NUMERICAL ANALYSIS

PROJECT – 2

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Taylor's Method: Consider the differential equation

$$\boldsymbol{y'(t) = f(t, y), \qquad y(t_0) = y_0}$$

If the existence of all higher order partial derivatives is assumed for y at $t = t_0$, then by Taylor series the value of y at any neighboring point $t_0 + \delta t$ can be written as

$$\boldsymbol{y(t_0 + \delta t) = y(t_0) + y'(t_0)\delta t + y''(t_0)\frac{\delta t^2}{2!} + y'''(t_0)\frac{\delta t^3}{3!} + \cdots}$$

Thus, $y(t_0 + \delta t)$ can be obtained by summing up the above series. However, in practical computation, the summation has to be terminated after some finite number of terms (say k). If the series is terminated after the 1^{st} derivative term then the approximation method is called **Euler's Method** and if it is terminated after k^{th} derivative term then it is called $\boldsymbol{k^{th}}$ **order Taylor's Method**. And thus,

$$\boldsymbol{y(t_0 + \delta t) = y(t_0) + y'(t_0)\delta t + y''(t_0)\frac{\delta t^2}{2!} + y'''(t_0)\frac{\delta t^3}{3!} + \cdots + y^{''(..k-times..)''}(t_0)\frac{\delta t^k}{k!}}$$

Now, to approximate $y(t)$ we start with $y_0 = y(t_0)$ and a small time-interval δt and a positive integer n. We approximate value of $y(t)$ at $t_1, t_2, \dots \dots, t_n$ using the above approximation where

$$\boldsymbol{t_i = t_{i-1} + \delta t}$$

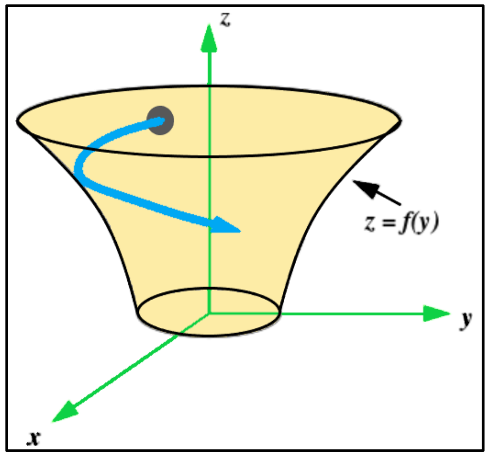
And,

$$\boldsymbol{y(t_i) = y(t_{i-1}) + y'(t_{i-1})\delta t + y''(t_{i-1})\frac{\delta t^2}{2!} + \cdots + y^{''(..k-times..)''}(t_{i-1})\frac{\delta t^k}{k!}, \qquad i = 1, 2, 3, \dots, n}$$

The points $\{t_i, y(t_i)\}$ may not lie exactly on the curve but if δt is small then this should lie close to it.

Gravity Well: Gravity Well Model is a model to demonstrate Einstein's theory of Gravitation. The model consists of some balls rolling on a large curved funnel and a big ball kept at the center of the funnel. The funnel represents the space-time warped by a heavy star, represented by the big ball, sitting at the center. The smaller balls represent asteroids and small moons that tend to roll into the cavity, but owing to their initial tangential velocities end up orbiting the star or the big ball.

Consider the following funnel like surface. It is obtained by rotating the curve $z = f(y)$ around the z-axis. For instance, $\boldsymbol{f(y) = \sqrt{y - 1}}$ would produce a surface like the following:



A ball is moving along the inner surface of the funnel. We shall ignore the radius of the ball and the friction of the surface. (Thus, the ball is a point mass slipping, not rolling, on the funnel.) We know the initial position and velocity of the ball. We want to find out the path that the ball will follow. There are **two forces** acting on the ball: **its weight and the reaction of the surface**. The first works downwards, and so is

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\boldsymbol{mg} \end{bmatrix}$$

