On the existence of the oblique asymptotes due to transformations to unknown functions

Jimmy Zhan

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1 Introduction

Consider the graph of $f(x)$ being

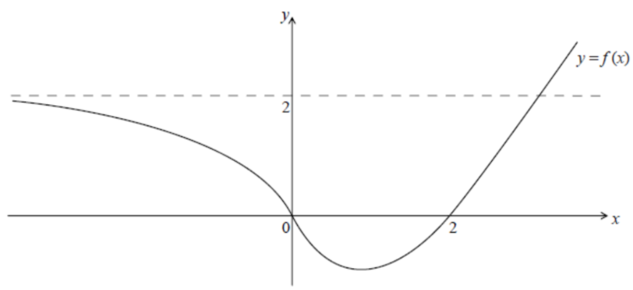

Figure 1: The question

It is asked to graph $g(x)$ defined by $g(x) = xf(x)$.

2 Result

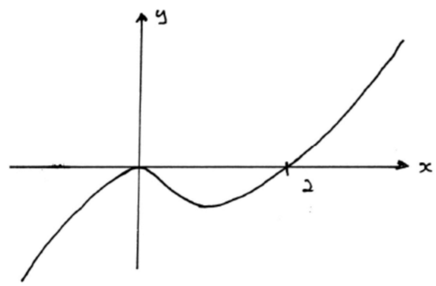

Figure 2: The answer

3 Hypothesis

If we view the multiplication of x as a transformation, then naturally, the asymptote $y = 2$ becomes, after the transformation, $y = 2x$. We suspect that $g(x)$ has a oblique asymptote of $y = 2x$.

However, we doubt if there a way to prove rigorously, the existence (or non-existence) of the oblique asymptote, without finding out the algrbraic expression.

4 Justification

From the graph it is obvious that $f(x)$ has an horizontal asymptote $y = 2$. That is

$$\lim_{x \to -\infty} f(x) = 2 \tag{1}$$

Assume the function $g : \mathbb{R} \to \mathbb{R}$ be continuous and differentiable for all x. $y = mx + c$ is an oblique asymptote of $g(x)$ if

$$\lim_{x \to -\infty} (g(x) - (mx + c)) = 0 \tag{2}$$

The slope m can be computed via

$$m = \lim_{x \to -\infty} \frac{g(x)}{x} \tag{3}$$

provided this limit exists and is finite and nonzero.

$$c = \lim_{x \to -\infty} (g(x) - mx) \tag{4}$$

If either of these limits does not exist or is infinite, there is no oblique asymptote.

Let $g(x) = xf(x)$. From equations 1 and 3,

$$m = \lim_{x \to -\infty} \frac{xf(x)}{x} = \lim_{x \to -\infty} f(x) = 2 \tag{5}$$

Therefore if an oblique asymptote exists, its slope is 2.

Now consider the y-intercept. Consider equation 4, using $g(x) = xf(x)$ and calculated slope $m = 2$,

$$c = \lim_{x \to -\infty} (xf(x) - 2x) = \lim_{x \to -\infty} x(f(x) - 2) \tag{6}$$

Analyze equation 6. As $x \to -\infty$, the term x approaches $-\infty$. Since $\lim_{x \to -\infty} f(x) = 2$, the term $(f(x) - 2)$ approaches 0. This gives an indeterminate form of type $(-\infty) \cdot 0$.

Therefore, it is not possible to determine the existence of the oblique asymptote without knowing the function. The limit of equation 6 could be finite, infinite, or zero depending on the function.

5 Last Attempt

The limit in equation 6 provides an indeterminate form. Therefore it is natural to think of using L'Hopital's Rule. We can rewrite equation 6 as

$$c = \lim_{x \to -\infty} \frac{f(x) - 2}{\frac{1}{x}} \tag{7}$$

As $x \to -\infty$, both the numerator and denominator approaches zero.

$$\begin{aligned} \frac{d}{dx}(f(x) - 2) &= f'(x) \\ \frac{d}{dx}(\frac{1}{x}) &= -\frac{1}{x^2} \end{aligned} \tag{8}$$

Equation 7 becomes

$$c = \lim_{x \to -\infty} \frac{f'(x)}{-\frac{1}{x^2}} = \lim_{x \to -\infty} -x^2 f'(x) \tag{9}$$

Seems promising. But no. Since $f(x)$ has a horizontal asymptote of $y = 2$, for the curve to approach this line, its slope must approach zero. Therefore $\lim_{x\to-\infty} f'(x) = 0$. While $\lim_{x\to-\infty} -x^2 = -\infty$. We are back in the indeterminate form we started with.

Appendix: Derivation of the slope and y-intercept limit equation

It should be obvious that

$$\lim_{x\to-\infty}(g(x) - (mx + c)) = 0$$

Factorizing

$$\lim_{x\to-\infty}(\frac{g(x)}{x} - \frac{mx}{x} - \frac{c}{x}) = 0$$
$$\lim_{x\to-\infty}(\frac{g(x)}{x} - m - \frac{c}{x}) = 0$$
$$(\lim_{x\to-\infty}\frac{g(x)}{x}) - m - 0 = 0$$
$$m = \lim_{x\to-\infty}\frac{g(x)}{x}$$

Additionally, from

$$\lim_{x\to-\infty}(g(x) - (mx + c)) = 0$$

There is

$$\lim_{x\to-\infty}((g(x) - mx) - c) = 0$$
$$\lim_{x\to-\infty}(g(x) - mx) = c$$