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Article *in* Optimization Methods and Software

DOI: 10.1080/10556780903448052

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Preprint ANL/MCS-P1243-0405

April 2005

[1]This work was supported by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Advanced Scientific Computing Research, Office of Science, U.S. Department of Energy, under Contract W-31-109-Eng-38. Any opinions or errors are the responsibility of the authors and not the sponsoring agency.

Solving Multi-Leader-Follower Games*

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April 22, 2005

Abstract

Multi-leader-follower games arise when modeling competition between two or more dominant firms and lead in a natural way to equilibrium problems with equilibrium constraints (EPECs). We examine a variety of nonlinear optimization and nonlinear complementarity formulations of EPECs. We distinguish two broad cases: problems where the leaders can cost-differentiate and problems with price-consistent followers. We demonstrate the practical viability of our approach by solving a range of medium-sized test problems.

Keywords: Nash games, Stackelberg games, nonlinear programming, nonlinear complementarity, NCP, MPEC, equilibrium problems with equilibrium constraints.

AMS-MSC2000: 90C30, 90C33, 90C55, 49M37, 65K10.

1 Introduction

Dominant firms in a market can exercise their power to manipulate the market to their own advantage. If all firms have the same market share, then the market can be modeled by a Nash game [23, 24], a noncooperative game in which all firms simultaneously compete against each other. When there is a single dominant firm, however, the market must be modeled as a Stackelberg (*single-leader-follower*) game [31, 20], in which the dominant firm, the *leader*, maximizes its profit subject to all other firms, the *followers*, being in a competitive equilibrium. Between these two extremes is the *multi-leader-follower* game that has multiple dominant firms and a number of followers. Multi-leader-follower games can be further differentiated into those in which the follower responses are constrained to be identical for each leader and those in which the followers are allowed to respond differently to each leader. We consider only the former, the *multi-leader-identical-follower* game. Problems of this type arise, for example, in the analysis of deregulated electricity markets [13, 4]. One formulation of multi-leader-follower games uses the novel modeling

*Preprint ANL/MCS-P1243-0405

paradigm of equilibrium problems with equilibrium constraints (EPECs). This paradigm was introduced in [27] and further developed in [7] in the context of modeling competition in European electricity markets. In our paper we describe practical approaches for solving EPECs and apply these techniques to several medium-sized problems.

Optimality conditions for EPECs are studied in [21] in the context of multiobjective optimization. Early algorithmic work on EPECs has focused on diagonalization techniques such as Gauss-Jacobi and Gauss-Seidel type methods. Such methods solve a cyclic sequence of single-leader-follower games until the decision variables of all leaders reach a fixed point. In [32], Su proposes a sequential nonlinear complementarity problem (NCP) approach for solving EPECs. The approach is related to the relaxation approach used in mathematical programs with equilibrium constraints (MPECs) [29] that relaxes the complementarity condition of each leader and drives the relaxation parameter to zero.

In this paper, we argue that EPECs should be solved by a single nonlinear optimization problem simultaneously for all leaders. Our approach exploits the insight gained from applying nonlinear solvers to MPECs. In particular, we derive an NCP formulation of the EPEC based on the equivalence between the Karush-Kuhn-Tucker (KKT) conditions of the MPEC and strong stationarity. This NCP formulation is analyzed further and we derive equivalent MPEC and nonlinear programming (NLP) formulations. We also introduce the notion of price consistency and show that it leads to a restricted square NCP that can be solved by applying standard NCP solvers.

This paper is organized as follows. The next section briefly reviews recent progress in solving Stackelberg games. Section 3 extends these ideas to multi-leader-follower games and introduces a new equilibrium concept. We also provide various formulations and show how equilibrium points can be computed reliably by solving nonlinear optimization problems. Section 4 introduces an alternative price-consistent formulation of the multi-leader-follower game that gives rise to a square complementarity problem. Section 5 explores the different formulations of equilibrium points and investigates the suitability of nonlinear solvers.

2 Single-Leader-Follower Games

In this section we briefly review some pertinent properties of single-leader-follower games. The Stackelberg game is played by a leader and a number of followers who compete noncooperatively. Given a strategy x of the leader, the ℓ followers choose their strategies such that

$$w_j^* \in \left\{ \begin{array}{ll} \underset{w_j \geq 0}{\operatorname{argmin}} & b_j(x, \hat{w}_j) \\ \text{subject to} & c_j(w_j) \geq 0 \end{array} \right\} \quad \forall j = 1, \ldots, \ell, \tag{2.1}$$

where $\hat{w}_j = (w_1^*, \ldots, w_{j-1}^*, w_j, w_{j+1}^*, \ldots, w_\ell^*)$. For fixed x, this problem is a Nash game. If (2.1) is convex, then the condition that the followers choose an optimal strategy can be written as an NCP parameterized by x:

$$\begin{array}{rcll} 0 \leq w_j & \perp & \nabla_{w_j} b_j(x, w) - \nabla_{w_j} c_j(w_j) z_j \geq 0 & \forall j = 1, \ldots, \ell \\ 0 \leq z_j & \perp & c_j(w_j) \geq 0 & \forall j = 1, \ldots, \ell, \end{array}$$

where the complementarity condition means that componentwise either the left or the right inequality is active. By defining variables $y_j = (w_j, z_j)$ and functions

$$h_j(x, y_j) = \begin{pmatrix} \nabla_{w_j} b_j(x, w) - \nabla_{w_j} c(w_j) z_j \\ c_j(w_j) \end{pmatrix}$$

and by introducing slack variables s, this NCP can be written as

$$\begin{array}{c} h(x, y) - s = 0 \\ 0 \le y \ \perp \ s \ge 0, \end{array} \tag{2.2}$$

where $h(x, y) = (h_1(x, y_1), \ldots, h_\ell(x, y_\ell))$.

Given that the followers are in equilibrium, the leader optimizes its objective $f(x, y)$ subject to its own set of constraints $g(x) \ge 0$ and the NCP (2.2):

$$\begin{array}{ll} \underset{x \ge 0, y, s}{\text{minimize}} & f(x, y) \\ \text{subject to} & g(x) \ge 0 \\ & h(x, y) - s = 0 \\ & 0 \le y \ \perp \ s \ge 0. \end{array} \tag{2.3}$$

Problem (2.3) is an MPEC. The left graph of Figure 1 shows an example of a Stackelberg game in which one large, dominant electricity producer acts as the leader, with a number of smaller producers acting as followers that play a Nash game with the independent system operator (ISO).

A number of limitations arise in writing a Nash game as complementarity constraints. If the followers' problems are not convex, then (2.1) and (2.2) are not equivalent. In fact, the solution set of (2.2) includes all stationary points of (2.1) and may lead the solver to a saddle point or a local maximum. Thus, a solution to (2.3) may not correspond to a solution to the Stackelberg game in these situations. We accept this limitation because it is not clear at present how this situation can be avoided in practice. Toward the end of this paper, we comment on how to interpret certain failures of the NLP approach in terms of the original problem.

