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The Basel Problem

Proof of the Basel Problem by Leonhard Euler

1 Introduction

The Basel Problem was first introduced in 1644 by an Italian mathematician Pietro Mengoli. The problem remained open for 90 years, until Leonhard Euler gave his first proof in 1734.

THe Basel Problem asks for the sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... \tag{1}$$

Mengoli calculated the sum of the first 1000 terms by himself, giving the results as follows:

$$1 + \frac{1}{2^2} + ... + \frac{1}{998^2} = 1.643932...$$
$$1 + \frac{1}{2^2} + ... + \frac{1}{998^2} + \frac{1}{999^2} = 1.643933... \tag{2}$$
$$1 + \frac{1}{2^2} + ... + \frac{1}{998^2} + \frac{1}{999^2} + \frac{1}{1000^2} = 1.643934...$$

He then proposed that the series is converging to 1.644934, yet not sure about any inaccuracies. Euler, when he first saw that number, immediately identified the value to be equivilant to $\frac{\pi^2}{6}$. The tale is now impossible to verify, what we are certain is that Euler gave the first ever proof for the Basel Problem.

2 The proof given by Euler

Since the "answer" contains π, why not start from trigonometric functions. From Maclaurin series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}... \tag{3}$$

From function's perspective, the sine function can also be expressed from its roots, such as:

$$...(x + 2\pi)(x + \pi)(x)(x - \pi)(x - 2\pi)...$$
$$= x(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)... \tag{4}$$

However, it is clear that for a lot of x-values, the product is divergent. In order for it to converge to $\sin(x)$, we can add a coefficient C, so that the equality holds:

$$\sin(x) = Cx(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)... \tag{5}$$

We can then find the value of C. From (5):

$$\frac{\sin(x)}{x} = C(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)...$$
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} C(x^2 - \pi^2)(x^2 - 2^2\pi^2)(x^2 - 3^2\pi^2)...$$
$$1 = C(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)... \tag{6}$$
$$C = \frac{1}{(-\pi^2)(-2^2\pi^2)(-3^2\pi^2)...}$$

Substituting back to (5):

$$\sin(x) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{2^2\pi^2})(1 - \frac{x^2}{3^2\pi^2})... \tag{7}$$

It is not difficult to verify that (7) is a valid factorisation of $\sin(x)$.

We now expand (7). It is impossible to fully expand an equation in an infinite product form. However we only need to find the coefficient of the x^2 term.

Consider a simpler version to investigate how the infinite product is expanded.

$$(1 - ax^2)(1 - bx^2)(1 - cx^2)...$$
$$= (1 - (a+b)x^2 + abx^4)(1 - cx^2)... \tag{8}$$
$$= 1 - (a+b+c)x^2 - (ab - ac - bc)x^4 + abcx^6...$$

Therefore, the expansion of (7) should give:

$$\sin(x) = x(1 - (\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2})x^2...) \tag{9}$$

From (3):

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}...$$
$$= x(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}...) \tag{10}$$

Observe the coefficient of x^2 term in (9) and (10). They both represent $\sin(x)$ and therefore the coefficients should be the same. Thus we have:

$$\frac{1}{\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + ... = \frac{1}{3!}$$
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ... = \frac{\pi^2}{6} \tag{11}$$

QED