

VECTORS

Vector : a one dimensional array of numbers

Examples:

row vector $[1 \quad 4 \quad 2]$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Identity vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATRICES

Matrix : a two dimensional array of numbers

Examples:

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

diagonal $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$,

Tridiagonal $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

MATRICES

Examples:

symmetric $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix},$

upper triangular

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Determinant of a MATRICES

Defined for square matrices only

Examples:

$$\det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2 \begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$
$$= 2(-25) - 1(12 + 5) - 1(15 - 0) = -82$$

Adding and Multiplying Matrices

The addition of two matrices A and B

- * Defined only if they have the same size
- * $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

- * The product $C = AB$ is defined only if $m = p$
- * $C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^m a_{ik} b_{kj} \quad \forall i, j$

Systems of Linear Equations

A system of linear equations can be presented in different forms

$$\left. \begin{array}{l} 2x_1 + 4x_2 - 3x_3 = 3 \\ 2.5x_1 - x_2 + 3x_3 = 5 \\ x_1 \quad \quad - 6x_3 = 7 \end{array} \right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form

Matrix form

Solutions of Linear Equations

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a solution to the following equations:

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$

Solutions of Linear Equations

- A set of equations is **inconsistent** if there exists no solution to the system of equations:

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 5$$

These equations are inconsistent

Solutions of Linear Equations

- Some systems of equations may have **infinite number of solutions**

$$x_1 + 2x_2 = 3$$

$$2x_1 + 4x_2 = 6$$

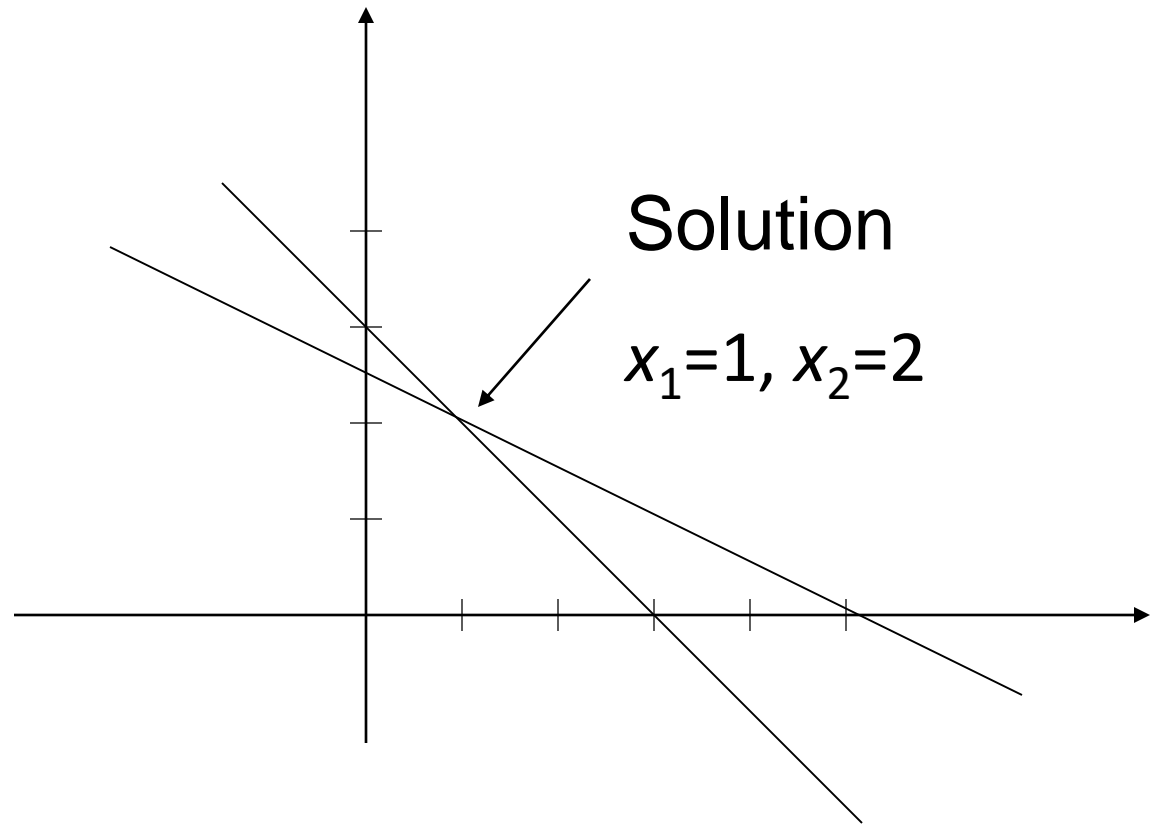
have infinite number of solutions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ 0.5(3-a) \end{bmatrix} \text{ is a solution for all } a$$