The formulation we have presented does not allow the constraints of each follower to depend on the strategy chosen by either the leader or other followers so that standard notions of Nash and Stackelberg games are recovered. Generalized Nash and Stackelberg games are obtained when the constraints are allowed to depend on the choices made by the other players. Thus, $c(w)$ becomes $\tilde{c}(x, w)$, and $g(x)$ can also depend on the followers' responses becoming $\tilde{g}(x, w)$. In this case, the MPEC (2.3) becomes

$$\begin{array}{ll} \underset{x \ge 0, y, s}{\text{minimize}} & f(x, y) \\ \text{subject to} & \tilde{g}(x, y) \ge 0 \\ & \tilde{h}(x, y) - s = 0 \\ & 0 \le y \ \perp \ s \ge 0, \end{array} \tag{2.4}$$

where $\tilde{h}$ is defined to take $\tilde{c}(x, w)$ into account. Some of the test problems in Section 5 are derived from generalized Nash and Stackelberg games.

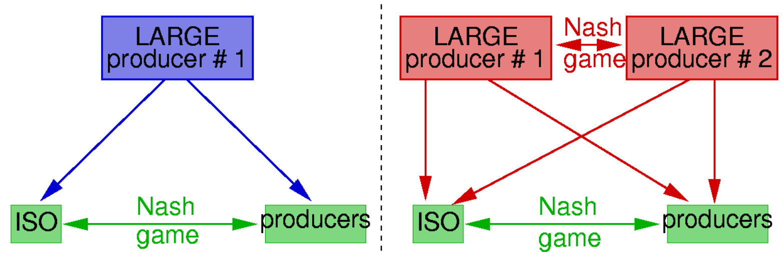

Figure 1: Structure of Stackelberg games and multi-leader multi-follower games

One attractive solution approach to (2.3) or (2.4) is to replace the complementarity condition by a nonlinear inequality, such as $y^T s \leq 0$ or $Ys \leq 0$, where Y is the diagonal matrix with y along its diagonal. This equivalent nonlinear program can then be solved by using standard NLP solvers. Unfortunately, this NLP violates the Mangasarian-Fromovitz constraint qualification (MFCQ) at *any* feasible point [29]. This failure of MFCQ implies that the multiplier set is unbounded, the central path fails to exist, the active constraint normals are linearly dependent, and linearizations can become inconsistent *arbitrarily close* to a solution [11]. In addition, early numerical experience with this approach had been disappointing [2]. As a consequence, solving MPECs as NLPs has been commonly regarded as numerically unsafe.

The failure of MFCQ in the equivalent NLP can be traced to the formulation of the complementarity constraint as $Ys \leq 0$. Consequently, algorithmic approaches have focused on avoiding this formulation. Instead, researchers have developed special-purpose algorithms for MPECs, such as branch-and-bound methods [2], implicit nonsmooth approaches [25], piecewise SQP methods [19], and perturbation and penalization approaches [6] analyzed in [30]. All of these techniques, however, require significantly more work than a standard NLP approach.

Recently, researchers have shown that MPECs can be solved reliably and efficiently [1, 3, 9, 11, 16, 17, 18, 28] using standard nonlinear optimization solvers by replacing the complementarity constraint with $Ys \leq 0$. The key observation in proving convergence of such an approach is that strong stationarity is equivalent to the KKT conditions of the equivalent NLP. We present strong stationarity in the context of the slightly more general MPEC (2.4) given by

$$\begin{array}{ll} \underset{x\geq 0,y,s}{\text{minimize}} & f(x,y) \\ \text{subject to} & g(x,y) \geq 0 \\ & h(x,y) - s = 0 \\ & y \geq 0,\ s \geq 0,\ Ys \leq 0. \end{array} \tag{2.5}$$

We will use strong stationarity to derive nonlinear formulations of EPECs. Strong stationarity is introduced in [29] and can be defined as follows.

Definition 2.1 *A point (x, y, s) is called a strongly stationary point of the MPEC (2.4)* *if and only if there exist multipliers* $\chi \geq 0, \lambda \geq 0, \mu, \psi, \sigma$ *such that*

$$\begin{array}{r} \nabla_x f(x,y) - \nabla_x g(x,y)\lambda - \nabla_x h(x,y)\mu - \chi = 0 \\ \nabla_y f(x,y) - \nabla_y g(x,y)\lambda - \nabla_y h(x,y)\mu - \psi = 0 \\ \mu - \sigma = 0 \\ g(x,y) \geq 0 \\ h(x,y) - s = 0 \\ x, y, s \geq 0 \\ x^T\chi = 0 \;\; \mathit{and} \;\; g(x,y)^T\lambda = 0 \;\; \mathit{and} \;\; y^T\psi = 0 \;\; \mathit{and} \;\; s^T\sigma = 0 \\ \mathit{if} \;\; y_i = s_i = 0 \;\; \mathit{then} \;\; \psi_i, \; \sigma_i \geq 0 \end{array} \quad . \tag{2.6}$$

The multipliers on the simple bounds ψ_i and σ_i are nonnegative only if $y_i = s_i = 0$, as is consistent with the observation that $y_i > 0$ implies $s_i = 0$, and σ_i is therefore the multiplier on an equality constraint whose sign is not restricted.

Fletcher et al. [11] have shown that strong stationarity is equivalent to the KKT conditions of the equivalent NLP (2.5). That is, there exist multipliers $\chi \geq 0, \lambda \geq 0, \mu$, $\psi \geq 0, \sigma \geq 0, \xi \geq 0$ such that

$$\begin{array}{r} \nabla_x f(x,y) - \nabla_x g(x,y)\lambda - \nabla_x h(x,y)\mu - \chi = 0 \\ \nabla_y f(x,y) - \nabla_y g(x,y)\lambda_y - \nabla h(x,y)\mu - \psi + S\xi = 0 \\ \mu - \sigma + Y\xi = 0 \\ 0 \leq g(x,y) \;\perp\; \lambda \geq 0 \\ h(x,y) - s = 0 \\ 0 \leq x \;\perp\; \chi \geq 0 \\ 0 \leq y \;\perp\; \psi \geq 0 \\ 0 \leq s \;\perp\; \sigma \geq 0 \\ 0 \leq -Ys \;\perp\; \xi \geq 0 \end{array} \quad . \tag{2.7}$$

One can show that the multipliers of the equivalent NLP (2.5) form a ray and that SQP methods converge to a minimum norm multiplier corresponding to the base of the ray [11]. The aim of this paper is to demonstrate that strong stationarity can be applied within the context of multi-leader-follower games to define equilibrium points, thus making EPECs amenable to approaches based on nonlinear optimization or nonlinear complementarity.

We conclude this section by recalling the definition of an MPEC constraint qualification.

Definition 2.2 *The MPEC (2.4) satisfies an MPEC linear independence constraint qualification (MPEC-LICQ), if the NLP (2.5) without the complementarity condition* $Ys \leq 0$ *satisfies an LICQ.*

3 Multi-Leader-Follower Games

Multi-leader-follower games arise when two or more Stackelberg players with identical followers compete noncooperatively. The right graph of Figure 1 shows an example of a multi-leader-follower game in which two large electricity producers act as leaders, with a number of smaller producers acting as followers that play a Nash game with the independent system operator. Such games can be modeled as equilibrium problems with equilibrium constraints. The aim is to find an equilibrium point where no player can improve the objective without degrading the objective of at least one other player. This goal is achieved by extending strong stationarity (Definition 2.1) to equilibrium problems with equilibrium constraints.

