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PQT ASSIGNMENT-I

1.) Given,

x	0	1	2	3	4
$P(x)$	k	$3k$	$5k$	$7k$	$9k$

i.) Find k
ii.) Find $P(x<3)$

i.) W.K.T

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

$$k + 3k + 5k + 7k + 9k = 1$$

$$\cancel{25k} \quad 25k = 1$$

$$k = \frac{1}{25}$$

ii.) $P(x<3) = P(x=0) + P(x=1) + P(x=2)$

$$= k + 3k + 5k.$$

$$= 9k$$

$$= \frac{9}{25}$$

2.) Given,

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$.

Find a, $P(x<5)$, $P(x \geq 3)$, $P(0<x<5)$ also the distribution function of x

WKT

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

$$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$$

$$81a = 1$$

$$a = \frac{1}{81}$$

$P(x<5) = P(x=0)+P(x=1)+P(x=2)+P(x=3)+P(x=4)$

$= a+3a+5a+7a+9a$

$= 25a$

$= \frac{25}{81}$

$P(x\geq 3) = P(x=3)+P(x=4)+P(x=5)+P(x=6)+P(x=7)+P(x=8)$

$= 7a+9a+11a+13a+15a+17a$

$= 71a$

$= \frac{71}{81}$

$P(0<x<5)$, Distrubution function of X

$F(1) = P(x\leq 1) = P(x=1) = 3a = \frac{\cancel{3}}{\cancel{81}_{27}} = \frac{1}{27}$

$F(2) = P(\cancel{x}\ 1\leq x\leq 2) = P(x=1)+P(x=2) = 3a+5a = 8a = \frac{8}{81}$

$F(3) = P(1\leq x\leq 3) = P(x=1)+P(x=2)+P(x=3) = 3a+5a+7a = 15a = \frac{\cancel{15}^{5}}{\cancel{81}_{27}} = \frac{5}{27}$

$F(4) = P(1\leq x\leq 4) = P(x=1)+P(x=2)+P(x=3)+P(x=4) = 3a+5a+7a+9a$

$= 24a = \frac{\cancel{24}^{8}}{\cancel{81}_{27}}$

$= \frac{8}{27}$

$\therefore\ a = \frac{1}{81}$

$P(x<5) = \frac{25}{81}$

$P(x\geq 3) = \frac{71}{81}$

3.)

x	0	1	2	3	4	5	6	7
$P(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i.) Find k

ii.) $P(1.5 < x < 4.5 \mid x > 2)$ find.

iii.) the smallest value of λ such that $P(x \leq \lambda) > \frac{1}{2}$

iv.) find the distribution function of x

i.) WKT

$$\sum_{i=1}^{\infty} P(x_i) = 1$$

$$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$10k^2 + 9k - 1 = 0.$$

$$10k^2 + 10k - k - 1$$

$$10k(k+1) - 1(k+1) = 0$$

$$k = \frac{1}{10}, \cancel{-1}.$$

$$\therefore, \quad k = \frac{1}{10}$$

ii.) $P(1.5 < x < 4.5 \mid x > 2) = P(x=3) + P(x=4)$

$$= 2k + 3k$$

$$= 5k$$

$$= 5 \cdot \frac{1}{10}$$

$$= \frac{1}{2}$$

iii.) $P(x \leq \lambda) > \frac{1}{2}$

$\therefore$, Case 1:- $P(x \leq 0) > \frac{1}{2}$

$$0 > \frac{1}{2} \times$$

Case 2:- $P(x \leq 1) > \frac{1}{2}$.

$$P(x=0) + P(x=1) > \frac{1}{2}$$

$$\frac{1}{10} > \frac{1}{2} \times$$

Case 3:-

$P(x \leq 2) > \frac{1}{2}$.

$P(x=0) + P(x=1) + P(x=2) > \frac{1}{2}$.

$3K > \frac{1}{2}$

$\frac{3}{10} > \frac{1}{2}$. ×

Case 4:-

$P(x \leq 3) > \frac{1}{2}$.

