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Unification of Functions

Composite of power-exponential functions

Consider function

$$f(x) = \frac{x^{x+1}}{(x+1)^x}$$

It is obvious that the domain of $f(x)$ is $x > 0$ when $f : \mathbb{R} \to \mathbb{R}$. And intuitively, $f(x)$ is continuous in the first quadrant, as $x \to \infty$. However, almost all plotting tools suggest otherwise.

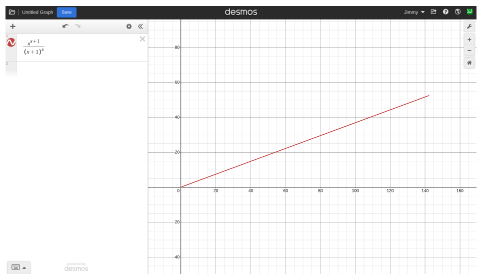

Figure 1: Desmos

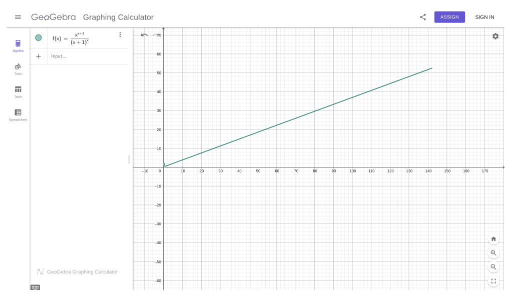

Figure 2: Geogebra

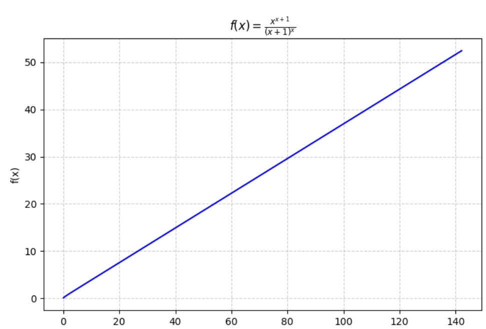

Figure 3: Matplotlib

Notice how all three tools stopped drawing the graph at a same point, around $x = 140$. In addition, though might not be obvious in the figures, the function appears to curve near the origin, and starts to merge to a straight line starting from some small x.

The **aim** of this document is to analyze the function, and figure out whether $f(x)$ stops at that certain point, if so, what is the value? And if not, why does it appear in such way?

Part 1 Behaviour around $x = 0$

Let $y = f(x)$.

$$\begin{aligned}
y &= x \cdot \frac{x^x}{(x+1)^x} \\
\ln y &= \ln \left[x \, (\frac{x}{x+1})^x \right] \\
&= \ln x + x \cdot \ln \, \frac{x}{x+1} \\
&= \ln x + x \left[\ln x - \ln(x+1) \right]
\end{aligned}$$

From Taylor,

$$\ln(x+1) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

Therefore,

$$\begin{aligned}
\ln y &= \ln x + x \left[\ln x - (x - \frac{x^2}{2} + \frac{x^3}{3} - o(x^4)) \right] \\
&= \ln x + x \ln x - x(x - \frac{x^2}{2} + \frac{x^3}{3} - o(x^4)) \\
&= \ln x + x \ln x - x^2 + \frac{x^3}{2} + o(x^4) \\
e^{\ln y} &= e^{\ln x + x \ln x - x^2 + \frac{x^3}{2} + o(x^4)} \\
y &= x \cdot e^{x \ln x - x^2 + o(x^3)} \\
f(x) &= x \cdot e^{x \ln x - x^2 + o(x^3)}
\end{aligned}$$

I am curious about the tangent line at $x = 0$.

$$f'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} e^{x \ln x - x^2 + o(x^3)} = e^0 = 1$$

Therefore, at $x = 0$, $f(x)$ has a tangent line of $y = x$.
But what about at larger values of x?

