)^2+y^2\right)^2}$$

$\therefore\ v_{xx}+v_{yy}=0$

Hence $v(x,y)$ is harmonic funⁿ. Thus, the harmonic conjugate of $v(x,y)$ exists.

(b) Given $u(x,y) = e^{-2xy}\sin(x^2-y^2)$ —— (1)

diffⁿ (1), with respect to x

$$\frac{\partial u}{\partial x} = e^{-2xy}\cos(x^2-y^2)\,2x - 2y\,e^{-2xy}\sin(x^2-y^2)$$

$$\frac{\partial u}{\partial y} = -2y\,e^{-2xy}\cos(x^2-y^2) - 2x\,e^{-2xy}\sin(x^2-y^2) \quad (2)$$

Now, we know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$\Rightarrow\ dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

by C-R eqⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow dv = +\left\{2y\, e^{-2xy} \cos(x^2-y^2) + 2x\, e^{-2xy} \sin(x^2-y^2)\right\} dx$

$+\left\{2x\, e^{-2xy} \cos(x^2-y^2) - 2y\, e^{-2xy} \sin(x^2-y^2)\right\} dy$ ——(3)

Eq(3) is an exact diffⁿ eqⁿ.

$\therefore \int dv = \int\left\{2y\, e^{-2xy} \cos(x^2-y^2) + 2x\, e^{-2xy} \sin(x^2-y^2)\right\} dx + 0$

~~$v = 2e^{-2y} \int y$~~

$\Rightarrow \quad v = -\int d\left(e^{-2xy} \cos(x^2-y^2)\right) + c$

$\therefore \boxed{v = -e^{-2xy} \cos(x^2-y^2) + c}$

$\therefore f(z) = e^{-2xy} \sin(x^2-y^2) - i\, e^{-2xy} \cos(x^2-y^2) + c$

$= -i\, e^{-2xy} \left(\cos(x^2-y^2) + i \sin(x^2-y^2)\right) + c$

$= -i\, e^{-2xy}\, e^{i(x^2-y^2)} + c$

$= -i\, e^{i(x^2-y^2-\frac{2xy}{i})} + c$

$= -i\, e^{i\{x^2+(iy)^2+2x\,iy\}} + c$

$= -i\, e^{i(x+iy)^2} + c$

$\boxed{f(z) = -i\, e^{iz^2} + c}$

Further.

$\therefore f'(z) = -i\, e^{iz^2}(i2z)$

$\boxed{f'(z) = 2z\, e^{iz^2}}$

(2) (a) $\phi(x,y) = x^2 - y^2 - 2xy - 2x - y - 1$ —(1) → velocity Potential

Let, the velocity field is as

$w = \phi + i\psi$, where ψ is the stream fun^n.

Now to get w, we can use Milne-Thomson's Technique.

$\therefore \quad w = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c.$ ——— (2)

$\cancel{\phi_1(x,y)}$

Diff^n (1), w.r.to x and y

$\frac{\partial \phi}{\partial x} = 2x - 2y - 2 = \phi_1(x,y)$ | $\frac{\partial \phi}{\partial y} = -2y - 2x - 1 = \phi_2(x,y)$

$\boxed{\therefore \phi_1(z,0) = 2z - 2}$ | $\phi_2(z,0) = (-2z - 1)$

$\therefore$ Eq (2) ⇒

$$w = \int (2z-2)\,dz + i\int (2z+1)\,dz + c$$

$$= z^2 - 2z + i(z^2 + z) + c$$

$$\boxed{w = z^2(1+i) + z(i-2) + c}$$ → Complex Potential

(b) $u(x,y) = c_1$ & $v(x,y) = c_2$

$\Rightarrow x^4 - 6x^2y^2 + y^4 = c_1$ ——(1)

$4x^3y - 4xy^3 = c_2$ ——(2)

