Jimmy Zhan
5 Nov, 2025
Algebraic Topology
Proofs about Topological Properties

Proof of Fundamental Definition of Closed Set in Metric Space

1 Proposition

A set D in $\mathbb{R}^2$ or $\mathbb{R}^3$ is closed if and only if its complement is an open set.

2 Definitions

Let D be a set in $\mathbb{R}^n$ where $n = 2$ or 3.
Definition 2.1: A set D is closed if it contains all of its boundary points. That is,

$$\text{Boundary}(D) \subseteq D \tag{1}$$

Definition 2.2.1: A set D is open if it contains none of its boundary points.
Definition 2.2.2: A set D is open if every point $p \in D$ is an interior point of D.
Definition 2.3.1: Interior point: $\exists r > 0$ s.t. $B_r(P) \subseteq D$.
Definition 2.3.2: Exterior point: $\exists r > 0$ s.t. $B_r(P) \subseteq D^c$.
Definition 2.3.3: Boundary point: $\forall r > 0, B_r(P)$ hits both D and D^c.

3 Proof

Similarly, in order to prove the "if and only if" statement, we must prove the following statements.
Statement 3.1: Given a closed set D in $\mathbb{R}^n$ where $n = 2$ or 3, the complement set D^c is an open set.
Statement 3.2: Given an open set D^c, the set D is a closed set.

Statement 1.1

1. Consider an arbitary point $p \in D^c$.
2. According to the definition of complement set, $p \notin D$.
3. From the corollary proven in previous documents, p is either an interior, exterior or boundary point.
4. From (2), p cannot be an interior point of D. What about boundary?
5. Since D is a closed set, from Definition 2.1, all boundary points are in D. $p \notin D$, therefore p cannot be a boundary point.
6. p then must be an exterior point.
7. From Definition 2.3.2, $\exists r > 0$ s.t. $B_r(P) \subseteq D^c$, which is the definition of an interior point.
8. p is an interior point of D^c.

We have therefore proven that any arbitary point $p \in D^c$ is an interior point of D^c, which is the definition of an open set. Therefore D^c is an open set.

Statement 1.2

1. Let p be any boundary point of D. In order to prove that D is closed, we only need to prove that p is an element of D, according to Definition 2.1.
2. **Assume** $p \notin D$, then p must be an element of D^c.
3. Since D^c is an open set, p must be an interior point of D^c.
4. From Definition 2.3.1, $B_r(p) \subseteq D^c$
5. This is precisely the definition of exterior point. p is an exterior point of D.

6. Contradiction. p cannot be a boundary point and exterior point. Therefore, the assumption $p \notin D$ must be false.

7. Therefore, the negation of the assumption must be true.

8. Thus, $p \in D$.

We have therefore proven that any boundary point p is an element of D, which is the definition of a closed set.