Graphical Solution of Systems of Linear Equations

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$



Cramer's Rule is Not Practical

Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

Cramer's Rule is not practical for large systems .

To solve N by N system requires $(N + 1)(N - 1)N!$ multiplications.

To solve a 30 by 30 system, 2.38×10^{35} multiplications are needed.

It can be used if the determinants are computed in efficient way

Naive Gaussian Elimination

- The method consists of two steps:
 - **Forward Elimination:** the system is reduced to **upper triangular form**. A sequence of **elementary operations** is used.
 - **Backward Substitution:** Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Elementary Row Operations

- Adding a multiple of one row to another
- Multiply any row by a non-zero constant

Example

Forward Elimination

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1 : Forward Elimination

Step 1 : Eliminate x_1 from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Example

Forward Elimination

Step2: Eliminate x_2 from equations 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

Step3: Eliminate x_3 from equation 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Forward Elimination

Summary of the Forward Elimination :

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Backward Substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for x_4 , then solve for x_3 ,...solve for x_1

$$x_4 = \frac{-3}{-3} = 1,$$

$$x_3 = \frac{-9 + 5}{2} = -2$$

$$x_2 = \frac{-6 - 2(-2) - 2(1)}{-4} = 1, \quad x_1 = \frac{16 + 2(1) - 2(-2) - 4(1)}{6} = 3$$

Forward Elimination

To eliminate x_1

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}} \right) a_{1j} & (1 \leq j \leq n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i1}}{a_{11}} \right) b_1 \end{aligned} \right\} 2 \leq i \leq n$$

To eliminate x_2

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}} \right) a_{2j} & (2 \leq j \leq n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i2}}{a_{22}} \right) b_2 \end{aligned} \right\} 3 \leq i \leq n$$

Forward Elimination

To eliminate x_k

$$\left. \begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}} \right) a_{kj} & (k \leq j \leq n) \\ b_i &\leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}} \right) b_k \end{aligned} \right\} k+1 \leq i \leq n$$

Continue until x_{n-1} is eliminated.

Backward Substitution

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{i,j}x_j}{a_{i,i}}$$

Naive Gaussian Elimination

- Summary of the Naive Gaussian Elimination

- Example

- Problems with Naive Gaussian Elimination

 - Failure due to zero pivot element

 - Error

- ▣ Pseudo-Code

Naive Gaussian Elimination

- o The method consists of two steps
 - o **Forward Elimination:** the system is reduced to **upper triangular form**. A sequence of **elementary operations** is used.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

- o **Backward Substitution:** Solve the system starting from the last variable. Solve for x_n, x_{n-1}, \dots, x_1 .

Example 1

Solve using Naive Gaussian Elimination :

Part1: Forward Elimination ____ Step1: Eliminate x_1 from equations 2, 3

$$x_1 + 2x_2 + 3x_3 = 8 \quad eq1 \text{ unchanged (pivot equation)}$$

$$2x_1 + 3x_2 + 2x_3 = 10 \quad eq2 \leftarrow eq2 - \left(\frac{2}{1}\right)eq1$$

$$3x_1 + x_2 + 2x_3 = 7 \quad eq3 \leftarrow eq3 - \left(\frac{3}{1}\right)eq1$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ -x_2 - 4x_3 &= -6 \\ -5x_2 - 7x_3 &= -17 \end{aligned}$$

Example 1

Part 1: Forward Elimination Step 2: Eliminate x_2 from equation 3

$$\begin{array}{ll} x_1 + 2x_2 + 3x_3 = 8 & eq1 \text{ unchanged} \\ -x_2 - 4x_3 = -6 & eq2 \text{ unchanged (pivot equation)} \\ -5x_2 - 7x_3 = -17 & eq3 \leftarrow eq3 - \left(\frac{-5}{-1} \right) eq2 \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} x_1 + 2x_2 + 3x_3 = 8 \\ -x_2 - 4x_3 = -6 \\ 13x_3 = 13 \end{array} \right.$$

