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27 Nov, 2025
The Basel Problem

Proof of the Basel Problem by Trigonometric Inequality and De Moivre's Theorem

1 Trigonometric Inequality

It suffices to show, that the following inequality holds within the interval $0 < x < \frac{\pi}{2}$:

$$\sin x < x < \tan x \tag{1}$$

Therefore we have

$$\begin{aligned} 1 <& \frac{x}{\sin x} < \frac{\tan x}{\sin x} \\ \cos x <& \frac{\sin x}{x} < 1 \\ \cos^2 x <& \frac{\sin^2 x}{x^2} < 1 \\ \frac{\cos^2 x}{\sin^2 x} <& \frac{1}{x^2} < \frac{1}{\sin^2 x} \\ \cot^2 x <& x^{-2} < \cot^2 x + 1 \end{aligned} \tag{2}$$

2 Main Section

Construct $x = \frac{n\pi}{2N+1}$ where $n, N \in \mathbb{N}$ such that $1 \leq n \leq N$. First, we should check if the criteria for the inequality to exist is met, that is, whether $0 < \frac{n\pi}{2N+1} < \frac{\pi}{2}$.

It is obvious that $\frac{n\pi}{2N+1} > 0$ for all n and N.
For all $1 \leq n \leq N$, there is

$$\frac{n\pi}{2N+1} \leq \frac{N\pi}{2N+1} \tag{3}$$

Consider the upper boundary only. Since $2N < 2N + 1$, there is

$$\begin{aligned} \frac{2N}{2N+1} &< 1 \\ \frac{N}{2N+1} &< \frac{1}{2} \\ \frac{N\pi}{2N+1} &< \frac{\pi}{2} \end{aligned} \tag{4}$$

Therefore we may proceed with the substitution. From (2):

$$
\begin{gathered}
\cot^2\left(\frac{n\pi}{2N+1}\right) < \frac{(2N+1)^2}{n^2\pi^2} < \cot^2\left(\frac{n\pi}{2N+1}\right) + 1 \\
\sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) < \sum_{n=1}^{N}\left(\frac{(2N+1)^2}{n^2\pi^2}\right) < \sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)+1\right) \\
\sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) < \frac{(2N+1)^2}{\pi^2}\sum_{n=1}^{N}\frac{1}{\pi^2} < \sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) + N \\
\frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) < \sum_{n=1}^{N}\frac{1}{\pi^2} < \frac{\pi^2}{(2N+1)^2}\sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) + \frac{N\pi^2}{(2N+1)^2}
\end{gathered}
\tag{5}
$$

Denote $A_n = \sum_{n=1}^{N}\left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right)$. The aim now is to show that

$$
\lim_{N\to\infty}\frac{A_n}{N^2} = \frac{2}{3} \tag{6}
$$

3 Expansion of the sum & Unpacking the inequality

Denote $\theta = \frac{n\pi}{2N+1}$. Therefore $\sin((2N+1)\theta) = 0$.
From De Moivre's Theorem

$$
\begin{aligned}
(\cos\theta + i\sin\theta)^{2N+1} &= \cos((2N+1)\theta) + i\sin((2N+1)\theta) \\
\sin((2N+1)\theta) &= \mathfrak{I}\left[(\cos\theta + i\sin\theta)^{2N+1}\right]
\end{aligned}
\tag{7}
$$

We use binomial expansion to obtain

$$
(\cos\theta + i\sin\theta)^{2N+1} = \sum_{k=0}^{2N+1}\binom{2N+1}{k}\left(\cos^{2N+1-k}\theta\right)(i\sin\theta)^k \tag{8}
$$

In (8), the imaginary part only exists for odd number k. Let $k = 2j+1$, $(j = 0, 1, ..., N)$. Note that $i^{2j+1} = i\cdot(-1)^j$. Taking the odd terms out, we obtain

$$
\begin{aligned}
\sin((2N+1)\theta) &= \sum_{j=0}^{N}(-1)^j\binom{2N+1}{2j+1}\left(\cos^{2N-2j}\theta\right)\left(\sin^{2j+1}\theta\right) \\
\frac{\sin((2N+1)\theta)}{\sin^{2N+1}\theta} &= \sum_{j=0}^{N}(-1)^j\binom{2N+1}{2j+1}\frac{\cos^{2N-2j}\theta}{\sin^{2N-2j}\theta}
\end{aligned}
\tag{9}
$$

Let $m = N - j$, $x = \cot^2\theta$. (9) becomes

$$
\begin{aligned}
\frac{\sin((2N+1)\theta)}{\sin^{2N+1}\theta} &= \sum_{n=0}^{N}(-1)^{N-m}\binom{2N+1}{2(N-m)+1}x^m \\
&= (2N+1)x^N - \binom{2N+1}{3}x^{N-1} + ... + (-1)^N\binom{2N+1}{2N+1}
\end{aligned}
\tag{10}
$$

It is established at the beginning of this section, that $\sin((2N+1)\theta) = 0$ when $\theta = \frac{n\pi}{2N+1}$. Therefore, at $\theta = \frac{n\pi}{2N+1}$, (10) is essentially a polynomial with respect to x, with roots in the form

$$
x_n = \cot^2\left(\frac{n\pi}{2N+1}\right),\ (n = 1, 2, ..., N) \tag{11}
$$

Denote

$$f(x) = (2N+1)x^N - \binom{2N+1}{3}x^{N-1} + ... + (-1)^N \binom{2N+1}{2N+1} \tag{12}$$

From Vieta's Theorem

$$x_1 + x_2 + ... + x_N = -\frac{b}{a} = \frac{\binom{2N+1}{3}}{2N+1}$$

$$A_n = \sum_{n=1}^{N} \left(\cot^2\left(\frac{n\pi}{2N+1}\right)\right) = \frac{\binom{2N+1}{3}}{2N+1} = \frac{N(2N-1)}{3} \tag{13}$$

$$\frac{A_n}{N^2} = \frac{N(2N-1)}{3N^2} = \frac{2}{3} - \frac{1}{3N}$$

Therefore, we are able to conclude that

$$\lim_{N\to\infty} \frac{A_n}{N^2} = \frac{2}{3} \tag{14}$$

4 Final Steps

From (14), we know that as $N \to \infty$, $\frac{A_n}{N^2} \to \frac{2}{3}$. Then

$$\frac{\pi^2}{(2N+1)^2}A_n = \pi^2 \cdot \frac{A_m}{N^2} \cdot \frac{N^2}{(2N+1)^2} \tag{15}$$

Therefore

$$\text{As } N \to \infty,\ \pi^2 \cdot \frac{A_m}{N^2} \cdot \frac{N^2}{(2N+1)^2} \to \pi^2 \cdot \frac{2}{3} \cdot \frac{1}{4} = \frac{\pi^2}{6} \tag{16}$$

Also

$$\text{As } N \to \infty,\ \frac{N}{(2N+1)^2} = \frac{1}{4N+4+\frac{1}{N}} \to 0 \tag{17}$$

Go back to the first inequality. From (5)

$$\frac{\pi^2}{(2N+1)^2}A_n < \sum_{n=1}^{N} \frac{1}{\pi^2} < \frac{\pi^2}{(2N+1)^2}A_n + \frac{N\pi^2}{(2N+1)^2} \tag{18}$$

We have stated that, as $N \to \infty$, both the upper and lower boundary approach $\frac{\pi^2}{6}$. From Squeeze Theorem,

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{\pi^2} = \frac{\pi^2}{6} \tag{19}$$

as desired.