Wiener- Ito Integrals

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Introduction

- By the first half of last century, many mathematicians, physicists and economist strongly felt that for a proper understanding of several phenomena in the respective disciplines, it was extremely necessary to have a theory of integration with respect to Brownian motion.
- However, the Brownian paths, except for being continuous, are otherwise extremely erratic. In particular, almost all Brownian trajectories are **of unbounded variation on every interval**.
- And that means that the classical theories of integration are of no use here and a completely new theory had to be developed.
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- To start with, lets look at the topics covered in the last semester.
- We defined a Standard Brownian motion and discussed its various properties.
- The two most important properties being the Markov and Martingale properties of SBM.
- Next, we have stated the law of SBM and defined a brownian motion starting at a point $x \neq 0$
- After this, we gave a canonical formulation of a Brownian motion as Markov process which provides a convenient setup for discussing the general theory of Markov processes.

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- Next we defined a Feller process and looked at the restricted generators which uniquely determines a feller process.
- Finally, we had concluded by showing that a Brownian motion has the Feller property with generator as half times the Laplacian, i.e., $Ag = \frac{1}{2}g'', \; g \in \tilde{R} = C_2(\mathbb{R})$.

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Topics to be Covered

- As it has already been observed that Brownian paths are extremely erratic in nature and are of unbounded variation on each interval, so we need a new theory of integration.
- First, we will see how to integrate deterministic integrands, that is, any real-valued function on $[0, \infty)$, with respect to a a SBM, called the Wiener Integration.
- Then, we move on to see how to integrate stochastic integrands, that is, any real valued random function with respect to a SBM, called the Ito Integration.
- Next , we will see a very useful analogue of the fundamental theorem of calculus in this new theory of integration called the Ito's formula.
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Wiener Integral: Definite integral

- Let $\{B_t, t \geq 0\}$ be a standard brownian motion (SBM) on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$.
- Let $\{\mathcal{A}_t,\ t \geq 0\}$ denote the natural filtration of the SBM, that is, $\mathcal{A}_t = \sigma(\{B_s,\ s \leq t\})$
- Fix $T \geq 0$.
- Let $L^2([0,T]) = L^2([0,T], \mathcal{B}([0,T]), \lambda)$, defined as,

  $$L^2([0,T]) = \{f : f \text{ m'ble on } [0,T], \int_0^T f^2(t)dt < \infty\}$$

  be the class of integrands.
- Let $\mathcal{L}^2$ denote the space of all square integrable random variables on $(\Omega, \mathcal{A}, \mathbf{P})$.
- **Aim:** To define the integral,

  $$I_T(f) = \int_0^T f(t)dB_t \text{ for } f \in L^2([0,T])$$

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- Further, we want to define $I_T(f)$ in a way so that $f \mapsto I_T(f)$ will be a linear isometry $I_T : L^2([0,T]) \to \mathcal{L}^2$.
- The idea is to first define $I_T(f)$ for some elementary class of f and then extend it by the following well-known result.

Theorem

Let H_1, H_2 be Hilbert spaces and A be a linear subspace of H_1. Then if $I : A \to H_2$ is a linear isometry on A and A is dense in H_1, I has a unique extension to an isometry on H_1

- **Note:** Both $L^2([0,T])$ and $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbf{P})$ are Hilbert spaces with the usual L^2 norm.
- **Idea:** The first job is to identify a class of "elementary integrands", which forms a linear subspace of $L^2([0,T])$ and is dense in $L^2([0,T])$.

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- **Idea:** Then we would try to define a linear isometry from this class to $\mathcal{L}^2$, so that by the previous theorem, we can uniquely extend this isometry to an isometry $I_T : L^2([0,T]) \to \mathcal{L}^2$.
- Here is our class of "elementary functions" :

$$\mathcal{E} = \{\sum_{i=1}^{n} c_i 1_{(t_{i-1},t_i]} : n \geq 1, 0 \leq t_0 < ... < t_n \leq T\}$$

- Now, let us make some observations on $\mathcal{E}$.
- It can proved that,

  (a) $\mathcal{E}$ is a vector subspace of $L^2[0,T]$.