$P(x=0) + P(x=1) + P(x=2) + P(x=3) > \frac{1}{2}$

$5K > \frac{1}{2}$

$5 \cdot \frac{1}{10} > \frac{1}{2}$

$\frac{1}{2} > \frac{1}{2}$ ×

Case 5:-

$P(x \leq 4) > \frac{1}{2}$

$P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) > \frac{1}{2}$

$8K > \frac{1}{2}$

$8 \cdot \frac{1}{10} > \frac{1}{2}$

$\frac{4}{5} > \frac{1}{2}$ ✓

∴ The minimum value of x is 4

iv.) The Distribution function of x

$F(0) = P(x=0) = 0$

$F(1) = P(x \leq 1) = P(x=0) + P(x=1) = K = \frac{1}{10}$

$F(2) = P(x \leq 2) = 3K = \frac{3}{10}$

$F(3) = P(x \leq 3) = 5K = \frac{5}{10} = \frac{1}{2}$

$F(4) = P(x \leq 4) = 8K = \frac{8}{10} = \frac{4}{5}$

$F(5) = P(x \leq 5) = K^2 + 8K = \frac{1}{100} + \frac{8}{10} = \frac{81}{100}$

$F(6) = P(x \leq 6) = 8K + 3K^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$

$F(7) = P(x \leq 7) = 10K^2 + 9K = \frac{1}{10} + \frac{9}{10} = 1$

4) Given,

$$f(x) = Ke^{-3x} \quad , x > 0$$

i.) Find K

ii.) Find $P(0.5 \le x \le 1)$

iii.) Find Mean of X

WKT

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} Ke^{-3x}\,dx = 1$$

$$K\int_{0}^{\infty} e^{-3x}\,dx = 1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_{0}^{\infty} = 1$$

$$-\frac{K}{3}\left[e^{-3\infty} - e^{0}\right] = 1$$

$$= -\frac{K}{3}[-1] = 1$$

$$K = 3$$

ii.) $$P(0.5 \le x \le 1) = \int_{1/2}^{1} 3e^{-3x}\,dx$$

$$= \frac{\cancel{3}e^{-3x}}{-\cancel{3}}\Big]_{1/2}^{1}$$

$$= -e^{-3x}\Big]_{1/2}^{1}$$

$$= -e^{-3} + e^{-3/2}$$

$$= e^{-3/2} - e^{-3}$$

$$\therefore P(0.5 \le x \le 1) = e^{-3/2} - e^{-3}$$

iii.) Mean of X

$$E(x) = \int_0^\infty x f(x)\, dx$$

$$= \int_0^\infty x(3e^{-3x})\, dx$$

$$= \int_0^\infty 3x e^{-3x}\, dx$$

$u = x \qquad v = e^{-3x}$

$u' = 1 \qquad v_1 = \frac{e^{-3x}}{-3}$

$v_2 = \frac{e^{-3x}}{9}$

$$= 3\left[\frac{xe^{-3x}}{-3} - \frac{e^{-3x}}{9}\right]_0^\infty$$

$$= 3\left[0 - 0 + 0 + \frac{1}{9}\right]$$

$$= \frac{1}{3}$$

$\therefore$ Mean $= \frac{1}{3}$

$\therefore$ $K = 3$

$\therefore$ $P(0.5 \leq x \leq 1) = e^{-3/2} - e^{-3}$ //

5) Given,

$$f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ 2k, & 2 \le x \le 4 \\ 6k - kx, & 4 \le x \le 6 \\ 0, & \text{elsewhere.} \end{cases}$$

find k, CDF of $F(x)$

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 6k - kx\,dx + \int_6^{\infty} 0\,dx = 1$$

$$\left.\frac{kx^2}{2}\right]_0^2 + \left.2kx\right]_2^4 + \left[6kx - \frac{kx^2}{2}\right]_4^6 + 0 = 1$$

$$\frac{4k}{2} - 0 + 8k - 4k + 36k - \frac{36k}{2} - 24k + \frac{16k}{2} = 1$$

$$2k + 8k - 4k + 36k - 18k - 24k + 8k = 1$$

$$54k - 46k = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

CDF $F(x) = P[X \le x]$

when $x < 0$, $F(x) = 0$

when $0 \le x < 2$, $F(x) = \int_0^x f(x)\,dx = \int_0^x kx\,dx = \left.\frac{kx^2}{2}\right]_0^x = \frac{x^2 k}{2} = \frac{x^2 \cdot 1}{8 \cdot 2} = \frac{x^2}{16}$

