Real-Time Distributed Control Systems 2017/18
Chapter 18
Solution of Distributed Optimization Problems: The consensus algorithm

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1 The global optimization problem

Our overal cost to optimize is:

$$\mathbf{d}^* = \underset{\mathbf{d}\in\mathcal{C}}{\operatorname{argmin}} \left\{ \frac{1}{2}\mathbf{d}^T Q \mathbf{d} + \mathbf{c}^T \mathbf{d} \right\}$$

with:

$$\mathbf{d} = \begin{bmatrix} d_1 & \cdots & d_N \end{bmatrix}^T$$

$$\mathbf{c} = \begin{bmatrix} c_1 & \cdots & c_N \end{bmatrix}^T$$

$$Q = \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_N \end{bmatrix}^T$$

and the constraints:

$$\mathcal{C} : \left\{ 0 \leq d_i \leq 100 \quad \text{and} \quad \sum_{j=1}^{N} k_{ij} d_j \geq L_i - o_i, \quad \forall i \right\}$$

2 The consensus algorithm

The consensus algorithm distributes the computation over a set of nodes in a network (see [1] for the details). Each node i holds a local copy $\mathbf{d}_i$ of the global variable $\mathbf{d}$.

$$\mathbf{d}_i = \begin{bmatrix} d_{i1} & \cdots & d_{ii} & \cdots & d_{iN} \end{bmatrix}^T$$

The cost function and the constraints of each node are partitions of the global cost function. Then it uses a method called ADMM (alternated direction method of multipliers) with the constraints that all local copies $\mathbf{d}_i$ should be identical (thus identical to their average). ADMM uses an augmented Lagrangian method (augments the Lagrangian with a quadratic term to promote the convergence of the method) to iterate the computation of the local solutions and the Lagrange multipliers. At each node i we have the following iterations in time t:

$$\begin{cases} \mathbf{d}_i(t+1) & = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2} + \mathbf{c}_i^T \mathbf{d}_i + \mathbf{y}_i^T(t)(\mathbf{d}_i - \bar{\mathbf{d}}_i(t)) + \frac{\rho}{2} \|\mathbf{d}_i - \bar{\mathbf{d}}_i(t)\|_2^2 \right\} \\ \bar{\mathbf{d}}_i(t+1) & = \sum_{j=1}^{N} \mathbf{d}_j(t) \\ \mathbf{y}_i(t+1) & = \mathbf{y}_i(t) + \rho \left(\mathbf{d}_i(t+1) - \bar{\mathbf{d}}_i(t+1) \right) \end{cases}$$

where $\mathbf{y}_i$ are Lagrange multipliers, ρ is the augmented Lagrangian method penalty parameter, and $\bar{\mathbf{d}}_i$ is the average of the solutions of the nodes. The parameters of the local cost function only consider the effect of the current node:

$$\mathbf{c}_i = \begin{bmatrix} 0 & \cdots & c_i & \cdots & 0 \end{bmatrix}^T$$

$$Q_i = \operatorname{diag}(0, \cdots, 0, q_i, 0, \cdots, 0) = \begin{bmatrix} 0 & & & & 0 \\ & \ddots & & & \\ & & q_i & & \\ & & & \ddots & \\ 0 & & & & 0 \end{bmatrix}^T$$

The local constraint also only considers a reduced number of constraints (in our case the ones that constrain the illuminance level at the node's desk), so that each local problem is easy to solve:

$$\mathcal{C}_i : \left\{ 0 \leq d_i \leq 100 \quad \text{and} \quad \mathbf{k}_i^T \mathbf{d}_i \geq L_i - o_i \right\}$$

with

$$\mathbf{k}_i = \begin{bmatrix} k_{i1} & \cdots & k_{ii} & \cdots & k_{iN} \end{bmatrix}^T$$

