Numerical Methods
Roots of Equations

Description

AIMS

This chapter is aimed to compute the root(s) of the equations by using graphical method and numerical methods.

EXPECTED OUTCOMES

1. Students should be able to find roots of the equations by using graphical approach and incremental search.
2. Students should be able to find the roots of the equations by using bracketing and open methods.
3. Students should be able to provide the comparison between bracketing and open methods.
4. Students should be able to calculate the approximate and true percent relative error.

REFERENCES

1. Norhayati Rosli, Nadirah Mohd Nasir, Mohd Zuki Salleh, Rozieana Khairuddin, Nurfatihah Mohamad Hanafi, Noraziah Adzhar. *Numerical Methods,* Second Edition, UMP, 2017 (Internal use)
2. Chapra, C. S. & Canale, R. P. *Numerical Methods for Engineers*, Sixth Edition, McGraw–Hill, 2010.

Content

INTRODUCTION

- Mathematical model in science and engineering involve equations that need to be solved.
- Equation of one variable can be formulated as

$$f(x) = 0 \qquad (1)$$

- Equation (1) can be in the form of linear and nonlinear.
- Solving equation (1) means that finding the values of *x* that satisfying equation (1).

INTRODUCTION (Cont.)

Equation (1) may belong to one of the following types of equations

1 **Algebraic equations**

2 **Polynomial equations**

3 **Transcendental equations**

A non-algebraic equation of trigonometric, exponential and logarithm function

INTRODUCTION (Cont.)

Example 1: Algebraic Equation

$$4x - 3x^2y - 15 = 0$$

Example 2: Polynomial Equation

$$x^2 + 2x - 4 = 0$$

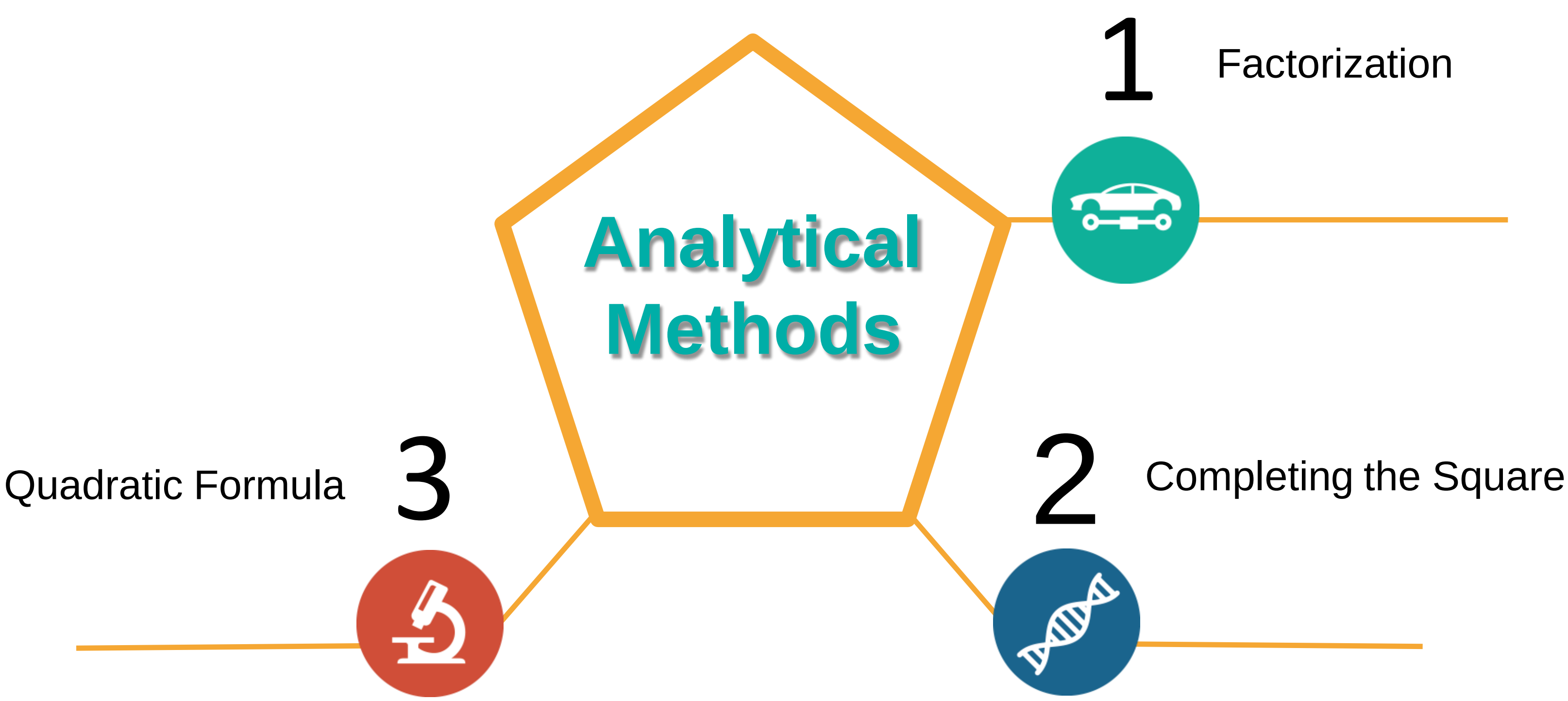
Example 3: Transcendental Equation

$$\sin(2x) - 3x = 0$$

INTRODUCTION (Cont.)

Finding Roots for Quadratic Equations

$$f(x) = ax^2 + bx + c$$



INTRODUCTION (Cont.)

- All above mentioned methods to solve quadratic equations are a**nalytical methods**
- The solution obtained by using analytical methods is called **exact solution**
- Due to the complexity of the equations in modelling the real life system, the exact solutions are often difficult to be found.
- Thus require the used of **numerical methods.**
- The solution that obtained by using numerical methods is called **numerical solution.**

INTRODUCTION (Cont.)

Three types of Numerical Methods shall be considered to find the roots of the equations:

Finding Roots using Numerical Methods

1 **Incremental Search**

2 **Bracketing Methods**

1. Bisection Method
2. False Position Method

3 **Open Methods**

1. Newton Raphson Method
2. Secant Method

Prior to the numerical methods, a **graphical method** of finding roots of the equations are presented.

GRAPHICAL METHOD

- Graphical method is the simplest method
- The given function is plotted on Cartesian coordinate and x −values (roots) that satisfying $f(x) = 0$ is identified.
- x −values (roots) satisfying $f(x) = 0$ provide approximation roots for the underlying equations.
- $f(x)$ can have one or possibly many root(s).

GRAPHICAL METHOD (Cont.)

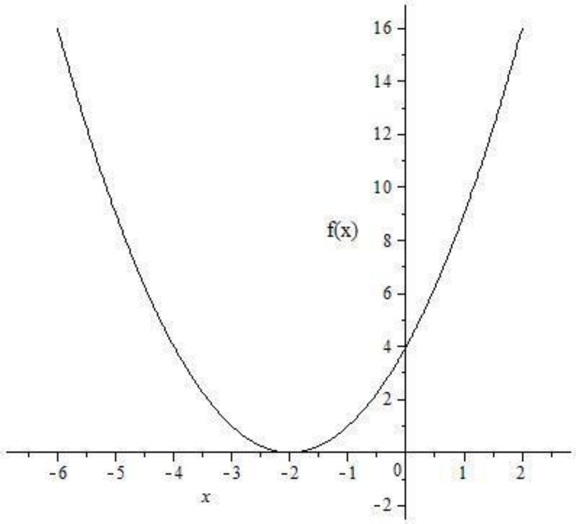


Figure 1 : One Solution

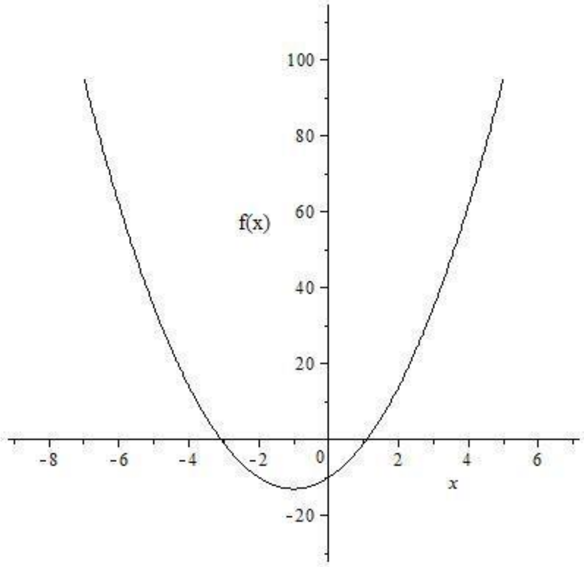


Figure 2 : Two Solutions

GRAPHICAL METHOD (Cont.)

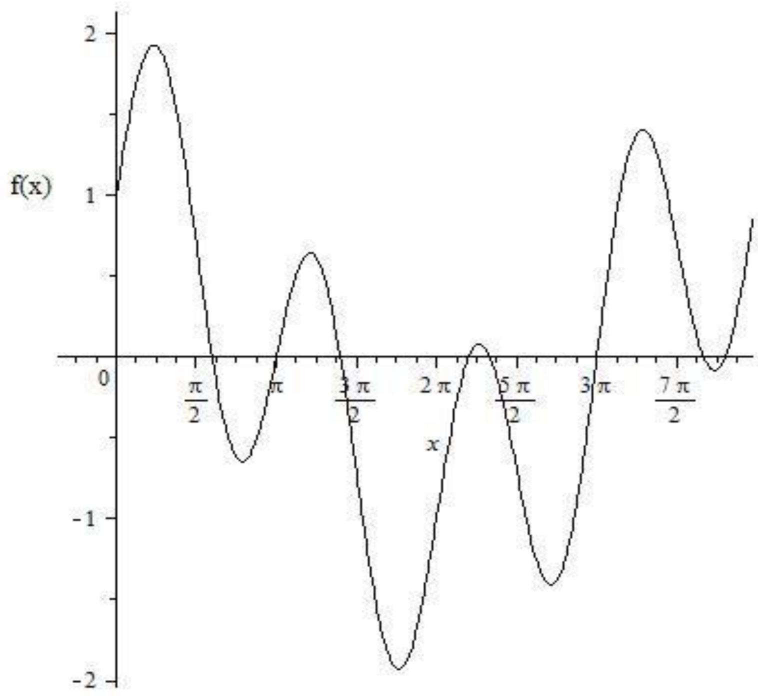


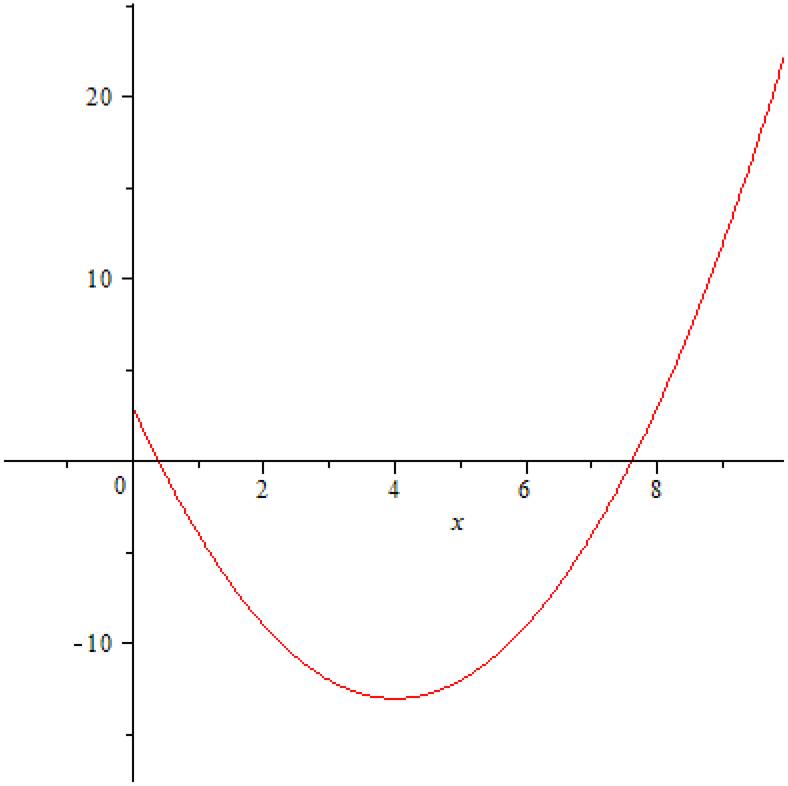
Figure 3 : Many Solutions

GRAPHICAL METHOD (Cont.)