when $2 \le x < 4$, $F(x) = \int_0^2 f(x)\,dx + \int_2^x f(x)\,dx$

$$= \int_0^2 kx\,dx + \int_2^x 2k\,dx$$

$$= \left.\frac{kx^2}{2}\right]_0^2 + \left.2kx\right]_2^x = \frac{4k}{2} - 0 + 2kx - 4k$$

$$= 2kx - 2k = \frac{2 \cdot 1}{8_4}x - \frac{2}{8_4}$$

$$F(x) = \frac{x}{4} - \frac{1}{4} = \frac{x-1}{4}$$

when $4 \le x \le 6$ $\quad F(x) = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^x f(x)\,dx$

$$= \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^x 6k - kx\,dx$$

$$= \left.\frac{kx^2}{2}\right]_0^2 + \left.2kx\right]_2^4 + \left.6kx - \frac{kx^2}{2}\right]_4^x$$

$$= \frac{4k}{2} - 0 + 8k - 4k + 6kx - \frac{kx^2}{2} - 24k + \frac{16k}{2}$$

$$= 2k + 8k - 4k + 6kx - \frac{kx^2}{2} - 24k + 8k$$

$$= 6kx - \frac{kx^2}{2} - 10k$$

$$= \frac{\cancel{6}^{3}x}{\cancel{8}_{4}} - \frac{x^2}{2\cdot 8} - \frac{\cancel{10}^{5}}{\cancel{8}_{4}}$$

$$= \frac{3x}{4} - \frac{x^2}{16} - \frac{5}{4}$$

$$= \frac{12x - x^2 - 20}{16}$$ //

when $\cancel{6 \le x \le}$ $x \ge 6$ $\quad F(x) = \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^x f(x)\,dx$

$$= \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 6k - kx\,dx + \int_6^x 0\,dx$$

$$F(x) = \left.\frac{kx^2}{2}\right]_0^2 + \left.2kx\right]_2^4 + \left.6kx - \frac{kx^2}{2}\right]_4^6 + 0$$

$$= \frac{4k}{2} - 0 + 8k - 4k + 36k - \frac{36k}{2} - 24k + \frac{16k}{2}$$

$$= 2k + 8k - 4k + 36k - 18k - 24k + 8k$$

$$= 8k$$

$$= \frac{8}{8}$$

$$F(x) = 1$$

7.) Given,

~~$F(x) = 1-(1+x)e^{-x}$, $x>0$~~

~~Find PDF of X, $P(1<x\leq 3)$~~

$f(x) = kx^2e^{-x}$, $x>0$

Find Mean, Variance.

WKT

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} kx^2e^{-x}\,dx = 1$$

$$k\int_{0}^{\infty} \underset{u}{x^2}\,\underset{v}{e^{-x}}\,dx = 1$$

$u = x^2$ $v = e^{-x}$

$u' = 2x$ $v_1 = -e^{-x}$

$u'' = 2$ $v_2 = e^{-x}$

$v_3 = -e^{-x}$

$$k\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k\left[0-0-0+0+0+2\right] = 1$$

$$k = \frac{1}{2}$$

$$\therefore f(x) = \frac{x^2e^{-x}}{2}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x\cdot\frac{x^2e^{-x}}{2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} \underset{u}{x^3}\,\underset{v}{e^{-x}}\,dx$$

$u = x^3$ $v = e^{-x}$

$u' = 3x^2$ $v_1 = -e^{-x}$

$u'' = 6x$ $v_2 = e^{-x}$

$u''' = 6$ $v_3 = -e^{-x}$

$v_4 = e^{-x}$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[0-0-0-0+0+0+0+6\right]$$

$$= \frac{6}{2}$$

$$= 3$$

$\therefore$ Mean $= 3$

~~variance~~ $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_0^{\infty} x^2 \cdot \frac{x^2 e^{-x}}{2}\,dx \quad = \frac{1}{2}\int_0^{\infty} \underset{u}{x^4}\,\underset{v}{e^{-x}}\,dx$

