Theorem 1: The maximum absolute error of an algebraic sum or difference of several approximate numbers does not exceed to the sum of absolute error of the numbers, that is, if x_i (i=1, 2 … n) be the n approximate numbers and u is their algebraic sum or difference, then

$$\Delta u \leq \Delta x_1 + \Delta x_2 + \ldots\ldots\ldots\ldots + \Delta x_n$$

We prove the results for two numbers:

(a)For sum:

- Let n_1, n_2 be approximate numbers to N_1, N_2 with errors E_1, E_2, so that $N_1= n_1+E_1$, $N_2= n_2+E_2$, then $N_1+N_2 = (n_1 + E_1) + (n_1 + E_2)$

$$= (n_1 + n_2) + (E_1 + E_2)$$

Let $N_1+N_2 = N$, $n_1 + n_2=n$ and $N= n + E$, then $N= n+ (E_1 + E_2)$

$N - n= E_1 + E_2$

$E= E_1 + E_2$

or, $|E| = | E_1 + E_2|$

or, $|E| \leq | E_1| + | E_2|$…………………………..(A)

(b) For difference:

$$N_1 - N_2 = (n_1 - n_2) + (E_1 - E_2)$$

Let $N_1 - N_2 = N$, $n_1 - n_2 = n$ and $N = n + E$,

then $N = n + (E_1 - E_2)$

$$N - n = E_1 - E_2$$

$$E = E_1 + (-E_2)$$

$$|E| \leq |E_1| + |-E_2|$$

$$|E| \leq |E_1| + |E_2| \text{..............................(B)}$$

(A) and (B) show that the absolute errors in the sum or difference of two numbers does not exceed the sum of their absolute errors. Similarly the result can be extended for three and more approximate numbers.

Theorem 2: The maximum relative error for both multiplication and division does not exceed to the algebraic sum of their relative errors.

(a) **For multiplication:** Let n_1, n_2 be approximate numbers to N_1, N_2 with errors E_1, E_2, so that $N_1N_2 = N$, $n_1n_2 = n$, $N = n + E$. Then,

$N_1N_2 = (n_1 + E_1)(n_2 + E_2)$

$\quad\quad = n_1n_2 + E_1n_2 + n_2E_2 + E_1E_2$

$N_1N_2 - n_1n_2 \equiv E = E_1n_2 + E_2n_2 + E_1E_2$

Neglecting the second order term E_1E_2 and dividing both sides by n_1n_2 i.e. n, one gets

$$\frac{E}{n} = \frac{E_1}{n_1} + \frac{E_2}{n_2}$$

$$\Rightarrow \left|\frac{E}{n}\right| \le \left|\frac{E_1}{n_1}\right| + \left|\frac{E_2}{n_2}\right| \quad \text{..............................(I)}$$

(b) **For division:** let $\frac{N_1}{N_2} = N$; $\frac{n_1}{n_2} = n$ and $N - n = E$, then

$$\frac{N_1}{N_2} = \frac{(n_{1+E_1})}{(n_2+E_2)} = \frac{n_1}{n_2}\left(1+\frac{E_1}{n_1}\right)\left(1+\frac{E_2}{n_2}\right)^{-1}$$

$$N \cong n\left(1+\frac{E_1}{n_1}\right)\left(1-\frac{E_2}{n_2}\right) = n\left(1+\frac{E_1}{n_1}-\frac{E_2}{n_2}\right)$$

where we have used the binomial theorem and neglected the second and higher order terms 9assuming they are small). Then

$$\frac{N-n}{n} \equiv \frac{E}{n} = \frac{E_1}{n_1} - \frac{E_2}{n_2}$$

$$\Rightarrow \quad \left|\frac{E}{n}\right| = \left|\frac{E_1}{n_1} + \left(-\frac{E_2}{n_2}\right)\right| \leq \left|\frac{E_1}{n_1}\right| + \left|\frac{E_2}{n_2}\right| \quad \text{..............................(II)}$$

(I) and (II) shows that relative error in the product of two approximate numbers is always less than the algebraic sum of their relative errors.

Example 1.2 If the number X is rounded to N decimal places, then

$$\Delta X = \frac{1}{2}(10^{-N}).$$

If $X = 0.51$ and is correct to 2 decimal places, then $\Delta X = 0.005$, and the relative accuracy is given by $0.005/0.51 = 0.98\%$.

Example 1.4 Three approximate values of the number 1/3 are given as 0.30, 0.33 and 0.34. Which of these three is the best approximation? We have

$$\left|\frac{1}{3} - 0.30\right| = \frac{1}{30}.$$

$$\left|\frac{1}{3} - 0.33\right| = \frac{0.01}{3} = \frac{1}{300}.$$

$$\left|\frac{1}{3} - 0.34\right| = \frac{0.02}{3} = \frac{1}{150}.$$

It follows that 0.33 is the best approximation for 1/3.

Example 1.5 Find the relative error of the number 8.6 if both of its digits are correct.

