Truncation Errors and the Taylor Series

Chapter 4

How does a CPU compute the following functions for a specific x value?

cos(x) ***sin(x)*** e^x ***log(x)*** etc.

- Non-elementary functions such as, ***trigonometric, exponential*** and others are expressed in an approximate fashion using **Taylor series** when their values, derivatives, and integrals are computed.

- **Taylor series** provides a means to predict the value of a function at one point in terms of the function value and its derivatives at another point.

Taylor Series (n^{th} *order* approximation):

$$f(x_{i+1}) \cong f(x_i) + \frac{f^{'}(x_i)}{1!}(x_{i+1} - x_i) + \frac{f^{''}(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

The Reminder term, R_n, accounts for all terms from (n+1) to infinity.

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

Define the *step size* as ***h=(x$_{i+1}$- x$_i$),*** the ***series*** becomes:

$$f(x_{i+1}) \cong f(x_i) + \frac{f^{'}(x_i)}{1!} h + \frac{f^{''}(x_i)}{2!} h^2 + \ldots + \frac{f^{(n)}(x_i)}{n!} h^n + R_n$$

Any smooth function can be approximated as a polynomial.

Take $\boldsymbol{x = x_{i+1}}$ Then $\boldsymbol{f(x) \approx f(x_i)}$ ***zero order*** approximation

$$f(x) \cong f(x_i) + f'(x_i)(x - x_i)$$

first order approximation

Second order approximation:

$$f(x) \cong f(x_i) + \frac{f'(x_i)}{1!}(x - x_i) + \frac{f''(x_i)}{2!}(x - x_i)^2$$

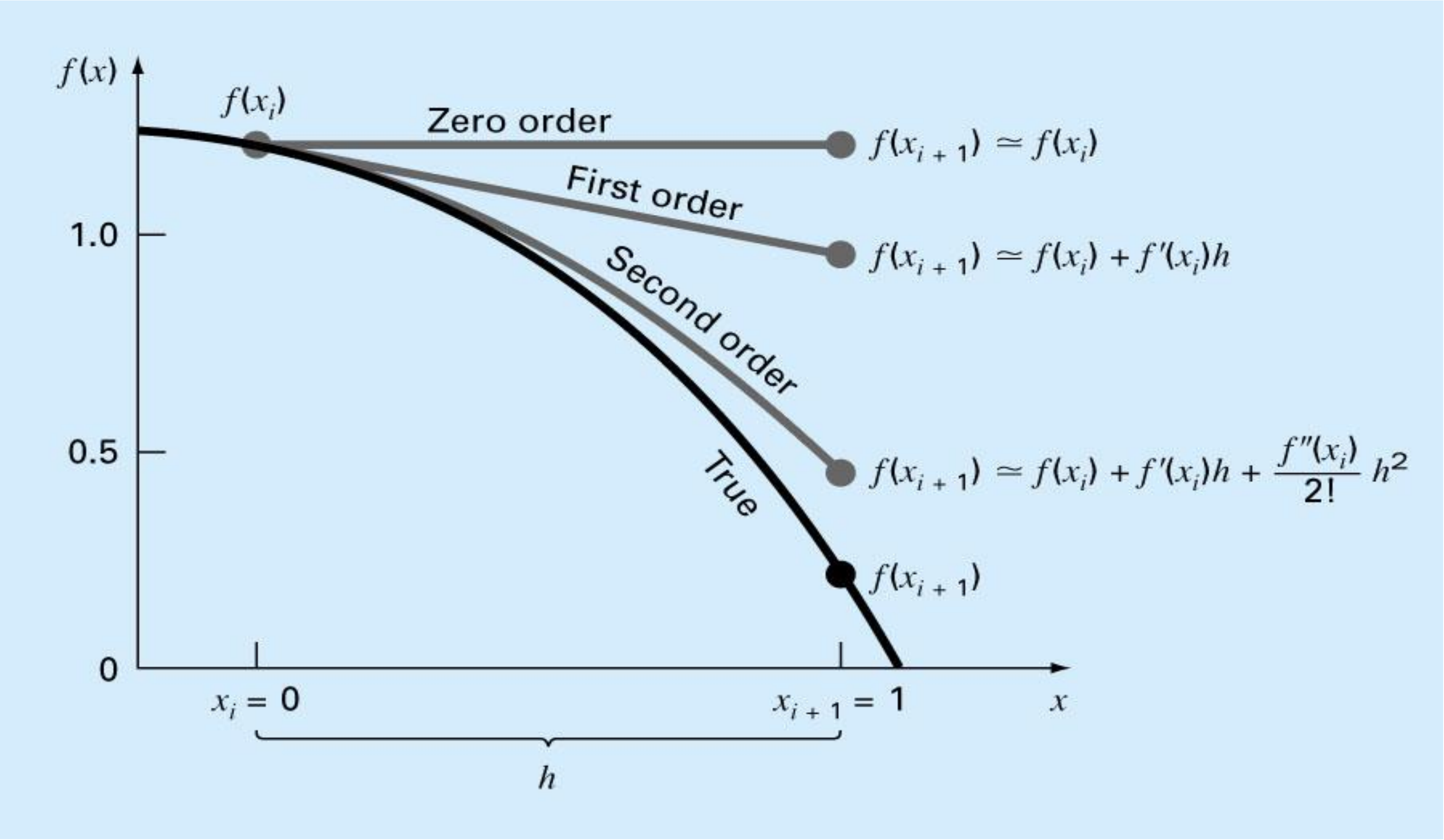
nth order approximation:

$$f(x) \cong f(x_i) + \frac{f'(x_i)}{1!}(x - x_i) + \frac{f''(x_i)}{2!}(x - x_i)^2 + \ldots + \frac{f^{(n)}(x_i)}{n!}(x - x_i)^n + R_n$$

- Each additional term will contribute some improvement to the approximation. Only if an infinite number of terms are added will the series yield an exact result.
- In most cases, only a few terms will result in an approximation that is close enough to the true value for practical purposes

Example

Approximate the function $f(x) = 1.2 - 0.25x - 0.5x^2 - 0.15x^3 - 0.1x^4$ from $x_i = 0$ with $h = 1$ and **predict** $f(x)$ at $x_{i+1} = 1$.



Example:
computing *f(x)* = e^x using Taylor Series expansion

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!}(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \ldots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

Choose $x = x_{i+1}$ and $x_i = 0$ Then $f(x_{i+1}) = f(x)$ and $(x_{i+1} - x_i) = x$

Since First Derivative of e^x is also e^x :

(2.) $(e^x)'' = e^x$ (3.) $(e^x)''' = e^x$, ... (n^{th}.) $(e^x)^{(n)} = e^x$

As a result we get:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Looks familiar?

***Maclaurin series for** e^x*

Yet another example:

computing $f(x)$ = cos(x) using Taylor Series expansion

Choose $x=x_{i+1}$ and $x_i=0$ Then $f(x_{i+1}) = f(x)$ and $(x_{i+1} - x_i) = x$

Derivatives of cos(x):

(1.) $(\cos(x))' = -\sin(x)$ (2.) $(\cos(x))'' = -\cos(x)$,

(3.) $(\cos(x))''' = \sin(x)$ (4.) $(\cos(x))'''' = \cos(x)$,

……

As a result we get:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Notes on Taylor expansion:

- Each additional term will contribute some improvement to the approximation. Only if an infinite number of terms are added will the series yield an exact result.

- In most cases, several terms will result in an approximation that is close enough to the true value for practical purposes

- Reminder value **R** represents the ***truncation error***
- The order of truncation error is h^{n+1} ➔ $R=O(h^{n+1})$,

  If $R=O(h)$, halving the step size will halve the error.

  If $R=O(h^2)$, halving the step size will quarter the error.