KUTAISI INTERNATIONAL UNIVERSITY

Numerical Analysis

Ramaz Botchorishvili

Kutaisi International University

1, 2025

Course Overview, Numerical Differentiation

- Course Overview
- Finite differences and truncation error
- Finite differences and rounding errors
- Richardson's extrapolation
- Method of undefind coefficients
- Stencil, nonuniform mesh, curvilinear mesh
- Compact finite differences
- Estimating expressions with finite differences
- Finite differences in multidimension
- Differentiation matrix, convolution, mask, mollifier, smoothing kernel
- Application to edge detection
- Q & A

Numerical Analysis, Course Overview 1

- Course format: 2L + 2TTF

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets

- Course format: 2L + 2TTF

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

- Midterm: 9-th week, 2.5 hours

- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;

- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;
- Two Computational Projects: weeks 6-8 and 12-14

- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF

- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;
- Two Computational Projects: weeks 6-8 and 12-14
- No make-up midterm, classwork or computational project!!!
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF
- Course content: **numerical differentiation and integration, root finding, fitting model functions to data**

- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;
- Two Computational Projects: weeks 6-8 and 12-14
- No make-up midterm, classwork or computational project!!!
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF
- Course content: **numerical differentiation and integration, root finding, fitting model functions to data**

- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;
- Two Computational Projects: weeks 6-8 and 12-14
- No make-up midterm, classwork or computational project!!!
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 1

- What is class material: syllabus, readers, lecture slides, homework sheets
- How often is class material uploaded and where can you find it
- Course format: 2L + 2TTF
- Course content: **numerical differentiation and integration, root finding, fitting model functions to data**
- Course pace: your feedback counts $\Rightarrow$ more efficient use of time = higher pace
- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Midterm: 9-th week, 2.5 hours
- Classwork: weekly;
- Two Computational Projects: weeks 6-8 and 12-14
- No make-up midterm, classwork or computational project!!!
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 2

- What is class material: syllabus, readers, lecture slides, homework sheets
- Course format: 2L + 2TTF
- Course content: **numerical differentiation, integration, root finding, fitting model functions to data**
- Course pace: your feedback counts $\Rightarrow$ more efficient use of time = higher pace
- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

Numerical Analysis, Course Overview 2

- What is class material: syllabus, readers, lecture slides, homework sheets
- Course format: 2L + 2TTF
- Course content: **numerical differentiation, integration, root finding, fitting model functions to data**
- Course pace: your feedback counts $\Rightarrow$ more efficient use of time = higher pace
- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam

- How to get help: Instructor's office hours appointment upon request, TA's office hours, student TA's

Numerical Analysis, Course Overview 2

- What is class material: syllabus, readers, lecture slides, homework sheets
- Course format: 2L + 2TTF
- Course content: **numerical differentiation, integration, root finding, fitting model functions to data**
- Course pace: your feedback counts $\Rightarrow$ more efficient use of time = higher pace
- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam
- Learning objectives: studying course content from the viewpoint of numerical mathematics

- How to get help: Instructor's office hours appointment upon request, TA's office hours, student TA's

Numerical Analysis, Course Overview 2

- What is class material: syllabus, readers, lecture slides, homework sheets
- Course format: 2L + 2TTF
- Course content: **numerical differentiation, integration, root finding, fitting model functions to data**
- Course pace: your feedback counts $\Rightarrow$ more efficient use of time = higher pace
- Interaction during lectures: based on your feedback checkpoints only, interaction will not be forced
- Assessment components: 12% Classwork + 12% Activity + 22% Midterm + 24% Computational projects + 30% Final Exam
- Learning objectives: studying course content from the viewpoint of numerical mathematics
- Learning outcomes: apply knowledge and skills
- How to get help: Instructor's office hours appointment upon request, TA's office hours, student TA's

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**
- Learning objectives: studying course content from the viewpoint of numerical mathematics

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**
- Learning objectives: studying course content from the viewpoint of numerical mathematics
- Learning outcomes: after taking this course students are able to

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**
- Learning objectives: studying course content from the viewpoint of numerical mathematics
- Learning outcomes: after taking this course students are able to
  - apply numerical methods for solving problems of rootfinding, numerical integration and differentiation, fitting model functions to given data

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**
- Learning objectives: studying course content from the viewpoint of numerical mathematics
- Learning outcomes: after taking this course students are able to
  - apply numerical methods for solving problems of rootfinding, numerical integration and differentiation, fitting model functions to given data
  - select suitable method of solution, analyze and prove their properties

Numerical Analysis, Course Overview 3

- Course content: **numerical differentiation, integration, root finding, fitting model functions to data points, to curves and surfaces, to given functions**
- Learning objectives: studying course content from the viewpoint of numerical mathematics
- Learning outcomes: after taking this course students are able to
  - apply numerical methods for solving problems of rootfinding, numerical integration and differentiation, fitting model functions to given data
  - select suitable method of solution, analyze and prove their properties
  - apply numerical methods to selected practical problems

Numerical Analysis, Course Overview related

QUESTIONS ?

