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The Basel Problem

Proof of the Basel Problem by Jesus Retamozo

1 Main Section

For the following proof, Retamozo constructed the integral

$$I = \int_0^\infty \int_0^\infty \frac{xdydx}{(1+x^2)(y^2+x^2)} \tag{1}$$

Integrating with respect to y followed by inegrating with respect to x.

$$I = \int_0^\infty \frac{x}{(1+x^2)} \left[\int_0^\infty \frac{dy}{(y^2+x^2)} \right] dx \tag{2}$$

Notice that the intergral $\int_0^\infty \frac{1}{(y^2+x^2)}dy$ is a standard arctan integral. We know

$$\int \frac{1}{(y^2+a^2)}dy = \frac{1}{a}\arctan(\frac{y}{a}) \tag{3}$$

In this case specifically, $x = a$.

$$\begin{gathered} \int_0^\infty \frac{1}{(y^2+x^2)}dy = \left[\frac{1}{x}\arctan(\frac{y}{x}) \right]_{y=0}^{y=\infty} \\ = \frac{1}{x}(\arctan(\infty) - \arctan(0)) = \frac{1}{x}(\frac{\pi}{2} - 0) = \frac{\pi}{2x} \end{gathered} \tag{4}$$

Substituting back to (2), we get

$$\begin{gathered} I = \int_0^\infty \frac{xdx}{(1+x^2)}(\frac{\pi}{2x}) \\ I = \frac{\pi}{2}\int_0^\infty \frac{dx}{1+x^2} \end{gathered} \tag{5}$$

Notice that the integral $\int \frac{dx}{1+x^2}$ is also a standard arctan integral. There is

$$I = \frac{\pi}{2}\left[\arctan(x)\right]_{x=0}^{x=\infty} = \frac{\pi}{2}(\arctan(\infty) - \arctan(0)) = \frac{\pi}{2}(\frac{\pi}{2} - 0) = \frac{\pi^2}{4} \tag{6}$$

Retamozo then proposed another way to unpack the integral. Let

$$I = \int_0^\infty \left[\int_0^\infty \frac{xdx}{(1+x^2)(y^2+x^2)} \right] dy \tag{7}$$

We need to use partial fractions here. Take x as the variable, and therefore y becomes a constant. Let

$$\begin{aligned} u &= x^2 \\ du &= 2xdx \\ xdx &= \frac{1}{2}du \end{aligned} \tag{8}$$

The integral in (7) within the brackets then becomes

$$\int_{x=0}^{x=\infty} \frac{\frac{1}{2}du}{(1+u)(y^2+u)} \tag{9}$$

Decompose the fraction

$$\begin{aligned}\frac{1}{(1+u)(y^2+u)} &= \frac{A}{1+u} + \frac{B}{y^2+u} \\ 1 &= A(y^2+u) + B(1+u)\end{aligned} \tag{10}$$

$$\text{Let } u = -1 \tag{11a}$$
$$1 = A(y^2+1) \tag{11b}$$
$$A = \frac{1}{y^2-1} \tag{11c}$$
$$\text{Let } u = -y^2 \tag{11d}$$
$$1 = B(1-y^2) \tag{11e}$$
$$B = \frac{1}{1-y^2} \tag{11f}$$

Therefore (9) becomes

$$\begin{aligned}&\frac{1}{2}\int_0^\infty \left(\frac{\frac{1}{y^2-1}}{1+u} + \frac{\frac{1}{1-y^2}}{y^2+u} \right) du \\ &= \frac{1}{2(1-y^2)} \int_0^\infty \left(\frac{1}{y^2+u} - \frac{1}{1+u} \right) du \\ &= \frac{1}{2(1-y^2)} \left[\ln(y^2+u) - \ln(1+u) \right]_{u=0}^{u=\infty} \\ &= \frac{1}{2(1-y^2)} \left[\ln \frac{(y^2+u)}{(1+u)} \right]_{u=0}^{u=\infty}\end{aligned} \tag{12}$$

Consider the upper and lower boundaries.

$$\begin{aligned}&\text{As } u \to \infty,\ \left(\ln \frac{(y^2+u)}{(1+u)} \right) \to \ln(1) = 0 \\ &\text{As } u \to 0,\ \ln \frac{(y^2+u)}{(1+u)} = \ln(y^2) = 2\ln(y)\end{aligned} \tag{13}$$

Therefore (12) becomes

$$\begin{aligned}&\frac{1}{2(1-y^2)}(-2\ln(y)) \\ &= -\frac{\ln(y)}{1-y^2}\end{aligned} \tag{14}$$

To reduce any confusion due to loads of calculation, let's go back to (7). We wish to unpack the integral

$$I = \int_0^\infty \left[\int_0^\infty \frac{x\,dx}{(1+x^2)(y^2+x^2)} \right] dy$$

We have now come to a point where the integral within the brackets is proven to be $\frac{-\ln(y)}{1-y^2}$.
Unpack the second integral.

