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Algebraic Topology
Proofs about Topological Properties

Proof of Fundamental Definition of Open Set in Metric Space

Proposition

A set D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is open if and only if every point on D is an interior point.

Definitions

Let D be a set in $\mathbb{R}^n$ where $n = 2$ or 3.

Definition 1.1: Consider an open ball centered at point P with radius $r < 0$. $B_r(P)$ denotes the set of all points whose distance to P is less than r.

$$B_r(x_0, y_0) = \{(x, y) \mid \text{distance from } (x, y) \text{ to } (x_0, y_0) \text{ is less than r}\} \tag{1}$$

Definition 1.2: A point $P \in D$ is an interior point of D if $\exists r > 0$ such that $B_r(P) \subseteq D$.

Definition 1.3: A point $P \in D$ is a boundary point of D if $\forall r > 0$ $B_r(P)$ contains points in D and

Definition 1.4: A set D is open if it contains none of its boundary points.

Proof

In order to prove the proposition with the phrase "if and only if", we need to prove the following statements:

1. Given an open set D, then every point on D is an interior point.
2. Given a set D with all the points in D being interior points, then D is an open set.

Statement 1

Corollary 1.1 For any set D, every point p in the space ($\mathbb{R}^2$ or $\mathbb{R}^3$) must be exactly one of the three categories: Interior point, Exterior point, or Boundary point. The three categories are mutually exclusive and exhaustive.

Proof:

1. Consider an arbitary point $p \in D$.
2. From Corollary 1.1, p must be an interior, exterior or boundary point. Since $p \in D$, p cannot be an exterior point.
3. Therefore p is either an interior or boundary point.
4. From Definition 1.4, any point in D must not be a boundary point.
5. p must be an interior point.
6. We have proven that any arbitary point p in an open set D must be an interior point.

Statement 2

1. To prove that D is an open set, we can show that D and the set of the boundary of D have no points in common. That is

$$D \cap \text{Boundary}(D) = \emptyset \tag{2}$$

2. Consider an arbitary point $p \in D$.
3. According to the assumption, p is an interior point of D.
4. From Corollary 1.1, p cannot be a boundary point.
5. Therefore, any arbitary point in D cannot be a boundary point, which is the definition of open set.