Example 1

Backward Substitution

$$x_3 = \frac{b_3}{a_{3,3}} = \frac{13}{13} = 1$$

$$x_2 = \frac{b_2 - a_{2,3}x_3}{a_{2,2}} = \frac{-6 + 4x_3}{-1} = 2$$

$$x_1 = \frac{b_1 - a_{1,2}x_2 - a_{1,3}x_3}{a_{1,1}} = \frac{8 - 2x_2 - 3x_3}{a_{1,1}} = 1$$

The solution is
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Determinant

The elementary operations do not affect the determinant

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\text{Elementary operations}} A' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = \det(A') = -13$$

How Many Solutions Does a System of Equations $AX=B$ Have?

Unique

$$\det(A) \neq 0$$

reduced matrix

has no zero rows

No solution

$$\det(A) = 0$$

reduced matrix

has one or more
zero rows

corresponding B
elements $\neq 0$

Infinite

$$\det(A) = 0$$

reduced matrix

has one or more
zero rows

corresponding B
elements $= 0$

Examples

Unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

solution:

$$X = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

No solution

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

No solution

0 = -1 impossible!

infinte # of solutions

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

↓

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Infinite# solutions

$$X = \begin{bmatrix} \alpha \\ 1 - .5\alpha \end{bmatrix}$$

Pseudo-Code: Forward Elimination

Do k = 1 to n-1

Do i = k+1 to n

factor = $a_{i,k} / a_{k,k}$

Do j = k+1 to n

$a_{i,j} = a_{i,j} - \text{factor} * a_{k,j}$

End Do

$b_i = b_i - \text{factor} * b_k$

End Do

End Do

Pseudo-Code: Back Substitution

$$x_n = b_n / a_{n,n}$$

Do $i = n-1$ downto 1

$$\text{sum} = b_i$$

Do $j = i+1$ to n

$$\text{sum} = \text{sum} - a_{i,j} * x_j$$

End Do

$$x_i = \text{sum} / a_{i,i}$$

End Do

Lectures 15-16:

Gaussian Elimination with Scaled Partial Pivoting

- ❑ Problems with Naive Gaussian Elimination
- ❑ Definitions and Initial step
- ❑ Forward Elimination
- ❑ Backward substitution
- ❑ Example

Problems with Naive Gaussian Elimination

- o The Naive Gaussian Elimination may fail for very simple cases.
(The pivoting element is zero).

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- o Very small pivoting element may result in serious computation errors

$$\begin{bmatrix} 10^{-10} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting :

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 2

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector:

disregard sign

find largest in
magnitude in
each row

Scale vector $S = [2 \quad 4 \quad 8 \quad 5]$

Index Vector $L = [1 \quad 2 \quad 3 \quad 4]$

Why Index Vector?

- Index vectors are used because it is much easier to exchange a single index element compared to exchanging the values of a complete row.
- In practical problems with very large N , exchanging the contents of rows may not be practical.

Example 2

Forward Elimination-- Step 1: eliminate x_1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 & 4 & 8 & 5] \\ L = [1 & 2 & 3 & 4] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \mid i = 1, 2, 3, 4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \frac{|4|}{5} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 and l_1

$$L = [4 \ 2 \ 3 \ 1]$$

Example 2

Forward Elimination-- Step 1: eliminate x_1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & 8 & 6 & 3 \\ \boxed{4} & \boxed{2} & \boxed{5} & \boxed{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \boxed{-1} \end{bmatrix}$$

First pivot equation



$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ \boxed{4} & \boxed{2} & \boxed{5} & \boxed{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ \boxed{-1} \end{bmatrix}$$

Example 2

Forward Elimination-- Step 2: eliminate x2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & 5.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 2 \ 3 \ 1]$$

$$\text{Ratios: } \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \mid i = 2,3,4 \right\} = \left\{ \frac{0.5}{4} \quad \frac{5.5}{8} \quad \frac{1.5}{2} \right\} \Rightarrow L = [\ 4 \ 1 \ 3 \ 2]$$