Example 4

Find root(s) of $f(x) = x^2 - 8x + 3$ by using graphical method.

Solution



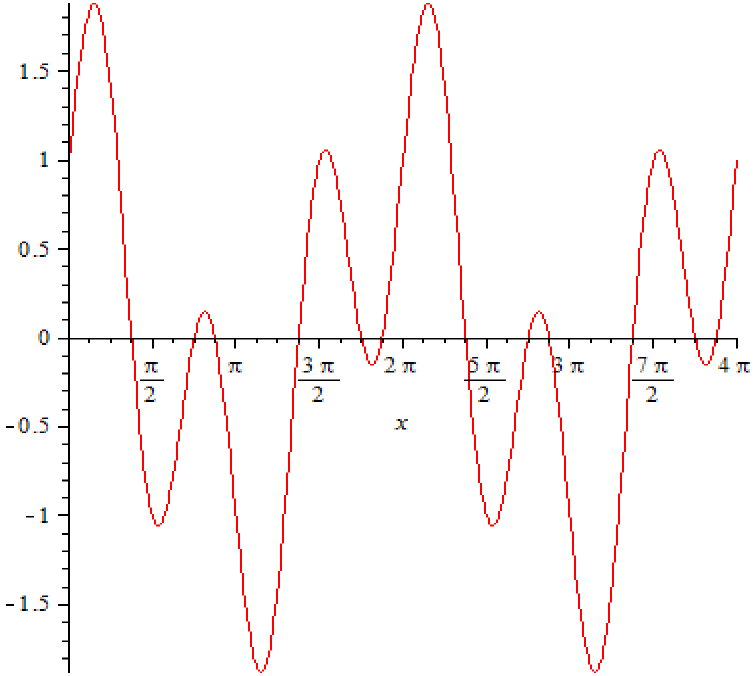
Based on the graph, the function $f(x)$ cross x −axis at two points.
Therefore there are two roots for $f(x)$
The approximate roots of $f(x)$ are 0.364 and 7.663

GRAPHICAL METHOD (Cont.)

Example 5

Find root(s) of $f(x) = \cos(x) + \sin(3x)$ for $0 \le x \le 4\pi$ by using graphical method.

Solution



There are twelve roots for $f(x)$ since the function cross x −axis at twelve points. The approximate roots of $f(x)$ are 1.238, 2.401, 2.701, 4.239, 5.439, 5.852, 7.39, 8.628, 8.966, 10.691, 11.704 and 12.154

GRAPHICAL METHOD (Cont.)

Example 6 [Chapra & Canale]

The velocity of a free falling parachutist is given as

$$v = \frac{gm}{c}\left(1 - e^{-\left(\frac{c}{m}\right)t}\right)$$

Use the graphical approach to determine the drag coefficient, c needed for a parachutist of mass, $m = 68.1$ kg to have a velocity of $40\ \mathrm{m}s^{-1}$ after free falling for time, 10s. Given also gravity is $9.8\ \mathrm{m}s^{-2}$

Solution

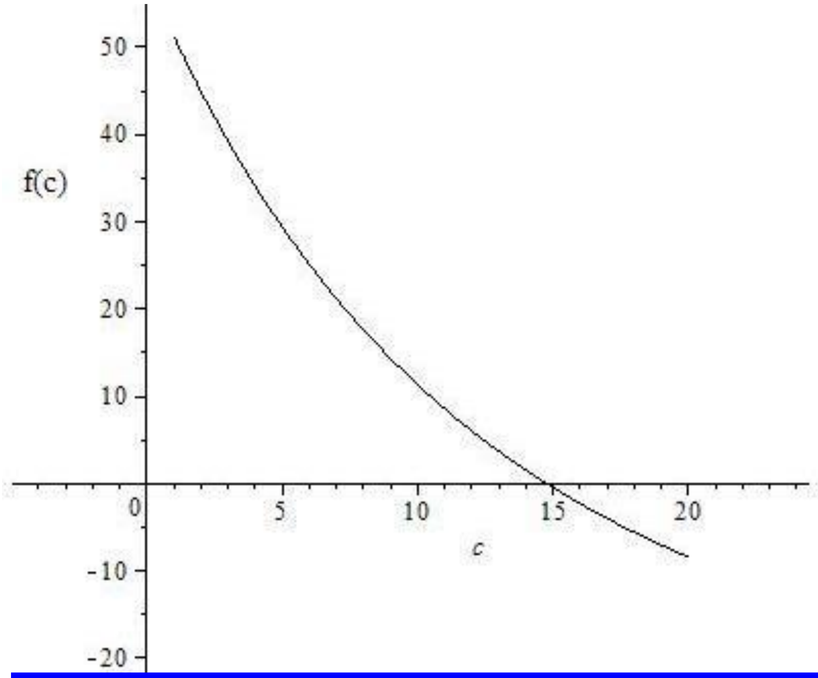
To determine the root of drag coefficient, c. we need to have a function $f(c) = 0$. Substituting the values given in the equation and rearranging yield

$$f(c) = \frac{9.8(68.1)}{c}\left(1 - e^{-\left(\frac{c}{68.1}\right)10}\right) - 40 = 0$$

GRAPHICAL METHOD (Cont.)

Solution (cont.)

Plot the function $f(c)$ and determine where the graph crosses the horizontal axis.



x	$f(x)$
4	34.115
8	17.653
12	6.0670
16	-2.2690
20	-8.4010

Functions have opposite sign

From the graphical view, the root exists between $c = 12$ and $c = 16$, where the functions $f(12)$ and $f(16)$ have opposite sign, that is $f(12) \times f(16) < 0$.

INCREMENTAL SEARCH

- Incremental search is a technique of calculating $f(x)$ for incremental values of x over the interval where the root lies.
- It starts with an initial value, x_0.
- The next value x_n for $n = 1,2,3, \dots$ is calculated by using

  $$x_n = x_{n-1} + h$$

  where h is referred to a step size.
- If the sign of two $f(x)$ changes or if

  $$f(x_n) \cdot f(x_{n-1}) < 0$$

  then the root exist over the prescribed interval of the lower bound, x_l and upper bound, x_u.
- The root is estimated by using

  $$x_r = \frac{x_l + x_u}{2}$$

INCREMENTAL SEARCH (Cont.)

Example 6

Find the first root of $f(x) = 4.15x^2 - 16x + 8$ by using incremental search. Start the procedure with the initial value, $x_0 = 0$ and the step size, $h = 0.1$. Perform three iterations of the incremental search to achieve the best approximation root.

Solution

Start the estimation with initial value $x_0 = 0$ and step size, $h = 0.1$.

x	$f(x)$
0	8
0.1	6.4415
0.2	4.966
0.3	3.5735
0.4	2.264
0.5	1.0375
0.6	-0.106

$$f(0.5) \cdot f(0.6) < 0$$

$$x_r = \frac{0.5 + 0.6}{2} = 0.55$$

INCREMENTAL SEARCH (Cont.)

Solution (Cont.)

Increasing the accuracy of root estimation with step size, $h = 0.01$ for $x \in [0.5, 0.6]$

x	$f(x)$
0.5	1.0375
0.51	0.919415
0.52	0.80216
0.53	0.685735
0.54	0.57014
0.56	0.455375
0.57	0.34144
0.58	0.11606
0.59	0.004615
0.60	-0.106

$$f(0.59) \cdot f(0.6) < 0$$

$$x_r = \frac{0.59 + 0.6}{2} = 0.5950$$

INCREMENTAL SEARCH (Cont.)

Solution (Cont.)

Increasing the accuracy of root estimation with step size, $h = 0.001$ for $x \in [0.59, 0.6]$

x	$f(x)$
0.59	0.004615
0.591	-0.0064385
0.592	-0.0175744
0.593	-0.02865665
0.594	-0.097306

$$f(0.59) \cdot f(0.591) < 0$$

$$x_r = \frac{0.59 + 0.591}{2} = 0.5905$$

$$\varepsilon_a = \left| \frac{0.5905 - 0.595}{0.5905} \right| \times 100\% = 0.76\%$$

For three iterations, the first root of $f(x) = 4.15x^2 - 16x + 8$ is 0.5905 with $\varepsilon_a = 0.76\%$

BRACKETING METHODS

- Figure 1 illustrates the basic idea of bracketing method–that is guessing an interval containing the root(s) of a function.
- Starting point of the interval is a lower bound, x_l. End point of the interval is an upper bound, x_u.
- By using bracketing methods, the interval will split into two subintervals and the size of the interval is successively reduced to a smaller interval.
- The subintervals will reduce the range of intervals until its distance is less than the desired accuracy of the solution

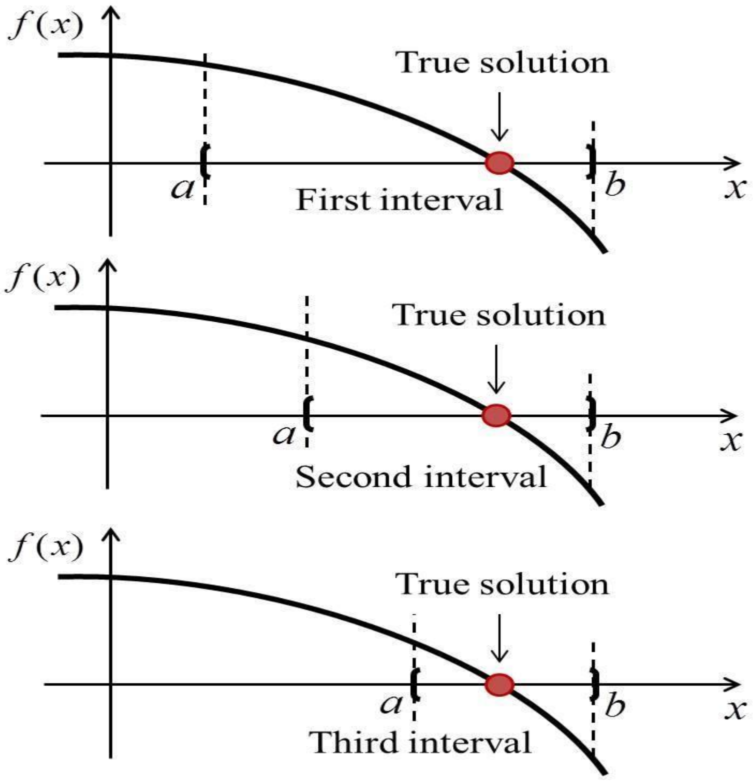
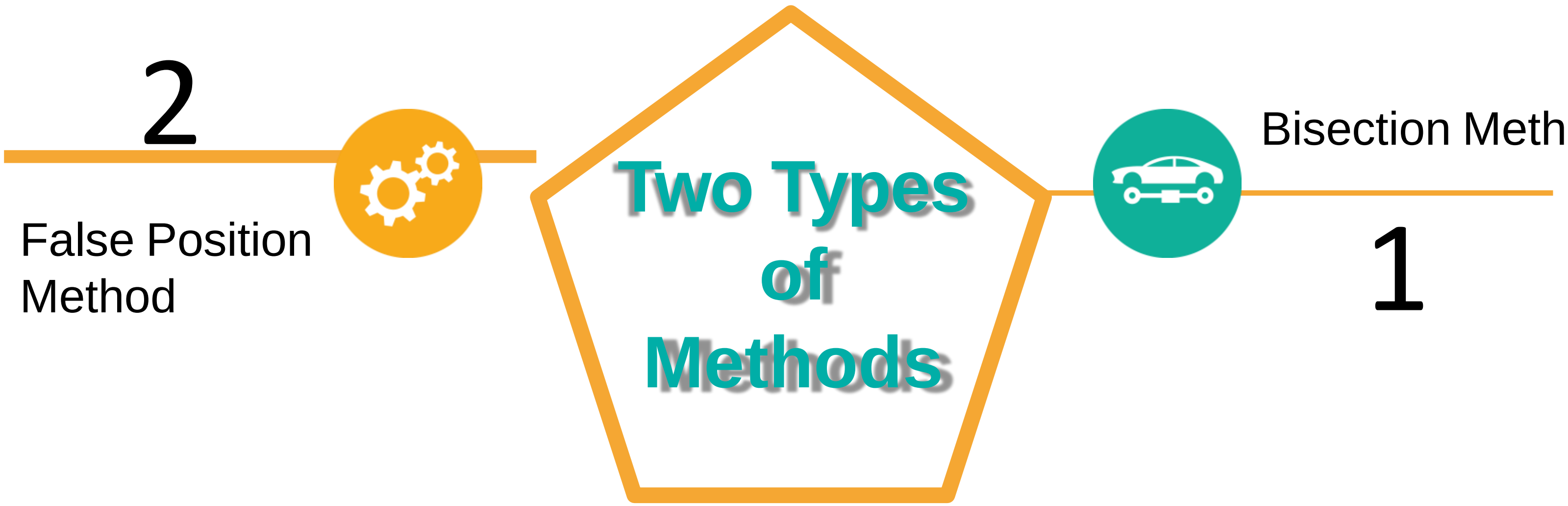


Figure 4: Graphical Illustration of Bracketing Method

BRACKETING METHODS

- Bracketing methods always converge to the true solution.
- There are two types bracketing methods; bisection method and false position method.