We let $k > 1$ be the number of leaders and denote by $x_i,\ i = 1, \ldots, k$, the decision variables for leader i. The leader variables are abbreviated by $x = (x_1, \ldots, x_k)$. The Stackelberg game played by leader i gives rise to the following MPEC:

$$\begin{array}{ll} \underset{x_i \geq 0, y, s}{\text{minimize}} & f_i(\hat{x}_i, y) \\ \text{subject to} & g_i(\hat{x}_i, y) \geq 0 \\ & h(\hat{x}_i, y) - s = 0 \\ & 0 \leq y \ \perp \ s \geq 0, \end{array}$$

where $\hat{x}_i = (x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_k^*)$. The multi-leader-follower game is defined as a solution to the following collection of MPECs:

$$(x_i^*, y^*, s^*) \in \left\{ \begin{array}{ll} \underset{x_i \geq 0, y, s}{\text{argmin}} & f_i(\hat{x}_i, y) \\ \text{subject to} & g_i(\hat{x}_i, y) \geq 0 \\ & h(\hat{x}_i, y) - s = 0 \\ & 0 \leq y \ \perp \ s \geq 0 \end{array} \right\} \ \forall i = 1, \ldots, k. \tag{3.1}$$

One attractive way to attack the multi-leader-follower game (3.1) is to follow the same formalism as in the derivation of the Nash game in the previous section. Formally, we replace each MPEC in (3.1) by its strong stationarity conditions and concatenate the equivalent KKT conditions (2.7) for all leaders $i = 1, \ldots, k$. This approach formulates the multi-leader-follower game as a nonlinear complementarity problem. A range of alternative formulations as nonlinear programming problems can also be found. We start by describing the derivation of NCP formulations.

3.1 NCP Formulations of Multi-Leader-Follower Games

The concatenation of the strong stationarity conditions for each leader produces the following NCP formulation of the multi-leader-follower game (3.1).

$$\nabla_{x_i} f_i(x, y) - \nabla_{x_i} g_i(x, y)\lambda_i - \nabla_{x_i} h(x, y)\mu_i - \chi_i = 0 \quad \forall i = 1, \ldots, k \tag{3.2a}$$
$$\nabla_y f_i(x, y) - \nabla_y g_i(x, y)\lambda_i - \nabla_y h(x, y)\mu_i - \psi_i + S\xi_i = 0 \quad \forall i = 1, \ldots, k \tag{3.2b}$$
$$\mu_i - \sigma_i + Y\xi_i = 0 \quad \forall i = 1, \ldots, k \tag{3.2c}$$
$$0 \leq g_i(x, y) \ \perp \ \lambda_i \geq 0 \quad \forall i = 1, \ldots, k \tag{3.2d}$$
$$h(x, y) - s = 0 \tag{3.2e}$$
$$0 \leq x_i \ \perp \ \chi_i \geq 0 \quad \forall i = 1, \ldots, k \tag{3.2f}$$
$$0 \leq y \ \perp \ \psi_i \geq 0 \quad \forall i = 1, \ldots, k \tag{3.2g}$$
$$0 \leq s \ \perp \ \sigma_i \geq 0 \quad \forall i = 1, \ldots, k \tag{3.2h}$$
$$0 \leq -Ys \ \perp \ \xi_i \geq 0 \quad \forall i = 1, \ldots, k \tag{3.2i}$$

The multipliers χ_i, ψ_i, and σ_i can be eliminated from the model. Moreover, the multipliers of the followers' constraints, ψ_i, σ_i, and ξ_i, can be different for each Stackelberg player. This formulation corresponds to a scenario in which the cost of the followers' actions can be different for each leader, that is the leaders *cost differentiate*. In contrast, in Section 4, we discuss conditions on the structure of the game (3.1) that allow us to enforce a *price-consistent formulation* in which μ_i, ψ_i, σ_i, and ξ_i are the same for every leader.

Formally, this approach to EPECs is analogous to the formulation of the Nash game (2.1) as a complementarity problem (2.2). Unlike a Nash game, however, the MPEC (2.3) is always nonconvex because of the presence of the complementarity constraint. Therefore, no simple equivalence exists between the solution set of the NCP defined by (3.2) and the solution set of the multi-leader-follower game (3.1). Moreover, this NCP is typically not square because the equations (3.2a), (3.2b), (3.2c), and (3.2e) cannot be uniquely matched to the free variables x_i, y, s, and μ_i, making it harder to solve than standard Nash games.

In [32], (3.2) is solved by adding a smoothing parameter to the original complementarity condition, replacing (3.2i) by

$$-te \leq -Ys \;\perp\; \xi_i \geq 0,$$

where $t \searrow 0$, and $e = (1, \ldots, 1)^T$. In contrast, we attack (3.2) directly by exploiting the recent advances in nonlinear solvers for MPECs.

We can simplify (3.2) by noting that the complementarity condition in (3.2i) always holds because y and s are nonnegative. In addition, the constraint $Ys \leq 0$ can be replaced by $0 \leq y \;\perp\; s \geq 0$. This formulation has the advantage that it makes the complementarity constraint transparent for the nonlinear solver, allowing, for example, different techniques to deal with the complementarity condition. Thus, we can replace the constraints (3.2g)–(3.2i) by the following set of inequalities:

$$\left.\begin{array}{rcr} 0 \leq y & \perp & \psi_i \geq 0 \\ 0 \leq s & \perp & \sigma_i \geq 0 \\ 0 \leq y & \perp & s \geq 0 \\ & & \xi_i \geq 0 \end{array}\right\} \forall i = 1, \ldots, k.$$

An alternative formulation is to replace (3.2g)–(3.2i) with the equivalent set of conditions:

$$\left.\begin{array}{rcr} 0 \leq \psi_i + s & \perp & y \geq 0 \\ 0 \leq \sigma_i + y & \perp & s \geq 0 \\ & & \psi_i, \sigma_i, \xi_i \geq 0 \end{array}\right\} \forall i = 1, \ldots, k.$$

Another source of degeneracy is the fact that the multipliers form a ray and are therefore not unique. This redundancy can be removed by adding a complementarity constraint that forces the multipliers of each Stackelberg game to be basic (and therefore unique), such as

$$0 \leq \psi_i + \sigma_i \;\perp\; \xi_i \geq 0.$$

Combining these observations, we arrive at the following NCP formulation of a multi-leader-follower game.

$$
\begin{aligned}
\nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i &= 0 \quad \forall i=1,\ldots,k && \text{(3.3a)}\\
\nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_i - \nabla_y h(x,y)\mu_i - \psi_i + S\xi_i &= 0 \quad \forall i=1,\ldots,k && \text{(3.3b)}\\
\mu_i - \sigma_i + Y\xi_i &= 0 \quad \forall i=1,\ldots,k && \text{(3.3c)}\\
0 \le g_i(x,y) \ \perp\ \lambda_i &\ge 0 \quad \forall i=1,\ldots,k && \text{(3.3d)}\\
h(x,y) - s &= 0 && \text{(3.3e)}\\
0 \le x_i \ \perp\ \chi_i &\ge 0 \quad \forall i=1,\ldots,k && \text{(3.3f)}\\
0 \le \psi_i + s \ \perp\ y &\ge 0 \quad \forall i=1,\ldots,k && \text{(3.3g)}\\
0 \le \sigma_i + y \ \perp\ s &\ge 0 \quad \forall i=1,\ldots,k && \text{(3.3h)}\\
0 \le \psi_i + \sigma_i \ \perp\ \xi_i &\ge 0 \quad \forall i=1,\ldots,k && \text{(3.3i)}
\end{aligned}
$$

This formulation is not a square NCP because y and s are matched with multiple inequalities in (3.3g) and (3.3h), respectively.