$u = x^4 \qquad v = e^{-x}$

$u' = 4x^3 \qquad v_1 = -e^{-x}$

$u'' = 12x^2 \qquad v_2 = e^{-x}$

$u''' = 24x \qquad v_3 = -e^{-x}$

$u'''' = 24 \qquad v_4 = e^{-x}$

$v_5 = -e^{-x}$

$E(x^2) = \frac{1}{2}\left[x^4(-e^{-x}) - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^{\infty}$

$E(x^2) = \frac{1}{2}\left[0+0+0+0+0+0+0+0+0+24\right]$

$= \frac{1}{\cancel{2}} \times 24$

$= 12$

$\boxed{E(x^2) = 12}$

$\text{variance} = E(x^2) - (E(x))^2$

$= 12 - 9$

$= 3$

~~8) Given, $M_x(t) = \frac{3}{3-t}$~~

9.) Given,

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

MGF

Mean = ? , variance = ?

$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{0} + \int_0^1 e^{tx} f(x)\,dx + \int_1^2 e^{tx} f(x)\,dx + \int_2^{\infty} e^{tx} f(x)\,dx$

$= \int_0^1 xe^{tx}\,dx + \int_1^2 \underset{u}{(2-x)}\,\underset{v}{e^{tx}}\,dx$

$u = x \qquad v = e^{tx} \qquad u = 2-x$

$u' = 1 \qquad v_1 = \frac{e^{tx}}{t} \qquad u' = -1$

$v_2 = \frac{e^{tx}}{t^2}$

$$= \left[\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2} \right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2} \right]_1^2$$

$$= \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} - 0 + \frac{1}{t^2} + \left[0 + \frac{e^{2t}}{t^2} - \cancel{\frac{e^t}{t}} - \frac{e^t}{t^2} \right)$$

$$= \frac{(e^{t})^2 - 2e^t + (1)^2}{t^2} \quad = \left(\frac{e^t}{t}\right)^2 - \frac{2e^t}{t^2} + \left(\frac{1}{t}\right)^2$$

$$\cancel{M_x(t) = \frac{(e^t - 1)^2}{t^2}} \qquad M_x(t) = \left(\frac{e^t}{t} - \frac{1}{t}\right)^2$$

$$\cancel{M_x^1(t)_{t=0} = \frac{t^2 \cdot 2(e^t - 1)e^t - (e^t-1)^2 \cdot 2t}{t^4}}$$

$$\cancel{= \frac{0 - 0}{t^4}}$$

$$\cancel{= 0}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^1 x \cdot x\,dx + \int_1^2 x \cdot (2-x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 2x\,dx - \int_1^2 x^2\,dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 + \left.\frac{2x^2}{2}\right]_1^2 - \left.\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} - 0 + 4 - 1 - \frac{8}{3} + \frac{1}{3}$$

$$= 3 - \frac{6}{3}$$

Mean = 1

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \int_0^1 x^3\,dx + \int_1^2 2x^2\,dx - \int_1^2 x^3\,dx$$

$$= \left.\frac{x^4}{4}\right]_0^1 + \left.\frac{2x^3}{3}\right]_1^2 - \left.\frac{x^4}{4}\right]_1^2$$

$$= 14\left[\frac{1}{3} - \frac{1}{4}\right] = \cancel{14}\,7\left[\frac{1}{\cancel{12}\,6}\right] = \frac{1}{4} - 0 + \frac{16}{3} - \frac{2}{3} - \frac{16}{4} + \frac{1}{4} = \frac{14}{3} - \frac{14}{4}$$

$$\therefore E(x^2) = 7/6$$

$$\text{Variance} = E(x^2) - (E(x))^2$$
$$= \frac{7}{6} - 1$$
$$= \frac{1}{6}$$ //

