

# The Liouville–Bratu–Gelfand Problem for Radial Operators

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We determine precise existence and multiplicity results for radial solutions of the Liouville–Bratu–Gelfand problem associated with a class of quasilinear radial operators, which includes perturbations of  $k$ -Hessian and  $p$ -Laplace operators. © 2002 Elsevier Science (USA)

*Key Words:* Liouville; Bratu; Gelfand;  $p$ -Laplacian;  $k$ -Hessian; radial solutions.

## 1. INTRODUCTION

The classical Liouville–Bratu–Gelfand problem is concerned with positive solutions of the equation:

$$\begin{cases} \Delta u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where  $\lambda > 0$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ . Equation (1) arises via the study of the quasilinear parabolic problem:

$$\begin{cases} v_t = \Delta v + \lambda(1 - \varepsilon v)^m e^{v/(1+\varepsilon v)}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega, \end{cases} \quad (2)$$

which is known as the *solid fuel ignition model*, and is derived as a model for the thermal reaction process in a combustible, nondeformable material

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of constant density during the ignition period. Here  $\lambda$  is known as the Frank–Kamenetskii parameter,  $v$  is a dimensionless temperature, and  $1/\varepsilon$  is the activation energy. One is interested in the question of what happens when a combustible medium is placed in a vessel whose walls are maintained at a fixed temperature. The derivation of Eq. (2) from general principles is accounted for in the comprehensive work of Frank–Kamenetskii [12, Chap. VI–VII]. See also [4, 11, 14, 23].

Nontrivial solutions of (1) arise as steady-state solutions of (2), within the approximation  $\varepsilon \ll 1$ . If such solutions exist, then the competing forces of cooling (due to diffusion from the boundary) and heating (due to the positive reaction term) must be in balance. Intuitively, one might expect that for large values of  $\lambda$  the reaction term will dominate and drive the temperature to infinity (*explosion*), whereas for smaller  $\lambda$ , steady states might be possible. Indeed, if we multiply (1) by a principal eigenfunction  $\phi$ , i.e.,  $\phi > 0$  satisfies

$$\begin{cases} \Delta\phi + \lambda_1\phi = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \quad (3)$$

with  $\lambda_1 > 0$ , then the inequality  $u \leq e^u$  together with an integration by parts shows that if  $u \geq 0$  solves (1), then  $\lambda \leq \lambda_1$ . This provides a rough upper bound on values of  $\lambda$  for which (1) may admit nontrivial solutions.

One may employ implicit function methods to establish a local solution curve  $(\lambda, u) \subset [0, \infty) \times C(\bar{\Omega})$  to (1), which emanates from  $\lambda = 0$ ,  $u = 0$ . Using Leray–Schauder continuation methods one may show that this curve is part of an unbounded continuum of solutions contained in  $[0, \lambda_1] \times C(\bar{\Omega})$ . To determine the precise multiplicity of solutions requires a complete understanding of this solution continuum.

For general domains this is difficult, however, if  $\Omega = B_1(0)$  is the unit ball in  $\mathbb{R}^N$ , then by the classical result of Gidas *et al.* [16] all solutions of (1) are radially symmetric and (1) is equivalent to the ordinary differential equation's boundary value problem

$$\begin{cases} u'' + \frac{N-1}{r} u' + \lambda e^u = 0, & r \in (0, 1), \\ u'(0) = u(1) = 0 \end{cases} \quad (4)$$

for the profile  $u(r) = u(|x|)$ . Note that the originally discrete parameter  $N$  is now allowed to vary continuously.

For  $N = 1$  this problem, amenable to reduction of order methods, was first studied by Liouville in 1853 [21] (see also [3]). For the free parameter  $u(0) = \mu$  one finds the boundary conditions  $u'(0) = u(1) = 0$  are satisfied

provided

$$\lambda = \lambda(\mu) = \frac{1}{2e^\mu} [\log(2e^\mu + 2\sqrt{e^\mu(e^\mu - 1)} - 1)]^2. \quad (5)$$

It follows that there is a unique  $\lambda^*$  (corresponding to the maximum value of the function defined by (5)) for which (4) has a unique solution and for each  $\lambda \in (0, \lambda^*)$ , Eq. (4) has precisely two solutions.