BISECTION METHOD

- Bisection method is the simplest bracketing method.
- The lower value, x_l and the upper value, x_u which bracket the root(s) are required.
- The procedure starts by finding the interval $[x_l, x_u]$ where the solution exist.
- As shown in **Figure 5**, at least one root exist in the interval $[x_l, x_u]$ if $f(x_l) \cdot f(x_u) < 0$

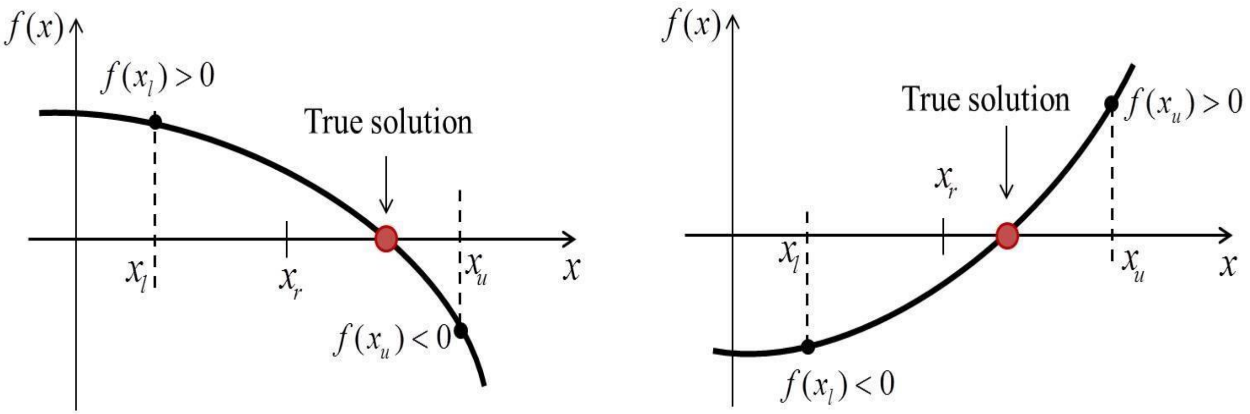


Figure 5: Solution of $f(x) = 0$

BISECTION METHOD (Cont.)

Algorithm

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

BISECTION METHOD (Cont.)

Example 7

Use three iterations of the bisection method to determine the root of $f(x) = -0.6x^2 + 2.4x + 5.5$. Employ initial guesses, $x_l = 5$ and $x_u = 10$. Compute the approximate percent relative error, ε_a and true percent relative error, ε_t after each iteration.

Solution

Calculate the true value for the given quadratic function $f(x) = -0.6x^2 + 2.4x + 5.5$ using quadratic formula (or you can calculate directly by using the calculator)

$$x = \frac{-2.4 \pm \sqrt{(2.4)^2 - 4(-0.6)(5.5)}}{2(-0.6)}$$

$$x = -1.6286, \; x = 5.6286$$

Choose the true value, $x = 5.6286$ for the highest root of $f(x)$. Estimate the root of $f(x)$ using bisection method with initial guess $x_l = 5$ and $x_u = 10$.

BISECTION METHOD (Cont.)

Solution (Cont.)

Estimate the root of $f(x)$ using bisection method with initial interval $[5,10]$.

- First iteration, $x \in [5,10]$

$$f(5) = 2.50$$

$$f(10) = -30.50$$

First estimate using bisection method formula

$$x_r = \frac{5+10}{2} = 7.5$$

$$f(7.5) = -10.25$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = 7.5$.

$$\varepsilon_t = \left| \frac{5.6286 - 7.5}{5.6286} \right| \times 100\% = 33.23\% \text{ and } \varepsilon_a = -$$

BISECTION METHOD (Cont.)

Solution (Cont.)

- Second iteration, $x \in [7.5,10]$

$$f(5) = -10.25$$

$$f(10) = -30.50$$

First estimate using bisection method formula

$$x_r = \frac{7.5+10}{2} = 6.25$$

$$f(6.25) = -2.9375$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = 6.25$.

$$\varepsilon_t = \left| \frac{5.6286 - 6.25}{5.6286} \right| \times 100\% = 11.04\% \text{ and } \varepsilon_a = 20\%$$

BISECTION METHOD (Cont.)

Solution (Cont.)

Continue the third iteration for $x \in [5,6.25]$. The results are summarized in the following table.

i	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l) \cdot f(x_r)$	s_t	s_a
1	5	10	7.5	2.5	-30.50	-10.25	-25.625	33.25	-
2	5	7.5	6.25	2.5	-10.25	-2.9375	-7.3438	11.04	20.00
3	5	6.25	5.625	2.5	-2.9375	0.0156	-0.0391	0.06	11.11

Therefore, after three iterations the approximate root of $f(x)$ is $x_r = 5.6250$ with $s_t = 0.06\%$ and $s_a = 11.11\%$.

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

Let, $a = 0$ and $b = 1$.

Now, $f(0) = (0)^3 + 4(0)^2 - 1 = -1 < 0$ and
$f(1) = (1)^3 + 4(1)^2 - 1 = 4 > 0$.
i.e., $f(a)$ and $f(b)$ has opposite signs.

Therefore, $f(x)$ has a root in the interval $[a, b] = [0, 1]$

$x_c = (0 + 1)/2 = 0.5$,
$f(0.5) = 0.125$. Now $f(a)$ and $f(x_c)$ has opposite signs

So, the next interval is $[0, 0.5]$

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

a	*b*	$x_c = (a+b)/2$	*f*(*a*)	*f*(*b*)	*f*(x_c)
0	1	0.5	**-1**	4	**0.125**
0	0.5	0.25	-1	**0.125**	**- 0.73438**
0.25	0.5	0.375	-0.73438	**0.125**	**- 0.38477**
0.375	0.5	0.4375	-0.38477	**0.125**	**- 0.15063**
0.4375	0.5	0.46875	-0.15063	**0.125**	**- 0.0181**
0.46875	0.5	0.484375	**-0.0181**	0.125	**0.05212**
0.46875	0.484375	0.476563	**- 0.0181**	0.05212	**0.01668**

... and so we approach the root 0.472834.

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

a	*b*	$x_c = (a+b)/2$	*f*(*a*)	*f*(*b*)	$f(x_c)$
0	1	0.5	**-1**	4	**0.125**
0	0.5	0.25	-1	**0.125**	**- 0.73438**
0.25	0.5	0.375	-0.73438	**0.125**	**- 0.38477**
0.375	0.5	0.4375	-0.38477	**0.125**	**- 0.15063**
0.4375	0.5	0.46875	-0.15063	**0.125**	**- 0.0181**
0.46875	0.5	0.484375	**-0.0181**	0.125	**0.05212**
0.46875	0.484375	0.476563	**- 0.0181**	0.05212	**0.01668**

... and so we approach the root 0.472834.

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval [0,2]?

Answer:

$f(x)$ is continuous on [0,2]

and $\mathrm{f}(0) * \mathrm{f}(2) = (1)(3) = 3 > 0$

$\Rightarrow$ Assumptions are not satisfied

$\Rightarrow$ Bisection method can not be used

Advantages

Simple and easy to implement

One function evaluation per iteration

The size of the interval containing the zero is reduced by 50% after each iteration

The number of iterations can be determined a priori

No knowledge of the derivative is needed

The function does not have to be differentiable

Disadvantage

Slow to converge

Good intermediate approximations may be discarded

We need two initial guesses a and b which bracket the root.

It is among the *slowest* methods to find the root.

When an interval contains more than one root, the bisection method can find *only* one of them.

Bisection Method: Class Work

Find the real root of the equation $f(x)=x3 - x - 1= 0$ correct to 2 decimal places.(ε=0.01).

Answer: 1.328125

Find the real root of the equation $f(x)=x4 - cos(x) + x = 0$ correct to 2 decimal places.(ε=0.01).

Answer: 0.637695

Iteration Method

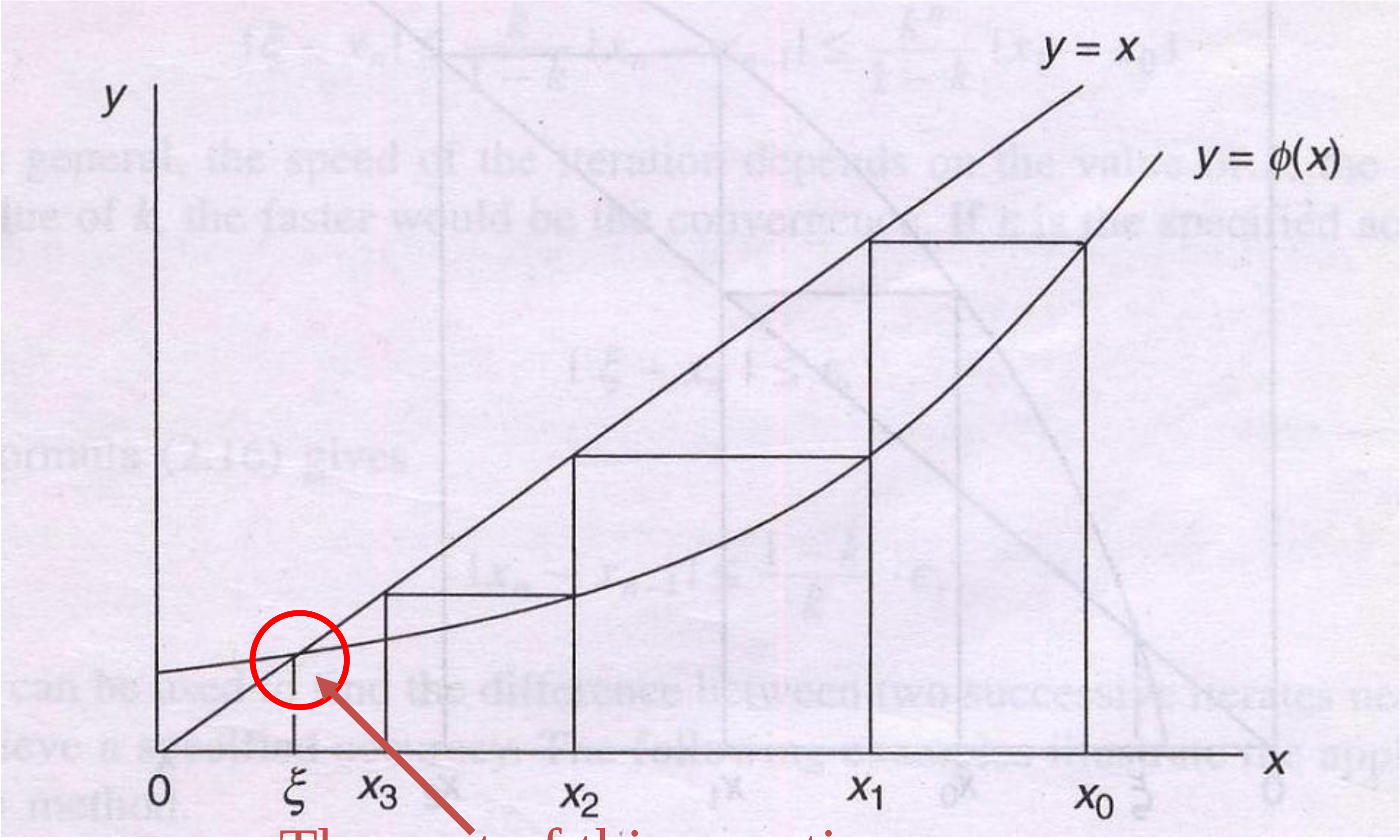
- Suppose we have an equation in the form $g(x) = 0$
- Rewrite the equation in the form $x = f(x)$.
- Start with an initial guess x_0, which is an ***approximation*** of the root.
- Calculate $x_1, \ldots, x_n, \ldots$ such that
  - $x_1 = f(x_0)$
  - $x_2 = f(x_1)$
  - $x_3 = f(x_2)$ ...
- Iterate the same process until $(x_n - x_{n-1})$ smaller than some specified tolerance.
- Geometrically, where the two graphs x and $f(x)$ intersects, that is the real root of the equation.

