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Multivariable Differential Calculus

Proof of the equation of curvature with parameter t

Statement

Notation. If $r(t)$ is any parametrization of C, and the point P on C corresponds to $t = t_0$, then the curvature κ at P will be denoted $\kappa(t = t_0)$.

Theorem 1. Let $r_1(t)$ be any smooth parametrization of the curve C, and $T_1(t)$ denote the unit tangent vector at t. Suppose the point P on C corresponds to $t = t_0$, and $T_1'(t_0)$ and $r_1''(t_0)$ exist. Then

$$\kappa(t = t_0) = \frac{\|T_1'(t_0)\|}{\|r_1'(t_0)\|} \tag{1}$$

Proof

Define g to be an arc length change of parameter, and set $r_2(s) = r_1(g(s))$.
From the definition of unit tangent vector

$$T(t) = \frac{r'(t)}{\|r'(t)\|} \tag{2}$$

Therefore

$$T_2(s) = \frac{r_2'(s)}{\|r_2'(s)\|} = r_2'(s) \tag{3}$$

Since $r_2(s) = r_1(g(s))$, differentiating using chain rule gives

$$r_2'(s) = r_1'(g(s)) \cdot g'(s) \tag{4}$$

Since $r_2(s)$ is an arc length parametrization

$$\|r_2'(s) = 1\| \tag{5}$$

Therefore

$$\begin{aligned} &\|r_1'(g(s))\| \cdot |g'(s)| \\ g'(s) &= \frac{1}{\|r_1'(g(s))\|} \end{aligned} \tag{6}$$

The reason for this is that $g'(s)$ must be larger than 0. The proof for this statement can be found in appendix.
We can then substitute equation 6 to equation 4.

$$r_2'(s) = r_1'(g(s)) \cdot \frac{1}{\|r_1'(g(s))\|} \tag{7}$$

From equation 3, $T_2(s) = r_2'(s)$, therefore

$$T_2(s) = \frac{r_1'(g(s))}{\|r_1'(g(s))\|} \tag{8}$$

which is the definition for $T_1(t)$.
QED

Appendix

Proof for $g'(s) > 0$

Consider $s(t)$

$$s = \int_{t_0}^{t} \|r'(u)\| \, du$$

in which $\|r'(u)\| \geq 0$. Therefore, when $t_2 > t_1$, $s(t_2) > s(t_1)$. $s(t)$ is a strictly monotone increasing function. The inverse of $s(t)$ is $g(s)$. $g(s)$ is therefore also a strictly increasing function. Therefore

$$g'(s) = \frac{1}{s'(t)}$$

Therefore $g'(s) > 0$.
QED