  (b) $\mathcal{E}$ is dense in $L^2[0,T]$.
- For $f \in \mathcal{E}$, we define a map $I_T : \mathcal{E} \to \mathcal{L}^2$ as

$$I_T(f) = I_T(\sum_{i=1}^{n} c_i 1_{(t_{i-1},t_i]}) = \sum_{i=1}^{n} c_i (B_{t_i} - B_{t_{i-1}})$$

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- From what we have done so far, we could actually take any $f \in L^2([0,\infty))$ and define $I_t(f) = \int_0^t f_s dB_s$, for each $t \geq 0$.
- However, $I_t(f)$ is defined separately for each t as an element of $\mathcal{L}^2$, but the connection among them is not clear at all.
- In the general theory of integration, it is usually desirable that an indefinite integral $\int_0^t f(s)ds$ should be continuous in t.
- **Main Concern:** For a given function $f \in L^2[0,\infty)$, is it possible to choose , for every $t \geq 0$, a version of $I_t(f)$, say, Y_t, such that, $t \to Y_t(\omega)$ is continuous, at least for almost every ω?
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- Now, when we extend this definition of integral from $f \in \mathcal{E}$ to $f \in L^2[0, \infty)$ via isometry, can we retain this continuity?
- The answer is :YES !
- And the main tool is Martingale Maximal Inequality!!

Wiener Integral: Indefinite case

- Here is the concluding result of this section.

Theorem

For every $f \in L^2[0, \infty)$, there exists a process $\{Y_t, 0 \leq t \geq 0\}$, such that $\forall\, t \in [0, \infty)$,

(a) $Y_t = \int_0^t f_s dB_s$

(b) Y_t is $\mathcal{A}_t$-measurable for each $t \in [0, \infty)$.

(c) $\exists$ a $\mathbf{P}$-null set N, such that , $\forall\, \omega \notin N, t \to Y_t(\omega)$ is continuous on $[0, \infty)$.

*Such a process is **unique** upto a $\mathbf{P}$-null set , i.e., if $\{Z_t, 0 \leq t \leq T\}$ is any process satisfying (a), (b) and (c), then $\exists$ a $\mathbf{P}$-null set set N such that , for $\omega \notin N, Z_t(\omega) = Y_t(\omega)$ for all t.*

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Wiener Integral: Indefinite case

- **Conclusion:** This process $\{Y_t = I_t(f), t \in [0, \infty)\}$ is formally known as the Wiener Integral of f.

Definition

*(**Wiener Integral**) Let $\{B_t, t \geq 0\}$ be a standard brownian motion (SBM) on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. For any $f \in L^2[0, \infty)$, the linear isometry $I_t : L^2[0, \infty) \to \mathcal{L}^2(\Omega, \mathcal{A}, \mathbf{P})$, defined as,*

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such that, the process $\{I_t(f), t \in [0, \infty)\}$ is $\mathcal{A}_t$-measurable, $t \to I_t(f)(\omega)$ is continuous in $[0, \infty)$ for almost every $\omega \in \Omega$ and it is unique upto a $\mathbf{P}$-null set, is known as the Wiener Integral of the function $f \in L^2[0, \infty)$ with respect to the SBM for any $t \in [0, \infty)$.

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- Let $\mathcal{L}^2$ denote the space of square integrable random variables on $(\Omega, \mathcal{A}, \mathbf{P})$.
- **Aim:** To define the integration,

$$\int_0^T f_s dB_s = I_T(f) \text{ for } f \in \mathcal{L}_T^2$$

- Thus, similar to the previous case, we want to define $I_T(f)$ in a way so that $f \mapsto I_T(f)$ will be a linear isometry $I_T : \mathcal{L}_T^2 \to \mathcal{L}^2$.
- **Note:** Both $\mathcal{L}_T^2$ and $\mathcal{L}^2(\Omega, \mathcal{A}, \mathbf{P})$ are Hilbert spaces with the usual L^2 norm.
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- In this case, our “elementary class” of functions is ,
  $\mathcal{E}_T = \{f : ([0,T] \times \Omega) \to \mathbb{R} : f(s,\omega) = \sum_{i=1}^{n} \xi_i(\omega) 1_{(t_{i-1},t_i]}(s)$, where $0 \le t_0 < t_1 < \cdots < t_n \le T$ and $\xi_1, \xi_2, \ldots, \xi_n$ are bounded random variables with ξ_i measurable w.r.t. $\mathcal{A}_{t_{i-1}}\}$
- Now, let us make some observations on $\mathcal{E}_T$.
- It can proved that,

  (a) $\mathcal{E}_T$ is a vector subspace of $\mathcal{L}^2_T$.

