Jimmy Zhan
5 Nov, 2025
Algebraic Topology
Proofs about Topological Properties

Appendix: Proof for Corollary 1.1

Statement

Corollary 1.1 For any set D, every point p in the space ($\mathbb{R}^2$ or $\mathbb{R}^3$) must be exactly one of the three categories: Interior point, Exterior point, or Boundary point. The three categories are mutually exclusive and exhaustive.

Proof

For **Exhausive**, we must show that any arbitary point p in the space must belong to at least one of the three categories.

For **Mutually Exclusive**, we must show that p must not belong to more than one category.

Mutual Exclusivity

Definition 2.1: A point p is called an exterior point if $\exists r > 0$ such that $B_r(p) \cap D \neq \emptyset$ **and** $B_r(p) \cap D^c \neq \emptyset$.

Case 1: Can an arbitary point p be both interior and exterior?

If p is interior, then $\exists r_1 > 0$ such that $B_{r_1}(p) \subseteq D$. This means this ball contains no points from D^c.

If p is exterior, then $\exists r_2 > 0$ such that $B_{r_2}(p) \subseteq D^c$. This means that this ball contains no points from D.

Pick $r = \min(r_1, r_2)$. Note that this new radius r is still positive. The smaller ball $B_r(p)$ must be entirely within D and entirely within D^c as well. This is a contradiction. A non-empty ball cannot be both inside and outside D.

Therefore a point cannot be both interior and exterior.

Case 2: Can p be both interior and boundary?

If p is interior, then $\exists r > 0$ such that $B_r(p) \subseteq D$. This means this ball contains no points from D^c. However, the definition of boundary point states that $\forall r > 0$, the ball $B_r(p)$ must contain at least one point from D^c. This is a contradiction.

Case 3 is essentially identical to case 2.

Therefore, we have proven the mutual exclusivity.

Exhausiveness

Let p be any arbitary point in metric space. Assume p is **not** an interior point.

The definition of interior is $\exists r > 0 \text{s.t.} B_r(p) \subseteq D$ (Definition 1.2).

The negation of this is: $\forall r > 0, B_r(p) \nsubseteq D$. If a ball $B_r(p)$ is not entirely inside D, it must contain at least one point from D^c. Therefore,

$$\forall r > 0, B_r(p) \cap D^C \neq \emptyset \tag{Condition 1}$$

Similarly, let's also assume p is **not** an exterior point. We can conclude that

$$\forall r > 0, B_r(p) \cap D \neq \emptyset \tag{Condition 2}$$

Combine the two assumptions.

From **Condition 1**, every ball $B_r(p)$ must contain a point from D^c.

From **Condition 2**, every ball $B_r(p)$ must contain a point from D.

which is exactly the definition of a boundary point!

Thus we have shown that any arbitary point p, if not interior nor exterior, is by definition a boundary point. This covers all possibilities.