Annual Performance Review 2024: Mark Simmons

by
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ABSTRACT

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Chapter 1

Introduction

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1.1 Context and motivations

Modeling three-phase flow has important application in petroleum engineering, and environmental science. Accurately capturing physical outcomes from fluids such as oil, gas, and water in an underground reservoir is crucial for the oil and gas industry. The following discussion and analysis of numerical solutions provide a robust model to solve these coupled non-linear partial differential equations in a way that accurately describes physical phenomenon we can observe.

1.2 APR Outline

This APR includes 4 chapters. Chapter 1 provides a brief overview of the model problem and equations on the 3-phase 3-component black oil model. Chapter 2 gives information and detail surrounding time discretization and a new stabilization method proof. Chapter 3 will give details surrounding space discretization. Chapter 4 will give convergence results for a test on manufactured solutions and simulation results on a 2-D domain. Finally, a conclusion including a summary of the APR and details surrounding future work.

Chapter 2

Model problem and Equations

Contents

2.1 Chapter 2-section 1

We will be showing stability of the following numerical scheme

$$\frac{S_a^{n+1} - S_a^n}{\Delta t} - \nabla \cdot (\tilde{K_a} \nabla (S_a^{n+1} - Sa^n)) - \nabla \cdot (K_a^n \nabla S_a^n) = 0$$

Proof: Our first hypothesis is that $K_a^n \leq \tilde{K}^n$. We multiply by $2\Delta t S_a^{n+1}$ and we integrate over our domain. We have

$$2\int S_a^{n+1}(S_a^{n+1} - S_a^n) - 2\Delta t \int \tilde{K_a} S_a^{n+1} \nabla \cdot (\nabla (S_a^{n+1} - S_a^n)) - 2\Delta t \int K_a^n S_a^{n+1} \nabla \cdot (\nabla S_a^n) = 0$$

Integrating by parts:

$$2\int S_a^{n+1}(S_a^{n+1} - S_a^n) + 2\Delta t \tilde{K} \int \nabla (S_a^{n+1} - S_a^n) \nabla S_a^{n+1} + 2\Delta t \int K_a^n \nabla S_a^{n+1} \nabla S_a^n = 0$$

Using the polar identity, $2a(a-b) = a^2 - b^2 + (a-b)^2$, on the first integral: we have

$$2\int S_a^{n+1}(S_a^{n+1} - S_a^n) = \int (S_a^{n+1})^2 - \int (S_a^n)^2 + \int (S_a^{n+1} - S_a^{n+1})^2$$

Doing the same for the second integral:

$$2\Delta t\tilde{K}\int \nabla(S_a^{n+1} - S_a^n)\nabla S_a^{n+1} = \Delta t\tilde{K}_a\int (\nabla S_a^{n+1})^2 - \Delta t\tilde{K}_a\int (\nabla S_a^n)^2 + \Delta t\tilde{K}_a\int (\nabla S_a^{n+1} - \nabla S_a^n)^2$$

Combining and Simplifying by using the definition of L_2 norm, we have

$$\|S_a^{n+1}\|^2 - \|S_a^n\|^2 + \|S_a^{n+1} - S_a^n\|^2 + \Delta t\tilde{K}_a\|\nabla S_a^{n+1}\|^2 - \Delta t\tilde{K}_a\|\nabla S_a^n\|^2 + \Delta t\tilde{K}\|\nabla S_a^{n+1} - \nabla S_a^n\|^2$$

$$+2\Delta t\int K_a^n\nabla S_a^{n+1}\nabla S_a^n = 0$$

Isolating the last integral term and adding and subtracting S_a^{n+1}, we have

$$\begin{aligned}2\Delta t\int K_a^n\nabla S_a^{n+1}\nabla S_a^n &= 2\Delta t\int K_a^n\nabla(S_a^n - \nabla S_a^{n+1} + \nabla S_a^{n+1})\nabla S_a^{n+1}\\ &= -2\Delta t\int K_a^n\nabla(S_a^{n+1} - S_a^n)\nabla S_a^{n+1} + 2\Delta t\int K_a^n(\nabla S_a^{n+1})^2\end{aligned}$$