The derivation in this section motivates the following definition. A similar definition regarding (3.2) can be found in [32].

Definition 3.1 *A solution of (3.1) is called an equilibrium point of the multi-leader-follower game. A solution* $(x^*, y^*, s^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ *of (3.2) or (3.3) is called a strongly stationary point of the multi-leader-follower game (3.1).*

The following proposition shows that equilibrium points are strongly stationary provided an MPEC-LICQ holds.

Proposition 3.1 *If* (x^*, y^*, s^*) *is an equilibrium point of (3.1) and if every MPEC of (3.1) satisfies an MPEC-LICQ, then there exist multipliers* $(\chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ *such that (3.2) and (3.3) hold.*

Proof. The statement follows directly from [29] and can also be found in [32]. If MPEC-LICQ holds, then there exist multipliers for every leader's Stackelberg game, and (3.2) and (3.3) follow. □

We note that both (3.2) and (3.3) are degenerate in the sense that the constraints violate any constraint qualification because of the presence of $Ys \le 0$. In addition, the Jacobian is singular whenever any component of both y and s is zero. This fact makes it difficult to tackle (3.3) with standard NCP solvers. In the next section we derive more robust formulations of the NCP that resolve the redundancy in (3.2) and can be solved by using standard nonlinear optimization techniques.

3.2 NLP Formulations of Multi-Leader-Follower Games

The redundancy inherent in the NCP formulation (3.2) of the previous section can be exploited to derive nonlinear programming formulations of the multi-leader-follower game. This section introduces two other formulations of the NCP (3.2). The first formulation is based on the idea of forcing the EPEC to identify the basic or minimal multiplier for each Stackelberg player. This formulation results in an MPEC. The second formulation penalizes the complementarity constraints and results in a well-behaved nonlinear optimization problem.

One difficulty with the NCP (3.2) is the existence of an infinite number of multipliers. Since the multipliers form a ray, however, there exists a minimum norm multiplier [11]. The first reformulation aims to find this particular multiplier by minimizing the ℓ_1-norm of the multiplier on the complementarity constraint, giving rise to the following MPEC:

$$
\begin{array}{lll}
\underset{x,y,s,\lambda,\mu,\psi,\sigma,\xi}{\text{minimize}} & \displaystyle\sum_{i=1}^{k} e^T\xi_i & \\
\text{subject to} & \nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 & \forall i = 1,\ldots,k \\
& \nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_i - \nabla_y h(x,y)\mu_i - \psi_i + S\xi_i = 0 & \forall i = 1,\ldots,k \\
& \mu_i - \sigma_i + Y\xi_i = 0 & \forall i = 1,\ldots,k \\
& 0 \le g_i(x,y) \ \perp\ \lambda_i \ge 0 & \forall i = 1,\ldots,k \\
& h(x,y) - s = 0 & \\
& 0 \le x_i \ \perp\ \chi_i \ge 0 & \forall i = 1,\ldots,k \\
& 0 \le y \ \perp\ \psi_i \ge 0 & \forall i = 1,\ldots,k \\
& 0 \le s \ \perp\ \sigma_i \ge 0 & \forall i = 1,\ldots,k \\
& 0 \le y \ \perp\ s \ge 0 & \\
& \xi_i \ge 0 & \forall i = 1,\ldots,k.
\end{array}
\tag{3.4}
$$

This reformulation violates standard constraint qualifications because it is an MPEC. However, recent developments show that MPECs like (3.4) can be solved reliably and efficiently by using standard NLP solvers [1, 10, 11]. We could also use the bounded multiplier reformulations of the complementarity constraints in the MPEC.

The next formulation aims to avoid this difficulty by minimizing the complementarity constraints. This formulation of the multi-leader-follower game follows an idea of Moré [22] and minimizes the complementarity conditions in (3.2d) and (3.2f)–(3.3i). After introducing slacks t_i to $g_i(x,y) \geq 0$, one can write this problem as follows:

$$\begin{array}{lll}
\underset{x,y,\nu,\mu,\xi}{\text{minimize}} & C_{pen} := \displaystyle\sum_{i=1}^{k} \left(x_i^T \chi_i + t_i^T \lambda_i + y^T \psi_i + s^T \sigma_i \right) + y^T s & \\
\text{subject to} & \nabla_{x_i} f_i(x,y) - \nabla_{x_i} g_i(x,y)\lambda_i - \nabla_{x_i} h(x,y)\mu_i - \chi_i = 0 & \forall i = 1,\ldots,k \\
& \nabla_y f_i(x,y) - \nabla_y g_i(x,y)\lambda_y - \nabla_i h(x,y)\mu_i - \psi_i + S\xi_i = 0 & \forall i = 1,\ldots,k \\
& \mu_i - \sigma_i + Y\xi_i = 0 & \forall i = 1,\ldots,k \\
& g_i(x,y) = t_i & \forall i = 1,\ldots,k \\
& h(x,y) = s & \\
& y \geq 0,\ s \geq 0 & \\
& \chi_i \geq 0,\ \lambda_i \geq 0,\ \psi_i \geq 0,\ \sigma_i \geq 0,\ \xi_i \geq 0 & \forall i = 1,\ldots,k \\
& x_i \geq 0,\ t_i \geq 0 & \forall i = 1,\ldots,k.
\end{array} \tag{3.5}$$

In this problem, the complementarity conditions have been moved into the objective by a penalty approach, and the remaining constraints are well-behaved. A penalty parameter of one is always adequate because the multi-leader-follower game has no objective function. We could also use the bounded multiplier reformulations of the complementarity constraints when deriving an NLP.

The following theorem summarizes the properties of the formulations introduced in the section.

Theorem 3.1 *If $(x^*, y^*, s^*, t^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ is a local solution of (3.4), then it follows that (x^*, y^*, s^*) is a strongly stationary point of the multi-leader-follower game (3.1). If $(x^*, y^*, s^*, t^*, \chi^*, \lambda^*, \mu^*, \psi^*, \sigma^*, \xi^*)$ is a local solution of (3.5) with $C_{pen} = 0$, then it follows that (x^*, y^*, s^*) is a strongly stationary point of the multi-leader-follower game (3.1).*

Proof. The proof follows directly from the developments above. □

4 Price-Consistent Formulations

For generalized Nash games, the constraints $c_j(x, \hat{w}_j)$ are typically identical functions. In this case, the multi-leader-follower game can be reduced to have only one copy of the NCP constraint (2.2). The first-order conditions for the generalized Nash game would be non-square in this case, however, because some multipliers cannot be matched to constraints. If we further assume the multipliers on identical joint constraints are the same, that is, the shadow prices are set by an independent entity that cannot price discriminate, then we need only one multiplier z, instead of one for each follower. We refer to problems having both modifications as being *price consistent*. Price-consistent generalized Nash games give rise to square complementarity problems and can be solved with standard NCP solvers.