10.) Given,

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3. \end{cases}$$

Find a, CDF of X

WKT

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 3a - ax\,dx = 1$$

$$= \left.\frac{ax^2}{2}\right]_0^1 + \left.ax\right]_1^2 + \left.3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$= \frac{a}{2} - 0 + 2a - a + 9a - \frac{9a}{2} - 6a + \frac{4a}{2} = 1$$

$$= 4a - \frac{4a}{2} = 1$$

$$= 2a = 1$$
$$a = \frac{1}{2}.$$

CDF of $f(X) = P[x \le X]$

when, $x < 0$, $F(x) = 0$

when, $0 \le x < 1$, $F(x) = \int_0^x f(x)\,dx = \int_0^x ax\,dx = \left.\frac{ax^2}{2}\right]_0^x = \frac{ax^2}{2}$
$= \frac{x^2}{4}$

when, $1 \le x < 2$, $F(x) = \int_0^1 f(x)\,dx + \int_1^x f(x)\,dx = \int_0^1 ax\,dx + \int_1^x a\,dx$

$$= \left.\frac{ax^2}{2}\right]_0^1 + \left.ax\right]_1^x = \frac{a}{2} - 0 + ax - a$$
$$= ax - \frac{a}{2} = \frac{x}{2} - \frac{1}{4}$$

$\therefore F(x) = \frac{2x-1}{4}$ //

when, $2 \le x \le 3$, $F(x) = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^x f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x 3a - ax\,dx$$

$$= \left.\frac{ax^2}{2}\right]_0^1 + \left.ax\right]_1^2 + \left.3ax - \frac{ax^2}{2}\right]_2^x$$

$$= \frac{a}{2} - 0 + 2a - a + 3ax - \frac{ax^2}{2} - 6a + \frac{4a}{\cancel{2}}^{2}$$

$$= \frac{a}{2} + a + 3ax - \frac{ax^2}{2} - 4a.$$

$$= \frac{a}{2} - 3a + 3ax - \frac{ax^2}{2}$$

$$= \frac{-5a}{2} + 3ax - \frac{ax^2}{2}$$

$$= -\frac{5}{4} + \frac{3x}{2} - \frac{a}{8}\frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} //$$

when, $x \ge 3$, $F(x) = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^x f(x)\,dx$

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 3a - ax\,dx + 0$$

$$= \left.\frac{ax^2}{2}\right]_0^1 + \left.ax\right]_1^2 + \left.3ax - \frac{ax^2}{2}\right]_2^3$$

$$= \frac{a}{2} - 0 + 2a - a + 9a - \frac{9a}{2} - 6a + \frac{4a}{2}.$$

$$= 4a - \frac{4a}{2}$$

$$= 2a$$

$$= 1$$

~~12) Given,~~

~~$f(x) = \frac{1}{2x}$~~

8.) Given,

$$M_x(t) = \frac{3}{3-t}$$

Find first four moments of x

$$M_x'(t)_{t=0} = \left(\frac{3}{3-t}\right)' = \frac{(3-t)0 + 3(1)}{(3-t)^2} = \frac{3}{(3-t)^2} = \frac{3}{9} = \frac{1}{3}$$

$$M_x''(t)_{t=0} = \left(\frac{3}{(3-t)^2}\right)' = \frac{(3-t)^2 0 - 3(2(3-t)(-1))}{(3-t)^4} = \frac{6(\cancel{3-t})}{(3-t)^{\cancel{4}3}} = \frac{6}{(3-t)^3}$$

$$= \frac{\cancel{18}\,2}{\cancel{81}\,9} = \frac{2}{9}$$

$$M_x'''(t)_{t=0} = \left(\frac{6}{(3-t)^3}\right)' = \frac{(3-t)^3 0 - 6(3(3-t)^2\cdot(-1))}{(3-t)^6}$$

$$= \frac{18\cancel{(3-t)^2}}{(3-t)^{\cancel{6}4}} = \frac{18}{(3-t)^4} = \frac{18}{81} = \frac{2}{9}$$

$$M_x^{IV}(t)_{t=0} = \left(\frac{18}{(3-t)^4}\right)' = \frac{(3-t)^4\cdot 0 - 18(4(3-t)^3\cdot(-1))}{(3-t)^8}$$

$$= \frac{72\cancel{(3-t)^3}}{(3-t)^{\cancel{8}5}} = \frac{72}{(3-t)^5}$$

$$= \frac{\cancel{72}\,\cancel{24}\,8}{\cancel{243}\,\cancel{81}\,27} = \frac{8}{27}$$

$\therefore$ $M_x'(t)_{t=0} = \frac{1}{3}$ (Mean (or) first moment)

$M_x''(t)_{t=0} = \frac{2}{9}$ (second moment)

$M_x'''(t)_{t=0} = \frac{2}{9}$ (third moment)

$M_x^{IV}(t)_{t=0} = 8/27$ (fourth moment)

13.) Given, $f(x) = \lambda e^{-\lambda x}, x>0$.

find mean, variance.