- which slide? please give slide number
- which theorem?
- which example?
- which ...?

Examples of Functions 1

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.

Examples of Functions 2

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Average temperature in the classroom is a function of time (Temperature - dependent, time independent variable)

Examples of Functions 3

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin

Examples of Functions 4

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin
- Lake on campus: depth is a function of Cartesian coordinates with origin e.g. ...

Examples of Functions 5

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin
- Lake on campus: depth is a function of Cartesian coordinates with origin e.g. ...
- Campus shuttle: position is a function of time and distance from main entrance

Examples of Functions 6

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin
- Lake on campus: depth is a function of Cartesian coordinates with origin e.g. ...
- Campus shuttle: position is a function of time and distance from main entrance
- Earth: surface elevation above see level is a function of spherical coordinates

Examples of Functions 7

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin
- Lake on campus: depth is a function of Cartesian coordinates with origin e.g. ...
- Campus shuttle: position is a function of time and distance from main entrance
- Earth: surface elevation above see level is a function of spherical coordinates
- Digital image: piece-wise constant function of Cartesian coordinates, origin - left lower corner

Examples of Functions 7

- Polynomial $P_n(x) = \sum_{i=0}^{n} a_i x^i, x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Trigonometric
  $T(x) = \sum_{i=0}^{n} (a_i \sin(x) + b_i \cos(x)), x, a_i \in \mathbb{R}, i = 0, 1, .., n$
- Rational $R_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}$, etc.
- Classroom: average temperature is a function of time (Temperature - dependent, time independent variable)
- Water channel on campus: cross section's average flow velocity is a function of time and distance from its origin
- Lake on campus: depth is a function of Cartesian coordinates with origin e.g. ...
- Campus shuttle: position is a function of time and distance from main entrance
- Earth: surface elevation above see level is a function of spherical coordinates
- Digital image: piece-wise constant function of Cartesian coordinates, origin - left lower corner

What do you do with functions ?

- Compute function value

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries
- Can you compute determinant accurate enough, fast and at low cost?

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries
- Can you compute determinant accurate enough, fast and at low cost?
- Cramer's rule solves the problem - slow and expensive: why?

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries
- Can you compute determinant accurate enough, fast and at low cost?
- Cramer's rule solves the problem - slow and expensive: why?
- Gaussian elimination - much better why?

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries
- Can you compute determinant accurate enough, fast and at low cost?
- Cramer's rule solves the problem - slow and expensive: why?
- Gaussian elimination - much better why?
- fast matrix multiplication based Strassen algorithm costs $O(n^{\log_2(7)})$ arithmetic operations only!

What do you do with functions ?

- Compute function value
  - accurate enough
  - fast
  - low cost
- Are these requirements really needed?

Example 1.1

- Determinant of a matrix is multivariate function of matrix entries
- Can you compute determinant accurate enough, fast and at low cost?
- Cramer's rule solves the problem - slow and expensive: why?
- Gaussian elimination - much better why?
- fast matrix multiplication based Strassen algorithm costs $O(n^{\log_2(7)})$ arithmetic operations only!
- Thomas algorithm for tridiagonal matrices - almost perfect why?

What do you do with functions ? cont.

- Compute function value

What do you do with functions ? cont.

- Compute function value
- Differentiate

What do you do with functions ? cont.

- Compute function value
- Differentiate
- Integrate

What do you do with functions ? cont.

- Compute function value
- Differentiate
- Integrate
- Manipulate, e.g. compress

What do you do with functions ? cont.

- Compute function value
- Differentiate
- Integrate
- Manipulate, e.g. compress
- Subject to requirements

What do you do with functions ? cont.

- Compute function value
- Differentiate
- Integrate
- Manipulate, e.g. compress
- Subject to requirements
  - accurate enough
  - fast enough
  - low cost, acceptable cost

Functions related

QUESTIONS ?

- which slide? please give slide number
- which theorem?
- which example?
- which ...?

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h}$$

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x-h)}{2h}$$
- Could these formulas used for computing derivatives?