Iteration Method: Convergence Conditions

- Any arbitrary approximation x_0, x_1, x_2 does not assure that it will converge to the actual root x of the equation.
  - E.g. $x = 10^x + 1$,
  - if $x_0 = 0$, $x_1 = 2$, $x_2 = 101$, …. that does not converge to the actual root x
  - As n increase, x_n increases without limit!
- The equation $x = f(x)$ converges to the real root x,
  - if $f(x)$ is continuous
  - If $| f'(x) | < 1$
- The equation $x = f(x)$ does not converges to the real root x if $| f'(x) | > 1$
- Therefore, $g(x) = 0$ has to be re-written as $x = f(x)$ in such a way that $| f'(x) | < 1$

Iteration Method: Convergence Conditions

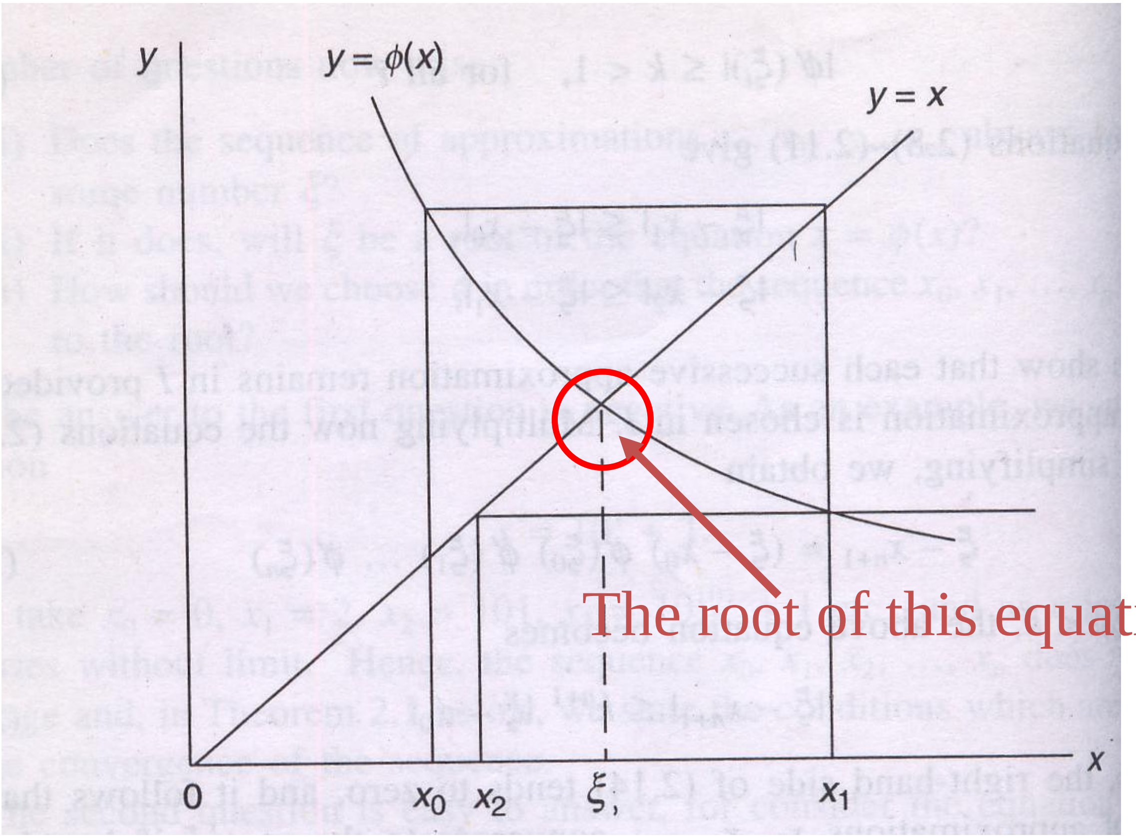
Convergence of $x_{n+1} = f(x_n)$, when $| f'(x) | < 1$



The root of this equation

Iteration Method: Convergence Conditions

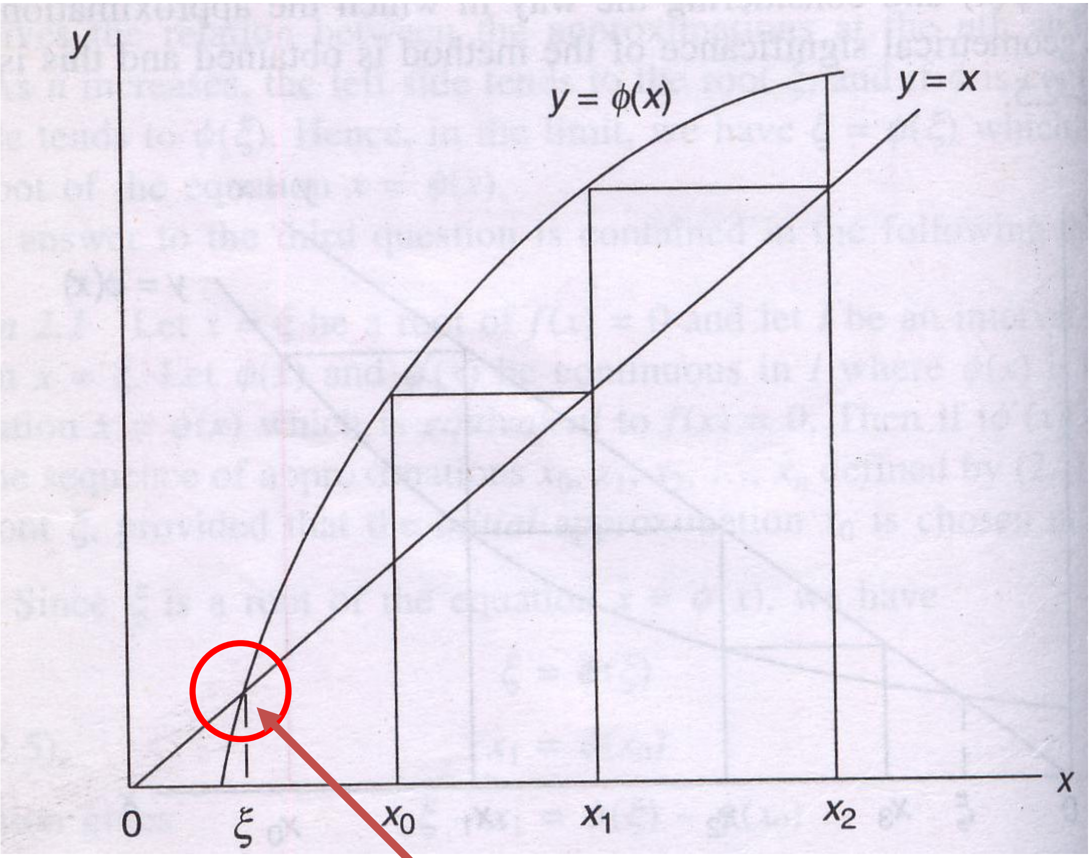
$x_{n+1} = f(x_n)$ oscillates but ultimately converges, when $|\, f\,'(x)\,| < 1$, but $f\,'(x) < 0$



The root of this equation

Iteration Method: Convergence Conditions

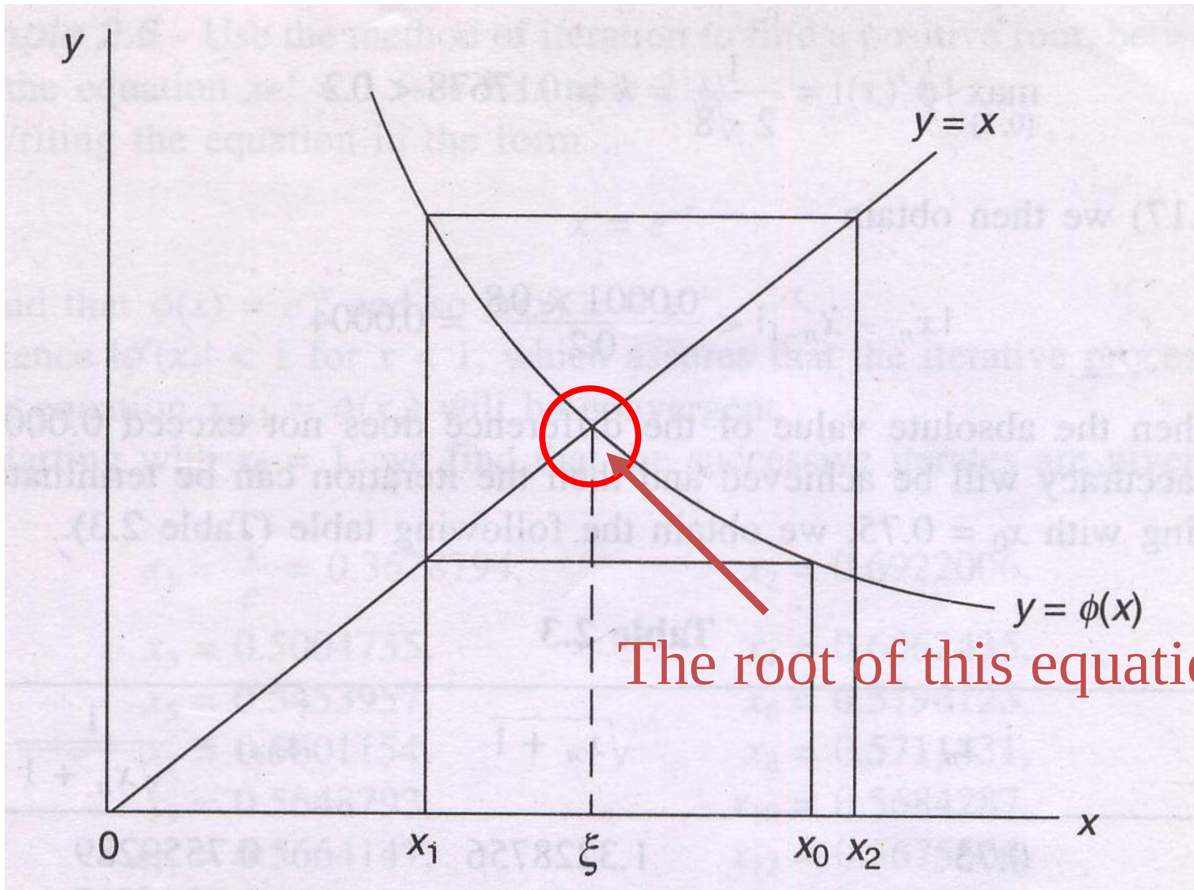
$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



The root of this equation

Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



The root of this equation

Iteration Method: Example

Solve $x = 2 + sin(x)/2$

Solution

Here $f(x) = 2 + sin(x)/2$

Starting with $x_0 = 2$ we calculate x_1, x_2,...

x_0	2
$x_1 = f(x_0)$	2.454648713
$x_2 = f(x_1)$	2.31708862
$x_3 = f(x_2)$	2.367105575
$x_4 = f(x_3)$	2.349674771
$x_5 = f(x_4)$	2.355850929
$x_6 = f(x_5)$	2.353674837
$x_7 = f(x_6)$	2.354443099
$x_8 = f(x_7)$	2.354172058
$x_9 = f(x_8)$	2.354267705
$x_{10} = f(x_9)$	2.354233955
$x_{11} = f(x_{10})$	2.354245864

Iteration Method: Example

Find the real root of the equation

$$g(x) = x^3 + x^2 - 1 = 0$$

Rewrite $g(x)$

$x^3 + x^2 - 1 = 0$

or, $x^3 + x^2 = 1$

or, $x^2(x+1) = 1$

or, $x^2 = 1/(x+1)$

or, $x = 1/\sqrt{(x+1)}$

Let, $x_0 = 0.75$

$x_0 = 0.7500000$
$x_1 = 0.7559289$
$x_2 = 0.7546517$
$x_3 = 0.7549263$
$x_4 = 0.7548672$
$x_5 = 0.7548799$
$x_6 = 0.7548772$
$x_7 = 0.7548778$
$x_8 = 0.7548776$
$x_9 = 0.7548777$
$x_{10} = 0.7548777$

Iteration Method: Class Work

Find the real root of the equation using iterative method (till 4 decimal places).

$$e^{-x} = 10x$$

Answer:
0.091276527

Iterative Method: Drawbacks

- We need an approximate initial guesses x_0.
- It is also a *slower* method to find the root.
- If the equation has more than one roots, then this method can find *only* one of them.