Diff^n (1) & (2), w.r.to x

$$4x^3 - 12xy^2 - 6x^2\,2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{12xy^2 - 4x^3}{4y^3 + 12x^2y}$$

$$\frac{dy}{dx} = \frac{3xy^2 - x^3}{y^3 + 3x^2y} = m_1$$

Again

$$12x^2y + 4x^3\frac{dy}{dx} - 4y^3 - 4x \cdot 3y^2y' = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{4y^3 - 12x^2y}{4x^3 - 12xy^2}$$

$$= \frac{y^3 - 3x^2y}{x^3 - 3xy^2} = m_2$$

$$\therefore m_1 \cdot m_2 = \frac{-(x^3 - 3xy^2)}{y^3 - 3x^2y} \times \frac{y^3 - 3x^2y}{x^3 - 3xy^2}$$

$$\boxed{m_1 \cdot m_2 = -1} \longrightarrow \text{Orthogonal}$$

3.

$$f(z) = \frac{x^3 + xy^2 + x}{x^2+y^2} + i\,\frac{x^2y + y^3 - y}{x^2+y^2}$$

$$= u + iv$$

$$u_x = \frac{(x^2+y^2)(3x^2+y^2+1) - (x^3+xy^2+x)2x}{(x^2+y^2)^2}$$

$$\cancel{v_y = \frac{(x^2+y^2)\; x^2+3y^2}{(x^2+y^2)^2}}$$

$$= \frac{3x^4 + x^2y^2 + x^2 + 3x^2y^2 + y^4 + y^2 - 2x^4 - 2x^2y^2 - 2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^4 + y^4 - x^2 + y^2 + 2x^2y^2}{(x^2+y^2)^2}$$

$$v_y = \frac{(x^2+y^2)(x^2+3y^2-1) - (x^2y+y^3-y)\,2y}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2 - x^2 + x^2y^2 + 3y^4 - y^2 - 2x^2y^2 - 2y^4 + 2y^2}{(x^2+y^2)^2}$$

$$v_y = \frac{x^4 + y^4 + y^2 - x^2 + x^2y^2}{(x^2+y^2)^2}$$

$$u_y = \frac{(x^2+y^2)2xy - (x^3+xy^2+x)2y}{(x^2+y^2)^2}$$

$$= \frac{2x^3y + 2xy^3 - 2x^3y - 2xy^3 - 2xy}{(x^2+y^2)^2}$$

$$= \frac{4x^3y + 4xy^3 - 2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$v_x = \frac{(x^2+y^2)\,2xy - (x^2y+y^3-y)\,2x}{(x^2+y^2)^2}$$

$$= \frac{2x^3y + 2xy^3 - 2x^3y - 2xy^3 + 2xy}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$\therefore \ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ also, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\longrightarrow$ C-R eqⁿ satisfy.

But at $(0,0)$, the differential coefficients $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are not continious.

$\therefore$ $f(z)$ is analytic in the entire complex plane except at $z=(0,0)$

Vellore Institute Of Technology, Chennai Campus
School of Advanced Sciences (Mathematics)
CAT 1, Module 2
BMAT102L-Complex Variables and Linear Algebra (A2 slot)

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1. Find the linear fractional transformation that maps the points $-1, 0, 1$ on the $z-$plane onto the points $-1, -i, 1$ respectively on the $w-$ plane. Also, find the image of the unit circle $|z| = 1$ under this transformation.

$$\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} = \frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)}$$

$$\frac{(z+1)(0-1)}{(z-1)(0+1)} = \frac{(w+1)(-i-1)}{(w-1)(-i+1)} \Rightarrow \frac{z+1}{z-1} = \frac{(w+1)(1+i)}{(w-1)(1-i)}$$

$$\Rightarrow (z+1)(w-1) = \frac{(1+i)^2}{2}(z-1)(w+1)$$

$$\Rightarrow zw - z + w - 1 = i(zw + z - w - 1) \rightsquigarrow z(w-1) + (w-1) = i(z(w+1) - (w+1))$$

$$\Rightarrow w(z+1) - (z+1) = i(w(z-1) + z - 1) \qquad \Rightarrow z[(w-1) - i(w+1)] = -i(w+1) - (w-1)$$

$$\Rightarrow w[(z+1) - i(z-1)] = iz - i + z + 1$$

$$\Rightarrow w = \frac{(1+i)z + (1-i)}{(1-i)z + (1+i)} \quad 5 \qquad \Rightarrow z = \frac{(1+i)w + (-1+i)}{(-1+i)w + (1+i)} \quad 2$$