$$\begin{aligned}I &= -\int_0^\infty \frac{\ln(y)}{1-y^2} dy \\ &= -\int_0^1 \frac{\ln(y)}{1-y^2} dy - \int_1^\infty \frac{\ln(y)}{1-y^2} dy\end{aligned} \tag{15}$$

For the second integral, let

$$
\begin{aligned}
y &= \frac{1}{u} \\
dy &= -\frac{du}{u^2} \\
\int_{y=1}^{\infty} \frac{\ln(y)}{1-y^2} dy &= \int_{u=1}^{0} \frac{\ln(\frac{1}{u})}{1-(\frac{1}{u})^2} (-\frac{1}{u^2}) du \\
&= \int_{0}^{1} \frac{-\ln(u)}{1-\frac{1}{u^2}} (\frac{1}{u^2}) du \\
&= \int_{0}^{1} \frac{-\ln(u)}{\frac{(u^2-1)}{u^2}} (\frac{1}{u^2}) du \\
&= \int_{0}^{1} \frac{\ln(u)}{1-u^2} du
\end{aligned} \tag{16}
$$

Amazing!

$$
\int_{0}^{1} \frac{\ln(y)}{1-y^2} dy = \int_{1}^{\infty} \frac{\ln(y)}{1-y^2} dy \tag{17}
$$

Therefore, from (15):

$$
I = -2 \int_{0}^{1} \frac{\ln(y)}{1-y^2} dy \tag{18}
$$

A geometric series is equivilant to $\frac{1}{1-y^2}$ for $y \in (0,1)$.
We know that, for any $|r| < 1$:

$$
(1-r)(1+r+r^2+r^3+...) = 1
$$

Therefore

$$
\frac{1}{1-r} = 1+r+r^2+r^3+...
$$

Replacing r with y^2, there is

$$
\frac{1}{1-y^2} = 1+y^2+y^4+y^6+... = \sum_{k=0}^{\infty} (y^2)^k = \sum_{k=0}^{\infty} y^{2k} \tag{19}
$$

Using (18), there is

$$
\begin{aligned}
I &= -2 \int_{0}^{1} \ln(y) \sum_{k=0}^{\infty} y^{2k} dy \\
I &= -2 \sum_{k=0}^{\infty} \int_{0}^{1} y^{2k} \ln(y) dy
\end{aligned} \tag{20}
$$

Using integration by parts,

$$
\text{Let } u = \ln(y) \tag{21a}
$$

$$
du = \frac{1}{y} dy \tag{21b}
$$

$$
\text{Let } dv = y^{2k} \tag{21c}
$$

$$
v = \frac{y^{2k+1}}{2k+1} \tag{21d}
$$

$$
\int_{0}^{1} y^{2k} \ln(y) dy = \left[\ln(y) \frac{y^{2k+1}}{2k+1} \right]_{y=0}^{y=1} - \int_{0}^{1} \frac{y^{2k+1}}{2k+1} \frac{1}{y} dy \tag{22}
$$

From L'Hopital, $[...]_0^1$ is 0 when $y = 1$, approaches 0 when $y \to 0$.

$$\begin{aligned}
\int_0^1 y^{2k}\ln(y)dy &= 0 - \int_0^1 \frac{y^{2k+1}}{2k+1}\frac{1}{y}dy \\
&= -\frac{1}{2k+1}\left[\frac{y^{2k+1}}{2k+1}\right]_0^1 = -\frac{1}{2k+1}\left(\frac{1}{2k+1} - 0\right) = -\frac{1}{(2k+1)^2} \\
I &= -2\sum_{k=0}^{\infty}\left[-\frac{1}{(2k+1)^2}\right] \\
I &= 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}
\end{aligned} \tag{23}$$

Let's take a step back to the first way of unpacking the integral. From (6), $I = \frac{\pi^2}{4}$. Therefore,

$$\begin{aligned}
\frac{\pi^2}{4} &= 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} \\
\frac{\pi^2}{8} &= \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}
\end{aligned} \tag{24}$$

Let

$$\begin{aligned}
S &= \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ... \\
S &= \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) + \left(\frac{1}{2^2} + \frac{1}{4^2} + ...\right) \\
\left(\frac{1}{2^2} + \frac{1}{4^2} + ...\right) &= \frac{1}{(2\cdot 1)^2} + \frac{1}{(2\cdot 2)^2} + \frac{1}{(2\cdot 3)^2} + ... = \frac{1}{(4\cdot 1^2} + \frac{1}{(4\cdot 2^2} + \frac{1}{(4\cdot 3^2} \\
&= \frac{1}{4}\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...\right) = \frac{S}{4} \\
S &= \frac{\pi^2}{8} + \frac{S}{4} \\
\frac{3}{4}S &= \frac{\pi^2}{8} \\
S &= \frac{\pi^2}{6}
\end{aligned} \tag{25}$$

QED!

For anyone reaching this point, congratulations! This pure calculus proof is indeed confusing and complicated, especially when it comes to unpacking I in a different manner. It takes not only perseverence but also passion for mathematics to finish reading the proof, not even mention deducing it from the start. It is encouraged, however, to review the proof once again by readers themselves, just to remove any minor confusion. But well done! You have conquered a complex proof!