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} \lambda e^{(t-\lambda)x}\, dx \qquad \because \int e^{ax} = \frac{e^{ax}}{a}$$

$$= \int_0^{\infty} \lambda e^{-(\lambda-t)x}\, dx \qquad a = -(\lambda-t)$$

$$= \left.\frac{\lambda e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty}$$

$$= 0 + \frac{\lambda e^0}{+(\lambda-t)}$$

$$M_x(t) = \frac{\lambda}{\lambda-t}$$

$$\text{Mean} = M_x'(t)_{t=0} = \left(\frac{\lambda}{\lambda-t}\right)' = \frac{(\lambda-t)0 - \lambda(-1)}{(\lambda-t)^2} = \frac{\lambda}{(\lambda-t)^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$M_x''(t)_{t=0} = \left(\frac{\lambda}{(\lambda-t)^2}\right)' = \frac{(\lambda-t)^2 0 - \lambda(2(\lambda-t)(-1))}{(\lambda-t)^4} = \frac{2\lambda(\cancel{\lambda-t})}{(\lambda-t)^{\cancel{4}\,3}}$$

$$= \frac{2\lambda}{(\lambda-t)^3}$$

$$= \frac{2\cancel{\lambda}}{\lambda^{\cancel{3}}} = \frac{2}{\lambda^2}.$$

$$\therefore \text{Variance} = M_x''(t)_{t=0} - \left(M_x'(t)_{t=0}\right)^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\therefore \text{Mean} = \frac{1}{\lambda}, \quad \text{variance} = \frac{1}{\lambda^2}$$

6) Given,

$$F(x) = 1 - (1+x)e^{-x} \quad , x > 0$$

Find pdf of X, $P(1<X<3)$

$$F(x) = 1 - (e^{-x} + xe^{-x}) \quad , x > 0$$

$$F(x) = 1 - e^{-x} - xe^{-x}, \quad x > 0$$

$$\text{Pdf} = \frac{d}{dx} F(x)$$

$$f(x) = \frac{d}{dx}(1 - e^{-x} - xe^{-x})$$

$$= 0 - \frac{e^{-x}}{-1} - [-xe^{-x} + e^{-x}]$$

$$= \cancel{e^{-x}} + xe^{-x} - \cancel{e^{-x}}$$

$$f(x) = xe^{-x} \quad , x > 0$$

$$P(1<X<3) = \int_1^3 f(x)\,dx$$

$$= \int_1^3 \underset{u}{x}\,\underset{v}{e^{-x}}\,dx$$

$u = x$ $\quad v = e^{-x}$

$u' = 1$ $\quad v_1 = \frac{e^{-x}}{-1}$

$v_2 = e^{-x}$

$$= \left[-xe^{-x} - e^{-x}\right]_1^3$$

$$= -3e^{-3} - e^{-3} + e^{-1} + e^{-1}$$

$$= 2e^{-1} - 4e^{-3} \;//$$

11) Given,

$$f_X(x) = 2x \quad , 0 < x < 1 ,$$

$$Y = 3x + 1$$

$$x = \frac{y-1}{3}$$

$$\frac{dx}{dy} = \frac{1}{3} \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{3}$$

$$f_Y(y) = f_X(x) = \left|\frac{dx}{dy}\right|$$

$$= 2x \cdot \frac{1}{3} \quad , 0 < x < 1$$

$$= \frac{2}{3}\left(\frac{y-1}{3}\right)$$

$$= \frac{2(y-1)}{9}$$

here $0 \leq x \leq 1$

$$0 \leq \frac{y-1}{3} \leq 1$$

$$f_Y(y) = \frac{2(y-1)}{9} \quad , 0 \leq y \leq 1$$

12) Given,

$$f(x) = \frac{1}{2^x} \quad , x = 1, 2, 3 \ldots$$

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x}$$

$$= \frac{e^t}{2} + \frac{e^{2t}}{4} + \frac{e^{3t}}{6} + \ldots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$= \frac{e^t}{2}\left[\frac{2}{2-e^t}\right]$$