In 1914 Bratu [5] found an explicit solution to (4) when  $N = 2$ , which obeys the same multiplicity criteria as the Liouville case (see also [3] where the explicit solution is given). Elusive to analytical solutions when  $N = 3$ , numerical progress for (4) was made by Frank–Kamenetskii (see [12]) in his study of thermal ignition problems. In particular, he was interested in steady-state solutions to thermal ignition problems in plane-parallel, cylindrical, and spherical vessels, corresponding to  $N = 1, 2$ , and 3, respectively. Of special concern was the maximal value  $\lambda^*$  of the parameter  $\lambda$  for which (4) admits a solution (to avoid explosion). Using various approximation schemes, he numerically integrated the equations and (correctly) approximated  $\lambda^*$  in each case ( $\lambda_{N=1}^* \approx 0.88$ ,  $\lambda_{N=2}^* = 2$ ,  $\lambda_{N=3}^* \approx 3.32$ ). Further progress for  $N = 3$  was made by Chandrasekhar [7, IV: Sects. 22–27], where (4) appears as a model for the temperature distribution of an isothermal gas sphere in gravitational equilibrium.

Gelfand [15] built upon Frank–Kamenetskii's work when  $N = 3$  and used Emden's transformation to prove the existence of a value of  $\lambda$  for which (4) has infinitely many nontrivial solutions. It is not clear if Frank–Kamenetskii was aware of this decisive change between multiplicity of solutions as  $N$  proceeds from 2 to 3. In 1973 Joseph and Lundgren [20] completely characterized the solution structure of (4) for all  $N$ . Of particular interest is the relationship they observed between the multiplicity of solutions and the space dimension  $N$  (see Fig. 1):

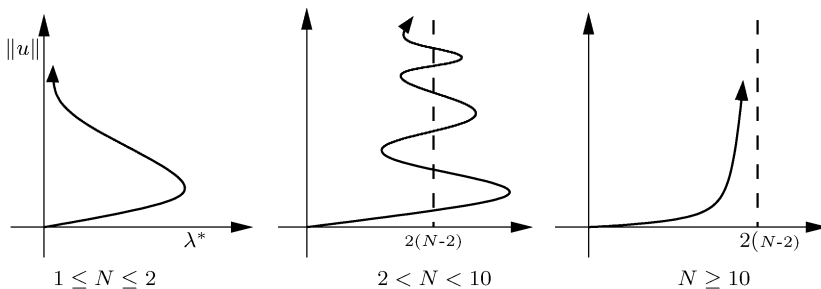


FIG. 1. Global continua for (4) depend on  $N$ .

*Case I:*  $1 \leq N \leq 2$ . There exists  $\lambda^* > 0$  such that (4) has exactly one solution for  $\lambda = \lambda^*$  and exactly two solutions for each  $\lambda \in (0, \lambda^*)$ .

*Case II:*  $2 < N < 10$ . Eq. (4) has a continuum of solutions which oscillates around the line  $\lambda = 2(N - 2)$ , with the amplitude of oscillations tending to zero, as  $u(0) = \|u\| \rightarrow \infty$ .

*Case III:*  $N \geq 10$ . Eq. (4) has a unique solution for each  $\lambda \in (0, 2(N - 2))$  and no solutions for  $\lambda \geq 2(N - 2)$ . Moreover,  $\|u\| \rightarrow \infty$  as  $\lambda \rightarrow 2(N - 2)$ .

In this paper, we study the Liouville–Bratu–Gelfand problem for a larger class of partial differential operators. Motivated by the recent work of Clément *et al.* [8] and Jacobsen [19], we are interested in existence and multiplicity results for the model equations

$$\begin{cases} \Delta_p u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (6)$$

where  $\Delta_p = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$  is the  $p$ -Laplace operator [18, 22] and

$$\begin{cases} S_k(D^2 u) + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (7)$$

where  $S_k(D^2 u)$  is the  $k$ -Hessian operator [6, 26], defined as the sum of all principal  $k \times k$  minors of the Hessian matrix  $D^2 u$ . For instance  $S_1(D^2 u) = \Delta u$  and  $S_N(D^2 u) = \det D^2 u$ , the Monge–Ampère operator.