  (b) $\mathcal{E}_T$ is dense in $\mathcal{L}^2_T$.
- For $f \in \mathcal{E}_T$, let us define a map I_T on $\mathcal{E}_T$ by,

  $$I_T(f) = I_T(\sum_{i=1}^{n} c_i 1_{(t_{i-1},t_i]}) = \sum_{i=1}^{n} c_i (B_{t_i} - B_{t_{i-1}})$$
- It is clear from definition that $I_T(f) \in \mathcal{L}^2$ and that $I_T(f)$ is $\mathcal{A}_T$-measurable, for all $f \in \mathcal{E}$.

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  $\mathcal{E}_T = \{f : ([0, T] \times \Omega) \to \mathbb{R} : f(s, \omega) = \sum_{i=1}^{n} \xi_i(\omega) 1_{(t_{i-1}, t_i]}(s),$ where $0 \leq t_0 < t_1 < \cdots < t_n \leq T$ and $\xi_1, \xi_2, \ldots, \xi_n$ are bounded random variables with ξ_i measurable w.r.t. $\mathcal{A}_{t_{i-1}}\}$
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- Also, it can be easily proved that I_T is a linear isometry on $\mathcal{E}_T$.
- So, now, using the theorem stated at the beginning, we can uniquely extend this linear isometry on $\mathcal{E}_T$ to a linear isometry I_T on $\mathcal{L}_T^2$.
- $I_T(f)$, thus defined, is called the Ito integral of $f \in \mathcal{L}_T^2$ and is written as $\int_0^T f_s dB_s$.
- From what we have done so far, we could actually take any $f \in \mathcal{L}_\infty^2$ and define $I_t(f) = \int_0^t f_s dB_s$, for each $t \geq 0$.
- Similar to the case of Wiener Integrals, our main concern here also is the continuity of the integral.
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Ito Integral : Indefinite Integral

- Here is an important result.

Theorem

For every $f \in \mathcal{L}^2_\infty$, $\exists$ a process $\{Y_t, t \in [0,\infty)\}$, such that,

(a) $Y_t = \int_0^t f_s dB_s = I_t(f) \;\; \forall\, t \in [0,\infty)$

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(c) $\exists$ a $\mathbb{P}$-null set N, such that , $\forall\, \omega \notin N, t \to Y_t(\omega)$ is continuous on [0,∞) $\forall\, t \in [0,\infty)$.

(d) $\{Y_t, t \in [0,\infty)\}$ is a square integrable martingale w.r.t. $(\mathcal{A}_t, t \in [0,\infty))$

*Further, such a process is **unique** upto a $\mathbb{P}$-null set , i.e., if $\{Z_t, t \in [0,\infty)\}$ is any process satisfying (a), (b) and (c), then $\exists$ a $\mathbb{P}$-null set set N such that , for $\omega \notin N$, $Z_t(\omega) = Y_t(\omega)$ for all t.*

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- **Conclusion:**This process $\{Y_t = I_t(f), t \in [0, \infty)\}$ is formally known as the Ito Integral of $f \in \mathcal{L}^2_\infty$.

Definition

*(**Ito Integral**)Let $\{B_t, t \geq 0\}$ be a standard brownian motion (SBM) on a probability space $(\Omega, \mathcal{A}, \mathbf{P})$. For any $f \in \mathcal{L}^2_\infty$, the linear isometry $I_t : \mathcal{L}^2_\infty \to \mathcal{L}^2(\Omega, \mathcal{A}, \mathbf{P})$, defined as,*

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such that,the process $\{I_t(f), t \in [0, \infty)\}$i is $\mathcal{A}_t$-measurable , $t \to I_t(f)(\omega)$ is continuous in $[0, \infty)$ for almost every $\omega \in \Omega$ and it is unique upto a $\mathbf{P}$-null set, is known as the Ito Integral of the function $f \in \mathcal{L}^2_\infty$ with respect to the SBM for any $t \in [0, \infty)$.

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- Given a continuous square integrable martingale Y_t, by Doob-Meyer decomposition,$\exists$ a continuous process A_t with non-decreasing paths, such that $\{Y_t^2 - A_t, t \in [0, \infty)\}$ is a martingale.
- This A_t is the Quadratic variation of Y_t and is written as $< Y >_t$.
- The next theorem identifies what is $< Y >_t$.

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Let $Y_t = I_t(f) = \int_0^t f(s)dB_s, t \in [0, \infty)\ \forall\ f \in \mathcal{L}^2_\infty$,that is, $\{Y_t, t \in [0, \infty)\}$ is a continuous square integrable martingale w.r.t. $(\mathcal{A}_t)$. . Then, $\{Y_t^2 - \int_0^t f_s^2 ds, t \in [0, \infty)\}$ is a martingale w.r.t. $(\mathcal{A}_t)$.