$$M_x(t) = \frac{e^t}{2-e^t}$$

$$\text{Mean} = M_x'(t)_{t=0} = e^{2t}(2-e^t)^{-2} + (2-e^t)^{-1}e^t. = \frac{2e^t}{(2-e^t)^2}$$

$$= 1 + 1$$

$$= 2.$$

$$\cancel{M_x''(t)_{t=0} = e^{2t}\left(-2(2-e^t)^{-3}\right) + (2-e^t)^{-2}\cdot 2e^{2t} + (2-e^t)^{-1}e^t + e^{2t}(2-e^t)^{-2}}$$

$$\cancel{= -2 + 2 + 1 + 1}$$

$$= 2$$

$$M_x''(t)_{t=0} = \frac{(2-e^t)^2 2e^t - 2e^t(2(2-e^t)(-e^t))}{(2-e^t)^{4}}$$

$$= \frac{4e^t - 2e^{2t} + 4e^{2t}}{(2-e^t)^2}$$

$$= \frac{4e^t + 2e^{2t}}{(2-e^t)^2}$$

Put $t=0$

$$= \frac{4+2}{(2-1)^2}$$

$$= \frac{6}{1}$$

$$M_x''(t)_{t=0} = 6$$

$$M_x'''(t) = \left(\frac{4e^t + 2e^{2t}}{(2-e^t)^2}\right)'$$

$$= \frac{(2-e^t)^{\cancel{2}}(4e^t + 4e^{2t}) + (4e^t + 2e^{2t})\left[2(\cancel{2-e^t})(e^t)\right)}{(2-e^t)^{\cancel{3}}}$$

$$= \frac{8e^t + 8e^{2t} - 4e^{2t} - \cancel{4e^{3t}} + 8e^{2t} + \cancel{4e^{3t}}}{(2-e^t)^2}$$

$$= \frac{8e^t + 12e^{2t}}{(2-e^t)^2}$$

Put $t = 0$

$$= \frac{8+12}{(2-1)^2} = \frac{20}{1}$$

$$M_x'''(t)_{t=0} = 20$$

$$M_x^{iv}(t) = \left(\frac{8e^t + 12e^{2t}}{(2-e^t)^2}\right)'$$

$$= \frac{(2-e^t)^{\cancel{2}}(8e^t + 24e^{2t}) + (8e^t + 12e^{2t})\left(2(\cancel{2-e^t})(-e^t)\right)}{(2-e^t)^{\cancel{3}}}$$

$$= \frac{16e^t - 24e^{2t} - 8e^{2t} - \cancel{24e^{3t}} + 16e^{2t} + \cancel{24e^{3t}}}{(2-e^t)^2}$$

$$= \frac{16e^t - 16e^{2t}}{(2-e^t)^2}$$

Put $t = 0$

$= 0$

$\therefore$ $M_x'(t)_{t=0} = 2$ (Mean or first moment)

$M_x''(t)_{t=0} = 6$ (second moment)

$M_x'''(t)_{t=0} = 20$ (third moment)

$M_x^{iv}(t)_{t=0} = 0$ (fourth moment

14) Given, $f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty \\ 0, & \text{otherwise.} \end{cases}$

Find k, Evaluate Probability $P(x \geq 0)$

WKT

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$k\left[\tan^{-1}x\right]_0^{\infty} = 1$$

$$k\left[\tan^{-1}\infty - \tan^{-1}0\right] = 1$$

$$k\left[\frac{\pi}{2}\right] = 1$$

$$k = \frac{2}{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$k\left[\tan^{-1}x\right]_{-\infty}^{\infty} = 1$$

$$k\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right] = 1$$

$$k\left[\frac{\pi}{2} - \left[-\tan^{-1}(\infty)\right]\right] = 1$$

$$k\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1$$

$$k\left[\pi\right] = 1$$

$$k = \frac{1}{\pi}$$

for Probability $x \geq 0$

$$\Rightarrow \int_{0}^{\infty} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}x\right]_0^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{1}{2}$$

$\therefore$ Probability $= \frac{1}{2}$.