The ideal generalized Nash game parameterized by x that we consider is

$$w_j^* \in \left\{ \begin{array}{ll} \underset{w_j \geq 0}{\operatorname{argmin}} & b_j(x, \hat{w}_j) \\ \text{subject to} & c(x, \hat{w}_j) \geq 0 \end{array} \right\} \quad \forall j = 1, \ldots, \ell, \tag{4.1}$$

where $\hat{w}_j = (w_1^*, \ldots, w_{j-1}^*, w_j, w_{j+1}^*, \ldots, w_\ell^*)$ as before and where we assume $c_j(x, \hat{w}_j) = c(x, \hat{w}_j)$ for all j. Price consistency is achieved by removing the joint constraints by introducing another set of players that choose the multipliers on the constraints and modifying the objective function for each original player. We refer to the players as the *independent system operators*. The standard Nash game parameterized by x after these changes are made is given by

$$\begin{aligned} w_j^* &\in \left\{ \underset{w_j \geq 0}{\operatorname{argmin}} \; b_j(x, \hat{w}_j) - c(x, \hat{w}_j)^T z_j^* \right\} \quad \forall j = 1, \ldots, \ell, \\ z_j^* &\in \left\{ \underset{z_j \geq 0}{\operatorname{argmin}} \; c(x, w^*)^T z_j \right\} \quad \forall j = 1, \ldots, \ell. \end{aligned} \tag{4.2}$$

The first-order conditions of (4.1) and (4.2) are identical. The independent system operators are allowed to discriminate in the price quoted to each player.

The price-consistent assumption states that the independent system operators must select a set of consistent prices for all original players, producing the restricted problem

$$\begin{aligned} w_j^* &\in \left\{ \underset{w_j \geq 0}{\operatorname{argmin}} \; b_j(x, \hat{w}_j) - c(x, \hat{w}_j)^T z^* \right\} \quad \forall j = 1, \ldots, \ell, \\ z^* &\in \left\{ \underset{z \geq 0}{\operatorname{argmin}} \; c(x, w^*)^T z \right\}, \end{aligned} \tag{4.3}$$

which contains only one set of multipliers and constraints; all other multipliers and constraints have been removed. A solution to the restricted problem is a critical point for the original problem.

A multi-leader-follower game is a generalized Nash game with shared decision variables because the follower needs to make an identical decision for each leader. These shared decision variables cause complications when demanding price consistency. We make the following assumptions on the structure of the multi-leader-follower game that will enable us to apply the price consistent restriction:

[**A1**] Leader i's general constraints are independent of the followers' decision variables; in other words, leader i's general constraints are of the form $g_i(x_i) \geq 0$.

[**A2**] Leader i's objective function is separable in x and y and is of the form $f_i(x, y) = f_i(x) + \tilde{f}(y)$.

Assumption [**A2**] enables us to add a third player that chooses values for the shared decision variables. This assumption can be relaxed to $\nabla_y f_i(x, y) = \nabla_y f_j(x, y)$ for all i and j.

Therefore, our idealized prototype for the multi-leader-follower game is of the form

$$(x_i^*, y^*, s^*) \in \left\{ \begin{array}{ll} \operatorname*{argmin}_{x_i \geq 0, y, s} & f_i(\hat{x}_i) + \tilde{f}(y) \\ \text{subject to} & g_i(x_i) \geq 0 \\ & h(\hat{x}_i, y) - s = 0 \\ & 0 \leq y \perp s \geq 0 \end{array} \right\} \forall i = 1, \ldots, k.$$

As before, we remove the joint NCP constraints by introducing an independent system operator and apply the price-consistent restriction to produce the following game.

$$\begin{array}{l} (x_i^*, y^*, s^*) \in \left\{ \begin{array}{ll} \operatorname*{argmin}_{x_i \geq 0, y, s} & f_i(\hat{x}_i) + \tilde{f}(y) \\ & -\left(h(\hat{x}_i, y) - s\right)^T \mu^* \\ & -y^T \psi^* - s^T \sigma^* + s^T Y \xi^* \\ \text{subject to} & g_i(x_i) \geq 0 \end{array} \right\} \quad \forall i = 1, \ldots, k, \\ \\ (\mu^*, \psi^*, \sigma^*, \xi^*) \in \left\{ \operatorname*{argmin}_{\mu, \psi \geq 0, \sigma \geq 0, \xi \geq 0} (h(x^*, y^*) - s^*)^T \mu + (y^*)^T \psi + (s^*)^T \sigma - (s^*)^T Y^* \xi \right\} \end{array} \tag{4.4}$$

This modification does not eliminate the shared decision variables. We now exploit the separability assumption on the objective functions, [**A2**], to produce an alternative game that contains an independent system operator and an auxiliary player having control over the shared decision variables.

$$\begin{array}{l} x_i^* \in \left\{ \begin{array}{ll} \operatorname*{argmin}_{x_i \geq 0} & f_i(\hat{x}_i) - \left(h(\hat{x}_i, y^*) - s^*\right)^T \mu^* \\ \text{subject to} & g_i(x_i) \geq 0 \end{array} \right\} \quad \forall i = 1, \ldots, k, \\ \\ (y^*, s^*) \in \left\{ \operatorname*{argmin}_{y, s} \tilde{f}(y) - \left(h(x^*, y) - s\right)^T \mu^* - y^T \psi^* - s^T \sigma^* + s^T Y \xi^* \right\}, \\ \\ (\mu^*, \psi^*, \sigma^*, \xi^*) \in \left\{ \operatorname*{argmin}_{\mu, \psi \geq 0, \sigma \geq 0, \xi \geq 0} (h(x^*, y^*) - s^*)^T \mu + (y^*)^T \psi + (s^*)^T \sigma - (s^*)^T Y^* \xi \right\}. \end{array} \tag{4.5}$$

Problems (4.4) and (4.5) are equivalent because they have the same set of solutions; we have removed only redundant equations. However, the first-order conditions for the first problem are not square, while the latter produces a square complementarity problem. Applying the inverse operation to ψ, σ, and ξ leads to the equivalent formulation:

$$\begin{array}{l} x_i^* \in \left\{ \begin{array}{ll} \operatorname*{argmin}_{x_i \geq 0} & f_i(\hat{x}_i) - \left(h(\hat{x}_i, y^*) - s^*\right)^T \mu^* \\ \text{subject to} & g_i(x_i) \geq 0 \end{array} \right\} \quad \forall i = 1, \ldots, k, \\ \\ (y^*, s^*) \in \left\{ \begin{array}{ll} \operatorname*{argmin}_{y, s} & \tilde{f}(y) - \left(h(x^*, y) - s\right)^T \mu^* \\ \text{subject to} & 0 \leq y \perp s \geq 0 \end{array} \right\} \\ \\ \mu^* \in \left\{ \operatorname*{argmin}_{\mu} (h(x^*, y^*) - s)^T \mu \right\}. \end{array} \tag{4.6}$$

This development motivates the following definition.