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Let $Y_t = I_t(f) = \int_0^t f(s)dB_s, t \in [0,\infty)\ \forall\ f \in \mathcal{L}^2_\infty$,that is, $\{Y_t, t \in [0,\infty)\}$ is a continuous square integrable martingale w.r.t. $(\mathcal{A}_t)$. . Then, $\{Y_t^2 - \int_0^t f_s^2 ds, t \in [0,\infty)\}$ is a martingale w.r.t. $(\mathcal{A}_t)$.

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- Given a continuous square integrable martingale Y_t, by Doob-Meyer decomposition,$\exists$ a continuous process A_t with non-decreasing paths, such that $\{Y_t^2 - A_t, t \in [0, \infty)\}$ is a martingale.
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- In this section , we state the analogues of the fundamental theorem of of calculus in case of stochastic calculus, called the Ito's formula.
- It has various versions.
- We give the statements of some well known versions of Ito's formula.

Theorem

*(**Ito's formula** I) Let $\{B_t : t \in [0, \infty)$ be a SBM and $\phi \in C_2(\mathbb{R})$. Then, $\forall\, t \geq 0$*

$$\phi(B_t) - \phi(0) = \int_0^t \phi'(B_s)\, dB_s + \frac{1}{2}\int_0^t \phi''(B_s) ds,$$

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Application : Dirichlet Problem

- Now, in the last section, we see the probabilistic methods of handling some problems in PDE theory .
- The first problem we look at is the Dirichlet Problem.
- Let D be a bounded domain in $\mathbb{R}^2$ and $f : \overline{D} \to \mathbb{R}$ a continuous function.
- Then the **Dirichlet Problem** is to find a continuous function $u : \overline{D} \to \mathbb{R}$
- Such that u is harmonic in D, that is , $u \in C_2(D)$ and ,

  $$\triangle u = 0 \text{ in D and } u = f \text{ on } \partial D$$
- Such a function , when it exists , will be called a *solution to the Dirichlet problem (D,f).*

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Proposition

Let $\{B_t, t \geq 0\}$ be a 2-dimensional SBM. For $x \in \overline{D}$, let $\{B_t^x, t \geq 0\}$ be the process $B_t^x = x + B_t, t \geq 0$, starting at x. Let $\tau_D^x = inf\{t > 0 : x + B_t^x \in D^c\}$. Then $B^x_{\tau_D^x} \in D^c$. And,

$$u(x) = E(f(B^x_{\tau_D^x})), x \in \overline{D}$$

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Application : Heat Equation

- Let $f : \mathbb{R} \to \mathbb{R}$ borel measurable.
- Let $u : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ in $C_2(\mathbb{R})$.
- Then, the heat equation is given by,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \;,\; u(0,x) = f(x)$$

- The physical interpretation here, is to imagine the real line as an insulated rod.
- f(x) represents the temperature at position x at time 0.
- And u(t,x) represents temperature at x time t.
- Then from the usual theory of physics, it can be seen that, u must satisfy the above parabolic PDE.
- Now, for $x \in \mathbb{R}$, the density of $x + B_t$, is given by,

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- f(x) represents the temperature at position x at time 0.
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- Then from the usual theory of physics, it can be seen that, u must satisfy the above parabolic PDE.
- Now, for $x \in \mathbb{R}$, the density of $x + B_t$, is given by,

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{(y-x)^2}{t}}, \quad y \in \mathbb{R}$$

Application : Heat Equation

- Let $f : \mathbb{R} \to \mathbb{R}$ borel measurable.
- Let $u : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ in $C_2(\mathbb{R})$.
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Application : Other parabolic PDEs

1. For $g:[0,\infty)\times\mathbb{R}\to\mathbb{R}, f:\mathbb{R}\to\mathbb{R}$,the PDE
$$\frac{\partial u}{\partial t}=\frac{1}{2}\triangle u+g\ ,\ u(0,x)=f(x)$$
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$u(t,x)=E(f(x+B_t))+E(\int_0^t g(t-s,x+B_s)\,ds)$
2. (Feynman-Kac) For $g:[0,\infty)\times\mathbb{R}\to\mathbb{R}, f:\mathbb{R}\to\mathbb{R}$, $c:\mathbb{R}\to\mathbb{R}$, the PDE
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3. For $g:[0,\infty)\times\mathbb{R}\to\mathbb{R}, f:\mathbb{R}\to\mathbb{R}, b:\mathbb{R}\to\mathbb{R}$, the PDE
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