3 The Primal Iterates

To solve the problem we have to iterate the computation of the primal variables $\mathbf{d}_i$, the computation of the average of the primal variables of all nodes, and the dual variables $\mathbf{y}_i$. The hard problem is the first one. Let us see how to solve it. The primal iteration is:

$$\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2} \mathbf{d}_i^T Q_i \mathbf{d}_i + \mathbf{c}_i^T \mathbf{d}_i + \mathbf{y}_i^T(t)(\mathbf{d}_i - \bar{\mathbf{d}}_i(t)) + \frac{\rho}{2} \|\mathbf{d}_i - \bar{\mathbf{d}}_i(t)\|_2^2 \right\}$$

It can be written as:

$$\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2} \mathbf{d}_i^T Q_i \mathbf{d}_i + \mathbf{d}_i^T \mathbf{c}_i + \mathbf{d}_i^T \mathbf{y}_i(t) - \bar{\mathbf{d}}_i^T(t) \mathbf{y}_i(t) + \frac{\rho}{2} (\mathbf{d}_i - \bar{\mathbf{d}}(t))^T (\mathbf{d}_i - \bar{\mathbf{d}}(t)) \right\}$$

Because we are minimizing in $\mathbf{d}_i$ we can remove from the cost all terms that do not depend in $\mathbf{d}_i$.

$$\mathbf{d}_i(t{+}1) = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2}\mathbf{d}_i^T Q_i \mathbf{d}_i + \mathbf{d}_i^T \mathbf{c}_i + \mathbf{d}_i^T \mathbf{y}_i(t) + \frac{\rho}{2}\mathbf{d}_i^T \mathbf{d}_i - \rho \mathbf{d}_i^T \bar{\mathbf{d}}_i(t) + \frac{\rho}{2}\bar{\mathbf{d}}_i^T(t)\bar{\mathbf{d}}_i(t) \right\}$$

Now, collecting terms, we have:

$$\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2}\mathbf{d}_i^T \left(Q_i + \rho I\right) \mathbf{d}_i + \mathbf{d}_i^T \left(\mathbf{c}_i + \mathbf{y}_i(t) - \rho \bar{\mathbf{d}}_i(t)\right) \right\}$$

where

$$R_i = Q_i + \rho I = \operatorname{diag}(\rho, \cdots, \rho, \rho + q_i, \rho, \cdots, \rho) = \begin{bmatrix} \rho & & & & 0 \\ & \ddots & & & \\ & & \rho + q_i & & \\ & & & \ddots & \\ 0 & & & & \rho \end{bmatrix}$$

and

$$\mathbf{z}_i(t) = -\mathbf{c}_i + -\mathbf{y}_i(t) + \rho \bar{\mathbf{d}}_i(t)$$

Finally:

$$\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i \in \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2}\mathbf{d}_i^T R_i \mathbf{d}_i - \mathbf{d}_i^T \mathbf{z}_i(t) \right\}$$

Let:

$$f^t(\mathbf{d}_i) = \frac{1}{2}\mathbf{d}_i^T R_i \mathbf{d}_i - \mathbf{d}_i^T \mathbf{z}_i(t)$$

This is an optimization problem with a convex function $f^t(\mathbf{d}_i)$ and convex set $\mathcal{C}_i$. The solution is in the interior of the feasible region or at its boundary.

4 Solution in the interior

Because our cost function is a convex quadratic form, if the solution is in the interior of the set, then it is the global solution. To compute the global solution we solve for $\nabla f^t(\mathbf{d}_i) = 0$:

$$\nabla f^t(\mathbf{d}_i) = R_i \mathbf{d}_i - \mathbf{z}_i(t) = 0$$

Thus we get:

$$\mathbf{d}_i^0(t+1) = R_i^{-1}\mathbf{z}_i(t)$$

where the superscript 0 indicates this is the global unconstrained solution, and:

$$P_i = R_i^{-1} = \operatorname{diag}\left(\frac{1}{\rho}, \cdots, \frac{1}{\rho}, \frac{1}{\rho + q_i}, \frac{1}{\rho}, \cdots, \frac{1}{\rho}\right) = \begin{bmatrix} \frac{1}{\rho} & & & & 0 \\ & \ddots & & & \\ & & \frac{1}{\rho+q_i} & & \\ & & & \ddots & \\ 0 & & & & \frac{1}{\rho} \end{bmatrix}$$