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x)-f(x-h)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x-h)}{2h}$$
- Could these formulas used for computing derivatives?
  - YES for calculating numerical approximation to function derivatives at x

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x)-f(x-h)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x-h)}{2h}$$
- Could these formulas used for computing derivatives?
  - YES for calculating numerical approximation to function derivatives at x
  - Do I still need approximate computations if exact formula for derivative is available?

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x)-f(x-h)}{h}$$
- $$f'(x) = \lim_{h\to 0+} \frac{f(x+h)-f(x-h)}{2h}$$
- Could these formulas used for computing derivatives?
  - YES for calculating numerical approximation to function derivatives at x
  - Do I still need approximate computations if exact formula for derivative is available?
  - Are these formulas good for any parameter h?

Computing function derivatives

Example 1.2

- $f(x)$ is continuously differentiable function (CALCULUS)
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h)-f(x)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x)-f(x-h)}{h}$$
- $$f'(x) = \lim_{h \to 0+} \frac{f(x+h)-f(x-h)}{2h}$$
- Could these formulas used for computing derivatives?
  - YES for calculating numerical approximation to function derivatives at x
  - Do I still need approximate computations if exact formula for derivative is available?
  - Are these formulas good for any parameter h?
  - Can I apriori estimate accuracy with respect to h?

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x+1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x + 1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $f'(0.25) = -9.066698770$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x+1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $f'(0.25) = -9.066698770$
- $$f'(x) \approx f'_{centralFD}(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$\begin{aligned} f'(x) \quad = \quad & \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x+1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \\ & \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}. \end{aligned}$$
- $f'(0.25) = -9.066698770$
- $$f'(x) \approx f'_{centralFD}(x) = \frac{f(x+h) - f(x-h)}{2h}$$
- $h = 0.005,\ f'_{centralFD}(0.25) = -9.067464295$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x + 1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $f'(0.25) = -9.066698770$
- $$f'(x) \approx f'_{centralFD}(x) = \frac{f(x+h) - f(x-h)}{2h}$$
- $h = 0.005,\ f'_{centralFD}(0.25) = -9.067464295$
- $|f'_{centralFD}(0.25) - f'(0.25)| \approx 7.7 \cdot 10^{-4}$

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x+1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $f'(0.25) = -9.066698770$
- $$f'(x) \approx f'_{centralFD}(x) = \frac{f(x+h)-f(x-h)}{2h}$$
- $h = 0.005,\ f'_{centralFD}(0.25) = -9.067464295$
- $|f'_{centralFD}(0.25) - f'(0.25)| \approx 7.7 \cdot 10^{-4}$
- Conclusions:

Computing function derivatives: exact vs approximate

Example 1.3

Source: J.V. Lambers, A.C.Summer, Explorations in Numerical Analysis, p.226

- $$f(x) = \frac{\sin^2\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$
- $$f'(x) = \frac{2\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\left[\frac{2x+1}{2\sqrt{x^2+1}(\cos x - x)} + \frac{\sqrt{x^2+1}(\sin x+1)}{(\cos x - x)^2}\right]}{\sin\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)} - \frac{\sin\left(\frac{\sqrt{x^2+x}}{\cos x - x}\right)\cos\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)\left[\frac{1}{2\sqrt{x}\sqrt{x^2+1}} - \frac{x(\sqrt{x}-1)}{(x^2+1)^{3/2}}\right]}{\sin^2\left(\frac{\sqrt{x}-1}{\sqrt{x^2+1}}\right)}.$$

- $f'(0.25) = -9.066698770$
- $$f'(x) \approx f'_{centralFD}(x) = \frac{f(x+h) - f(x-h)}{2h}$$
- $h = 0.005,\ f'_{centralFD}(0.25) = -9.067464295$
- $|f'_{centralFD}(0.25) - f'(0.25)| \approx 7.7 \cdot 10^{-4}$
- Conclusions: fast? accurate enough? low cost?

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2}f''(x_0)+\ldots+\frac{h^n}{n!}f^{(n)}(x_0)+O(h^{n+1})$$

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + ... + \frac{h^n}{n!} f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.5

- $$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) + \frac{h}{2} f''(x_0) + O(h^2)$$

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.5

- $$\frac{f(x_0+h) - f(x_0)}{h} = f'(x_0) + \frac{h}{2}f''(x_0) + O(h^2)$$
- $$|f'(x_0) - \frac{f(x_0+h) - f(x_0)}{h}| \approx O(h)$$
- Conclusions:

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \ldots + \frac{h^n}{n!} f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.5