Newton-Raphson Method

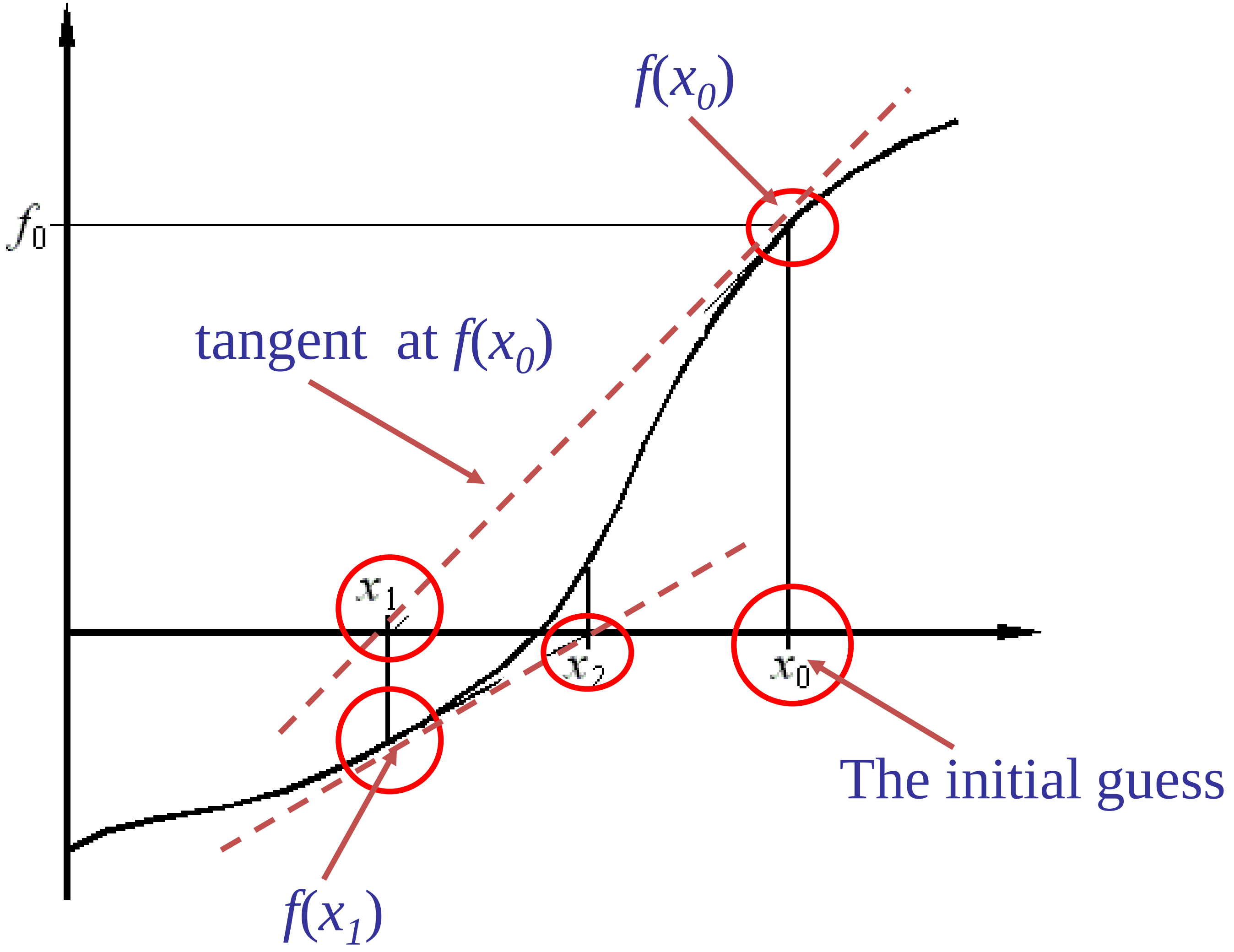
- This method is more efficient than the Bisection and Iteration methods.

If

- x is the real root and x_0 is an initial approximation of the real root of an equation $f(x) = 0$,
- $f'(x_0) \neq 0$,
- $f(x)$ has the same sign between x_0 and x,

Then, the tangent at $f(x_0)$ can lead to the real root x.

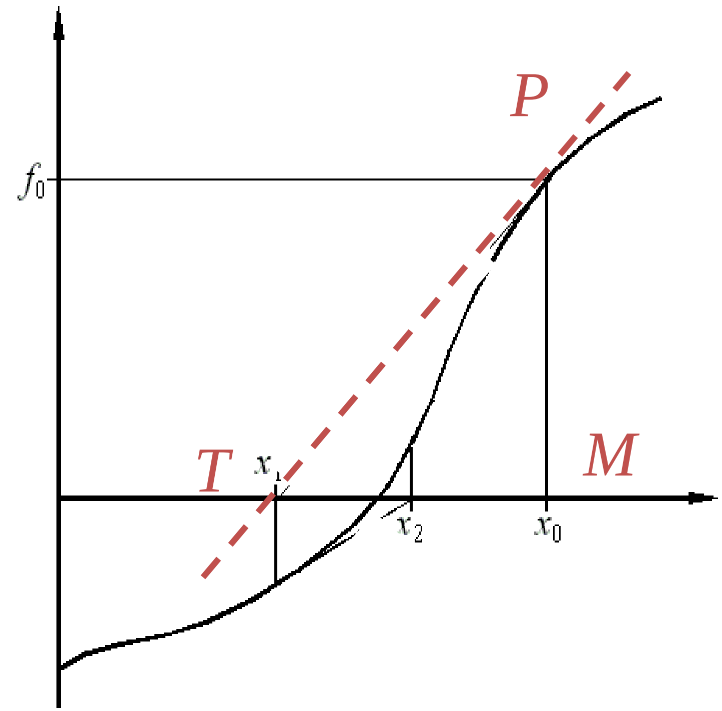
Newton-Raphson Method: Geometric Significance



Newton-Raphson Method: Geometric Significance

Here,

- The slope at x_1 is $tan\ (PTM)$
- $tan\ (PTM) = PM/\mathrm{TM}$
- $tan\ (PTM) = f\,(x_0)/h$
- Again, $tan\ (PTM) = f\,'(x_0)$
- Therefore, $f\,'(x_0) = f\,(x_0)/h$
- Or, $h = f\,(x_0)/\,f\,'(x_0)$
- $x_1 = x_0 - h$
- Therefore, $x_1 = x_0 - f\,(x_0)/\,f\,'(x_0)$
- Similarly, $x_2 = x_1 - f\,(x_1)/\,f\,'(x_1)$



Newton-Raphson Method

Methodology

- Let x_0 be an approximate root of $f(x) = 0$ and
- Let, x_1 is the correct root such that $x_1 = x_0 + h$ and $f(x_1) = 0$.
- Expanding $f(x_0+h)$ by Taylor's series, we obtain,

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

- Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0$$

- Which gives

$$h = -\frac{f(x_0)}{f'(x_0)}$$

Newton-Raphson Method (Cont'd.)

- A better approximation than x_0 is therefore given by x_1 where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Successive approximation are given by x_2, x_3, ….., x_n, x_{n+1} where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- This formula is known as the Newton-Raphson formula.

Newton-Raphson Method: Example

Find the real root of the equation using Newton-Raphson's Method

$f(x) = x^3 + 4x^2 - 1 = 0, \quad f'(x) = 3x^2 + 4*2x - 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{\ 3} + 4x_n^{\ 2} - 1}{3x_n^{\ 2} + 8x}$$

x_0	0.5
x_1	0.473684211
x_2	0.472834787
x_3	0.472833909
x_4	0.472833909

Newton-Raphson Method: Example

Class work

Use Newton-Raphson's Method to find a root of the equation correct to 2 decimal places.(ε=0.01)

$$x^3 - 2x - 5 = 0$$

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2$

Result 2.094551482

Newton-Raphson Method: Example

Class work

Use Newton-Raphson’s Method to find a root of the equation correct to 2 decimal places.(ε=0.01)

$$x \sin x = - \cos x$$

$$f(x) = x \sin x + \cos x$$

$$f'(x) = x \cos x$$

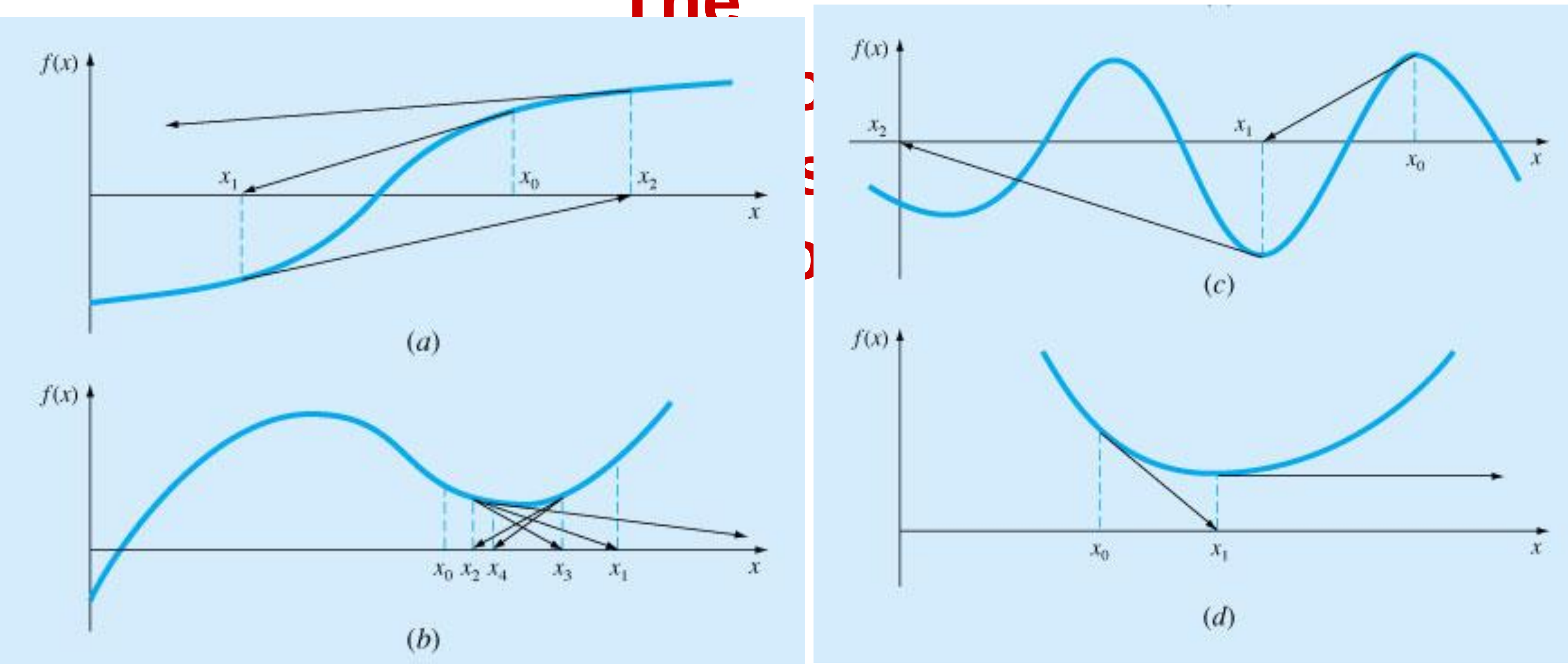
Result 2.798386046

Converges fast (quadratic convergence), if it converges.
Requires only one guess

Newton-Raphson Method: Drawbacks

- The Newton-Raphson method requires the calculation of the *derivative* of a function, which is not always easy.
- If f' vanishes at an iteration point, then the method will fail to converge.
- When the step is too large or the value is oscillating, other more conservative methods should take over the case.

Pitfalls of The



Cases where Newton Raphson method diverges or exhibit poor convergence.

a) Reflection point **b) oscillating around a local optimum**

c) Near zero slop , and **d) zero slop**

The Method of False Position Or Regula Falsi

- Like the bisection method, Method of False Position requires two initial guesses x_a and x_b such that $f(x) = 0$ and $f(x_a)$ and $f(x_b)$ has opposite signs.
- Since the graph of $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.
- The difference between these two methods is, instead of simply dividing the region in two, it obtains a new point x_1 which is (hopefully, but not necessarily) closer to the root.
- If $f(x_a)$ and $f(x_1)$ has opposite signs, then the new interval to be explored is $[x_a, x_1]$.
- Otherwise, the new interval is $[x_b, x_1]$.
- The procedure is repeated till the root is obtained to the desired accuracy.

