Jimmy Zhan
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The Basel Problem

Proof of the Basel Problem by Fourier Transform

1 Fourier Transform

We first consider a single-variable function f which is continuous in $[0,1]$ and $f(0) = f(1)$. The criteria for point-wise convergence of a periodic function f are as follows:

If f satisfies a Holder condition, then its Fourier series converges uniformly.

If f is of bounded variation, then its Fourier series converges everywhere. If f is additionally continuous, the convergence is uniform.

If f is continuous and its Fourier coefficients are absolutely summable, then the Fourier series converges uniformly.

2 Unpacking the integral

Let f be $f(x) = x(1-x)$. It satisfies all requirements mentioned above.

For a function $f(x)$ on an interval $[0, L]$, the Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right) \tag{1}$$

Since the period $L = 1$, we can find out each of the terms:

Constant term is $\frac{a_0}{2}$ where

$$\begin{aligned} a_0 &= \frac{2}{L}\int_0^L f(x)dx \\ a_0 &= 2\int_0^1 (x - x^2)dx \\ a_0 &= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3} \\ \frac{a_0}{2} &= \frac{1}{6} \end{aligned} \tag{2}$$

Cosine terms

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2\pi n x}{L}\right)dx \\ a_n &= 2\int_0^1 (x - x^2)\cos(2\pi n x)dx \end{aligned} \tag{3}$$

Apply integration by parts:

$$\text{Let } u = x - x^2 \tag{4a}$$

$$du = (1-2x)dx \tag{4b}$$

$$\text{Let } dv = \cos(2\pi n x)dx \tag{4c}$$

$$v = \frac{\sin(2\pi n x)}{2\pi n} \tag{4d}$$

$$a_n = 2\left(\left[(x - x^2)\frac{\sin(2\pi n x)}{2\pi n}\right]_0^1 - \int_0^1 \frac{\sin(2\pi n x)}{2\pi n}(1-2x)dx\right) \tag{5}$$

The first term is zero at both $x = 1$ and $x = 0$, therefore we obtain

$$\begin{aligned} a_n &= -2\int_0^1 \frac{\sin(2\pi nx)}{2\pi n}(1-2x)dx \\ a_n &= -\frac{1}{\pi n}\int_0^1 \sin(2\pi nx)(1-2x)dx \end{aligned} \tag{6}$$

Use integration by parts again:

$$\text{Let } u = 1 - 2x \tag{7a}$$

$$du = -2dx \tag{7b}$$

$$\text{Let } dv = \sin(2\pi nx)dx \tag{7c}$$

$$v = -\frac{\cos(2\pi nx)}{2\pi n} \tag{7d}$$

$$a_n = \frac{1}{\pi n}\left(\left[-(1-2x)\frac{\cos(2\pi nx)}{2\pi n}\right]_0^1 - \int_0^1 -\frac{\cos(2\pi nx)}{2\pi n}(-2)dx\right) \tag{8}$$

Evaluate the first part:
At $x = 1$:

$$(1-2)\left(-\frac{\cos(2\pi n)}{2\pi n}\right) = (-1)(-\frac{1}{2\pi n}) = \frac{1}{2\pi n} \tag{9}$$

At $x = 0$:

$$(1-0)\left(-\frac{\cos(0)}{2\pi n}\right) = (1)(-\frac{1}{2\pi n}) = -\frac{1}{2\pi n} \tag{10}$$

Therefore

$$\left[-(1-2x)\frac{\cos(2\pi nx)}{2\pi n}\right]_0^1 = \frac{1}{2\pi n} + \frac{1}{2\pi n} = \frac{1}{\pi n} \tag{11}$$

Look at the remaining integral:

$$\begin{aligned} -\int_0^1 -\frac{\cos(2\pi nx)}{2\pi n}(-2)dx &= -\int_0^1 \frac{2\cos(2\pi nx)}{2\pi n}dx = -\frac{1}{\pi n}\int_0^1 \cos(2\pi nx)dx \\ &= -\frac{1}{\pi n}\left[\frac{\sin(2\pi nx)}{2\pi n}\right]_0^1 \end{aligned} \tag{12}$$

This is zero at both $x = 1$ and $x = 0$. Putting it all back together we obtain:

$$a_n = -\frac{1}{\pi n}\left(\left[\frac{1}{\pi n}\right] - 0\right) = -\frac{1}{\pi^2 n^2} \tag{13}$$

Sine terms

This is special as $f(x)$ is stmmetric around $x = \frac{1}{2}$. This causes all the b_n terms to be zero. Let's prove it with integration.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2\pi nx}{L}\right)dx = 2\int_0^1 (x-x^2)\sin(2\pi nx)dx \tag{14}$$

Using integration by parts (similar with before) we obtain

$$b_n = 2\left(\left[(x-x^2)\left(-\frac{\cos(2\pi nx)}{2\pi n}\right)\right]_0^1 - \int_0^1 \left(-\frac{\cos(2\pi nv)}{2\pi n}\right)(1-2x)dx\right) \tag{15}$$

Similarly, the first term is zero at both $x = 1$ and $x = 0$.

$$b_n = 2\int_0^1 \frac{\cos(2\pi nx)}{2\pi n}(1-2x)dx = \frac{1}{\pi n}\int_0^1 \cos(2\pi nx)(1-2x)dx \tag{16}$$

Integrate by parts again:

$$b_n = \frac{1}{\pi n}\left(\left[(1-2x)\frac{\sin(2\pi nx)}{2\pi n}\right]_0^1 - \int_0^1 \frac{\sin(2\pi nx)}{2\pi n}(-2)dx\right) \tag{17}$$

The first part is zero at both $x = 1$ and $x = 0$. Evaluate the second integral gives:

$$\begin{aligned}\int_0^1 \frac{\sin(2\pi nx)}{2\pi n}dx &= \frac{1}{\pi n}\int_0^1 \sin(2\pi nx)dx = \frac{1}{\pi n}\left[-\frac{\cos(2\pi nx)}{2\pi n}\right]_0^1 \\ &= -\frac{1}{2\pi^2 n^2}(\cos(2\pi n) - \cos(0)) = 0\end{aligned} \tag{18}$$

Therefore $b_n = 0$ for all n.

3 Assembly

To sum up, we have:

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{6} \\ a_n &= -\frac{1}{\pi^2 n^2} \\ b_n &= 0\end{aligned} \tag{19}$$

From (1):

$$x(1-x) = \frac{1}{6} - \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{\pi^2 n^2} \tag{20}$$

Substituing $x = 0$ gives:

$$\begin{aligned}\frac{1}{6} - \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} &= 0 \\ \frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{6} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}\end{aligned} \tag{21}$$

as required.