Finally, we can write:

$$\mathbf{d}_i^0(t+1) = P_i \mathbf{z}_i(t) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} = \begin{bmatrix} \frac{\rho \bar{d}_{i1}(t) - y_{i1}(t)}{\rho} \\ \vdots \\ \frac{\rho \bar{d}_{ii}(t) - y_{ii}(t) - c_i}{\rho+q_i} \\ \vdots \\ \frac{\rho \bar{d}_{iN}(t) - y_{iN}(t)}{\rho} \end{bmatrix}$$

In summary, the variables of the solution that are not controlled directly by the node are computed as:

$$d_{ij}^0(t+1) = \bar{d}_j(t) - y_{ij}(t)/\rho, \quad j \neq i$$

The variable that is controlled by the node itself is given by:

$$d_{ii}^0(t+1) = \frac{1}{\rho + q_i}(\rho \bar{d}_i(t) - y_{ii}(t) - c_i)$$

Now we must check for feasibility. The feasibility set is an affine inequality:

$$\mathcal{C}_i : \{C_i \mathbf{d}_i \leq \mathbf{b}_i\}$$

with

$$C_i = \begin{bmatrix} -\mathbf{k}_i^T \\ -\mathbf{e}_i^T \\ \mathbf{e}_i^T \end{bmatrix}$$

where

$$\mathbf{e}_i^T = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \end{bmatrix}$$

and

$$\mathbf{b}_i = \begin{bmatrix} o_i - L_i \\ 0 \\ 100 \end{bmatrix}$$

If the unconstrained minimum is not feasible, we have to look for the solutions on the boundary.

5 Solutions in the boundary

To look for the solutions on the boundary, we must solve the following optimization problem:

$$\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i \in \partial \mathcal{C}_i}{\operatorname{argmin}} \left\{ \frac{1}{2} \mathbf{d}_i^T R_i \mathbf{d}_i - \mathbf{d}_i^T \mathbf{z}_i(t) \right\}$$

How to characterize the boundary $\partial \mathcal{C}_i$ of the feasible region $\mathcal{C}_i$?

$$\mathcal{C}_i : \{C_i \mathbf{d}_i \leq \mathbf{b}_i\}$$

Our feasible region is bounded by three constraints:

CT1 The linear constraint on the minimum illuminance of the desk

$$\mathbf{k}_i \mathbf{d}_o = L_i - o_i$$

CT2 The lower bound on the luminaire intensity (0)

$$d_{ii} = 0$$

CT3 The upper bound on the luminaire intensity (100)

$$d_{ii} = 100$$

So, the boundary of the feasible region is composed of the boundaries of the individual constraints CT1, CT2 and CT3 (hyperplanes of dimension $N-1$ - lines in a 2D problem) and the boundaries of the intersection of constraints CT1 with CT2, and CT1 with CT3 (hyperplanes of dimension $N-2$ - points in a 2D problem). Note that CT2 and CT3 are impossible to hold at the same time. Mathematically the boundary of $\mathcal{C}_i$ can then be defined by the union of these five sets intersected with the feasible region:

$$\partial \mathcal{C}_i = \bigcup_{j=1}^{5} \mathcal{S}_i^j \cap \mathcal{C}_i$$

where

$$\mathcal{S}_i^j : \left\{ A_i^j \mathbf{d}_i = \mathbf{u}_i^j \right\}$$

with

$$A_i^1 = -\mathbf{k}_i^T, \quad \mathbf{u}_i^1 = o_i - L_i$$
$$A_i^2 = -\mathbf{e}_i^T, \quad \mathbf{u}_i^2 = 0$$
$$A_i^3 = \mathbf{e}_i^T, \quad \mathbf{u}_i^3 = 100$$
$$A_i^4 = \begin{bmatrix} -\mathbf{k}_i^T \\ -\mathbf{e}_i^T \end{bmatrix}, \quad \mathbf{u}_i^4 = \begin{bmatrix} o_i - L_i \\ 0 \end{bmatrix}$$
$$A_i^5 = \begin{bmatrix} -\mathbf{k}_i^T \\ \mathbf{e}_i^T \end{bmatrix}, \quad \mathbf{u}_i^5 = \begin{bmatrix} o_i - L_i \\ 100 \end{bmatrix}$$

The strategy to find the solution is to compute the optimum $\mathbf{d}_i^j$ of the cost function subject to each of the sets $\mathcal{S}_i^j$, then check if the solution is feasible $C_i \mathbf{d}_i^j \leq \mathbf{b}_i$, and finally select the coordinate of minimum of the feasible optima $\mathbf{d}_i^j$.