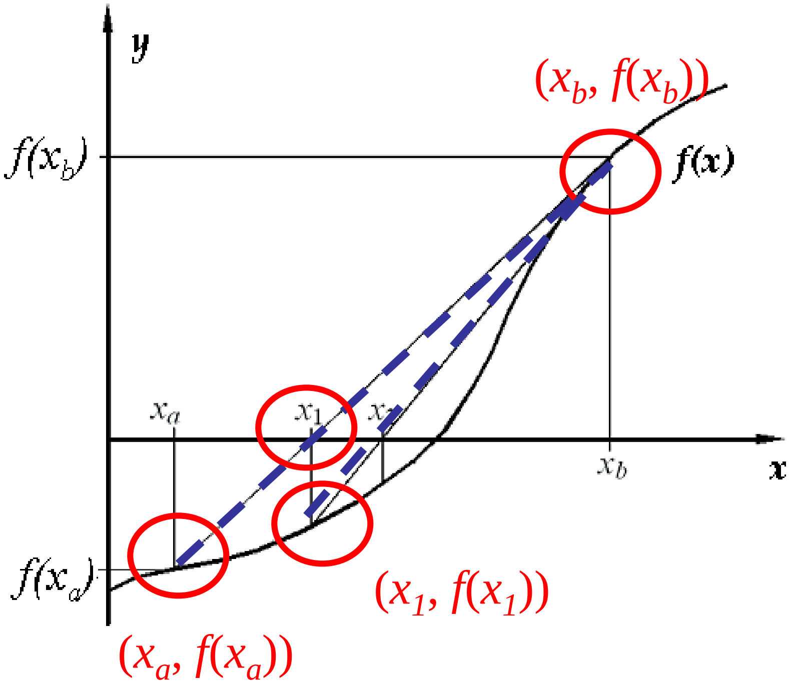
The Method of False Position

- This method consists in replacing the part of the curve between the points $(x_a, f(x_a))$ and $(x_b, f(x_b))$.

- The equation of the chord joining the two points $(x_a, f(x_a))$ and $(x_b, f(x_b))$ is

$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

- It takes the point of intersection of the chord with the x-axis as an approximation to the root (here, x_1).



The Method of False Position

- The point of intersection in the present case is given by putting $y = 0$ in the equation

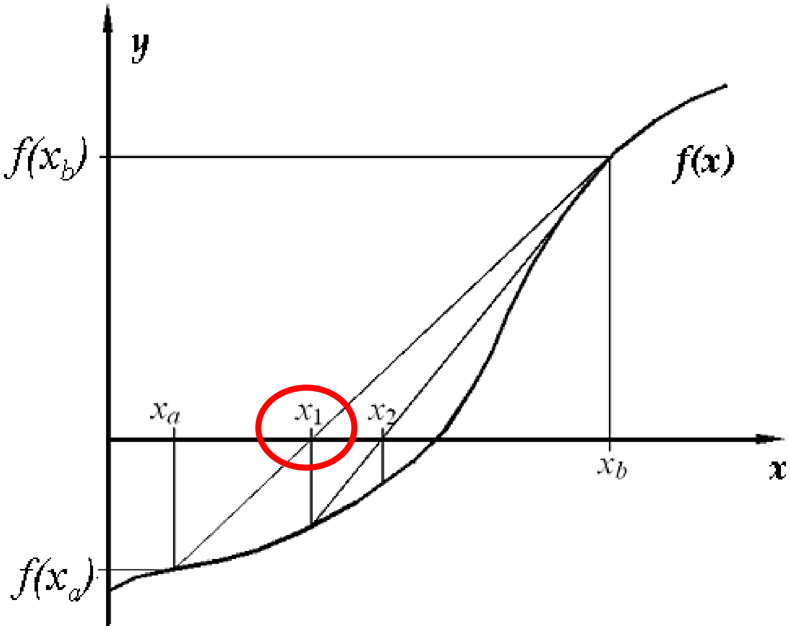
$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

- Thus we obtain

$$x = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$

- Hence, the approximate root is

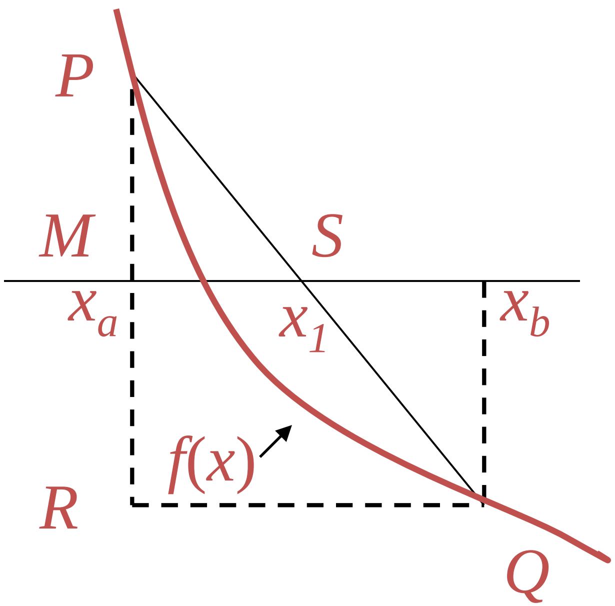
$$x_1 = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$



The Method of False Position : Geometric Significance

Here, for Δ PMS and Δ PRQ

- MS/MP = RQ/RP
- $(x_1 - x_a)/f(x_a) = (x_b - x_a)/(f(x_b)+-f(x_a))$
- $x_1 - x_a = f(x_a)(x_b - x_a)/(f(x_a)+-f(x_b))$
- $x_1 = x_a - f(x_a)(x_b - x_a)/(f(x_b) - f(x_a))$



The Method of False Position: Example

Find the real root of the equation till 2 decimal place

$$f(x) = x^3 - 2x - 5 = 0$$

We observe that $f(2) = -1$ and $f(3) = 16$

And hence a root lies between 2 and 3. Then

x0	x1	x2	f(x0)	f(x1)	f(x2)
2	3	2.058824	-1	16	-0.3908
2.058824	3	2.081264	-0.3908	16	-0.1472
2.081264	3	2.089639	-0.1472	16	-0.05468
2.089639	3	2.09274	-0.05468	16	-0.0202
2.09274	3	2.093884	-0.0202	16	-0.00745

x1	2.059
x2	2.081
x3	2.090
x4	2.093

x_4 is correct to 2 decimal places.

The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place
$x^3 - 2x^2 + 3x = 5$ between the points 1 and 2.

Result 1.843734

The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place

$$\sin x + x - 1 = 0.$$

Result 0.510973

Pitfalls of the False Position Method

Although a method such as false position is often superior to bisection, there are some cases (when function has significant curvature) that violate this general conclusion.

In such cases, the approximate error might be misleading and the results should always be checked by substituting the root estimate into the original equation and determining whether the result is close to zero.

major weakness of the false-position method: its one sidedness That is, as iterations are proceeding, one of the bracketing points will tend stay fixed which lead to poor convergence.

Advantages:

1. Simple
2. Brackets the Root

Disadvantages:

1. Can be VERY slow
2. Like Bisection, need an initial interval around the root.

FALSE POSITION METHOD (Cont.)

- It is an improvement of the Bisection method.
- The bisection method converges slowly due to its behavior in redefined the size of interval that containing the root.
- The procedure begins by finding an initial interval $[x_l, x_u]$ that bracket the root.
- $f(x_l)$ and $f(x_u)$ are then connected using a straight line.
- The estimated root, x_r is the x-value where the straight line crosses x-axis.
- **Figure 6** indicates the graphical illustration of False Position method.

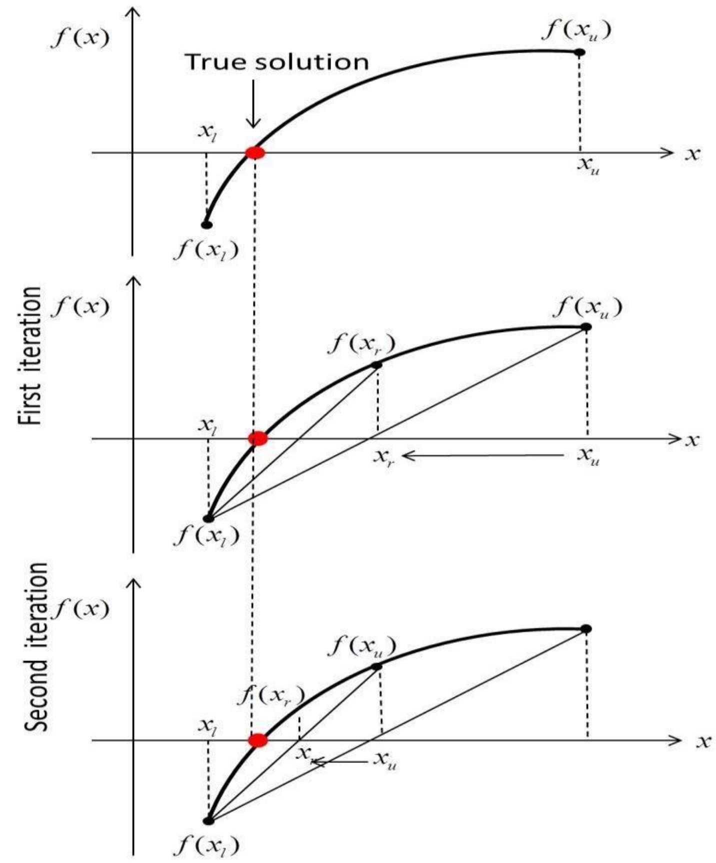


Figure 6: Graphical Illustration of False Position Method

FALSE POSITION METHOD (Cont.)

False Position Method Formula

Straight line joining the two points $(x_l, f(x_l))$ and $(x_u, f(x_u))$ is given by

$$\frac{f(x_u) - f(x_l)}{x_u - x_l} = \frac{y - f(x_u)}{x - x_u}$$

Since the line intersect the x-axis at x_r, so for $x = x_r, y = 0$, the following is obtained

$$x_r - x_u = -\frac{f(x_u)(x_u - x_l)}{f(x_u) - f(x_l)}$$

Rearranging the second equation yields the **False Position Method Formula**

$$x_r = x_u - \left[\frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}\right]$$

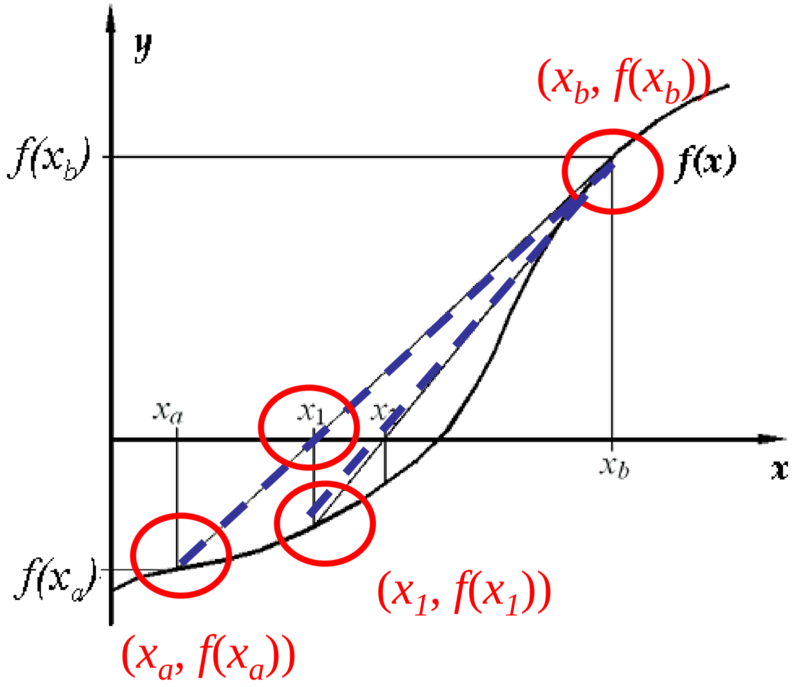
The Method of False Position

- This method consists in replacing the part of the curve between the points $(x_l, f(x_l))$ and $(x_u, f(x_u))$.

- The equation of the chord joining the two points $(x_l , f(x_l))$ and $(x_u , f(x_u))$ is

$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

- It takes the point of intersection of the chord with the x-axis as an approximation to the root (here, x_1).