Example 2

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0.25 & 1.6667 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 6.8333 \\ -1 \end{bmatrix}$$

**Third pivot
equation**

$$L = [4 \ 1 \ 2 \ 3]$$

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix}$$

Example 2

Backward Substitution

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0 & -2.5 & 1.8333 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 2.1667 \\ 9 \\ -1 \end{bmatrix} \quad L = [4 \ 1 \ 2 \ 3]$$

$$x_4 = \frac{b_3}{a_{3,4}} = \frac{9}{2} = 4.5, \quad x_3 = \frac{b_2 - a_{2,4}x_4}{a_{2,3}} = \frac{2.1667 - 1.8333x_4}{-2.5} = 2.4327$$

$$x_2 = \frac{b_1 - a_{1,4}x_4 - a_{1,3}x_3}{a_{1,2}} = \frac{1.25 - 0.25x_4 - 0.75x_3}{-1.5} = 1.1333$$

$$x_1 = \frac{b_4 - a_{4,4}x_4 - a_{4,3}x_3 - a_{4,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -7.2333$$

Example 3

Solve the following system using Gaussian Elimination with Scaled Partial Pivoting

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Example 3

Initialization step

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Scale vector $S = [2 \quad 4 \quad 8 \quad 5]$

Index Vector $L = [1 \quad 2 \quad 3 \quad 4]$

Example 3

Forward Elimination-- Step 1: eliminate x_1

Selection of the pivot equation

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} S = [2 & 4 & 8 & 5] \\ L = [1 & 2 & 3 & 4] \end{cases}$$

$$Ratios = \left\{ \frac{|a_{l_i,1}|}{S_{l_i}} \mid i = 1, 2, 3, 4 \right\} = \left\{ \frac{|1|}{2}, \frac{|3|}{4}, \frac{|5|}{8}, \frac{|4|}{5} \right\} \Rightarrow \text{max corresponds to } l_4$$

equation 4 is the first pivot equation Exchange l_4 and l_1

$$L = [4 \ 2 \ 3 \ 1]$$

Example 3

Forward Elimination-- Step 1: eliminate x_1

Update A and B

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 3 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 2: eliminate x_2

Selection of the second pivot equation

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 2 \ 3 \ 1]$$

$$\text{Ratios: } \left\{ \frac{|a_{l_i,2}|}{S_{l_i}} \mid i = 2,3,4 \right\} = \left\{ \frac{0.5}{4}, \frac{10.5}{8}, \frac{1.5}{2} \right\} \Rightarrow L = [\ 4 \ 3 \ 2 \ 1]$$

Example 3

Forward Elimination-- Step 2: eliminate x_2

Updating A and B

$$\begin{bmatrix} 0 & -1.5 & 0.75 & 0.25 \\ 0 & 0.5 & -2.75 & 1.75 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & 1 & 3 & 2 \end{bmatrix}$$

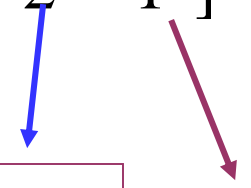
$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Forward Elimination-- Step 3: eliminate x3

Selection of the third pivot equation

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$S = [2 \ 4 \ 8 \ 5] \quad L = [\ 4 \ 3 \ 2 \ 1]$$


$$\text{Ratios: } \left\{ \frac{|a_{l_i,3}|}{S_{l_i}} \mid i = 3,4 \right\} = \left\{ \frac{2.7619}{4}, \frac{0.7857}{2} \right\} \Rightarrow L = [\ 4 \ 3 \ 2 \ 1]$$

Example 3

Forward Elimination-- Step 3: eliminate x3

$$\begin{bmatrix} 0 & 0 & 0.7857 & 0.3571 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.9286 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix}$$