5.1 Solution general form

Let us consider the general form of the solution to the problem:

$$
\begin{aligned}
&\mathbf{d}_i(t+1) = \underset{\mathbf{d}_i}{\operatorname{argmin}} \left\{ \tfrac{1}{2}\mathbf{d}_i^T R_i \mathbf{d}_i - \mathbf{d}_i^T \mathbf{z}_i(t) \right\} \\
&s.t. \\
&A_i \mathbf{d}_i = \mathbf{u}_i
\end{aligned}
$$

Let us form the Lagrangian:

$$
L(\mathbf{d}_i, \lambda) = \frac{1}{2}\mathbf{d}_i^T R_i \mathbf{d}_i - \mathbf{d}_i^T \mathbf{z}_i(t) + \lambda^{\mathbf{T}} \left(A_i \mathbf{d}_i - \mathbf{u}_i \right)
$$

The solution is obtained at a stationary point of the Lagrangian restricted to the constraint:

$$
\begin{cases}
\nabla L = 0 \\
A_i \mathbf{d}_i = \mathbf{u}_i
\end{cases}
$$

$$
\begin{cases}
R_i \mathbf{d}_i - \mathbf{z}_i(t) + A_i^T \lambda = 0 \\
A_i \mathbf{d}_i = \mathbf{u}_i
\end{cases}
$$

$$
\begin{bmatrix} R_i & A_i^T \\ A_i & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{d}_i \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{z}_i(t) \\ \mathbf{u}_i \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{d}_i \\ \lambda \end{bmatrix} = \begin{bmatrix} R_i & A_i^T \\ A_i & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z}_i(t) \\ \mathbf{u}_i \end{bmatrix}
$$

To compute the matrix inverse in the previous equation let us use the following lemma:

$$
\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - B^T A^{-1} B)^{-1} B^T A^{-1} & A^{-1}B(D - B^T A^{-1} B)^{-1} \\ (D - B^T A^{-1} B)^{-1} B^T A^{-1} & (D - B^T A^{-1} B)^{-1} \end{bmatrix}
$$

Replacing A by R_i, B by A_i^T, D by 0, and recalling that $P_i = R_i^{-1}$, we have:

$$
\begin{bmatrix} R_i & A_i^T \\ A_i & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} P_i - P_i A_i^T (A_i P_i A_i^T)^{-1} A_i P_i & P_i A_i^T (A_i P_i A_i^T)^{-1} \\ (A_i P_i A_i^T)^{-1} A_i P_i & -(A_i P_i A_i^T)^{-1} \end{bmatrix}
$$

Thus, $\mathbf{d}_i^*$ can be computed by:

$$
\mathbf{d}_i^*(t+1) = P_i \mathbf{z}_i(t) - P_i A_i^T (A_i P_i A_i^T)^{-1} A_i P_i \mathbf{z}_i(t) + P_i A_i^T (A_i P_i A_i^T)^{-1} \mathbf{u}_i
$$

Let:

$$
X_i = A_i P_i A_i^T
$$

$$
\mathbf{w}_i(t) = A_i P_i \mathbf{z}_i(t)
$$

Then we have the slightly simpler form:

$$
\mathbf{d}_i^*(t+1) = P_i \mathbf{z}_i(t) + P_i A_i^T X_i^{-1} (\mathbf{u}_i - \mathbf{w}_i(t))
$$