Note that both equations are extensions of (1). In particular, the results of Joseph and Lundgren explain the radial case of (6) for  $p = 2$  and of (7) when  $k = 1$ . For  $p \neq 2$  or  $k > 1$ , only fragmentary information is known. In [8], the authors consider (among other topics) the radial case of both (6) for  $p = N$  and (7) for  $k = n$ , where  $N = 2n$ . Surprisingly, they find the twofold multiplicity (as one sees in the case  $k = 1, N = 2$ ) holds for all  $p$  or  $k$ . In [19], the author considers a variation of (7) for the Monge–Ampère operator ( $k = N$ ) with  $\Omega$  any strictly convex domain, and finds qualitative results similar to the case  $k = N = 1$ . In particular, unlike the Joseph and Lundgren result, it is shown that regardless of the space dimension, all continua eventually tend to infinity with  $\lambda \rightarrow 0$ .

It is our purpose here to extend and unify all of these results to the general Liouville–Bratu–Gelfand problem for the class of quasilinear elliptic equations defined by

$$\begin{cases} r^{-\gamma} (r^\alpha |u'|^\beta u')' + \lambda e^u = 0, & r \in (0, 1), \\ u > 0, & r \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (8)$$

where the inequalities

$$\alpha \geq 0, \quad (9)$$

$$\gamma + 1 > \alpha, \quad (10)$$

$$\beta + 1 > 0 \quad (11)$$

hold. For instance, the following operators are included in this class:

Operator	$\alpha$	$\beta$	$\gamma$
Laplacian	$N - 1$	0	$N - 1$
$p$ -Laplacian ( $p > 1$ )	$N - 1$	$p - 2$	$N - 1$
$k$ -Hessian	$N - k$	$k - 1$	$N - 1$

If  $\Omega = B_1(0)$  is the unit ball, then Eq. (8) arises from (6) and (7) as a consequence of a priori symmetry results. For (7), this is essentially a consequence of the Gidas–Ni–Nirenberg result, suitably extended by Delanoë [10]. The moving plane method of Alexandrov [1] and Serrin [25] was extended to (6) by Badiale and Nabana [2] under the additional assumption that the origin is the only interior critical point. Indeed, the fundamental difficulty in applying the moving plane method to quasilinear elliptic equations involving the  $p$ -Laplace operator is the possibility of the loss of ellipticity at interior critical points. Here the positive nonlinearity prevents such a situation.

## 2. GLOBAL CONTINUA

In this section, we prove that Eq. (8) has a global continuum of solutions which emanates from  $\lambda = 0$ ,  $u = 0$  and is bounded in the  $\lambda$ -direction.

Let  $E = C[0, 1]$  and consider the boundary value problem (8), i.e.,

$$\begin{cases} r^{-\gamma}(r^\alpha|u'|^\beta u')' + \lambda e^u = 0, & r \in (0, 1), \\ u > 0, & r \in (0, 1), \\ u'(0) = u(1) = 0. \end{cases} \quad (12)$$

It follows that  $u \in E$  is a solution of (12) if and only if  $u$  is a solution of the integral equation

$$u(r) = \int_r^1 \left( \frac{\lambda}{t^\alpha} \int_0^t s^\gamma e^{u(s)} ds \right)^{1/(1+\beta)} dt. \quad (13)$$

We define the operator

$$T : [0, \infty) \times E \rightarrow E$$

by

$$T(\lambda, u)(t) = \int_r^1 \left( \frac{\lambda}{t^\alpha} \int_0^t s^\gamma e^{u(s)} ds \right)^{1/(1+\beta)} dt. \quad (14)$$

It follows from the assumptions made on the constants  $\alpha, \beta, \gamma$ , that  $T$  is a completely continuous operator with

$$T(0, u) = 0, \quad \forall u \in E.$$

Using Leray–Schauder continuation arguments (see [9, 24]) we conclude that (13) has a continuum  $\mathcal{C} \subset [0, \infty) \times E$  of solutions satisfying:

- $\mathcal{C}$  is unbounded,
- $(0, u) \in \mathcal{C}$  if and only if  $u = 0$ .

We next observe that if the pair  $(\lambda, u)$  solves (13), then

$$\begin{aligned} u(r) &= \int_r^1 \left( \frac{\lambda}{t^\alpha} \int_0^t s^\gamma e^{u(s)} ds \right)^{1/(1+\beta)} dt \\ &\geq \int_r^1 \left( \frac{\lambda}{t^\alpha} \int_0^r s^\gamma e^{u(r)} ds \right)^{1/(1+\beta)} dt. \end{aligned} \quad (15)$$

Hence for (say)  $r \in [\frac{1}{4}, \frac{3}{4}]$ , we conclude that

$$u(r)e^{-u(r)/(\beta+1)} \geq c\lambda^{1/(\beta+1)}, \quad (16)$$

where  $c$  is a positive constant depending only on  $\alpha, \beta$ , and  $\gamma$ . Since the left-hand side of (16) is bounded (independently of the value of  $u(r)$ ) we obtain a bound on  $\lambda$ . We hence have proved the following result.