Definition 4.1 *The game (4.6) is called a* price-consistent *multi-leader-follower game.*

Next, we derive the NCP formulation of the price-consistent multi-leader-follower game (4.6) as

$$\begin{array}{rl} \nabla_{x_i} f_i(x) - \nabla_{x_i} g_i(x_i)\lambda_i - \nabla_{x_i} h(x,y)\mu - \chi_i = 0 & \forall i = 1,\ldots,k \\ \nabla_y \tilde{f}(y) - \nabla_y h(x,y)\mu - \psi + S\xi = 0 & \\ \mu - \sigma + Y\xi = 0 & \\ 0 \le g_i(x_i) \perp \lambda_i \ge 0 & \forall i = 1,\ldots,k \\ h(x,y) - s = 0 & \\ 0 \le x_i \perp \chi_i \ge 0 & \forall i = 1,\ldots,k \\ 0 \le y \perp \psi \ge 0 & \\ 0 \le s \perp \sigma \ge 0 & \\ 0 \le -Ys \perp \xi \ge 0 \quad . & \end{array} \tag{4.7}$$

Other versions of this NCP can also be posed. For example, the bounded multiplier version would be

$$\begin{array}{rl} \nabla_{x_i} f_i(x) - \nabla_{x_i} g_i(x_i)\lambda_i - \nabla_{x_i} h(x,y)\mu - \chi_i = 0 & \forall i = 1,\ldots,k \\ \nabla_y \tilde{f}(y) - \nabla_y h(x,y)\mu - \psi + S\xi = 0 & \\ \mu - \sigma + Y\xi = 0 & \\ 0 \le g_i(x_i) \perp \lambda_i \ge 0 & \forall i = 1,\ldots,k \\ h(x,y) - s = 0 & \\ 0 \le x_i \perp \chi_i \ge 0 & \forall i = 1,\ldots,k \\ 0 \le \psi + s \perp y \ge 0 & \\ 0 \le \sigma + y \perp s \ge 0 & \\ 0 \le \psi + \sigma \perp \xi \ge 0 \quad . & \end{array} \tag{4.8}$$

Both problems are square NCPs without side constraints. Moreover, ψ and σ can be eliminated to produce a reduced model.

By making a further simplifying assumption on the underlying model, we can establish an interesting relationship between price consistency and a multiobjective optimization problem. We start by defining *complete separability.*

Definition 4.2 *We say that the multi-leader-follower game (3.1) is* completely separable *if the constraints are separable, that is, $g_i(\hat{x}_i, y) = g_i(x_i)$, and the objective is separable, that is, $f_i(\hat{x}_i, y) = \bar{f}_i(x_i) + \tilde{f}(y)$ for every leader $i = 1, \ldots, k$.*

The following proposition relates price-consistent multi-leader-follower games to a standard MPEC.

Proposition 4.1 *Assume that the multi-leader-follower game is completely separable. Then it follows that the first-order conditions (4.7) of the game (4.6) are equivalent to the strong stationarity conditions of the following MPEC:*

$$
\begin{array}{lll}
\underset{x \geq 0}{\text{minimize}} & \displaystyle\sum_{i=1}^{k} \bar{f}_i(x_i) + \tilde{f}(y) & \\
\text{subject to} & g_i(x_i) \geq 0 & \forall i = 1, \ldots, k \\
& h(x, y) - s = 0 & \\
& 0 \leq y \ \perp \ s \geq 0. &
\end{array}
\tag{4.9}
$$

Proof. The proof follows by comparing the strong-stationarity conditions of the MPEC (4.9) with the first-order conditions of (4.7). □

Problem (4.9) minimizes the collective losses for the leaders and can be interpreted as finding a particular solution to a multiobjective optimization problem by minimizing a convex combination of all leaders' objectives. This observation is readily extended to multi-leader-follower games in which the leaders have common identical constraints of the form $c(x, y) \geq 0$.

As a consequence, existence results can be derived for completely separable EPECs by showing the existence of a solution to the price-restricted MPEC (4.9). This observation provides a starting point for deriving existence results for certain classes of EPECs.

By introducing the price-consistent restriction, we produce a model that may be easier to solve than the original. Because price consistency is a restriction, any solution to the restricted model is a solution to the unrestricted version. However, the restricted model may not have a solution, while the unrestricted model may have a solution as the following example illustrates.

The example, a generalized Nash game, shows the different possible results for the price-consistent formulation:

$$
\begin{array}{lll}
\underset{x_1}{\text{minimize}} & x_1^2 + a x_1 x_2 & \text{subject to } x_1 + x_2 = c \\
\underset{x_2}{\text{minimize}} & x_2^2 + b x_1 x_2 & \text{subject to } x_1 + x_2 = c,
\end{array}
$$

where a, b, and c are parameters. One can show that every point $(x_1, c - x_1)$ is an equilibrium point of the multi-leader-follower game (3.1). However, the price-consistent game can have zero, one, or an infinite number of solutions. In particular, the price-consistent game has the following:

1. A unique equilibrium if $a + b \neq 4$.

2. An infinite number of equilibria $(x_1, c - x_1, 2c)$ when $a = b = 2$.

3. An infinite number of equilibria $(x_1, c - x_1, 2x_1 - ax_1)$ when $a + b = 4$, $a \neq b$, and $c = 0$.

4. No equilibrium when $a + b = 4$, $a \neq b$, and $c \neq 0$.

In the last case, one player would make an infinite profit, while the other an infinite loss as x_1 goes to infinity and x_2 goes to minus infinity. Moreover, if $a = b$, the price-consistent game is computing a first-order critical point for the single optimization problem:

$$\underset{x_1,x_2}{\text{minimize}} \quad x_1^2 + x_2^2 + a x_1 x_2 \quad \text{subject to } x_1 + x_2 = c.$$

This problem is unbounded below but has the unique critical point $(\frac{c}{2}, \frac{c}{2}, c + \frac{ca}{2})$ corresponding to a maximizer when $a > 2$, an infinite number of solutions $(x_1, c - x_1, 2c)$ when $a = 2$, and a unique solution $(\frac{c}{2}, \frac{c}{2}, c + \frac{ca}{2})$ when $a < 2$.

5 Numerical Experience

This section provides some preliminary numerical experience in solving medium-scale EPECs with up to a few hundred variables. The numerical solution of EPECs is a novel area: there are no established test problem libraries and few numerical studies. We begin by describing the test problems and then provide a detailed comparison of our formulations with the diagonalization approach and the approach of [32]. All problems are available online at `http://www-unix.mcs.anl.gov/~leyffer/MacEPEC/`.

5.1 Description of Test Problems

The test problems fall into three broad classes: randomly generated problems, academic test problems, and a more realistic model that arose out of a case study of the interaction of electric power and NO_x allowance markets in the eastern United States [4].

The `AMPL` models of all test problems identify the NCPs (3.2) and (3.3), the MPEC (3.4), and the NLP (3.5) formulations as `*-NCP.mod, *-NCPa.mod, *-MPEC.mod`, and `*-NLP.mod`, respectively. The price-consistent models (4.7) and (4.8) are labeled `*-PC.mod` and `*-PCa.mod`. The diagonalization techniques are also implemented in `AMPL`: the Gauss-Jacobi iteration is identified by `*-GJ.mod` and the Gauss-Seidel iteration by `*-GS.mod`. Finally, the NCP smoothing technique of [32] is identified by the addition of `*-NCPt.mod`.

5.1.1 Randomly Generated EPECs

Randomly generated test problems are usually a poor substitute for numerical experiments. However, the fact that solving EPECs is a relatively new area means there are few realistic test problems. Thus, in order to demonstrate the efficiency of our approach on medium-sized problems, we decided to include results on randomly generated problems.