Part 2 Behaviour as value of x increases

From Part 1, there is

$$\begin{aligned}
\ln y &= \ln x + x \left[\ln x - \ln(x+1) \right] \\
&= \ln x + x \ln x - x \ln(x+1) \\
&= \ln x + x \ln x - x \left[\ln x + \ln(1 + \frac{1}{x}) \right]
\end{aligned}$$

From Taylor,

$$\ln(1+\frac{1}{x}) = 0 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - ...$$

Therefore,

$$\begin{aligned}\ln y = \ln x + x\ln x - x\left[\ln x + 0 + \frac{1}{x} - \frac{1}{2x^2} + o(\frac{1}{x^3})\right] \\ = \ln x - 1 + \frac{1}{2x} - o(\frac{1}{x^2}) \\ e^{\ln y} = e^{\ln x - 1 + \frac{1}{2x} - o(\frac{1}{x^2})} \\ y = x \cdot \frac{1}{e} \cdot e^{\frac{1}{2x} - o(\frac{1}{x^2})} \\ y = \frac{x}{e} \cdot e^{\frac{1}{2x} - o(\frac{1}{x^2})}\end{aligned}$$

From Taylor,

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

Therefore

$$e^{\frac{1}{2x} - o(\frac{1}{x^2})} = 1 + \frac{1}{2x} - o(\frac{1}{x^2}) + \frac{1}{2}(\frac{1}{2x} - o(\frac{1}{x^2}))^2 + ...$$

Consider all the order of magnitude for each term:
1 is a constant.
$\frac{1}{2x}$ belongs to $\frac{1}{x}$.
$o(\frac{1}{x^2})$ is smaller than $\frac{1}{x^2}$, therefore can be merged into $o(\frac{1}{x})$.
$\frac{1}{2}(\frac{1}{2x} - o(\frac{1}{x^2}))^2$ is of order $\frac{1}{x^2}$, therefore can also be merged into $o(\frac{1}{x})$.
And the remaining terms are smaller than $\frac{1}{x^2}$, therefore can also be merged into $o(\frac{1}{x})$.
Therefore, we can say that

$$e^{\frac{1}{2x} - o(\frac{1}{x^2})} = 1 + \frac{1}{2x} + o(\frac{1}{x})$$

Therefore

$$f'(x) = \frac{x}{e} \cdot (1 + \frac{1}{2x} + o(\frac{1}{x})) = \frac{x}{e} + \frac{1}{2e} + o(1)$$

Therefore the tangent line has gradient $\frac{1}{e}$, y-intecept $\frac{1}{2e}$.

Part 3 Where does the graph go after a certain point?

From Part 2, we know that there exists an oblique asymptote, which has the equation

$$g(x) = \frac{x}{e} + \frac{1}{2e}$$

I wonder how the graph of $f(x)$ follows the asymptote as $x \to \infty$.

$$I = \lim_{x\to\infty} \frac{\frac{x^{x+1}}{(x+1)^x}}{\frac{x}{e} + \frac{1}{2e}}$$

Let $n = x + 1$.

$$I = \lim_{n\to\infty} \frac{\frac{(n-1)^n}{n^{n-1}}}{\frac{n-\frac{1}{2}}{e}}$$

$$I = \lim_{n\to\infty} e \cdot \frac{(n-1)^n}{n^{n-1} \cdot (n-\frac{1}{2})}$$

$$I = \lim_{n\to\infty} e \cdot \frac{n^n(1-\frac{1}{n})^n}{n^{n-1} \cdot (n-\frac{1}{2})}$$

$$I = \lim_{n\to\infty} e \cdot \frac{n}{n-\frac{1}{2}} \cdot (1-\frac{1}{n})^n$$

$$I = e \cdot 1 \cdot \frac{1}{e} = 1$$

So the graph of $f(x)$ should always follow the asymptote. We have proven mathematically that the graph seen in the figures above are undoubtly wrong!

Part 4 Why is the plotting wrong?

We've demonstrated, in previous sections, that $f(x)$ possesses a well-defined oblique asymptote, and never "stops" or "breaks" for any finite $x > 0$, most plotting software displays an apparent discontinuity near $x \approx 140$. I believe that this visual "cut-off" has no mathematical significance; it originates purely from floating-point overflow and sampling behavior within the numerical implementation.

When computed directly as

$$f(x) = \frac{x^{x+1}}{(x+1)^x}$$

both numerator and denominator become astronomically large even though their ratio remains moderate.

In double-precision arithmetic, the largest representable finite number is roughly 1×10^{308}, corresponding to $\ln(10^{308}) \approx 709$.

The term x^x exceeds this limit when $x \ln x > 709$, which occurs at $x \approx 142.9$.

Ultimately, to avoid overflow, one should evaluate

$$\ln f(x) = (x+1)\ln x - x\ln(x+1)$$

then only exponentiate only at the end

$$f(x) = e^{(x+1)\ln x - x\ln(x+1)}$$