The reaction acts inwards along the normal to the surface at the current position of the ball. Let the current position of the ball be

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{f}(\boldsymbol{u}) \end{bmatrix}$$

where $\boldsymbol{f}(\boldsymbol{u}) = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2}$. A little coordinate geometry shows that the normal lies along

$$\begin{bmatrix} -\boldsymbol{x} \\ -\boldsymbol{y} \\ \dfrac{\boldsymbol{u}}{\boldsymbol{f}'(\boldsymbol{u})} \end{bmatrix}$$

So, the reaction force is

$$\boldsymbol{R}\begin{bmatrix} -\boldsymbol{x} \\ -\boldsymbol{y} \\ \dfrac{\boldsymbol{u}}{\boldsymbol{f}'(\boldsymbol{u})} \end{bmatrix}$$

for some unknown function R. So, we have equation of motion:

$$\boldsymbol{m}\begin{bmatrix} \boldsymbol{x}'' \\ \boldsymbol{y}'' \\ \boldsymbol{z}'' \end{bmatrix} = \boldsymbol{R}\begin{bmatrix} -\boldsymbol{x} \\ -\boldsymbol{y} \\ \dfrac{\boldsymbol{u}}{\boldsymbol{f}'(\boldsymbol{u})} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ -\boldsymbol{mg} \end{bmatrix}$$

Thus, we have 3 equations in 3 unknowns: x, y and R. Notice that z is a known function of x and y. To simplify the equations first find z'' in terms of x, y and their derivatives. Then eliminate R to get two equations in two unknowns:

$$\boldsymbol{x}'' = -\boldsymbol{x}\boldsymbol{R}'$$

$$y'' = -yR'$$

where

$$R' = \frac{\frac{f'(u)}{u}\left(x'^2 + y'^2 - u'^2\right) + u'^2 f''(u) + g}{u\left(f'(u) + \frac{1}{f'(u)}\right)}$$

Project: Use 2^{nd} order Taylor method to solve the above problem for the initial condition

$$x(0) = 10,\ \ y(0) = 0,\ \ x'(0) = 0,\ \ y'(0) = 5$$

Take $g = 9.8$.

Procedure:

1.) Select a suitable δt and n and using the initial condition find $z(0),\ x''(0),\ y''(0)$.
2.) Set $t_1 = \delta t$ and apply 2^{nd} order Taylor's Method to find $x(t_1)$ and $y(t_1)$ and using these find $z(t_1)$.
3.) Using Euler's Method calculate $x'(t_1)$ and $y'(t_1)$.
4.) Set $t_2 = t_1 + \delta t$ and repeat the process again to get $x(t_2)$, $y(t_2)$ and $z(t_2)$.
5.) Keep on repeating the process until $t = t_n$.
6.) Plot the all the $\{x(t_i),\ y(t_i),\ z(t_i)\}$ triplets for $i = 0, 1, 2, 3, \ldots, n$, to visualize the path followed by the object.

Python Code:

```
# PROJECT - 2

## Importing Libraries
import numpy as np
import matplotlib.pyplot as plt

## Intial Conditions
x = 10.00
y = 0.00
dx = 0.00
dy = 5.00
print("Initial Condition:\nx=%d, y=%d, x'=%d & y'=%d"%(int(x),int(y),int(dx),int(dy)))

## Finding the path of the object using the intial conditions
dt = float(input("Time interval = "))
n = float(input("Number of intervals = "))
print("Final time = ",dt*n)

path=np.empty((0,3),float)

for i in range(0,int(n)+1):
    u = (x**2 + y**2)**0.5
    du = (x*dx + y*dy)/u
    z = (u - 1)**0.5
    dz = 1/(2*((u-1)**0.5))
    d2z = -1/(4*((u-1)**1.5))
    path=np.append(path,[[x,y,z]],axis=0)
    R = (((dx**2 + dy**2 - du**2)*dz/u + (du**2)*d2z + 9.8)/(u*(dz + 1/dz)))
    d2x = -x*R
    d2y = -y*R
    x = x + dx*dt + d2x*(dt**2/2)
    y = y + dy*dt + d2y*(dt**2/2)
    dx = dx + d2x*dt
    dy = dy + d2y*dt
```