**THEOREM 2.1.** *There exists a positive number  $\lambda^*$ , depending only on  $\alpha, \beta, \gamma$ , such that any solution  $(\lambda, u)$ ,  $\lambda \geq 0$ , of (12) satisfies  $\lambda \leq \lambda^*$ , and there exists a continuum  $\mathcal{C} \subset [0, \lambda^*] \times E$  of solutions of (12) which is unbounded in  $[0, \lambda^*] \times E$ .*

It follows from the contraction mapping principle that for  $\lambda$ , sufficiently small, the continuum  $\mathcal{C}$  is a continuous curve emanating from the origin  $(0, 0) \in [0, \lambda^*] \times E$ .

We also observe that initial value problems

$$\begin{cases} r^{-\gamma}(r^\alpha|u'|^\beta u')' + \lambda e^u = 0, & r \in (0, 1), \\ u > 0, & r \in (0, 1), \\ u(0) = \mu > 0, u'(0) = 0 \end{cases} \quad (17)$$

are uniquely solvable. Hence, for  $(\lambda, u) \in \mathcal{C}$  we may parameterize  $\lambda$  as follows:

$$\lambda(\mu) = \frac{\mu^{\beta+1}}{(\int_0^1 (\frac{1}{t^\alpha} \int_0^t s^\gamma e^{u(s)} ds)^{1/(1+\beta)} dt)^{\beta+1}}, \quad (18)$$

where

$$\mu = u(0).$$

It is the subject of the following sections to give a more precise description of the continuum  $\mathcal{C}$ .

### 3. REFINEMENT

The purpose of this section is to elucidate the precise structure of the solution continuum of Theorem 2.1. We achieve this through a change of variables that relates solutions of (12) with solutions of a first-order system in the plane.

Define two distinguished constants  $\xi$  and  $\delta$  by

$$\xi = \gamma + 1 - \alpha, \quad (19)$$

$$\delta = \frac{\gamma + \beta - \alpha + 2}{\xi}. \quad (20)$$

It follows from assumptions (10) and (11) that both  $\xi$  and  $\delta$  are positive. For reference we note the values of these constants for two special cases:

Operator	$\xi$	$\delta$
$p$ -Laplacian	1	$p$
$k$ -Hessian	$k$	2

Assume  $(\lambda_0, u) \in \mathcal{C}$ , i.e.,  $u$  solves (12) with  $\lambda = \lambda_0$ , and consider the following change of variables:

$$\begin{cases} s = \xi \ln r, \\ v = -\frac{du}{ds}, \\ w = \lambda_0 e^{\delta s} e^u. \end{cases}$$

If  $\dot{v}$  and  $\dot{w}$  denote differentiation with respect to  $s$ , then

$$\dot{w} = \lambda_0 \delta e^{\delta s} e^u + \lambda_0 e^{\delta s} e^u \dot{u} = w(\delta - v) \quad (21)$$

and

$$\dot{v} = -\frac{d}{ds} \left( u'(r) \frac{r}{\xi} \right) = -u'(r) \frac{r}{\xi^2} - u''(r) \frac{r^2}{\xi^2}, \quad (22)$$

which we may rewrite as

$$\xi^2 \dot{v} = -r^2 u''(r) - r u'(r). \quad (23)$$

Developing (12) one finds

$$(\beta + 1) r^\alpha |u'|^\beta u'' + \alpha r^{\alpha-1} |u'|^\beta u' + \lambda_0 r^\gamma e^u = 0. \quad (24)$$

Multiplying (24) by  $r^{2-\alpha}$  and dividing by  $|u'|^\beta$  yields

$$r^2 u'' + \frac{\alpha}{\beta + 1} r u' + \frac{1}{(\beta + 1) \xi^\beta v^\beta} w = 0. \quad (25)$$

Next, observe from (25) and (23) that

$$\xi^2 \dot{v} = \left( \frac{\alpha}{\beta + 1} - 1 \right) r u' + \frac{1}{(\beta + 1) \xi^\beta v^\beta} w. \quad (26)$$