- $$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) + \frac{h}{2} f''(x_0) + O(h^2)$$
- $$|f'(x_0) - \frac{f(x_0 + h) - f(x_0)}{h}| \approx O(h)$$
- Conclusions:
  - Formula is first order accurate wrt h

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + ... + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.5

- $$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0) + \frac{h}{2}f''(x_0) + O(h^2)$$
- $$|f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h}| \approx O(h)$$
- Conclusions:
  - Formula is first order accurate wrt h
  - For estimating first order derivative f should be twice continuously differentiable

Estimating accuracy of finite differences

Definition 1.4

Taylor's formula

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.5

- $$\frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) + \frac{h}{2}f''(x_0) + O(h^2)$$
- $$|f'(x_0) - \frac{f(x_0 + h) - f(x_0)}{h}| \approx O(h)$$
- Conclusions:
  - Formula is first order accurate wrt h
  - For estimating first order derivative f should be twice continuously differentiable
- Question: what if f is continuously differentiable only?

Finite differences for second order derivatives

Definition 1.6

Taylor's formula

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \ldots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Finite differences for second order derivatives

Definition 1.6

Taylor's formula

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \dots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.7

- $$\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0) + \frac{2h^2}{4!}f^{(4)}(x_0) + O(h^4)$$

Finite differences for second order derivatives

Definition 1.6

Taylor's formula

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x_0) + \dots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Example 1.7

- $$\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} = f''(x_0) + \frac{2h^2}{4!}f^{(4)}(x_0) + O(h^4)$$
- Question: what is accuracy of $\frac{f(x_0)-2f(x_0+h)+f(x_0+2h)}{h^2}$ at x_0?

Finite differences related

QUESTIONS ?

- which slide? please give slide number
- which theorem?
- which example?
- which ...?

Richaradson's extrapolation 1

Definition 1.8

Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

Richaradson's extrapolation 1

Definition 1.8

Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) + O(h^6)$$

Richaradson's extrapolation 1

Definition 1.8

Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \ldots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x,h) + b_2 h^2 + b_4 h^4 + O(h^6)$

Richaradson's extrapolation 1

Definition 1.8

Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \dots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x,h) + b_2h^2 + b_4h^4 + O(h^6)$
- $Error(h) = E(h) = b_2h^2 + b_4h^4 + O(h^6)$

Richaradson's extrapolation 1

Definition 1.8

Taylor's formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) + \frac{h^5}{5!}f^{(5)}(x) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f^{(3)}(x_0) + \frac{h^4}{4!}f^{(4)}(x) - \frac{h^5}{5!}f^{(5)}(x) + \dots + \frac{(-h)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x,h) + b_2h^2 + b_4h^4 + O(h^6)$
- $Error(h) = E(h) = b_2h^2 + b_4h^4 + O(h^6)$
- b_2, b_4 does not depend on h

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x, h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x, h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 (2h)^2 + b_4 (2h)^4 + O(h^6)$

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x, h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 (2h)^2 + b_4 (2h)^4 + O(h^6)$
- $Error(2h) = E(2h) = b_2 4 h^2 + b_4 16 h^4 + O(h^6)$

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x,h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x,2h) + b_2 (2h)^2 + b_4 (2h)^4 + O(h^6)$
- $Error(2h) = E(2h) = b_2 4h^2 + b_4 16h^4 + O(h^6)$
- $4f'(x) = 4f'_{centralFD}(x,h) + b_2 4h^2 + b_4 4h^4 + O(h^6)$

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x,h) + b_2h^2 + b_4h^4 + O(h^6)$
- $Error(h) = E(h) = b_2h^2 + b_4h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x,2h) + b_2(2h)^2 + b_4(2h)^4 + O(h^6)$
- $Error(2h) = E(2h) = b_24h^2 + b_416h^4 + O(h^6)$
- $4f'(x) = 4f'_{centralFD}(x,h) + b_24h^2 + b_44h^4 + O(h^6)$
- $f'(x) = f'_{centralFD}(x,2h) + b_24h^2 + b_416h^4 + O(h^6)$

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x, h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 (2h)^2 + b_4 (2h)^4 + O(h^6)$
- $Error(2h) = E(2h) = b_2 4h^2 + b_4 16h^4 + O(h^6)$
- $4f'(x) = 4f'_{centralFD}(x, h) + b_2 4h^2 + b_4 4h^4 + O(h^6)$
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 4h^2 + b_4 16h^4 + O(h^6)$
- $3f'(x) = 4f'_{centralFD}(x, h) - f'_{centralFD}(x, 2h) - b_4 12h^4 + O(h^6)$

