Path-dependent PDEs with signature kernels

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We are interested in developing a probabilistic numerical method based on the signature kernel to solve numerically linear (and non-linear) path-dependent PDEs. We explain the general idea looking at the example of the path-dependent heat equation. Then, if time allows, we will consider a linear path-dependent PDE arising from option pricing under rough volatility.

1 The signature kernel

In what follows $\mathcal{X} = \mathbb{R}^2$, but this applies to generic Polish spaces. Let $k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a characteristic kernel on $\mathcal{X}$ and for any $x \in \mathcal{X}$ denote by $k_x = k(x, \cdot)$ the corresponding feature map, which is injective and is an element of the RKHS $\mathcal{H}_k$.

Consider the set $C^1(\mathcal{X})$ of tree-reduced, continuous paths of bounded variation with values in $\mathcal{X}$ equipped with the quotient topology. For any path $X \in C^1(\mathcal{X})$ define the lifted $\mathcal{H}_k$-valued path $X^k \in C^1(\mathcal{H}_k)$ as $X^k_t = k_{X_t}$.

For any two paths $X, Y \in C^1(\mathcal{X})$ define the k-signature kernel as

$$k_{sig}^{X,Y}(s,t) = \left\langle S(X^k_{[0,s]}), S(Y^k_{[0,t]}) \right\rangle_{T(\mathcal{H}_k)} \tag{1}$$

where $\langle \cdot, \cdot \rangle_{T(\mathcal{H}_k)}$ is an inner product of choice on the tensor algebra $T(\mathcal{H}_k)$. For simplicity we consider the Euclidean inner product in $\mathbb{R}^2$, but other choices are possible, see [CLX21].

It was shown in [SCF+21] that the signature kernel in eq. (1) solves the following integral equation

$$k_{sig}^{X,Y}(s,t) = 1 + \int_0^s \int_0^t k_{sig}^{X,Y}(u,v) \langle dX^k_u, dY^k_v \rangle_{\mathcal{H}_k}$$

Using a first order approximation we get the following numerical scheme

$$\begin{aligned} k_{sig}^{X,Y}(s,t) &= 1 + \lim_{\Delta_s, \Delta_t \to 0} \sum_{i=0}^{N_s} \sum_{j=0}^{N_t} k_{sig}^{X,Y}(i\Delta_s, j\Delta_t) \langle X^k_{i\Delta_s} - X^k_{(i-1)\Delta_s}, Y^k_{j\Delta_t} - X^k_{(j-1)\Delta_t} \rangle_{\mathcal{H}_k} \\ &= 1 + \lim_{\Delta_s, \Delta_t \to 0} \sum_{i=0}^{N_s} \sum_{j=0}^{N_t} k_{sig}^{X,Y}(i\Delta_s, j\Delta_t) \Big(k(X_{i\Delta_s}, Y_{j\Delta_t}) - k(X_{(i-1)\Delta_s}, Y_{j\Delta_t}) \\ &\qquad - k(X_{i\Delta_s}, Y_{(j-1)\Delta_t}) + k(X_{(i-1)\Delta_s}, Y_{(j-1)\Delta_t}) \Big) \end{aligned}$$

where $N_s \Delta_s = s, N_t \Delta_t = t$ and where the last equality is due to the reproducing property of k, i.e.

$$\langle X^k_s, Y^k_t \rangle = \langle k_{X_s}, k_{Y_t} \rangle = k(X_s, Y_t).$$

This means that the lifted signature kernel k_{sig} can be computed via an iterative procedure using only evaluation of the original kernel k.

Directional derivatives of the signature kernel For any path $\boldsymbol{\gamma} \in \mathcal{X}$ the directional derivative of the (Euclidean) signature kernel along $\boldsymbol{\gamma}$ is defined as

$$k_{sig,\boldsymbol{\gamma}}^{X,Y} := \left(\frac{\partial}{\partial \epsilon} k_{sig}^{X+\epsilon\boldsymbol{\gamma},Y} \right)_{\epsilon=0}.$$

[LSC+21] show that the directional derivative $k_{sig,\boldsymbol{\gamma}}^{X,Y}$ solves the following integral equation

$$k_{sig,\boldsymbol{\gamma}}^{X,Y}(s,t) = \int_0^s \int_0^t \left(k_{sig,\boldsymbol{\gamma}}^{X,Y}(u,v) \left\langle dX_s^k, dY_t^k \right\rangle + k_{sig}^{X,Y}(u,v) \left\langle d\gamma_s^k, dY_t^k \right\rangle \right), \tag{2}$$

with boundary conditions $u_{\boldsymbol{\gamma}}(0,\cdot) = 0$ and $u_{\boldsymbol{\gamma}}(\cdot,0) = 0$. An approximation of the above integral equation lead to a similar numerical scheme as for the forward dynamics.