5.2 Solution in $\mathcal{S}_i^1$

$$A_i^1 = -\mathbf{k}_i^T, \quad \mathbf{u}_i^1 = o_i - L_i$$

$$A_i^1 P_i = \left[\begin{array}{ccccc} -\frac{k_{i1}}{\rho} & \cdots - \frac{k_{ii}}{\rho+q_i} & \cdots & -\frac{k_{iN}}{\rho} \end{array} \right]$$

$$\mathbf{w}_i^1(t) = -\frac{k_{i1}}{\rho} z_{i1}(t) - \cdots - \frac{k_{ii}}{\rho + q_i} z_{ii}(t) - \cdots - \frac{k_{iN}}{\rho} z_{iN}(t)$$

$$X_i^1 = \frac{k_{i1}^2}{\rho} + \cdots + \frac{k_{ii}^2}{\rho + q_i} + \cdots + \frac{k_{iN}^2}{\rho}$$

The right hand side of the previous equation is the squared P_i norm of $\mathbf{k_i}$. Let us denote it by n_i:

$$n_i = \|\mathbf{k}_i\|_{P_i}^2 = \frac{k_{i1}^2}{\rho} + \cdots + \frac{k_{ii}^2}{\rho + q_i} + \cdots + \frac{k_{iN}^2}{\rho}$$

Thus

$$(X_i^1)^{-1} = \frac{1}{n_1}$$

Finally:

$$\mathbf{d}_i^1(t+1) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} - \frac{1}{n_i} \begin{bmatrix} \frac{k_{i1}}{\rho} \\ \vdots \\ \frac{k_{ii}}{\rho+q_i} \\ \vdots \\ \frac{k_{iN}}{\rho} \end{bmatrix} (o_i - L_i + \frac{k_{i1}}{\rho} z_{i1}(t) + \cdots + \frac{k_{ii}}{\rho + q_i} z_{ii}(t) + \cdots + \frac{k_{iN}}{\rho} z_{iN}(t))$$

5.3 Solution in $\mathcal{S}_i^2$

$$A_i^2 = -\mathbf{e}_i^T, \quad \mathbf{u}_i^2 = 0$$

$$A_i^2 P_i = \left[\begin{array}{cccc} 0 & \cdots - \frac{1}{\rho+q_i} & \cdots & 0 \end{array} \right]$$

$$\mathbf{w}_i^2(t) = -\frac{z_{ii}(t)}{\rho + q_i}$$

$$X_i^2 = \frac{1}{\rho + q_i}$$

$$(X_i^2)^{-1} = \rho + q_i$$

Finally:

$$\mathbf{d}_i^2(t+1) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ 0 \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix}$$

5.4 Solution in $\mathcal{S}_i^3$

$$A_i^3 = \mathbf{e}_i^T, \quad \mathbf{u}_i^3 = 100$$

$$A_i^3 P_i = \begin{bmatrix} 0 & \cdots \frac{1}{\rho+q_i} & \cdots & 0 \end{bmatrix}$$

$$\mathbf{w}_i^3(t) = \frac{z_{ii}(t)}{\rho+q_i}$$

$$X_i^3 = \frac{1}{\rho+q_i}$$

$$(X_i^3)^{-1} = \rho + q_i$$

Finally:

$$\mathbf{d}_i^3(t+1) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} (100 - \frac{1}{\rho+q_i} z_{ii}(t)) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ 100 \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix}$$

5.5 Solution in $\mathcal{S}_i^4$

$$A_i^4 = \begin{bmatrix} -\mathbf{k}_i^T \\ -\mathbf{e}_i^T \end{bmatrix} \quad \mathbf{u}_i^4 = \begin{bmatrix} o_i - L_i \\ 0 \end{bmatrix}$$

$$A_i^4 P_i = \begin{bmatrix} -\frac{k_{i1}}{\rho} & \cdots & -\frac{k_{ii}}{\rho+q_i} & \cdots & -\frac{k_{iN}}{\rho} \\ 0 & \cdots & -\frac{1}{\rho+q_i} & \cdots & 0 \end{bmatrix}$$