Here

$$E_A = 0.05$$

Hence

$$E_R = \frac{0.05}{8.6} = 0.0058.$$

Example 1.6 Evaluate the sum $S = \sqrt{3} + \sqrt{5} + \sqrt{7}$ to 4 significant digits and find its absolute and relative errors.

We have

$$\sqrt{3} = 1.732, \sqrt{5} = 2.236 \quad \text{and} \quad \sqrt{7} = 2.646$$

Hence $S = 6.614$. Then

$$E_A = 0.0005 + 0.0005 + 0.0005 = 0.0015$$

The total absolute error shows that the sum is correct to 3 significant figures only. Hence we take $S = 6.61$ and then

$$E_R = \frac{0.0015}{6.61} = 0.0002.$$

Mathematical Preliminaries

Theorem 1.1 If $f(x)$ is continuous in $a \le x \le b$, and if $f(a)$ and $f(b)$ are of opposite signs, then $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

Theorem 1.2 (*Rolle's theorem*) If $f(x)$ is continuous in $a \le x \le b$, $f'(x)$ exists in $a < x < b$ and $f(a) = f(b) = 0$, then, there exists at least one value of x, say ξ, such that $f'(\xi) = 0$, $a < \xi < b$.

Theorem 1.3 (*Generalized Rolle's theorem*) Let $f(x)$ be a function which is n times differentiable on $[a, b]$. If $f(x)$ vanishes at the $(n + 1)$ distinct points $x_0, x_1, \ldots, x_n$ in (a, b), then there exists a number ξ in (a, b) such that $f^{(n)}(\xi) = 0$.

Theorem 1.4 (*Intermediate value theorem*) Let $f(x)$ be continuous in $[a, b]$ and let k be any number between $f(a)$ and $f(b)$. Then there exists a number ξ in (a, b) such that $f(\xi) = k$ (see Fig. 1.1).

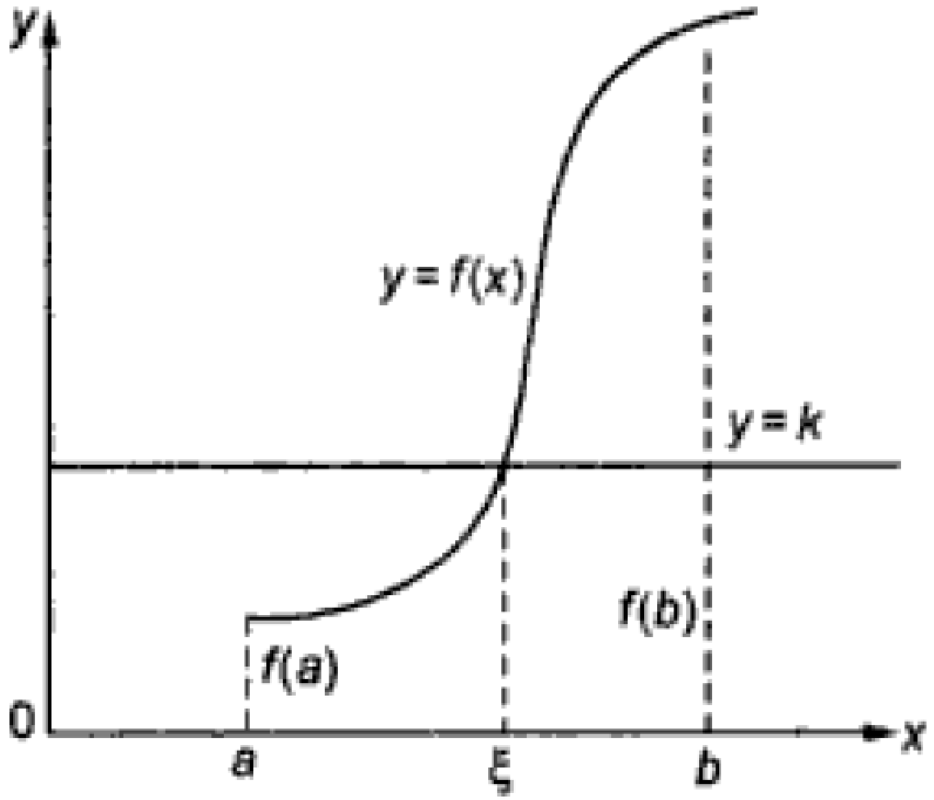

Figure 1.1

Theorem 1.5 (*Mean-value theorem for derivatives*) If $f(x)$ is continuous in $[a, b]$ and $f'(x)$ exists in (a, b), then there exists at least one value of x, say ξ, between a and b such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}, \qquad a < \xi < b.$$

Setting $b = a + h$, this theorem takes the form

$$f(a+h) = f(a) + hf'(a + \theta h), \qquad 0 < \theta < 1.$$

Theorem 1.6 (*Taylor's series for a function of one variable*) If $f(x)$ is continuous and possesses continuous derivatives of order n in an interval that includes $x = a$, then in that interval

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n(x),$$

where $R_n(x)$, the ***remainder term***, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!}f^{(n)}(\xi), \qquad a < \xi < x.$$