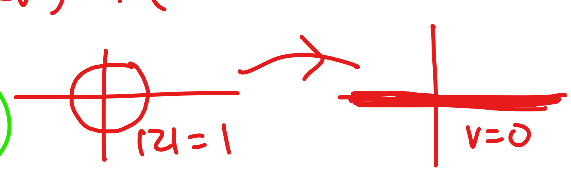
$$|z| = 1 \Rightarrow |(1+i)w + (-1+i)| = |(-1+i)w + (1+i)|$$

$$\Rightarrow |u + iv + iu - v - 1 + i| = |-u - iv + iu - v + 1 + i|$$

2

$$\Rightarrow (u-v-1)^2 + (u+v+1)^2 = (1-u-v)^2 + (u-v+1)^2$$

$$\Rightarrow 4v = -4v \Rightarrow v = 0. \quad 1$$



2. Find the image of the rectangle $-1 \le x \le 2$, $-\pi < y < \pi$ under the maps (i) $w = e^z$ and (ii) $w = \frac{1}{z}$. Sketch the regions.

$w = e^z \Rightarrow R = e^x \,, \quad \phi = y.$

$x = -1 \Rightarrow R = e^{-1}$ C_1

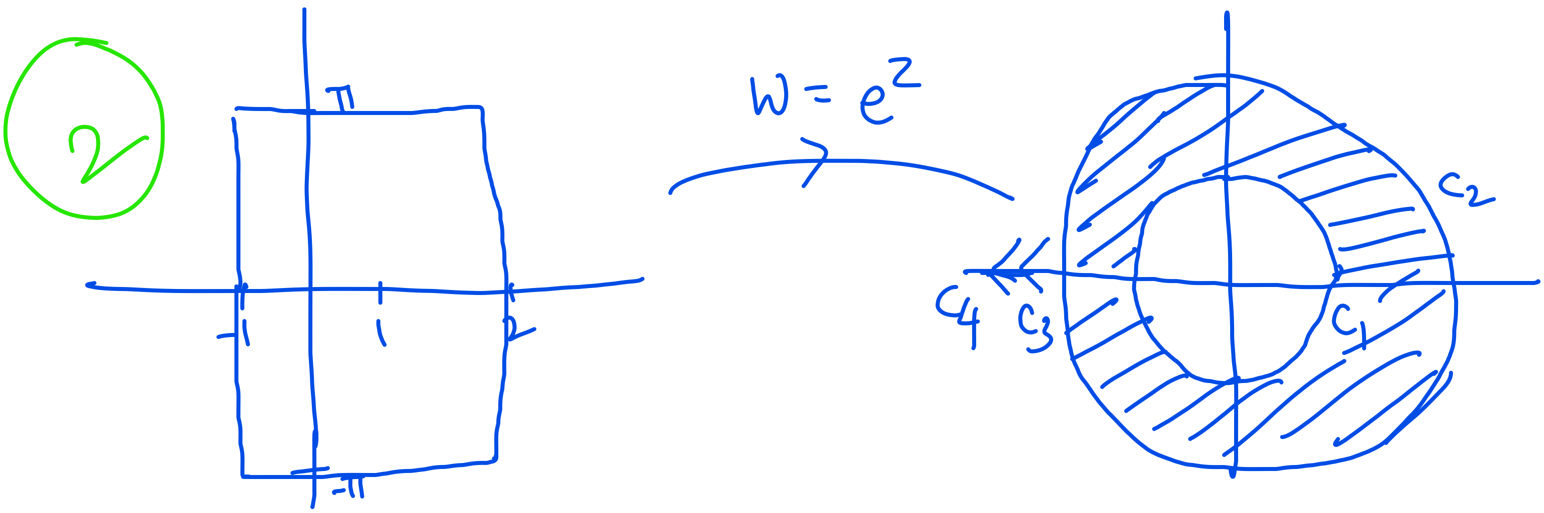
$x = 2 \Rightarrow R = e^2$ C_2

2

$y = \phi = -\pi$ C_3

$y = \phi = \pi.$ C_4

2



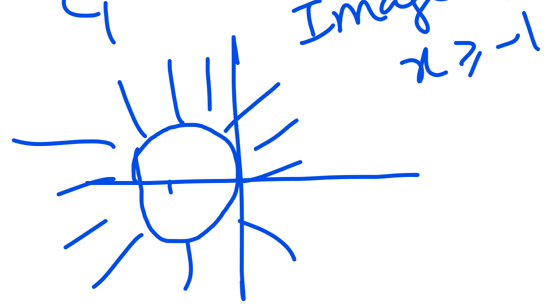
$w = \frac{1}{z} \Rightarrow x = \frac{u}{u^2+v^2} \,, \quad y = \frac{-v}{u^2+v^2}$