$$\mathbf{w}_i^4(t) = \begin{bmatrix} -\frac{k_{i1}}{\rho} z_{i1}(t) - \cdots - \frac{k_{ii}}{\rho+q_i} z_{ii}(t) - \cdots - \frac{k_{iN}}{\rho} z_{iN}(t) \\ -\frac{z_{ii}(t)}{\rho+q_i} \end{bmatrix}$$

$$X_i^4 = \begin{bmatrix} n_i & \frac{k_{ii}}{\rho+q_i} \\ \frac{k_{ii}}{\rho+q_i} & \frac{1}{\rho+q_i} \end{bmatrix}$$

$$(X_i^4)^{-1} = \frac{1}{\frac{n_i}{\rho+q_i} - \frac{k_{ii}^2}{(\rho+q_i)^2}} \begin{bmatrix} \frac{1}{\rho+q_i} & \frac{-k_{ii}}{\rho+q_i} \\ \frac{-k_{ii}}{\rho+q_i} & n_i \end{bmatrix}$$

$$(X_i^4)^{-1} = \frac{(\rho+q_i)^2}{n_i(\rho+q_i) - k_{ii}^2} \begin{bmatrix} \frac{1}{\rho+q_i} & \frac{-k_{ii}}{\rho+q_i} \\ \frac{-k_{ii}}{\rho+q_i} & n_i \end{bmatrix}$$

$$(X_i^4)^{-1} = \frac{\rho+q_i}{n_i(\rho+q_i) - k_{ii}^2} \begin{bmatrix} 1 & -k_{ii} \\ -k_{ii} & n_i(\rho+q_i) \end{bmatrix}$$

Let:

$$g_i = \frac{\rho+q_i}{n_i(\rho+q_i) - k_{ii}^2}$$

$$P_i(A_i^4)^T(X_i^4)^{-1} = g_i \begin{bmatrix} -\frac{k_{i1}}{\rho} & \frac{k_{i1}k_{ii}}{\rho} \\ \vdots & \vdots \\ 0 & \frac{k_{ii}^2}{\rho+q_1} - n_i \\ \vdots & \vdots \\ -\frac{k_{iN}}{\rho} & \frac{k_{iN}k_{ii}}{\rho} \end{bmatrix}$$

Recall that:

$$n_i = \frac{k_{i1}^2}{\rho} + \cdots + \frac{k_{ii}^2}{\rho+q_i} + \cdots + \frac{k_{iN}^2}{\rho}$$

Let:

$$\mathbf{k}_i^r = \begin{bmatrix} k_{i1} & \cdots & 0 & \cdots & k_{iN} \end{bmatrix}^T$$

We can write:

$$P_i(A_i^4)^T(X_i^4)^{-1} = \frac{g_i}{\rho} \begin{bmatrix} -k_{i1} & k_{i1}k_{ii} \\ \vdots & \vdots \\ 0 & -\|\mathbf{k}_i^r\|_2^2 \\ \vdots & \vdots \\ -k_{iN} & k_{iN}k_{ii} \end{bmatrix}$$

$$P_i(A_i^4)^T(X_i^4)^{-1}\mathbf{w}_i^4(t) = \frac{g_i}{\rho} \begin{bmatrix} \frac{k_{i1}k_{i1}z_{i1}(t)}{\rho} + \cdots + \frac{k_{i1}k_{ii}z_{ii}(t)}{\rho+q_i} + \cdots + \frac{k_{i1}k_{iN}z_{iN}(t)}{\rho} - \frac{k_{i1}k_{ii}z_{ii}(t)}{\rho} \\ \vdots \\ -\frac{\|\mathbf{k}_i^r\|_2^2 z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{k_{iN}k_{i1}z_{i1}(t)}{\rho} + \cdots + \frac{k_{iN}k_{ii}z_{ii}(t)}{\rho+q_i} + \cdots + \frac{k_{iN}k_{iN}z_{iN}(t)}{\rho} - \frac{k_{iN}k_{ii}z_{ii}(t)}{\rho} \end{bmatrix}$$