Example 3

Backward Substitution

$$\begin{bmatrix} 0 & 0 & 0 & 0.8448 \\ 0 & 0 & -2.7619 & 1.7143 \\ 0 & -10.5 & -0.25 & -0.75 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1.4569 \\ 1.8571 \\ 2.25 \\ -1 \end{bmatrix} \quad L = [4 \ 3 \ 2 \ 1]$$

$$x_4 = \frac{b_{l_4}}{a_{l_4,4}} = \frac{1.4569}{0.8448} = 1.7245, \quad x_3 = \frac{b_{l_3} - a_{l_3,4}x_4}{a_{l_3,3}} = \frac{1.8571 - 1.7143x_4}{-2.7619} = 0.3980$$

$$x_2 = \frac{b_{l_2} - a_{l_2,4}x_4 - a_{l_2,3}x_3}{a_{l_2,2}} = -0.3469$$

$$x_1 = \frac{b_{l_1} - a_{l_1,4}x_4 - a_{l_1,3}x_3 - a_{l_1,2}x_2}{a_{l_1,1}} = \frac{-1 - 3x_4 - 5x_3 - 2x_2}{4} = -1.8673$$

How Do We Know If a Solution is Good or Not

Given $AX=B$

X is a solution if $AX-B=0$

Compute the residual vector $R= AX-B$

Due to rounding error, R may not be zero

The solution is acceptable if $\max_i |r_i| \leq \varepsilon$

How Good is the Solution?

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 4 \\ 5 & -8 & 6 & 3 \\ 4 & 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{solution} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1.8673 \\ -0.3469 \\ 0.3980 \\ 1.7245 \end{bmatrix}$$

$$\text{Residues: } R = \begin{bmatrix} 0.005 \\ 0.002 \\ 0.003 \\ 0.001 \end{bmatrix}$$

Remarks:

- We use index vector to avoid the need to move the rows which may not be practical for large problems.
- If we order the equation as in the last value of the index vector, we have a triangular form.
- Scale vector is formed by taking maximum in magnitude in each row.
- Scale vector does not change.
- The original matrices A and B are used in checking the residuals.

Lecture 17

Tridiagonal & Banded Systems and Gauss-Jordan Method

- ❑ Tridiagonal Systems
- ❑ Diagonal Dominance
- ❑ Tridiagonal Algorithm
- ❑ Examples
- ❑ Gauss-Jordan Algorithm

Tridiagonal Systems

Tridiagonal Systems:

- The non-zero elements are in the **main diagonal**, **super diagonal** and **subdiagonal**.
- $a_{ij}=0$ if $|i-j| > 1$

$$\begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 2 & 6 & 2 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

Tridiagonal Systems

- Occur in many applications
- Needs less storage ($4n-2$ compared to $n^2 + n$ for the general cases)
- Selection of pivoting rows is unnecessary (under some conditions)
- Efficiently solved by Gaussian elimination

Algorithm to Solve Tridiagonal Systems

- Based on Naive Gaussian elimination.
- As in previous Gaussian elimination algorithms
 - Forward elimination step
 - Backward substitution step
- Elements in the **super diagonal** are not affected.
- Elements in the **main diagonal**, and **B** need updating

Tridiagonal System

All the a elements will be zeros, need to update the d and b elements

The c elements are not updated

$$\begin{bmatrix} d_1 & c_1 & & & \\ a_1 & d_2 & c_2 & & \\ & a_2 & d_3 & \ddots & \\ & & \ddots & \ddots & c_{n-1} \\ & & & a_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{bmatrix} d_1 & c_1 & & & \\ & d'_2 & c_2 & & \\ & & d'_3 & \ddots & \\ & & & \ddots & c_{n-1} \\ & & & & d'_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \\ \vdots \\ b'_n \end{bmatrix}$$

Diagonal Dominance

A matrix A is diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^n |a_{ij}| \quad \text{for } (1 \leq i \leq n)$$

The magnitude of each diagonal element is larger than the sum of elements in the corresponding row.

Diagonal Dominance

Examples:

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 6 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

Diagonally dominant

$$\begin{bmatrix} -3 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Not Diagonally dominant

Diagonally Dominant Tridiagonal System

- A tridiagonal system is diagonally dominant if

$$|d_i| > |c_i| + |a_{i-1}| \quad (1 \leq i \leq n)$$

- Forward Elimination preserves diagonal dominance

Solving Tridiagonal System

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}$$

$$b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \quad 2 \leq i \leq n$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}$$

$$x_i = \frac{1}{d_i} (b_i - c_i x_{i+1}) \quad \text{for } i = n-1, n-2, \dots, 1$$