We have written a random EPEC generator in `matlab` that generates a random EPEC instance and writes the data to an `AMPL` file. Each leader is a quadratic program with equilibrium constraints and follows ideas from [12]. We note that [32] has a more sophisticated generator that follows ideas from [14]. Each leader $i = 1, \ldots, k$ has variables $x_i \in R^n$, where we assume for simplicity that all leaders have the same number of variables. Leader

i's problem is the QPEC

$$\begin{array}{ll} \underset{x_i \geq 0, y, s}{\text{minimize}} & \frac{1}{2} x^T G_i x + g_i^T x \\ \text{subject to} & b_i - A_i x_i \geq 0 \\ & Nz + My + q = s \\ & 0 \leq y \ \perp \ s \geq 0, \end{array}$$

where the data for leader i is given by the following randomly generated vectors and matrices: g_i, the objective gradient; G_i, the objective Hessian, a positive definite $nk \times nk$ matrix; A_i, the $p \times n$ constraint matrix on the controls x_i; and b_i, a random vector for which $b_i - A_i e \geq 0$. The data for the follower is given by N, an $m \times nk$ matrix; M, an $m \times m$ diagonally dominant matrix; and the vector q. These problems satisfy Assumptions [**A1**] and [**A2**], so a price-consistent version can be generated, but the constraints are not completely separable.

Table 1: Details of Randomly Generated Datasets

	Datasets		
Parameter	01–10	11–20	21–30
Number of leaders, l	2	2	4
Number of leader constraints, p	4	8	8
Number of follower variables, m	16	32	32
Number of leader variables, n	8	16	16
Coefficient range of A_i	$[-4, 4]$	$[-4, 4]$	$[-4, 4]$
Coefficient range of N	$[0, 8]$	$[0, 8]$	$[0, 8]$
Coefficient range of b	$[0, 8]$	$[0, 8]$	$[0, 8]$
Coefficient range of g_i	$[-6, 6]$	$[-6, 6]$	$[-6, 6]$
Coefficient range of q	$[-4, 4]$	$[-4, 4]$	$[-4, 4]$
Density of G	0.2	0.2	0.05
Density of A_i	0.4	0.4	0.2
Density of N	0.2	0.2	0.1
Density of M	0.2	0.2	0.1

The data is generated randomly from a uniform $(0, 1)$ distribution and is scaled and shifted to lie in a user-defined interval. The generated problems are sparse so that large-scale EPECs can be generated and solved. The AMPL model files are `EPEC-*.mod`. We have generated three datasets, each containing ten instances. The characteristics of each dataset is shown in Table 1. Note that the data is deliberately output in single precision because, in our experience, this heuristic usually increases degeneracy.

5.1.2 Academic Test Problems

ex-001 is a small EPEC having an equilibrium point. Assumption [**A2**] is satisfied by this problem, so price-consistent formulations exist. The `AMPL` models are `ex-001-*.mod`.

example-4 is a small EPEC from [27] similar to **ex-001**, but designed to illustrate a situation where each Stackelberg game has a solution but no solution exists for the

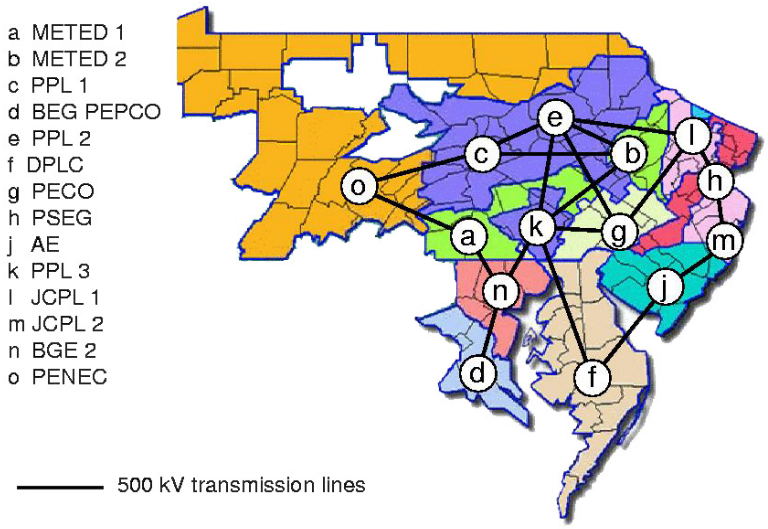

Figure 2: Network Topology of `electric-3.dat`

multi-leader-follower game. In particular, this problem is infeasible. Moreover, it violates Assumption [**A2**], so there is no square price-consistent model. The `AMPL` models are `ex-4-*.mod`.

outrata3 is an EPEC generated from the `outrata3*.mod` MPEC models [15, 26]. The control variables from all leaders enter the lower-level problem by averaging over the leaders. This trick ensures that the EPEC does not separate into individual MPECs. This problem has no solution and illustrates the behavior of the solvers when tackling a problem without solution. All Stackelberg players have the same constraints but different objective functions, so the problem violates Assumption [**A2**]. The `AMPL` models are `outrata3-*.mod`.

outrata4 is derived from **outrata3** so that the objective functions satisfy Assumption [**A2**]. A price-consistent solution to this model exists. The `AMPL` models are contained in `outrata4-*.mod`.

5.1.3 Electricity Market Models

This model is an electric power market example from [27]. It has two electricity firms (leaders) competing in a market. There is an arbitrager (follower) that exploits the price differential between regions and an independent system operator (ISO). However, unlike the formulation in [27], we enforce the response of the followers to be identical for all leaders.

Each leader maximizes profit subject to capacity constraints, the arbitrager's optimality conditions, and a market clearing condition. This optimization problem is an MPEC. The competition between leaders gives rise to a complementarity problem obtained by writing down the strong-stationariy conditions for each leader, the ISO's optimality conditions, and the market clearing condition.

The models are denoted by `electric-*.mod`. There are three data instances. The first, `electric-1.dat` is the small 3-node example from [27]. This model is infeasible because we enforce the response of the followers to be identical for all leaders. The next problem, `electric-2.dat`, is another small 3-node example that has a solution. The data for `electric-3.dat` is a larger 14-node example derived from real data of the PJM model [4]. The network of `electric-3.dat` is shown in Figure 2. The resulting multi-leader-follower game has two Stackelberg players. The 14-node example results in a game with approximately 150 constraints and 160 variables; the resulting NCP formulation has approximately 700 variables and constraints, making this the largest EPEC solved to date. There exists no price-consistent formulation for these problems because the objective functions of the leaders violate Assumption [**A2**].

5.2 Comparison of EPEC Solution Approaches

The different formulations give rise to problems of differing size, which are summarized in Tables 2 and 3. Each triple in these tables shows the number of variables, the number of constraints, and the number of complementarity conditions *after AMPL's presolve*, which may eliminate variables and constraints. The benefit of the diagonalization techniques is that they do not require gradients to be computed, a process that is potentially error prone. In addition, the resulting problems are much smaller. On the other hand, each sweep of a diagonalization technique solves k MPECs.

Table 2: Test Problem Sizes for Different Formulations

	Formulation			
Problem	NCP	NCPa	MPEC	NLP
ex-001	12/ 14/ 6	14/ 16/ 6	14/ 16/ 6	12/ 8/-
example-4	14/ 16/ 8	16/ 18/ 8	15/ 16/ 7	16/ 8/-
outrata3	76/100/ 52	76/ 100/ 52	80/ 92/ 40	84/ 52/-
outrata4	80/104/ 52	80/ 104/ 52	80/ 92/ 36	84/ 52/-
electric-1	153/180/110	228/ 255/110	165/168/ 86	225/117/-
electric-2	157/182/114	232/ 257/114	169/170/ 90	229/118/-
electric-3	734/806/560	1022/1094/560	806/806/488	990/502/-
random-[01-10]	193/199/ 76	204/ 231/ 97	193/199/ 76	208/120/-
random-[11-20]	375/379/137	393/ 443/183	375/379/137	416/240/-
random-[21-30]	567/494/112	592/ 642/235	567/494/112	750/472/-

We use `filterMPEC` [9], a sequential quadratic programming (SQP) method adapted to solving MPECs to solve the `AMPL` models `*-MPEC.mod, *-NCP.mod, *-NCPa.mod` and `*-NLP.mod`. In addition, we use the NCP solver `PATH` [5, 8], a generalized Newton method that solves a linear complementarity problem to compute the direction, for the price-consistent models. We also experimented with using `PATH` to solve the other NCP formulations, but our experience with these nonsquare and degenerate NCPs was rather disappointing.