Let:

$$v_i(t) = \mathbf{z}_i^T(t)\mathbf{k}_i^r$$

We can write:

$$P_i(A_i^4)^T(X_i^4)^{-1}\mathbf{w}_i^4(t) = \frac{g_i}{\rho}\begin{bmatrix} \frac{k_{i1}}{\rho}v_i(t) \\ \vdots \\ -\frac{\|\mathbf{k}_i^r\|_2^2}{\rho+q_i}z_{ii}(t) \\ \vdots \\ \frac{k_{iN}}{\rho}v_i(t) \end{bmatrix}$$

$$P_i(A_i^4)^T(X_i^4)^{-1}\mathbf{u}_i(t) = \frac{g_i}{\rho}\begin{bmatrix} -k_{i1}(o_i - L_i) \\ \vdots \\ 0 \\ \vdots \\ -k_{iN}(o_i - L_i) \end{bmatrix}$$

Finally:

$$\mathbf{d}_i^4(t+1) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} + \frac{g_i}{\rho}\begin{bmatrix} k_{i1}(L_i - o_i) - \frac{k_{i1}}{\rho}v_i(t) \\ \vdots \\ \frac{\|\mathbf{k}_i^r\|_2^2}{\rho+q_i}z_{ii}(t) \\ \\ \vdots \\ k_{iN}(L_i - o_i) - \frac{k_{iN}}{\rho}v_i(t) \end{bmatrix}$$

5.6 Solution in $\mathcal{S}_i^5$

$$A_i^5 = \begin{bmatrix} -\mathbf{k}_i^T \\ \mathbf{e}_i^T \end{bmatrix} \quad \mathbf{u}_i^5 = \begin{bmatrix} o_i - L_i \\ 100 \end{bmatrix}$$

$$A_i^5 P_i = \begin{bmatrix} -\frac{k_{i1}}{\rho} & \cdots & -\frac{k_{ii}}{\rho+q_i} & \cdots & -\frac{k_{iN}}{\rho} \\ 0 & \cdots & \frac{1}{\rho+q_i} & \cdots & 0 \end{bmatrix}$$

$$\mathbf{w}_i^5(t) = \begin{bmatrix} -\frac{k_{i1}}{\rho}z_{i1}(t) - \cdots - \frac{k_{ii}}{\rho+q_i}z_{ii}(t) - \cdots - \frac{k_{iN}}{\rho}z_{iN}(t) \\ \frac{z_{ii}(t)}{\rho+q_i} \end{bmatrix}$$

$$X_i^5 = \begin{bmatrix} n_i & -\frac{k_{ii}}{\rho+q_i} \\ -\frac{k_{ii}}{\rho+q_i} & \frac{1}{\rho+q_i} \end{bmatrix}$$

$$(X_i^5)^{-1} = \frac{1}{\frac{n_i}{\rho+q_i} - \frac{k_{ii}^2}{(\rho+q_i)^2}}\begin{bmatrix} \frac{1}{\rho+q_i} & \frac{k_{ii}}{\rho+q_i} \\ \frac{k_{ii}}{\rho+q_i} & n_i \end{bmatrix}$$

$$(X_i^5)^{-1} = \frac{(\rho+q_i)^2}{n_i(\rho+q_i) - k_{ii}^2}\begin{bmatrix} \frac{1}{\rho+q_i} & \frac{k_{ii}}{\rho+q_i} \\ \frac{k_{ii}}{\rho+q_i} & n_i \end{bmatrix}$$

$$(X_i^5)^{-1} = \frac{\rho + q_i}{n_i(\rho + q_i) - k_{ii}^2} \begin{bmatrix} 1 & k_{ii} \\ k_{ii} & n_i(\rho + q_i) \end{bmatrix}$$

Let:

$$g_i = \frac{\rho + q_i}{n_i(\rho + q_i) - k_{ii}^2}$$

$$P_i(A_i^5)^T(X_i^5)^{-1} = g_i \begin{bmatrix} -\frac{k_{i1}}{\rho} & -\frac{k_{i1}k_{ii}}{\rho} \\ \vdots & \vdots \\ 0 & -\frac{k_{ii}^2}{\rho+q_1} + n_i \\ \vdots & \vdots \\ -\frac{k_{iN}}{\rho} & -\frac{k_{iN}k_{ii}}{\rho} \end{bmatrix}$$