2 The path-dependent heat equation

Let W be a one-dimensional Brownian motion on $[0,T]$ and consider the following three, one-dimensional processes

$$X_t = \int_0^t K(t,s)\ \mathrm{d}W_s, \qquad K_t^{t_0} = K(t_0,t), \qquad \Theta_t^{t_0} = \begin{cases} \int_0^{t_0} K(t,s)\ \mathrm{d}W_s & t > t_0 \\ 0 & t \leq t_0 \end{cases}$$

where $K(s,t) = \sqrt{2H}(s-t)^{H-\frac{1}{2}}$, for some $H \in (0,1)$.

In addition, define the time-augmented version of the above one-dimensional processes to be following two-dimensional processes

$$\widehat{X}_t = (t, X_t), \quad \widehat{K}_t^{t_0} = (t, K_t^{t_0}), \quad \widehat{\Theta}_t^{t_0} = (t, \Theta_t^{t_0})$$

Denote by $\Omega := C([t,T],\mathbb{R}^2)$ the space of continuous paths on $[t,T]$ (extra regularity?).

It can be shown that the price of the European option $\mathbb{E}[f(X_T)|\mathcal{F}_{t_0}]$ is a function of time and paths

$$u : [0,T] \times \Omega \to \mathbb{R}$$

that solves the following path-dependent heat equation

$$\left(\partial_{t_0} + \frac{1}{2} \partial^2_{\widehat{K}^{t_0}\widehat{K}^{t_0}} \right) u = 0, \qquad u(T,\gamma) = f(\gamma_T), \quad \text{for any } \gamma \in \Omega. \tag{3}$$

where $\partial_{\widehat{K}^{t_0}} u$ denotes the directional derivative of u along the direction of the path $\widehat{K}^{t_0}$.

Example 1 *Computations of state-dependent payoffs are quite simple since $X_T \sim \mathcal{N}(0, T^{2H})$. Let Φ denote the cumulative distribution function of a standard normal distribution $\mathcal{N}(0,1)$. The following are a few examples:*

$$f = \mathrm{Id}, \qquad \mathbb{E}[f(X_T)|\mathcal{F}_t] = \int_0^t \sqrt{2H}(T-r)^{H-\frac{1}{2}}\ \mathrm{d}W_r = \Theta_T^t;$$

$$f(x) = |x|, \qquad \mathbb{E}[f(X_T)|\mathcal{F}_t] = |\Theta_T^t| + (T-t)^H \sqrt{\frac{2}{\pi}};$$

$$f(x) = \mathrm{e}^{\nu x},\ \nu \in \mathbb{R}, \qquad \mathbb{E}[f(X_T)|\mathcal{F}_t] = \exp\left(\nu\Theta_T^t + \frac{\nu^2 (T-t)^{2H}}{2} \right);$$

$$f(x) = (x-K)_+,\ K \in \mathbb{R}, \qquad \mathbb{E}[f(X_T)|\mathcal{F}_t] = \frac{(T-t)^H}{\sqrt{2\pi}} \exp\left(-\frac{(K-\Theta_T^t)^2}{2(T-t)^{2H}} \right) - (K - \Theta_T^t)\Phi\left(+ \frac{\Theta_T^t - K}{(T-t)^H} \right);$$

$$f(x) = (\mathrm{e}^x - K)_+,\ K > 0, \qquad \mathbb{E}[f(X_T)|\mathcal{F}_t] = \exp\left(\frac{(T-t)^{2H}}{2} + \Theta_T^t \right) \Phi\left((T-t)^H - \frac{\log(K) - \Theta_T^t}{(T-t)^H} \right) - K\Phi\left(-\frac{\log(K) - \Theta_T^t}{(T-t)^H} \right).$$

Remark 1 *We can also consider path-dependent payoffs* $\mathbb{E}\left[f(X_{[0,T]})|\mathcal{F}_t\right] = u(t, X \otimes_t \Theta^t)$. *Now the PDE* (3) *is unchanged except for the terminal condition* $u(T,\gamma) = f(\gamma)$ *is a function on paths.*

3 A probabilistic numerical method for solving PDEs with kernels

In what follows, we present the numerical method applied to the specific example of the path-dependent heat equation, but the method can be generalised to any linear path-dependent PDE. This is based mainly on the material in [COSG17] where the chosen kernel is the signature kernel.