$\underline{x = -1}$: $u^2 + v^2 + u = 0$

$\left(u + \frac{1}{2}\right)^2 - \frac{1}{4} + v^2 = 0$

$\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$

1



$\underline{x = 2}$: $\quad u^2 + v^2 - \frac{u}{2} = 0$

1 $\quad \left(u - \frac{1}{4}\right)^2 + v^2 = \frac{1}{16}$

C_2

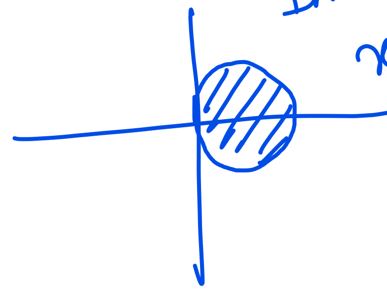


Image of $x \le 2$

$\underline{y = -\pi}$: $\quad u^2 + v^2 - \frac{v}{\pi} = 0$

1 $\quad u^2 + \left(v - \frac{1}{2\pi}\right)^2 = \frac{1}{4\pi^2}$

C_3

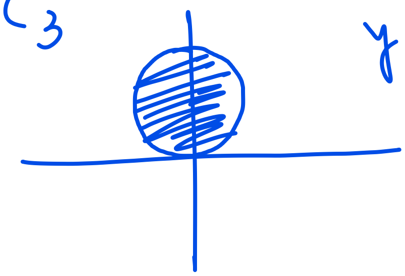


Image of $y > -\pi$

$\underline{y = \pi}$: $\quad u^2 + v^2 + \frac{v}{\pi} = 0$

1 $\quad u^2 + \left(v + \frac{1}{2\pi}\right)^2 = \frac{1}{4\pi^2}$

C_4

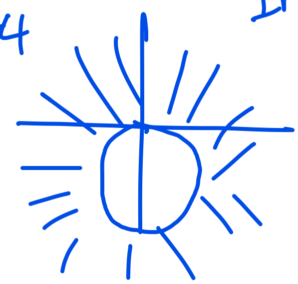
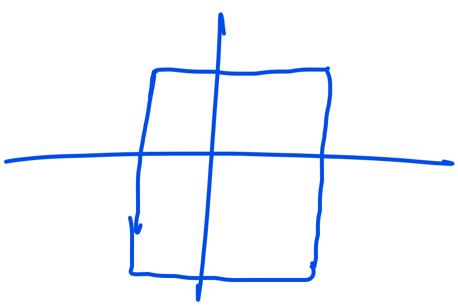
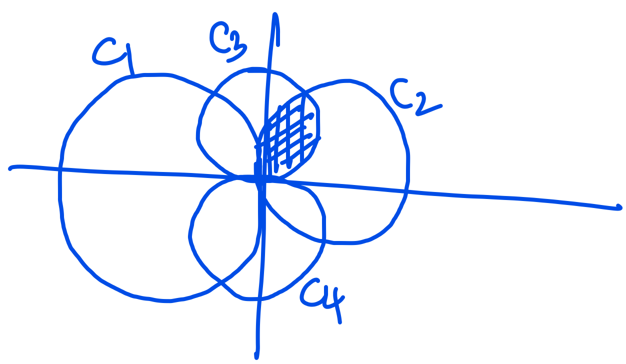