From (26) and (21) we obtain the first-order system

$$\begin{cases} \dot{w} = w(\delta - v), \\ \dot{v} = \left( \frac{\beta + 1 - \alpha}{\xi(\beta + 1)} \right) v + \left( \frac{1}{(\beta + 1) \xi^{\beta+2}} \right) w v^{-\beta} \end{cases} \quad (27)$$



with the additional conditions

$$w(0) = \lambda_0, \quad (28)$$

$$v(0) = -u'(1)/\xi, \quad (29)$$

$$w(-\infty) = 0, \quad (30)$$

$$v(-\infty) = 0. \quad (31)$$

The systems corresponding to the  $p$ -Laplacian and  $k$ -Hessian are

$$\begin{cases} \dot{w} = w(p-v), \\ \dot{v} = \left(\frac{p-N}{p-1}\right)v + \frac{wv^{2-p}}{p-1} \end{cases} \quad \text{and} \quad \begin{cases} \dot{w} = w(2-v), \\ \dot{v} = \left(\frac{2k-N}{k^2}\right)v + \frac{wv^{1-k}}{k^{k+2}}, \end{cases} \quad (32)$$

respectively.

An element  $(\lambda_0, u) \in \mathcal{C}$ , therefore, corresponds to an integral curve of (27) which emanates from  $(0, 0) = (w(-\infty), v(-\infty))$  and passes through  $(\lambda_0, \theta) = (w(0), v(0))$ . Moreover,  $u'(1) = -\xi\theta$  and  $u(0) = \mu = \int_{-\infty}^0 v(s) ds$ . In particular, each crossing of this integral curve with the line  $w = \lambda$  corresponds to a solution of (12). It follows that the multiplicity of solutions to (12) is determined by the structural properties of this curve, whose existence is guaranteed from Theorem 2.1.

System (27) has a critical point at

$$w = (\alpha - \beta - 1)(\delta\xi)^{\beta+1}, \quad v = \delta. \quad (33)$$

If  $\beta \in (-1, 0]$ , then the origin is also a critical point, otherwise it is a singular point, and (33) is the unique critical point of (27). In the borderline case when  $\alpha - \beta - 1 = 0$  (corresponding to  $\Delta_p$  in  $\mathbb{R}^p$  or  $S_k$  in  $\mathbb{R}^{2k}$ ), the entire  $v$  axis is singular. The following lemma characterizes the basic properties of integral curves of (27) that relate to  $\mathcal{C}$ :

**LEMMA 3.1.** *Let  $\mathcal{U}$  denote the integral curve of (27) corresponding to  $\mathcal{C}$ . If*

$$\alpha - \beta - 1 > 0, \quad (34)$$

*then  $\mathcal{U}$  connects  $(0, 0)$  to  $((\alpha - \beta - 1)(\delta\xi)^{\beta+1}, \delta)$ . If (34) fails, then  $\mathcal{U}$  connects  $(0, 0)$  to either  $(0, \infty)$  when  $\alpha - \beta - 1 < 0$ , or possibly to a singular point on the  $v$  axis when  $\alpha - \beta - 1 = 0$ .*

*Proof.* It follows from Theorem 2.1 and the remarks thereafter that  $\mathcal{U}$  exists and is bounded in the  $w$  direction. It is also clear from (27) that  $\mathcal{U}$  is confined to the first quadrant. The behavior as  $s \rightarrow \infty$  falls into two main cases, depending on (34).

Assume (34) fails strictly, i.e.,  $\alpha - \beta - 1 < 0$ . In this case the critical point lies in the second quadrant. From (11) and (27) one sees that  $\dot{v} > 0$  for all  $s$ , while  $\dot{w}$  is positive for  $v < \delta$  and negative otherwise. It follows that  $\mathcal{U}$  leaves the origin with both  $w$  and  $v$  increasing. When  $v = \delta$  the behavior of  $w$  shifts from increasing to decreasing, and  $\mathcal{U}$  now approaches the  $v$ -axis. Since the critical point lies on the line  $v = \delta$  and  $\mathcal{U}$  has already passed this level, it continues to increase in  $v$  and decrease in  $w$ , approaching  $(0, \infty)$  asymptotically to the  $v$ -axis. If  $\alpha - \beta - 1 = 0$ , then the  $v$ -axis becomes singular and it is possible for  $\mathcal{U}$  to approach a singular point on the  $v$ -axis (e.g., see the case  $k = 2, N = 4$  in Fig. 2).