Recall the path-dependent heat equation 3. Let $Z = [0,T] \times \Omega$ and the boundary $\partial Z = \{T\} \times \Omega$. Define the mixed kernel $\kappa : Z \times Z \to \mathbb{R}$ as follows

$$\kappa(z, z') = \exp\left(-\frac{1}{2\sigma^2}|t - t'|^2\right) k_{sig}^{\gamma,\gamma'}(t, t') \tag{4}$$

for any $z = (t, \gamma)$ and $z' = (t', \gamma')$ in Z.

Claim: The RKHS $\mathcal{H}_\kappa$ associated to the kernel κ is large enough to contain solutions to the path-dependent heat equation.

Consider M points $z_1 = (t_1, \gamma^1), ..., z_M = (t_M, \gamma^M) \in Z$ ordered in such a way that $z_1, ..., z_{M_\Omega}$ are in the interior of Z and $z_{M_\Omega+1}, ..., z_M$ are on the boundary ∂Z. Denote by $\mathcal{L}_\gamma$ the pathwise heat operator along the path γ

$$\mathcal{L}_\gamma = \partial_t + \frac{1}{2}\partial^2_{\gamma,\gamma}$$

We consider the following optimal recovery problem that can also be interpreted as maximum a posterior (MAP) estimation for a GP constrained by a PDE [CHOS21]

$$\min_{u \in \mathcal{H}_\kappa} \|u\|_{\mathcal{H}_\kappa} \quad \text{s.t.} \quad \begin{cases} (\mathcal{L}_\gamma u)(t_m, \gamma^m) = 0 & \text{if } m = 1, ..., M_\Omega \\ u(T, \gamma^m) = f(\gamma^m_T) & \text{if } m = M_{\Omega+1}, ..., M. \end{cases} \tag{5}$$

In what follows we'll denote $\bar{\mathcal{L}_\gamma}$ the operator $\mathcal{L}_\gamma$ acting on the second variable of the function it acts on.

Remark 2 *An element* $u \in \mathcal{H}_\kappa$ *is of the form*

$$u = \sum_m \alpha_m \kappa(\cdot, z_m), \quad z_m \in Z, \alpha_m \in \mathbb{R}$$

Hence, the optimisation in eq. (5) *could in principle be tackled using a gradient descent approach for example, although a thorough convergence analysis should be done.*

3.1 Explicit solution via the Representer Theorem

The *Representer Theorem* [OS19] allows to get an explicit solution of the above optimisation

$$u(t, \tau) = xG^{-1}y$$

where $x \in \mathbb{R}^{1 \times M}$ is the following vector

$$x_m = \begin{cases} \bar{\mathcal{L}}_\gamma \kappa((t,\tau), z_m) & \text{if } m = 1, ..., M_\Omega \\ \kappa((t,\tau), z_m) & \text{if } m = M_{\Omega+1}, ..., M. \end{cases}$$

$x \in \mathbb{R}^{M\times 1}$ is the following vector

$$y_m = \begin{cases} 0 & \text{if } m = 1, ..., M_\Omega \\ f(\tau_T) & \text{if } m = M_{\Omega+1}, ..., M. \end{cases}$$

and $G \in \mathbb{R}^{M\times M}$ is a matrix defined by block

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the 3 matrices $A \in \mathbb{R}^{M_\Omega \times M_\Omega}, B \in \mathbb{R}^{M_\Omega \times (M-M_\Omega)}, C \in \mathbb{R}^{(M-M_\Omega)\times M_\Omega}, D \in \mathbb{R}^{(M-M_\Omega)\times(M-M_\Omega)}$ are defined as follows

$$\begin{aligned} A_{i,j} &= \mathcal{L}_\gamma \bar{\mathcal{L}}_\gamma \kappa(z_i, z_j), && i = 1, ..., M_\Omega, \quad j = 1, ..., M_\Omega \\ B_{i,j} &= \mathcal{L}_\gamma \kappa(z_i, z_j), && i = 1, ..., M_\Omega, \quad j = M_\Omega, ..., M \\ C_{i,j} &= \bar{\mathcal{L}}_\gamma \kappa(z_i, z_j), && i = M_\Omega, ..., M, \quad j = 1, ..., M_\Omega \\ D_{i,j} &= \kappa(z_i, z_j), && i = M_\Omega, ..., M, \quad j = M_\Omega, ..., M \end{aligned}$$