Image of $y < \pi$



$\xrightarrow{w = 1/2}$

2



3. Determine the points where e^{z^5-80z} is not conformal.

e^{z^5-80z} is not conformal at its critical pts.

ie., $\left\{ z \in \mathbb{C} : \frac{d}{dz}\left(e^{z^5-80z}\right) = 0 \right\}$ — 1

$= \left\{ z \in \mathbb{C} : \left(e^{z^5-80z}\right)\left(5z^4-80\right) = 0 \right\}$ — 1

$= \left\{ z \in \mathbb{C} : z^4 = 16 \right\} \quad \left(\because e^{z^5-80z} \neq 0\right)$ — 1

$= \left\{ z \in \mathbb{C} : z^2 = \pm 4 \right\}$

$= \left\{ 2, -2, 2i, -2i \right\}$. — 2

Reg. No.:

Name :

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Class No.	**:**	CH2022231001185, 1186, 1187, 1188, 1189, 1190, 1191, 1192, 1193, 1194, 1195, 1196, 1197	**Slot**	**:**	**A2+TA2+TAA2**
Faculty (s)	**:**	Dr Jaganathan B, Dr Manivannan A, Dr. Felix A Dr Dhivya M, Dr Sudip Debnath, Dr Durga N Dr Prasanna Lakshmi M, Dr Harshavarthini, Dr Ashish Kumar, Dr Kamalesh, Dr Sushmitha, Dr Amit Kumar Rahul, Dr Balaji S	**Max. Marks** **Time**	**:** **:**	**50** **90 Minutes**

Answer ALL questions

Q.No.	**Sub. Sec.**	**Question Description**	**Marks**
1.	a)	Does the harmonic conjugate of the function $v(x, y) = \log_e((x-1)^2 + y^2)$ exist?	**4**
	b)	If $u(x, y) = e^{-2xy}\sin(x^2 - y^2)$ is the real part of an analytic function $f(z) = u + iv$, then find the imaginary part. Also, determine $f'(z)$.	**6**
2.	a)	If $\phi(x, y) = x^2 - y^2 - 2xy - 2x - y - 1$ is the velocity potential of a incompressible fluid flow through a conduit, then calculate the complex potential $w = \phi(x, y) + i\psi(x, y)$.	**5**
	b)	Check the condition for orthogonally for the family of curves $u(x, y) = C_1$ and $v(x, y) = C_2$, when $f(z) = u + iv = (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$.	**5**
3.		Test the analyticity for the given function, $f(z) = \frac{x^3+xy^2+x}{x^2+y^2} + i\,\frac{x^2y+y^3-y}{x^2+y^2}$.	**5**
4.		Find the linear fractional transformation that maps the points $-1, 0, 1$ on the z-plane onto the points $-1, -i, 1$ respectively on the w-plane. Also, find the image of the unit circle $\lvert z\rvert = 1$ under this transformation.	**10**
5.		Find the image of the rectangle $-1 \le x \le 2, -\pi < y < \pi$ under the maps (i) $w = e^z$ and (ii) $w = \frac{1}{z}$. Sketch the regions.	**10**
6.		Determine the points where e^{z^5-80z} is not conformal.	**5**

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