The Method of False Position

- The point of intersection in the present case is given by putting $y = 0$ in the equation

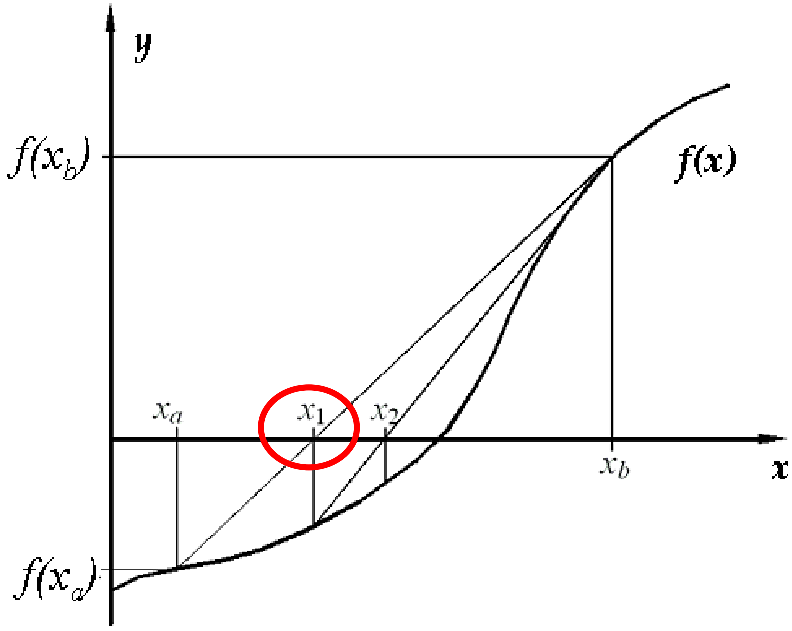
$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

- Thus we obtain

$$x = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$

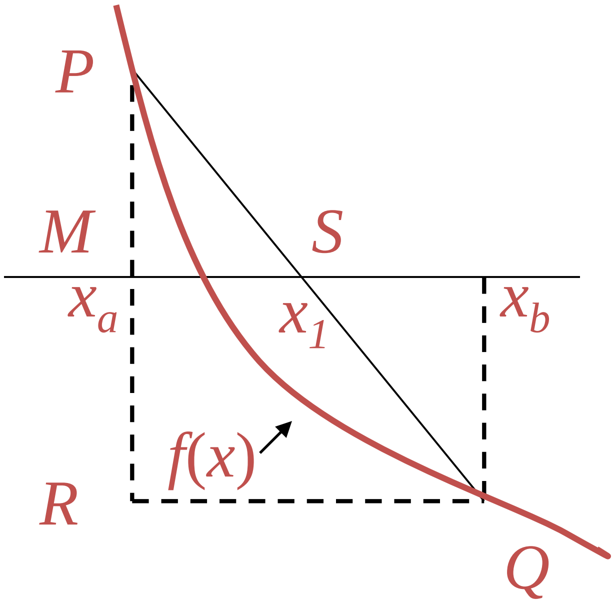
- Hence, the approximate root is

$$x_1 = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$



Here, for Δ PMS and Δ PRQ

- MS/MP = RQ/RP
- $(x_1 - x_a)/f(x_a) = (x_b - x_a)\,/(f(x_b) +- f(x_a))$
- $x_1 - x_a = f(x_a)(x_b - x_a)\,/(f(x_a) +- f(x_b))$
- $x_1 = x_a - f(x_a)(x_b - x_a)\,/(f(x_b) - f(x_a))$



FALSE POSITION METHOD (Cont.)

Algorithm

For the continuous equation of one variable, $f(x) = 0$,

Step 1: Choose the lower guess, x_l and the upper guess, x_u that bracket the root such that the function has opposite sign over the interval, $x_l \leq x \leq x_u$.

Step 2: The estimation root, x_r is computed by using

$$x_r = x_u - \left[\frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \right]$$

Step 3: Use the following evaluations to identify the subinterval that the root lies

- ✓ If $f(x_l) \cdot f(x_r) < 0$, then the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) > 0$, then the root lies in the upper subinterval. Therefore set $x_l = x_r$ and repeat **Step 2.**
- ✓ If $f(x_l) \cdot f(x_r) = 0$, then the root is equal to x_r. Terminate the computation.

Step 4: Calculate the approximate percent relative error,

$$\varepsilon_a = \left| \frac{x_r^{present} - x_r^{previous}}{x_r^{present}} \right| \times 100\%$$

Step 5: Compare with. If $\varepsilon_a < \varepsilon_s$, then stop the computation. Otherwise go to **Step 2** and repeat the process by using the new interval.

FALSE POSITION METHOD (Cont.)

Example 8

Determine the first root $f(x) = -3x^3 + 19x^2 - 20x - 13$ by using False position method. Use the initial guesses of $x_l = -1$ and $x_u = 0$ with stopping criterion, ε_s=1%.

Solution

- First iteration, $x \in [-1,0]$

$$f(-1) = 29$$

$$f(0) = -13$$

First estimate using False position method is

$$x_r = 0 - \frac{(-13)(-1-0)}{29-(-13)} = -0.3095$$

$$f(-0.3095) = -4.9010$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = -0.3095$.

$$\varepsilon_a = -$$

FALSE POSITION METHOD (Cont.)

Solution

- Second iteration, $x \in [-1, -0.3095]$.

Second estimate is

$$x_r = -0.3095 - \frac{(-4.9010)(-1+0.3095)}{29-(-4.9010)} = -0.4093$$

$$f(-0.4093) = -1.4253$$

Since $f(x_l) \cdot f(x_r) < 0$, the root lies in the lower subinterval. Then set $x_u = -0.4093$.

$$\varepsilon_a = 24.38\%$$

FALSE POSITION METHOD (Cont.)

Solution (Cont.)

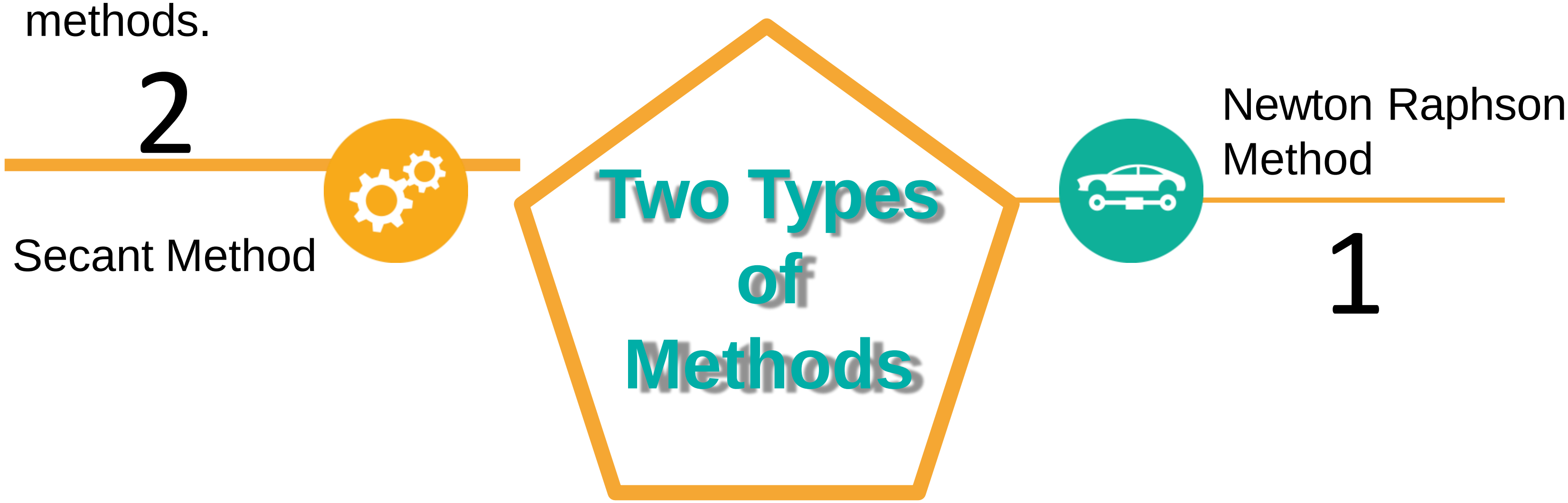
Continue the third iteration for $x \in [-1, -0.4093]$. The results are summarized in the following table.

i	x_l	x_u	x_r	$f(x_l)$	$f(x_u)$	$f(x_r)$	$f(x_l) \cdot f(x_r)$	ε_a
1	-1	0	-0.3095	29	-13	-4.0910	-142.1290	-
2	-1	-0.3095	-0.4093	29	-4.9003	-1.4253	-41.3337	24.38
3	-1	-0.4093	-0.4370	29	-1.4241	-0.3812	-11.0548	6.33
4	-1	-0.4370	-0.4443	29	-0.3820	-0.1002	-2.0907	1.65
5	-1	-0.4443	-0.4462	29	-0.1002	-0.0267	-0.7743	0.43

Therefore, after fifth iterations the approximate root of $f(x)$ is $x_r = -0.4462$ with $\varepsilon_a = 0.43\%$.

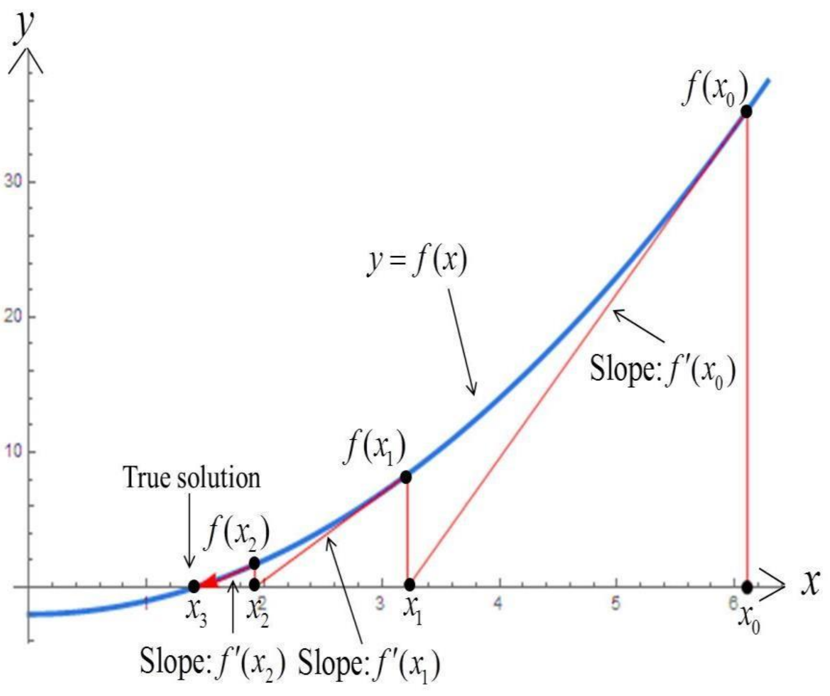
OPEN METHODS

- The idea of this method is to consider at least one initial guess which is not necessarily bracket the root.
- Normally, the chosen initial value(s) must be close to the actual root that can be found by plotting the given function against its independent variable.
- In every step of root improvement, x_r of previous step is considered as the previous value for the present step.
- In general, open methods provides no guarantee of convergence to the true value, but once it is converge, it will converge faster than bracketing methods.



NEWTON RAPHSON METHOD

- It is an open method for finding roots of $f(x) = 0$ by using the successive slope of the tangent line.
- The Newton Raphson method is applicable if $f(x)$ is continuous and differentiable.
- **Figure 6** shows the graphical illustration of Newton Raphson methoc
- Numerical scheme starts by choosing the initial point, x_0 as the first estimation of the solution.
- The improvement of the estimation of x_1 is obtained by taking the tangent line to $f(x)$ at the point $(x_0, f(x_0))$ and extrapolate the tangent line to find the point of intersection with an x–axis.



NEWTON RAPHSON METHOD (Cont.)

Slope for the first iteration is:

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} \quad (1)$$

Rearranging equation (1) yields:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- The next estimation, x_2 is the intersection of the tangent line $f(x)$ at the point $(x_1, f(x_1))$.
- The estimation, x_{i+1} is the intersection of the tangent line $f(x)$ at the point $(x_i, f(x_i))$. The slope of the i^{th} iteration is

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \quad (2)$$

Rearranging equation (2) gives **Newton Raphson Formula**:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

NEWTON RAPHSON METHOD (Cont.)

Algorithm

For the continuous and differentiable function, $f(x) = 0$:

Step 1: Choose initial value, x_0 and find $f'(x_0)$.

Step 2: Compute the next estimate, x_{i+1} by using Newton Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 3: Calculate the approximate percent relative error, ε_a

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

Step 4: Compare ε_s with ε_a. If $\varepsilon_a < \varepsilon_s$, the computation is stopped. Otherwise, repeat **Step 2**.

NEWTON RAPHSON METHOD (Cont.)

