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Multivariable Differential Calculus

The Nature of Dot Product and its Uniqueness

Part 1 The geometric and algebraic interpretation of dot product

Algebra

Algebraically, the dot product is defined as

$$u \cdot v = \sum_i u_i v_i$$

From this perspective, the dot product measures **how much the two vectors vary in direction**.
If u_i and v_i point in the same direction (whether positive or negative), then $u_i v_i > 0$.
If they point in opposite directions, $u_i v_i < 0$.
And the dot product sums up all these directional agreements across dimensions.

Geometry

Geometrically, the dot product is defined as

$$u \cdot v = \|u\| \|v\| \cos\theta$$

where θ is the angle between u and v.
In this manner, the dot product expresses **how the magnitudes of two vectors align geometrically**.
When u and v point in the same direction, $\cos\theta = 1$, and the dot product is maximal.
When they are orthogonal, $\cos\theta = 0$, and the dot product is also zero.
When they point in opposite directions, $\cos\theta = -1$, and the dot product is negative.
In short, **the dot product measures how much two vectors aligh in both magnitude and direction.**

Part 2 A formal proof that dot product must be the way it is

Before the proof, we must first acknowledge several key assumptions of the inner product under Euclidean Space. For the time being, let's denote the inner product as $\langle \cdot, \cdot \rangle$. It must satisfy:

1. Symmetry: $\langle v, w \rangle = \langle w, v \rangle$
2. Bilinearity: for any vector u,v,w and scalar a, there is

$$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ and } \langle au, v \rangle = a \langle u, v \rangle$$

3. Positive definite: $\langle v, v \rangle = \|v\|^2 \geq 0$ and the equality only applies when $v = 0$.
4. The length squared in standard coordinates is the sum of the squares of the components.

$$\|v\|^2 = \sum_k v_k^2$$

Proof by Basis Expansion

Let $\{e_1, e_2, ..., e_n\}$ be the standard orthonormal basis, so that

$$v = \sum_i v_i e_i, \quad w = \sum_j w_j e_j$$

Using bilinearity and symmetry of inner product,

$$\langle v, w\rangle = \left\langle \sum_i v_i e_i, \sum_j w_j e_j \right\rangle = \sum_i \sum_j v_i w_j \langle e_i, e_j\rangle$$

Now we find $\langle e_i, e_j\rangle$.
If $i = j$, then

$$\langle e_i, e_i\rangle = \|e_i\|^2 = 1$$

If $i \neq j$, then

$$\|e_i + e_j\|^2 = 1^2 + 1^2 = 2$$

Expanding gives

$$\|e_i + e_j\|^2 = \langle e_i + e_j, e_i + e_j\rangle = \langle e_i, e_i\rangle + 2\langle e_i, e_j\rangle + \langle e_j, e_j\rangle$$

Therefore

$$\begin{aligned} 2 &= 2 + 2\langle e_i, e_j\rangle \\ \langle e_i, e_j\rangle &= 0 \end{aligned}$$

To conclude

$$\langle e_i, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

Hence,

$$\langle e_i, e_j\rangle = \delta_{ij}$$

Then

$$\langle v, w\rangle = \sum_i \sum_j v_i w_j \delta_{ij} = \sum_i v_i w_i$$

QED.

Part 3 Why dot product?

We agree that the statement

$$\|v\|^2 = v \cdot v$$

holds true. But why the dot product, not the cross product or other multiplication?

Proof 1

We want some operation $\odot$ such that

$$\|v\|^2 = v \odot v$$

to define the square of a vector's length. What properties should such an operation have if it's to represent length?
1. Result must be a scalar.
And the only multiplication that creates a scalar is the dot product.
2. Length must be independent of change in coordinate.

The dot product has the property

$$(Ru) \cdot (Rv) = u \cdot v$$

for any rotation matrix R.
The dot product preserves geometric invariance.

Proof 2

Without losing generality, we define the dot product to be

$$u \cdot v = u_1 v_1 + u_2 v_2, \quad u, v \in \mathbb{R}^2$$

Then, for any vector $v = \langle v_1, v_2 \rangle$,

$$\begin{aligned} \|v\| &= \sqrt{v_1^2 + v_2^2} \\ \|v\|^2 &= v_1^2 + v_2^2 = v_1 v_1 + v_2 v_2 = v \cdot v \end{aligned}$$

Appendix: Derivation from the algebraic expression to geometric one

Consider vectors u and v being vectors in 2-space, head-to-head, with angle θ. From cosine rule,

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta$$

Algebraically,

$$\|u - v\|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 = \|u\|^2 + \|v\|^2 - 2(u_1 v_1 + u_2 v_2)$$

QED.