Recall that:

$$n_i = \frac{k_{i1}^2}{\rho} + \cdots + \frac{k_{ii}^2}{\rho + q_i} + \cdots + \frac{k_{iN}^2}{\rho}$$

Let:

$$\mathbf{k}_i^r = \begin{bmatrix} k_{i1} & \cdots & 0 & \cdots & k_{iN} \end{bmatrix}^T$$

We can write:

$$P_i(A_i^5)^T(X_i^5)^{-1} = \frac{g_i}{\rho} \begin{bmatrix} -k_{i1} & -k_{i1}k_{ii} \\ \vdots & \vdots \\ 0 & \|\mathbf{k}_i^r\|_2^2 \\ \vdots & \vdots \\ -k_{iN} & -k_{iN}k_{ii} \end{bmatrix}$$

$$P_i(A_i^5)^T(X_i^5)^{-1}\mathbf{w}_i^5(t) = \frac{g_i}{\rho} \begin{bmatrix} \frac{k_{i1}k_{i1}z_{i1}(t)}{\rho} + \cdots + \frac{k_{i1}k_{ii}z_{ii}(t)}{\rho+q_i} + \cdots + \frac{k_{i1}k_{iN}z_{iN}(t)}{\rho} - \frac{k_{i1}k_{ii}z_{ii}(t)}{\rho} \\ \vdots \\ \frac{\|\mathbf{k}_i^r\|_2^2 z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{k_{iN}k_{i1}z_{i1}(t)}{\rho} + \cdots + \frac{k_{iN}k_{ii}z_{ii}(t)}{\rho+q_i} + \cdots + \frac{k_{iN}k_{iN}z_{iN}(t)}{\rho} - \frac{k_{iN}k_{ii}z_{ii}(t)}{\rho} \end{bmatrix}$$

Let:

$$v_i(t) = \mathbf{z}_i^T(t)\mathbf{k}_i^r$$

We can write:

$$P_i(A_i^5)^T(X_i^5)^{-1}\mathbf{w}_i^5(t) = \frac{g_i}{\rho} \begin{bmatrix} \frac{k_{i1}}{\rho} v_i(t) \\ \vdots \\ \frac{\|\mathbf{k}_i^r\|_2^2}{\rho+q_i} z_{ii}(t) \\ \vdots \\ \frac{k_{iN}}{\rho} v_i(t) \end{bmatrix}$$

$$P_i(A_i^5)^T(X_i^5)^{-1}\mathbf{u}_i(t) = \frac{g_i}{\rho}\begin{bmatrix} -k_{i1}(o_i - L_i) - 100k_{i1}k_{ii} \\ \vdots \\ 100\|\mathbf{k}_i^r\|_2^2 \\ \vdots \\ -k_{iN}(o_i - L_i) - 100k_{iN}k_{ii} \end{bmatrix}$$

Finally:

$$\mathbf{d}_i^5(t+1) = \begin{bmatrix} \frac{z_{i1}(t)}{\rho} \\ \vdots \\ \frac{z_{ii}(t)}{\rho+q_i} \\ \vdots \\ \frac{z_{iN}(t)}{\rho} \end{bmatrix} + \frac{g_i}{\rho}\begin{bmatrix} k_{i1}(L_i - o_i) - 100k_{i1}k_{ii} - \frac{k_{i1}}{\rho}v_i(t) \\ \vdots \\ 100\|\mathbf{k}_i^r\|_2^2 - \frac{\|\mathbf{k}_i^r\|_2^2}{\rho+q_i}z_{ii}(t) \\ \vdots \\ k_{iN}(L_i - o_i) - 100k_{iN}k_{ii} - \frac{k_{iN}}{\rho}v_i(t) \end{bmatrix}$$

References

[1] Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato and Jonathan Eckstein. Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers. *Foundations and Trends in Machine Learning, Vol. 3, No. 1 (2010) 1122.*