Now assume (34) holds. From (27) it follows that  $\mathcal{U}$  is now also bounded in  $v$  and must either converge to the critical point (33) or a limit cycle surrounding the critical point. To rule out periodic orbits we employ Dulac's criterion [17] with the function  $w^{-1}v^\beta$ . Indeed, if a periodic orbit  $\sigma$  exists, enclosing a region  $\Sigma$ , then by Green's Theorem we obtain the contradiction

$$0 = \oint_{\sigma} \frac{v^\beta}{w} (\dot{w} dv - \dot{v} dw) \quad (35)$$

$$= \int_{\Sigma} \left( \frac{\partial}{\partial w} \left( \frac{\dot{w} v^\beta}{w} \right) + \frac{\partial}{\partial v} \left( \frac{\dot{v} v^\beta}{w} \right) \right) dw dv \quad (36)$$

$$= \int_{\Sigma} \frac{\partial}{\partial v} \left( \frac{(\beta + 1 - \alpha)}{\xi(\beta + 1)} \frac{v^{\beta+1}}{w} + \frac{1}{(\beta + 1)\xi^{\beta+2}} \right) dw dv \quad (37)$$

$$= \int_{\Sigma} \left( \frac{\beta + 1 - \alpha}{\xi} \right) \frac{v^\beta}{w} dw dv \quad (38)$$

$$< 0, \quad (39)$$

since the integrand in (38) is negative throughout the first quadrant. Therefore when (34) holds,  $\mathcal{U}$  converges to the critical point as  $s \rightarrow \infty$ . ■

Figure 2 contains several plots of typical integral curves for systems corresponding to the  $p$ -Laplace and  $k$ -Hessian operators. In the top row the origin is a saddle point and the curve  $\mathcal{U}$  corresponds to a heteroclinic orbit connecting the unstable manifold of the origin to the stable manifold of the

