Discretization of the Advection Equation

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We consider the 1D linear advection equation:

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

Step 1: Discretize Space and Time

Let:

$$x_i = i\Delta x, \quad i = 0, 1, \ldots, N-1$$
$$t_j = j\Delta t, \quad j = 0, 1, 2, \ldots$$
$$u_i^j \approx u(x_i, t_j)$$

Step 2: Discretize the PDE

Using implicit (backward Euler) time discretization at time step $j+1$:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

Using second-order central difference in space at time $j+1$:

$$\frac{\partial u}{\partial x} \approx \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta x}$$

Plug into the PDE:

$$\frac{u_i^{j+1} - u_i^j}{\Delta t} + C \cdot \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta x} = 0$$

Multiply through by Δt:

$$u_i^{j+1} - u_i^j + \frac{C\Delta t}{2\Delta x}(u_{i+1}^{j+1} - u_{i-1}^{j+1}) = 0$$

Rewriting:

$$u_i^{j+1} + r(u_{i+1}^{j+1} - u_{i-1}^{j+1}) = u_i^j \quad \text{where } r = \frac{C\Delta t}{2\Delta x}$$

Step 3: Matrix Form

This leads to the tridiagonal system:

$$A\mathbf{u}^{j+1} = \mathbf{u}^j$$

Where $\mathbf{u}^j \in \mathbb{R}^N$ and the matrix $A \in \mathbb{R}^{N\times N}$ is:

$$A = \begin{bmatrix} 1 & r & 0 & \cdots & 0 \\ -r & 1 & r & \cdots & 0 \\ 0 & -r & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & r \\ 0 & 0 & 0 & -r & 1 \end{bmatrix}$$