Example

Solve

$$\begin{bmatrix} 5 & 2 & & \\ 1 & 5 & 2 & \\ & 1 & 5 & 2 \\ & & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_i \leftarrow d_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) c_{i-1}, \quad b_i \leftarrow b_i - \left(\frac{a_{i-1}}{d_{i-1}} \right) b_{i-1} \quad 2 \leq i \leq 4$$

Backward Substitution

$$x_n = \frac{b_n}{d_n}, \quad x_i = \frac{1}{d_i} (b_i - c_i x_{i+1}) \quad \text{for } i = 3, 2, 1$$

Example

$$D = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 12 \\ 9 \\ 8 \\ 6 \end{bmatrix}$$

Forward Elimination

$$d_2 = d_2 - \left(\frac{a_1}{d_1} \right) c_1 = 5 - \frac{1 \times 2}{5} = 4.6, \quad b_2 = b_2 - \left(\frac{a_1}{d_1} \right) b_1 = 9 - \frac{1 \times 12}{5} = 6.6$$

$$d_3 = d_3 - \left(\frac{a_2}{d_2} \right) c_2 = 5 - \frac{1 \times 2}{4.6} = 4.5652, \quad b_3 = b_3 - \left(\frac{a_2}{d_2} \right) b_2 = 8 - \frac{1 \times 6.6}{4.6} = 6.5652$$

$$d_4 = d_4 - \left(\frac{a_3}{d_3} \right) c_3 = 5 - \frac{1 \times 2}{4.5652} = 4.5619, \quad b_4 = b_4 - \left(\frac{a_3}{d_3} \right) b_3 = 6 - \frac{1 \times 6.5652}{4.5652} = 4.5619$$

Example

Backward Substitution

- After the Forward Elimination:

$$D^T = [5 \quad 4.6 \quad 4.5652 \quad 4.5619], \quad B^T = [12 \quad 6.6 \quad 6.5652 \quad 4.5619]$$

- Backward Substitution:

$$x_4 = \frac{b_4}{d_4} = \frac{4.5619}{4.5619} = 1,$$

$$x_3 = \frac{b_3 - c_3 x_4}{d_3} = \frac{6.5652 - 2 \times 1}{4.5652} = 1$$

$$x_2 = \frac{b_2 - c_2 x_3}{d_2} = \frac{6.6 - 2 \times 1}{4.6} = 1$$

$$x_1 = \frac{b_1 - c_1 x_2}{d_1} = \frac{12 - 2 \times 1}{5} = 2$$

Gauss-Jordan Method

- The method reduces the general system of equations $AX=B$ to $IX=B$ where I is an identity matrix.
- Only Forward elimination is done and no backward substitution is needed.
- It has the same problems as Naive Gaussian elimination and can be modified to do partial scaled pivoting.
- It takes 50% more time than Naive Gaussian method.

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 1 Eliminate x_1 from *equations 2 and 3*

$$\left. \begin{array}{l} eq1 \leftarrow eq1 / 2 \\ eq2 \leftarrow eq2 - \left(\frac{4}{1}\right)eq1 \\ eq3 \leftarrow eq3 - \left(\frac{2}{1}\right)eq1 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 6 & -5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

Step 2 Eliminate x_2 from equations 1 and 3

$$\left. \begin{array}{l} eq2 \leftarrow eq2 / 6 \\ eq1 \leftarrow eq1 - \left(\frac{-1}{1} \right) eq2 \\ eq3 \leftarrow eq3 - \left(\frac{0}{1} \right) eq2 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 1 & 0 & 0.1667 \\ 0 & 1 & -0.8333 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.1667 \\ 1.1667 \\ 2 \end{bmatrix}$$

Step 3 Eliminate x_3 from *equations 1 and 2*

$$\left. \begin{array}{l} eq3 \leftarrow eq3 / 2 \\ eq1 \leftarrow eq1 - \left(\frac{0.1667}{1} \right) eq3 \\ eq2 \leftarrow eq2 - \left(\frac{-0.8333}{1} \right) eq3 \end{array} \right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Gauss-Jordan Method

Example

$$\begin{bmatrix} 2 & -2 & 2 \\ 4 & 2 & -1 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$$

is transformed to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \Rightarrow \text{solution is } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$