Example 8

Determine the first root $f(x) = 8e^{-x}\sin(x) - 1$ by using Newton Rapshon method. Use the initial guesses of $x_0 = 0.3$ and perform the computation up to three iterations.
(*Use radian mode in your calculator*)

Solution

Step 1

$$f(x) = 8e^{-x}\sin(x) - 1$$

$$f'(x) = 8e^{-x}\left(\cos(x) - \sin(x)\right)$$

Step 2

First iteration, $x_0 = 0.3$

$f(0.3) = 8e^{-0.3}\sin(0.3) - 1 = 0.7514,$

$f'(0.3) = 8e^{-0.3}(\cos(0.3) - \sin(0.3)) = 3.9104,$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.3 - \frac{0.7514}{3.9104} = 0.1078$$

$$\varepsilon_a = \left|\frac{0.1078 - 0.3}{0.1078}\right| \times 100\% = 178.18\%$$

NEWTON RAPHSON METHOD (Cont.)

Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

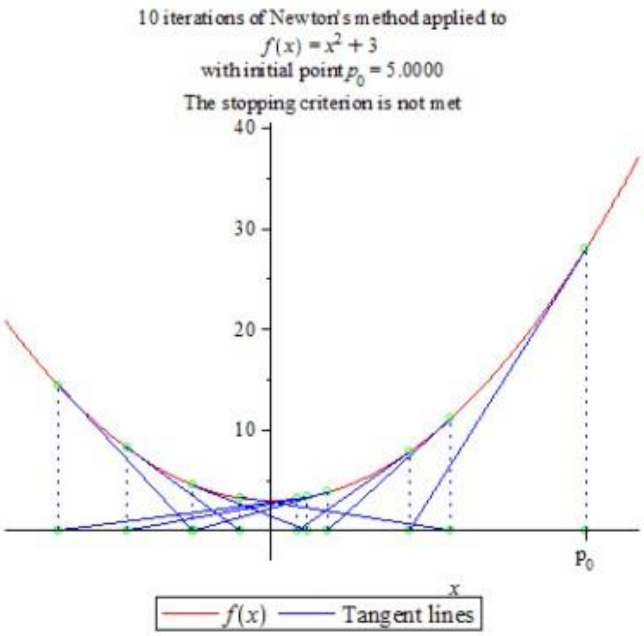
No. of iteration	i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}	ε_a (%)
1	0	0.3	0.7514	3.9104	0.1078	178.18
2	1	0.1078	-0.2270	6.3674	0.1435	24.84
3	2	0.1435	-0.0090	5.8684	0.1450	1.05

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 0.1450$ with $\varepsilon_a = 1.05\%$.

NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

Case 1: The tendency of the results obtained from the Newton Raphson method to oscillate around the local maximum or minimum without converge to the actual root.

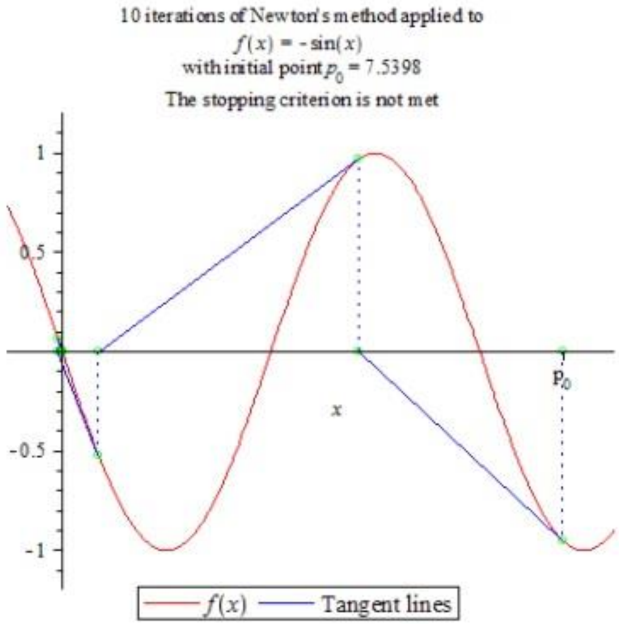


Case 2: Division by zero involve in the Newton Raphson formula when $f'(x) = 0$.

NEWTON RAPHSON METHOD (Cont.)

Pitfalls of the Newton Raphson Method

Case 3: In some cases where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. The guesses may jump and converge to some other roots and the process become oscillatory, which leads to endless cycle of fluctuations between x_i and x_{i+1} without converge to the desired root.



SECANT METHOD

Introduction

- In many cases, the derivative of a function is very difficult to find or even is not differentiable.
- Alternative approach is by using secant method.

- The slope in Newton's Rapshon method is substituted with backward finite divided difference

$$f'(x_i) = \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

- The secant method formula is:

$$x_{i+1} = x_i - \left[\frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \right]$$

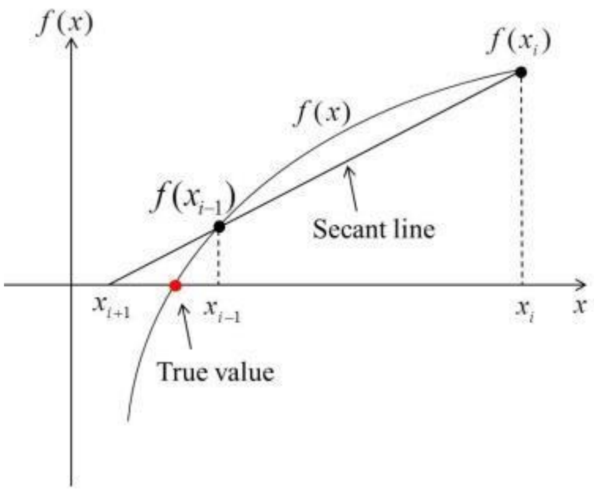
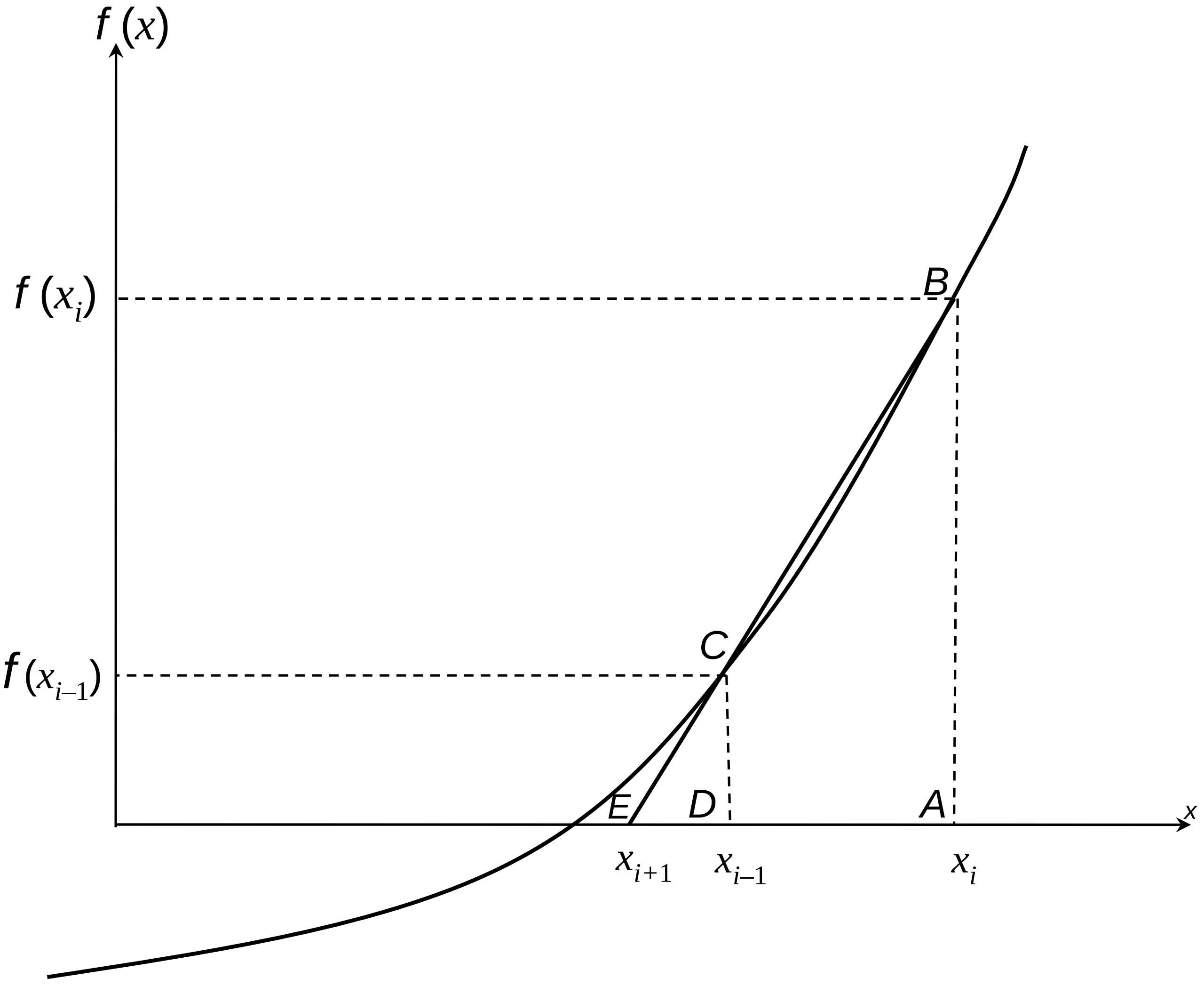


Figure 7: Graphical Illustration of Secant Method

Figure 1 Geometrical representation of the secant method



Derivation of Secant Method

The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, and , one draws a straight line between and passing through the x-axis at x_i. *ABE* and *DCE* are similar triangles.

Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

SECANT METHOD (Cont.)

Algorithm

For the continuous function, $f(x) = 0$:

Step 1: Choose initial values, x_{-1} and x_0. Find $f(x_{-1})$ and $f(x_0)$.

Step 2: Compute the next estimate, x_{i+1} by using secant method formula

$$x_{i+1} = x_i - \left[\frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \right]$$

Step 3: Calculate the approximate percent relative error, ε_a

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100\%$$

Step 4: Compare ε_s with ε_a. If $\varepsilon_a < \varepsilon_s$, the computation is stopped. Otherwise, repeat **Step 2**.

SECANT METHOD (Cont.)

Example 9

Determine one of the real root(s) of $f(x) = -12 - 21x + 18x^2 - 2.4x^3$ by using secant method with initial guesses of $x_{-1} = 1.0$ and $x_0 = 1.3$. Perform the computation until $\varepsilon_a < 5\%$.

Solution

First iteration, $x_{-1} = 1.0$ and $x_0 = 1.3$

$f(1.0)$ =-17.4

$f(1.3) = -14.528$

$$x_1 = x_0 - \left[\frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)}\right]$$

$$= 1.3 - \left[\frac{-14.1528(1-1.3)}{-17.4+14.1528}\right] = 2.6075$$

$$\varepsilon_a = \left|\frac{2.6075-1.3}{2.6075}\right| \times 100\% = 50.14\% > \varepsilon_s$$

SECANT METHOD (Cont.)

Solution (Cont.)

Continue the second iteration and the results are summarised as follows.

No. of Iteration	i	x_{i-1}	x_i	$f(x_{i-1})$	$f(x_i)$	x_{i+1}	$\varepsilon_a(\%)$
1	0	1	1.3	-17.4	-14.1527	2.6075	50.14
2	1	1.3	2.6075	-14.1528	13.0780	1.9796	31.72
3	2	2.6075	1.9796	13.0780	-1.6519	2.0500	3.44

Therefore, after three iterations the approximated root of $f(x)$ is $x_3 = 2.0500$ with $\varepsilon_a = 3.44\%$.

Advantages of Secant Method

1. It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton’s method which requires two.

Disadvantages of Secant Method

1. It may not converge.
2. There is no guaranteed error bound for the computed iterates.
3. It is likely to have difficulty if $f'(\alpha) = 0$. This means the x-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.

Conclusion

Bracketing Method	Open Method
Need two initial guesses	Can involve one or more initial guesses
The root is located within an interval prescribed by a lower and an upper bound.	Not necessarily bracket the root.
Always work but converge slowly	Do not always work (can diverge) but when they do they usually converge much more quickly.

Summary

Method	Advantages	Disadvantages
Bisection	- Easy, Reliable, Convergent - One function evaluation per iteration - No knowledge of derivative is needed	- Slow - Needs an interval [a,b] containing the root, i.e., $f(a)f(b)<0$
Newton	- Fast (if near the root) - Two function evaluations per iteration	- May diverge - Needs derivative and an initial guess x_0 such that $f'(x_0)$ is nonzero
Secant	- Fast (slower than Newton) - One function evaluation per iteration - No knowledge of derivative is needed	- May diverge - Needs two initial points guess x_0, x_1 such that $f(x_0)- f(x_1)$ is nonzero