Richaradson's extrapolation 2

- $$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{3!} f^{(3)}(x) - \frac{h^4}{5!} f^{(5)}(x) + O(h^6)$$
- $f'(x) = f'_{centralFD}(x, h) + b_2 h^2 + b_4 h^4 + O(h^6)$
- $Error(h) = E(h) = b_2 h^2 + b_4 h^4 + O(h^6)$
- b_2, b_4 does not depend on h
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 (2h)^2 + b_4 (2h)^4 + O(h^6)$
- $Error(2h) = E(2h) = b_2 4h^2 + b_4 16h^4 + O(h^6)$
- $4f'(x) = 4f'_{centralFD}(x, h) + b_2 4h^2 + b_4 4h^4 + O(h^6)$
- $f'(x) = f'_{centralFD}(x, 2h) + b_2 4h^2 + b_4 16h^4 + O(h^6)$
- $3f'(x) = 4f'_{centralFD}(x, h) - f'_{centralFD}(x, 2h) - b_4 12h^4 + O(h^6)$
- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h)+f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4)$

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h)+f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4)$
- Conclusions

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h)+f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4)$
- Conclusions
  - Richardson's extrapolation achieves high order accuracy by means of combining lower order formulas

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h)+f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4)$
- Conclusions
  - Richardson's extrapolation achieves high order accuracy by means of combining lower order formulas
  - Formal high order accuracy requires high order derivatives from the function under consideration
- Questions

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h) + f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4)$
- Conclusions
  - Richardson's extrapolation achieves high order accuracy by means of combining lower order formulas
  - Formal high order accuracy requires high order derivatives from the function under consideration
- Questions
  - Using Richardson's extrapolation can you obtain FD formula having accuracy $O(h^{2k}), k > 2$

Richaradson's extrapolation 3

- $f'(x) = \frac{4f'_{centralFD}(x,h) - f'_{centralFD}(x,2h)}{3} + O(h^4)$
- $f'(x) = \frac{-f(x+2h)+f(x+h)-8f(x-h)+f(x-2h)}{12h} + O(h^4)$
- Conclusions
  - Richardson's extrapolation achieves high order accuracy by means of combining lower order formulas
  - Formal high order accuracy requires high order derivatives from the function under consideration
- Questions
  - Using Richardson's extrapolation can you obtain FD formula having accuracy $O(h^{2k}), k > 2$
  - Is Richardson's extrapolation valid for computing second order derivatives?

Richardson's extrapolation related

QUESTIONS ?

- which slide? please give slide number
- which theorem?
- which example?
- which ...?

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula

$$f'(x) \approx \frac{1}{h} \sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h} \sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h}\sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i
3. Expand $f(x + p_i h)$ in Taylor's series

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h} \sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i
3. Expand $f(x + p_i h)$ in Taylor's series
4. Substitute $f(x + p_i h)$ by its Taylor's series in (1)

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h}\sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i
3. Expand $f(x + p_i h)$ in Taylor's series
4. Substitute $f(x + p_i h)$ by its Taylor's series in (1)
5. Consider equation
$$f'(x) - \frac{1}{h}\sum_{i=1}^{m} c_i f(x + p_i h) = O(h^k) \tag{2}$$

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h} \sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i
3. Expand $f(x + p_i h)$ in Taylor's series
4. Substitute $f(x + p_i h)$ by its Taylor's series in (1)
5. Consider equation
$$f'(x) - \frac{1}{h} \sum_{i=1}^{m} c_i f(x + p_i h) = O(h^k) \tag{2}$$
6. From (2) extract system of equations wrt unknown coefficients by means of grouping them with $h^0, h^1, h^2, ...$ etc.

Finding undetermined coefficients 1

Considered approaches can be generalized in the following algorithm

Algorithm 1.9

Finding undetermined coefficients

1. Consider formula
$$f'(x) \approx \frac{1}{h}\sum_{i=1}^{m} c_i f(x + p_i h) \tag{1}$$
2. $c_i, p_i, i = 1, 2, ..., n$ are undetermined; define target stencil and set p_i
3. Expand $f(x + p_i h)$ in Taylor's series
4. Substitute $f(x + p_i h)$ by its Taylor's series in (1)
5. Consider equation
$$f'(x) - \frac{1}{h}\sum_{i=1}^{m} c_i f(x + p_i h) = O(h^k) \tag{2}$$
6. From (2) extract system of equations wrt unknown coefficients by means of grouping them with $h^0, h^1, h^2, ...$ etc.

Undetermined coefficients related

QUESTIONS ?

- which slide? please give slide number
- which theorem?
- which example?
- which ...?

Q & A