4 The rough Heston case (for later, if time allows)

We are chiefly motivated by an example of PPDE arising in rough volatility modeling. The rough Heston model reads

$$\begin{aligned} S_t &= S_0 + \int_0^t S_r \sqrt{V_r}\big(\rho \,\mathrm{d}W_t^1 + \bar{\rho}\,\mathrm{d}W_t^2\big), \\ V_t &= V_0 + \int_0^t \frac{(t-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\big(\lambda(\theta - V_r)\,\mathrm{d}r + \nu\sqrt{V_r}\,\mathrm{d}W_r^2\big), \end{aligned}$$

where $S_0, V_0 > 0$, $\rho \in (-1,1)$, $\bar{\rho} = \sqrt{1-\rho^2}$, $H \in (0, \frac{1}{2}]$, $\lambda, \theta, \nu > 0$, and W^1, W^2 are two independent Brownian motions. It turns out that, in this framework, there exists a function $u : [0,T] \times \mathbb{R}_+ \times \mathrm{C}([0,T], \mathbb{R}) \to \mathbb{R}$ such that the price of an European option

$$C_t := \mathbb{E}[g(S_T)|\mathcal{F}_t] = u(t, S_t, \Theta^t_{[t,T]}), \tag{6}$$

where

$$\Theta_s^t := V_0 + \int_0^t \frac{(s-r)^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}\big(\lambda(\theta - V_r)\,\mathrm{d}r + \nu\sqrt{V_r}\,\mathrm{d}W_r^2\big), \quad t < s.$$

Furthermore, this function $u(t,x,\omega)$ that we denote u from now on is the (unique?) solution to the following PPDE

$$\partial_t u + \frac{\lambda(\theta-\omega_t)}{\Gamma(H+\frac{1}{2})}\langle \partial_\omega u, \mathrm{K}^t\rangle + \frac{x^2\omega_t}{2}\partial^2_{xx}u + \frac{\rho\nu x\omega_t}{\Gamma(H+\frac{1}{2})}\langle \partial_\omega(\partial_x u), \mathrm{K}^t\rangle + \frac{\nu^2\omega_t}{2\Gamma(H+\frac{1}{2})}\langle \partial^2_{\omega\omega}u, (\mathrm{K}^t, \mathrm{K}^t)\rangle = 0, \tag{7}$$

where $\mathrm{K}_s^t := (s-t)^{H-\frac{1}{2}}$ and $\langle \partial_\omega u, \eta\rangle$ is a Fréchet derivative with respect to ω in the direction of the path η. For details on the derivatives, see [VZ19, Section 3.1].

Remark 3 *How general is the representation* (6) *? Does it also apply to path-dependent payoffs?*

4.1 Benchmark by Fourier transform

The affine structure of this model gives us access to its Fourier transform in semi-closed form and therefore to Fourier methods for computing conditional expectations of the type (6), hence providing us a benchmark for our method [EER19]. Following [JO19, Section 5.1], for all $t \geq 0$ we denote by $\Phi_t : \mathbb{R} \to \mathbb{C}$ the Fourier transform

$$\Phi_t(v) := \mathbb{E}\left[\left(\frac{S_t}{S_0}\right)^{\mathrm{i}v}\right].$$

The Call price is then obtained by inverse Fourier transform as

$$\mathbb{E}[(S_T - K)_+] = S_0 - \frac{\sqrt{S_0 K}}{\pi}\int_0^\infty \Re\left(\mathrm{e}^{\mathrm{i}vk}\Phi_t\left(p - \frac{\mathrm{i}}{2}\right)\right)\frac{\mathrm{d}v}{v^2 + \frac{1}{4}}.$$

Regarding the Fourier transform itself, it can be written as

$$\log \Phi_t(v) = \theta \mathfrak{I}^1 h(v,t) + V_0 \mathfrak{I}^{1-\alpha} h(v,t),$$

where $\mathfrak{I}^\gamma$ denotes the fractional integral and $h(v,t)$ solves a fractional Riccati equation for all $v \in \mathbb{R}$ which can be solved using the standard Adams scheme. The details are written better than I could in [JO19, Section 5.1].

Remark 4 *They only consider expectations but the conditional expectation is also known [AJLP19, Theorem 4.3].*

References

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