Modeling the Old Man River Problem

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Introduction

In this project, we analyze and model the path of a swimmer attempting to cross a river with a constant current velocity. The swimmer aims to reach the west shore of a one-mile-wide river, which flows northward with a constant velocity. Our goal is to derive differential equations describing the swimmer's path, solve these equations, and explore various cases based on the swimmer's speed relative to the river's speed.

We divide the problem into two main parts: the swimmer aiming at a specific point on the west shore, and the swimmer swimming directly west. Python code is used to illustrate these paths and generate plots of the solutions.

Problem 1: Swimmer Aiming Toward a Specific Point

The swimmer enters the river at point $(1, 0)$ on the east shore and attempts to swim to point $(0, 0)$ on the west shore, adjusting his swimming direction as he goes to keep moving toward $(0, 0)$.

Modeling the Path of the Swimmer

Given the river velocity $\mathbf{v_r}$ directed northward and the swimmer's velocity $\mathbf{v_s}$ directed toward $(0, 0)$, the resulting differential equation for the swimmer's path is:

$$\frac{dy}{dx} = \frac{v_s y - v_r \sqrt{x^2 + y^2}}{v_s x}$$

Using $k = \frac{v_r}{v_s}$, we simplify the equation to:

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$

Solution of the Differential Equation

To solve this equation, we apply the substitution $y = x\mu$, leading to:

$$\mu + x\frac{d\mu}{dx} = \mu - k\sqrt{1 + \mu^2}$$

Separating variables and integrating, we find:

$$\int \frac{1}{\sqrt{1+\mu^2}} d\mu = -k \int \frac{1}{x} dx$$

which yields:

$$\mu = \sinh(-k \ln|x| + C)$$

Converting back, we obtain:

$$y = x \sinh(-k \ln|x| + C)$$

Given $y(1) = 0$, we conclude $C = 0$, resulting in:

$$y = x \sinh(-k \ln|x|).$$

Case Analysis

We analyze cases based on k:

Case $k = 1$:

$$y = x \sinh(-\ln|x|) = x \left(\frac{e^{-\ln|x|} - e^{\ln|x|}}{2} \right)$$

$$y = x \left(\frac{1/x - x}{2} \right) = x \left(\frac{1 - x^2}{2x} \right) = \frac{1}{2} - \frac{x^2}{2}.$$

$$\lim_{x \to 0} \left(\frac{1}{2} - \frac{x^2}{2} \right) = \frac{1}{2}.$$

Conclusion: When $k = 1$, the swimmer approaches the point $(0, \frac{1}{2})$ and not the origin $(0, 0)$.

Case $k > 1$:

$$y = x \sinh(-k \ln|x|) = x \left(\frac{e^{-k \ln|x|} - e^{k \ln|x|}}{2} \right)$$

$$y = x \left(\frac{x^{-k} - x^k}{2} \right) = \frac{x^{1-k} - x^{1+k}}{2}.$$

$$\lim_{x \to 0} \left[\frac{1}{2}(x^{1-k} - x^{1+k}) \right] = \lim_{x \to 0} \left[\frac{1}{2} \left(\frac{1}{x^k} - x^{1+k} \right) \right] = \infty.$$

Conclusion: The swimmer does not reach the origin point $(0, 0)$.

Case $0 < k < 1$:

$$y = x \sinh(-k \ln|x|) = \frac{1}{2}(x^{1-k} - x^{1+k}).$$

$$\lim_{x \to 0} \left[\frac{1}{2}(x^{1-k} + x^{1+k}) \right] = 0.$$

Conclusion: The swimmer reaches the origin point $(0, 0)$.

Since $k = \frac{v_r}{v_s}$ and $0 < k < 1$ implies $v_r < v_s$, results that the swimmer reaches the origin point for every $v_s > v_r$.