Table 4 provides a comparison between the different solution approaches or EPECs. For the NCP/MPEC/NLP formulations of Section 3 we report the number of major (SQP) iterations. For the price-consistent NCP formulations of Section 4 we count the number of major iterations (roughly equivalent to an SQP iteration). For the sequential

Table 3: Test Problem Sizes for Different Formulations

Problem	Formulation NCP(t)	PC	PCa	GJ/GS
ex-001	12/ 14/ 6	8/ 8/ 3	8/ 8/ 3	3/ 2/ 1
example-4	14/ 16/ 8	n/a	n/a	4/ 3/ 2
outrata3	76/100/ 52	n/a	n/a	8/ 7/ 4
outrata4	80/104/ 52	28/ 28/ 16	28/ 28/ 16	8/ 7/ 4
electric-1	153/146/ 74	n/a	n/a	42/ 33/12
electric-2	157/148/ 78	n/a	n/a	45/ 34/12
electric-3	734/664/416	n/a	n/a	230/184/72
random-[01-10]	187/219/107	136/136/ 72	136/136/ 72	29/ 19/13
random-[11-20]	371/435/211	272/272/144	272/272/144	55/ 39/23
random-[21-30]	639/824/375	352/352/192	352/352/192	70/ 31/ 6

NCP approach we report the total number of major (SQP) iterations. Finally, for the diagonalization methods we sum the average number of major iterations to solve each MPEC during the Gauss-Jacobi or Gauss-Seidel process. We believe this sum provides an accurate picture of the relative performance of the diagonalization methods, which solve smaller (single leader) subproblems. The iteration counts for the randomly generated EPECs are averaged over the ten problem instances. Comparing CPU times would have been problematic, because some of the algorithms are implemented as AMPL scripts, which run significantly slower than Fortran or C.

The column headers in Table 4 refer to the problem name and the solution approach, where NCP, NCPa, MPEC, and NLP refer to formulations (3.2), (3.3), (3.4), and (3.5), respectively; PC and PCa refer to the price-consistent formulations (4.7) and (4.8); NCP(t) refers to the approach in [32] with the standard sequence of smoothing parameters $t = 1, 10^{-1}, \ldots, 10^{-8}$ (except that we may terminate early if the solution is complementary); and GJ and GS refer to the Gauss-Jacobi and Gauss-Seidel method, respectively.

Table 4: Comparison of Iteration Counts for EPEC Methods

Problem	Solution Methods NCP	NCPa	MPEC	NLP	NCP(t)	GJ	GS	PC	PCa
ex-001	1	3	1	1	19	3.5	3	2	2
example-4	7[I]	16[I]	10[I]	35[I]	80[I]	29.5[S]	28[S]	n/a	n/a
outrata3	28[I]	22[I]	22[I]	41[I]	129[I]	FAIL	15.75	n/a	n/a
outrata4	14	33	81	10	52	30.25	10.5	8	7
electric-1	43[I]	56[I]	59[I]	112[I]	225[I]	FAIL	3.5[S]	n/a	n/a
electric-2	9[I]	12[I]	15	17	37	91.5	32.0	n/a	n/a
electric-3	20	28	78	107	28	1.0	1.0	n/a	n/a
random[01-10]	113.8	31.3	13.0	2.1	642.2	22.7	13.85	8.4	8.2
random[11-20]	96.2	51.5	16.4	8.8	712.8	60.4	34.25	11.3	9.2
random[21-30]	17.5	21.1	17.1	3.8	184.7	17.3	10.95	10.1	8.3

We start by commenting on the ability of the formulation to detect infeasible problem instances. Convergence to locally infeasible points is indicated by [I] in Table 4. We note, that for `example-4-*`, `outrata3`, and `electric-1`, the NLP/NCP approaches successfully detect infeasibility. On the other hand, the diagonalization techniques converge to spurious stationary points (indicated by [S]), such as C- or M-stationary points, where one or both players have trivial descent directions. Thus, even though Gauss-Seidel and Gauss-Jacobi converge, the result is misleading because it does not correspond to a solution. This result is not surprising because the underlying MPEC solver is not guaranteed to converge to B-stationary points. Currently, no MPEC solver guarantees B-stationarity under reasonable assumptions.

In our view, one of the advantages of the NLP/NCP approach is its ability to identify infeasible problem instances. However, this process is not always guaranteed to be global, as `electric-2` indicates, where both NCP and NCPa erroneously conclude that the problem is infeasible.

Regarding the performance of the different solvers and formulations, there appears to be no clear overall winner, though the NLP approach is often the fastest. In particular, for the randomly generated problems, the NLP solver can be orders of magnitude faster than the other approaches.

We also note that the price-consistent formulations are very competitive when they exist and have a solution. This result indicates that more research is needed to identify robust formulations and solution tools for EPECs. A related open question concerns the "correct" formulation and presolve for NCPs. A bad formulation can often hide structure such as skew-symmetry that `PATH` can exploit during the solution.

When we compare the direct NCP approach (3.3) with the sequential NCP(t) approach, we observe that there is no benefit in smoothing the NCP formulation. The direct approach is typically an order of magnitude faster than the sequential approach. Contrary to intuition, the sequential NCP approach does not benefit from warm starts: each NCP takes a similar number of iterations as t is reduced. This observation is consistent with the situation in MPECs, where SQP methods are much faster than sequential relaxation approaches that solve one NLP per iteration for a decreasing sequence of regularization parameters. In our view, the only way in which the relaxation approach makes sense is if it is used in conjunction with inexact solves, such as in the context of interior-point methods [16, 28].

6 Conclusions

We have presented two novel approaches for solving multi-leader-follower games or EPECs. The first approach is based on the strong-stationarity conditions of each leader, and we derive a family of NCP, NLP, and MPEC formulations. The second approach to multi-leader-follower games imposes an additional restriction, called price consistency, that results in a square nonlinear complementarity problem. Both approaches allow the use of standard nonlinear optimization software to be extended to EPECs. In both approaches, also, the EPEC is solved by a single optimization problem, unlike traditional approaches that solve a sequence of related optimization problems.

We provide some preliminary numerical results demonstrating that our new approaches are competitive with existing methods in terms of both robustness and efficiency. In particular, the new approaches allow us to provide an indication when the multi-leader-follower game has no solution. In addition, the new approaches are shown to be more efficient than traditional diagonalization techniques.

A number of important open research questions remain. For example, all solution techniques for EPEC rely on efficient and robust MPEC solvers. To our knowledge, however, no MPEC solver can guarantee convergence to stationary points under reasonable conditions. Another open question is how to formulate and presolve NCPs, MPECs, and EPECs so that the solvers can take advantage of underlying structure such as skew symmetry.

Acknowledgments

This work was supported by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Advanced Scientific Computing Research, Office of Science, U.S. Department of Energy, under Contract W-31-109-ENG-38.

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