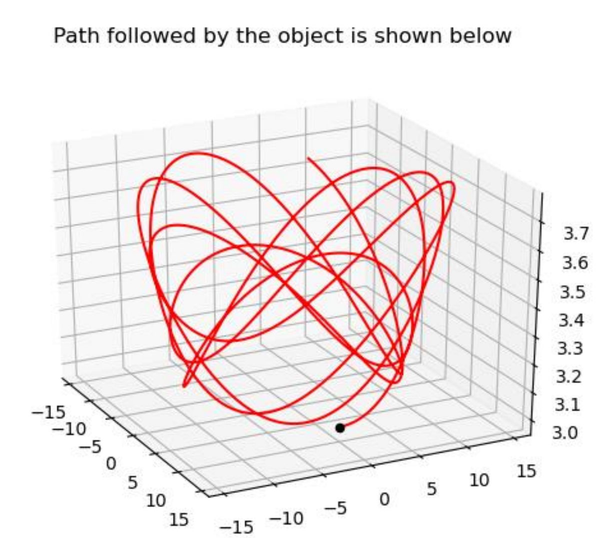
```python
print("\nPath taken By the object\n",path)

## Visualising the path taken the by the object in a 3D graph
fig = plt.figure()
ax = fig.gca(projection = '3d')
ax.plot(path[:,0],path[:,1],path[:,2],linewidth=1.5,color='red')
ax.scatter(path[0,0],path[0,1],path[0,2], color = 'black')
plt.title('Path followed by the object is shown below\n\n')
plt.show()
```

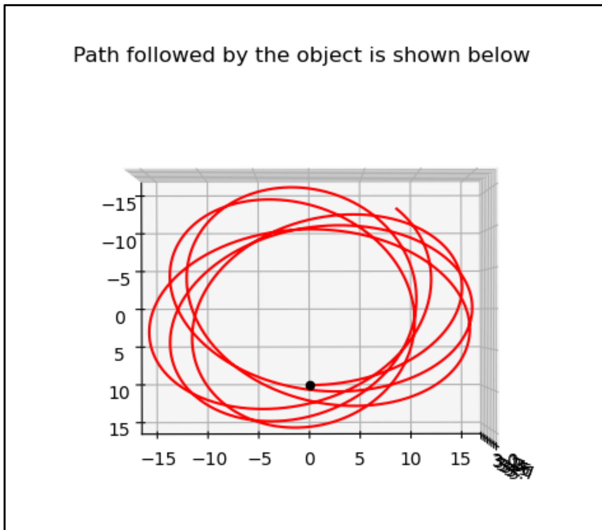
Output:

```
Python 3.8.2 (tags/v3.8.2:7b3ab59, Feb 25 2020, 23:03:10) [MSC v.1916 64 bit (AMD64)] on win32
Type "help", "copyright", "credits" or "license()" for more information.
>>>
= RESTART: D:\Notes\Numerical Analysis\Projects\Project2_GravityWell\Project2_GravityWell.py
Initial Condition:
x=10, y=0, x'=0 & y'=5
Time interval = 0.01
Number of intervals = 10000
Final time =  100.0

Path taken By the object
 [[ 10.          0.          3.        ]
 [  9.99991716   0.05        3.00000703]
 [  9.99966865   0.09999959   3.00002811]
 ...
 [-12.05697698   7.97974279   3.66857702]
 [-12.08335527   7.95321368   3.6695857 ]
 [-12.10962274   7.92661143   3.67058867]]
>>>
```



Side View



Top View