Python Code and Plot

To visualize the solution, we provide Python code (Figure 1) and a plot (Figure 2) representing the swimmer's path.

```python
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt


def swimmer_ode(x, y, vs, vr):
    return (vs * y - vr * np.sqrt(x**2 + y**2))/(vs * x)


vs = 1
vr = 2


y0 = [0]


solution = solve_ivp(swimmer_ode, [1, 0], y0, args=(vs, vr), dense_output=True)


x_vals = np.linspace(1, 0, 1000)
y_vals = solution.sol(x_vals)

# Plot the solution
plt.figure(figsize=(10, 8))
plt.plot(x_vals, y_vals[0], 'b-', label=f'vs={vs}, vr={vr}')
plt.grid(True)
plt.xlabel('Distance from West Shore (miles)')
plt.ylabel('Distance Northward (miles)')
plt.title("Swimmer's Path Across River with Initial Condition y(1)=0")
plt.legend()
plt.axis('equal')
plt.show()

# Print final position
print("Final position (x,y) = ("+str(x_vals[-1])+", "+str(y_vals[0][-1])+")")
```

Figure 1: Python code for solving Problem 1.

Problem 2: Swimmer Swimming West

This time, the swimmer maintains a constant westward direction. Given the river's northward current, the differential equation for the swimmer's path is:

$$\frac{dy}{dx} = -\frac{v_r}{v_s}.$$

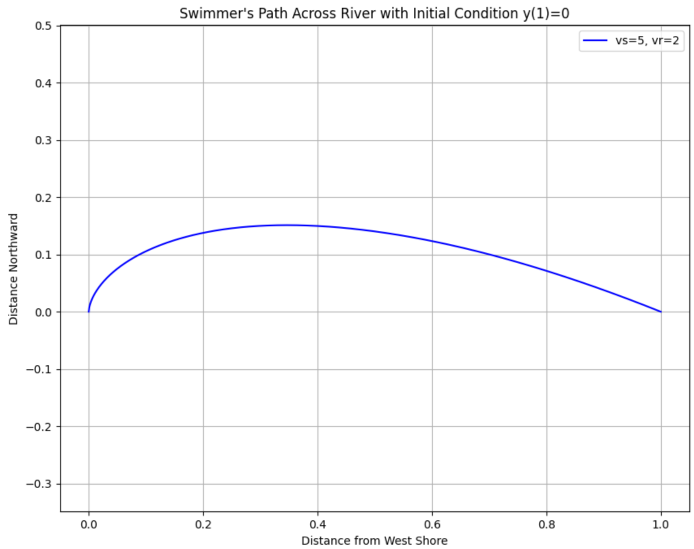

Figure 2: Plot of the swimmer's path.

Solution of the Differential Equation

Integrating gives:

$$y = -\frac{v_r}{v_s}x + C.$$

Using $y(1) = 0$, we find $C = \frac{v_r}{v_s}$, leading to:

$$y = -\frac{v_r}{v_s}x + \frac{v_r}{v_s}.$$

Variable Current Analysis

With a variable current $v_r(x) = 30x(1-x)$, the differential equation becomes:

$$\frac{dy}{dx} = -\frac{30x(1-x)}{2} = -15x(1-x).$$

Integrating this yields:

$$y = 5x^3 - \frac{15x^2}{2} + \frac{5}{2}.$$

Thus, upon reaching the west shore at $x = 0$, $y(0) = \frac{5}{2}$ miles north of $(0,0)$, requiring the swimmer to walk $\frac{5}{2}$ miles south.

Python Code

Below is the Python code used to solve Problem 2 (Figure 3).

```python
from sympy import symbols, integrate, solve, Eq

x = symbols('x')

v_s_value = 2
v_r_function = 30 * x * (1 - x)

integrated_y = integrate(-v_r_function / v_s_value, x)

C = symbols('C')
y_solution = integrated_y + C
C_value = solve(Eq(y_solution.subs(x, 1), 0), C)[0]

final_y_solution = y_solution.subs(C, C_value)
print(final_y_solution)
```

Figure 3: Python code for solving Problem 2.

Conclusion

In this project, we examined how a swimmer's path is influenced by river currents. Our analysis highlights the importance of the relationship between swimmer and river velocities. We also saw how changes in current affect the path and demonstrated how differential equations can model complex real-world scenarios.