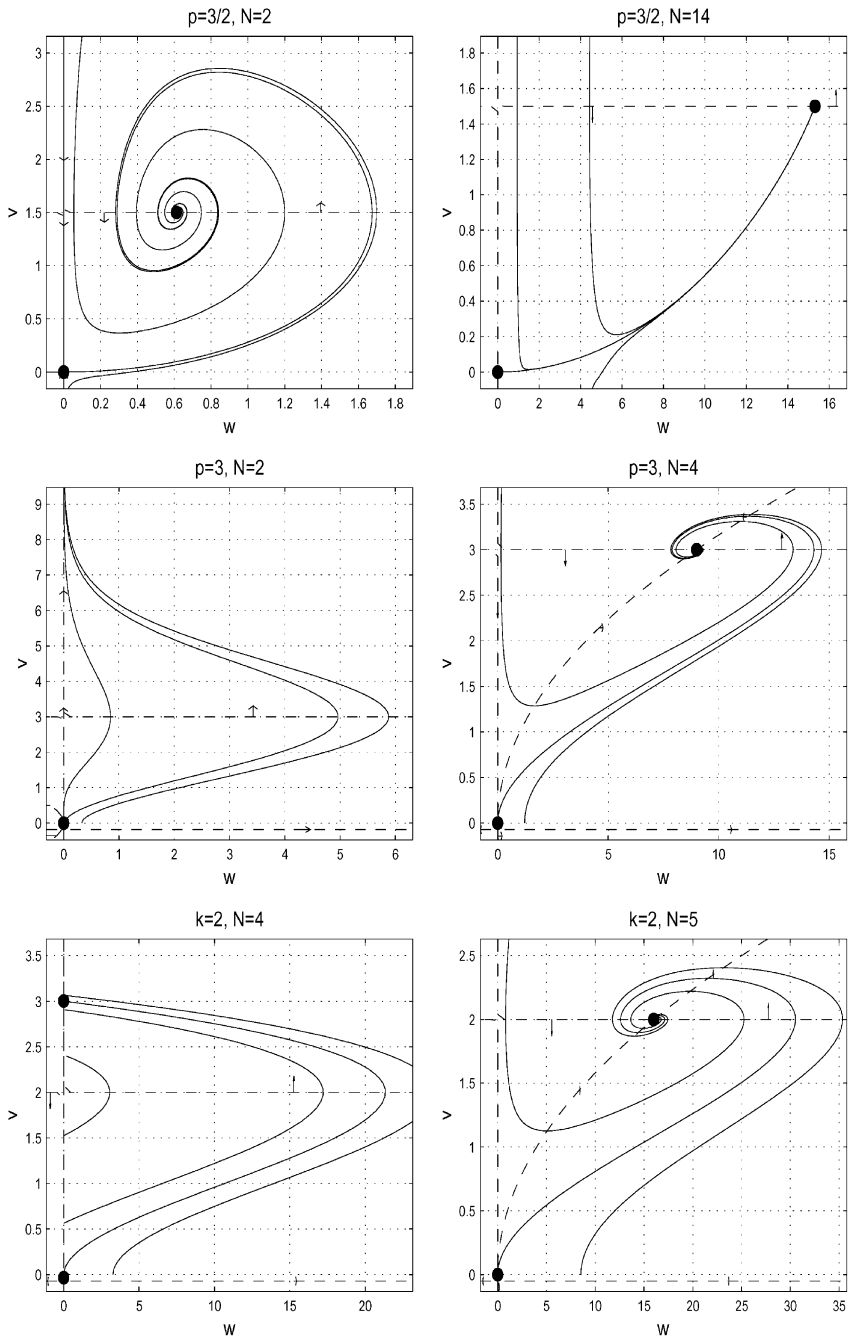


FIG. 2. Typical orbits for cases of (32).

critical point (33). The middle row demonstrates the transition from  $N < p$  to  $N > p$ , representing the changing dynamics due to the sign condition (34). The borderline case is illustrated in the bottom row (note that  $N = 2k$  corresponds to the case  $N = p$ ). Note also, that when (34) holds, there is a unique integral curve emanating from the origin which must therefore correspond to  $\mathcal{U}$ , whereas when it fails strictly, the origin becomes a node for which one of the curves must correspond to  $\mathcal{U}$ .

From Lemma 3.1 we conclude that if (34) fails, then the multiplicity of solutions to (12) behaves as in Case I of the Joseph–Lundgren result, i.e., there exists a constant  $\lambda^*$  (corresponding to the value of  $w$  in  $(w, \delta) \in \mathcal{U}$ ) for which (12) has a unique solution, and for each  $\lambda \in (0, \lambda^*)$ , (12) has precisely two distinct solutions.

It remains to determine the multiplicity of solutions when (34) holds. Consider the linearization at the critical point (33). The trace and determinant of the Jacobian matrix at the critical point are given by

$$\operatorname{tr} J = \frac{\beta + 1 - \alpha}{\xi}, \quad \det J = -\frac{\delta}{\beta + 1} \frac{\beta + 1 - \alpha}{\xi} = -\frac{\delta}{\beta + 1} \operatorname{tr} J. \quad (40)$$

Notice that the sign condition (34) may now be restated as assuming  $\operatorname{tr} J < 0$ . This effectively makes the critical point an attractor. To see this explicitly, first note that the eigenvalues of the linearized system are given by the equation

$$\lambda = \frac{\operatorname{tr} J}{2} \pm \frac{\sqrt{(\operatorname{tr} J)^2 - 4 \det J}}{2}. \quad (41)$$

Second, using the identities in (40), we may rewrite (41) as

$$\lambda = \frac{\operatorname{tr} J}{2} \pm \frac{\sqrt{(\operatorname{tr} J)^2 + \frac{4\delta}{\beta+1} \operatorname{tr} J}}{2}. \quad (42)$$

Thus, we see when  $\operatorname{tr} J < 0$ , we will either have two negative real eigenvalues or a pair of complex eigenvalues with negative real part. Complex eigenvalues occur when

$$(\beta + 1)(\beta + 1 - \alpha) > -4\delta\xi. \quad (43)$$

These observations, coupled with Lemma 3.1, establish the following theorem, completely characterizing the multiplicity of solutions to (12):

THEOREM 3.1. Consider Eq. (8), i.e.,

$$\begin{cases} r^{-\gamma}(r^\alpha|u'|^\beta u')' + \lambda e^u = 0, & r \in (0, 1), \\ u > 0, & r \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (44)$$

with inequalities (9)–(11) satisfied. The solution structure to (44) is as follows (see Fig. 3):

Case I:  $\alpha - \beta - 1 \leq 0$ . There exists a unique  $\lambda^* > 0$  such that (44) has exactly one solution for  $\lambda = \lambda^*$  and exactly two solutions for  $0 < \lambda < \lambda^*$ .

