Neural Network Approximation Theory

Introduction

This document delves into the theoretical capabilities of neural networks in approximating mathematical functions. The core idea is to explore whether a neural network can not only replicate a given function but also potentially enhance its performance.

Definitions

1. **Function** f: A predefined mathematical function that we aim to approximate.

2. **Function** g: A hypothetical function that outperforms f based on a specific performance metric.

3. **Neural Network** N: A feedforward neural network with parameters θ that aims to approximate f.

4. **Performance Metric** L: A metric (e.g., Mean Squared Error) that measures the difference between the outputs of a function and the true values.

Assumptions

1. N perfectly replicates f, i.e., for all inputs x,

$$N(x; \theta) = f(x)$$

2. There exists a g such that for all x,

$$L(g, x) < L(f, x)$$

Objective

To demonstrate that there exists a set of parameters θ' for which N not only approximates f but also has the potential to approximate g, thereby outperforming f.

Proof

1. **Neural Network's Approximation Ability**:

    - By the Universal Approximation Theorem, for any continuous function g and any $\epsilon > 0$, there exists a neural network N' and parameters θ' such that:

    $$|N'(x; \theta') - g(x)| < \epsilon$$

    for all x.

2. **Constructing a Composite Network**:

   - Define a new neural network N'' as a combination of N and N'. Specifically, for any input x, the output of N'' is:

   $$N''(x; \theta, \theta') = N(x; \theta) + N'(x; \theta')$$

3. **Performance Analysis**:

   - Given the assumption that N perfectly replicates f, for all x:

   $$N''(x; \theta, \theta') = N(x; \theta) + N'(x; \theta')$$

   - Using the approximation ability of N', we deduce:

   $$|N(x; \theta) + N'(x; \theta') - g(x)| < \epsilon$$

   This implies that N'' can potentially approximate g and thereby outperform f.

4. **Conclusion**:

   - Through the composite network N'', constructed from N and N', we've demonstrated the potential of neural networks to not only replicate but also enhance the performance of a given function f.