Case II:  $0 < \alpha - \beta - 1 < \frac{4\delta\xi}{\beta+1}$ . Equation (44) has a continuum of solutions which oscillates around the line  $\lambda = (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$ , with the amplitude of oscillations tending to zero as  $\|u\| \rightarrow \infty$ .

Case III:  $\alpha - \beta - 1 \geq \frac{4\delta\xi}{\beta+1}$ . Equation (44) has a unique solution for each  $\lambda \in (0, (\alpha - \beta - 1)(\delta\xi)^{\beta+1})$  and no solutions for  $\lambda \geq (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$ . Moreover,  $\|u\| \rightarrow \infty$  as  $\lambda \rightarrow (\alpha - \beta - 1)(\delta\xi)^{\beta+1}$ .

Theorem 3.1 provides a complete description for the multiplicity of solutions for the generalized Liouville–Bratu–Gelfand problem (44). In particular, we recover both the Joseph–Lundgren result when  $k = 1$  or  $p = 2$  [20] and the Clément–DeFigueiredo–Mitidieri result when  $N = 2k$  or  $p = N$  [8].

We conclude with an illustration of Theorem 3.1 for the  $p$ -Laplacian and  $k$ -Hessian.

EXAMPLE 3.1. Consider the equation

$$\begin{cases} \Delta_p u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (45)$$

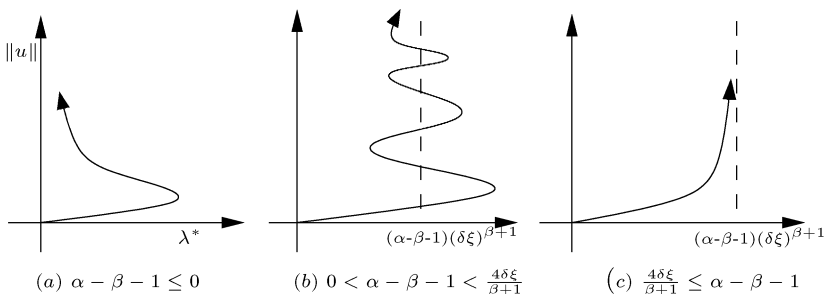


FIG. 3. Illustration of Theorem 3.1.

with  $\Omega$  the unit ball in  $\mathbb{R}^N$ . In this case  $\alpha - \beta - 1 = N - p$ ,  $\beta + 1 = p - 1$ ,  $\delta\xi = p$ , and the oscillation condition (43) becomes

$$(p - 1)(p - N) > -4p. \quad (46)$$

Thus we can replace captions (a)–(c) in Fig. 3 with

$$(a) \ N \leq p, \quad (b) \ p < N < \frac{p^2 + 3p}{p - 1}, \quad (c) \ \frac{p^2 + 3p}{p - 1} \leq N, \quad (47)$$

where the critical line in (b) and (c) occurs at  $\lambda = (N - p)p^{p-1}$ .

It is interesting to note how the multiplicity of solutions varies as  $p$  varies; i.e., as  $p \rightarrow 1$ , the number of space dimensions where the continua are oscillatory tends to infinity, whereas, as  $p \rightarrow \infty$ , the number of oscillatory dimensions tends to 3, given by those  $N$  such that  $p < N < p + 4$ .

EXAMPLE 3.2. Consider the equation

$$\begin{cases} S_k(D^2u) + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \quad (48)$$

with  $\Omega$  the unit ball in  $\mathbb{R}^N$ . In this case one finds  $\alpha - \beta - 1 = N - 2k$ ,  $\beta + 1 = k$ ,  $\delta\xi = 2k$ , and the oscillation condition (43) becomes

$$2k < N < 2k + 8. \quad (49)$$

Therefore, we may replace captions (a)–(c) in Fig. 3 with

$$(a) \ N \leq 2k, \quad (b) \ 2k < N < 2k + 8, \quad (c) \ 2k + 8 \leq N,$$

with the critical line in (b) and (c) occurring at  $\lambda = (N - 2k)(2k)^k$ . For instance we see that when  $k = N$ , the case of the Monge–Ampère operator, for *no* dimensions will the continua be oscillatory.

*Note Added in Proof.* The authors would like to thank Professor F. Gazzola for drawing their attention to the paper [13], which includes several interesting and related developments concerning the  $p$ -Laplacian equation (6).

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