using the polar identity again and using the definition of the L^2 norm, the last term we get is:

$$\begin{aligned}2\Delta t\int K_a^n\nabla S_a^{n+1}\nabla S_a^n &=\\ -2\Delta t\int K_a^n\nabla(S_a^{n+1} - S_a^n)\nabla S_a^{n+1} + 2\Delta t\int K_a^n(\nabla S_a^{n+1})^2 &=\\ -\Delta t\|K_a^n\nabla S_a^{n+1}\|^2 + \Delta t\|K_a^n\nabla S_a^n\|^2 - \Delta t\|K_a^n(\nabla S_a^{n+1} - \nabla S_a^n)\|^2 + 2\Delta t\|K_a^n\nabla S_a^{n+1}\|^2 &=\\ \Delta t\|K_a^n\nabla S_a^{n+1}\|^2 + \Delta t\|K_a^n\nabla S_a^n\|^2 - \Delta t\|K_a(\nabla S_a^{n+1} - \nabla S_a^n)\|^2\end{aligned}$$

Adding back into original,

$$\|S_a^{n+1}\|^2 - \|S_a^n\|^2 + \|S_a^{n+1} - S_a^n\|^2 + \Delta t\tilde{K}_a\|\nabla S_a^{n+1}\|^2 - \Delta t\tilde{K}_a\|\nabla S_a^n\|^2 + \Delta t\tilde{K}\|\nabla S_a^{n+1} - \nabla S_a^n\|^2$$

$$+\Delta t\|K_a^n \nabla S_a^{n+1}\|^2 + \Delta t\|K_a^n \nabla S_a^n\|^2 - \Delta t\|K_a^n(\nabla S_a^{n+1} - \nabla S_a^n)\|^2 = 0$$

since we assumed that $K_a^n \leq \tilde{K}_a$, we can say that

$$\Delta t\tilde{K}\|\nabla S_a^{n+1} - \nabla S_a^n\|^2 - \Delta t\|K_a^n(\nabla S_a^{n+1} - \nabla S_a^n)\|^2 \geq 0$$

Also we know that $\|S_a^{n+1} - S_a^n\|^2 + \|K_a^n \nabla S_a^{n+1}\|^2 + \|K_a^n \nabla S_a^n\|^2 \geq 0$,

Thus by rearanging terms,

$$\|S_a^{n+1}\|^2 + \Delta t\tilde{K}_a\|\nabla S_a^{n+1}\|^2 \leq \|S_a^n\|^2 + \Delta t\tilde{K}_a\|\nabla S_a^n\|^2$$

This concludes the proof. $\square$ REMEMBER TO ADD PUNCTUATION AND FORMAT MUTLILINE BETTER

Chapter 3

Chapter 3 Title

Contents

3.1 Chapter 3 - Section 1

3.1.1 Quarter of Five Spot Problem

The domain is $\Omega = [0, 100]^2 \backslash \{[0, 5]^2 \cup [95, 100]^2\} \, m^2$, i.e., it is the square $[0, 100]^2 \; m^2$ where two corners have been cut out; see following figure.

The boundary conditions are

$$\text{Inflow on } \Gamma_1 : p_\ell = 3 \times 10^5 \; Pa, \;\; s_a = 0.7$$

$$\text{Outflow on } \Gamma_4 : p_\ell = 10^5 \, Pa, \;\; \lambda_a \kappa \nabla p_c \cdot \mathbf{n} = 0$$

$$\text{No-Flow on } \Gamma_2 \cup \Gamma_3 \cup \Gamma_5 \cup \Gamma_6 : \lambda_\ell \kappa \nabla p_\ell \cdot \mathbf{n} = 0, \;\; \lambda_a \kappa \nabla p_c \cdot \mathbf{n} = 0.$$

The initial conditions are $s_a = 0.2$ and $p_\ell = 10^5\ Pa$. The rest of the flow properties are given by:

$$\phi = 0.2, \quad \mu_\ell = 2 \times 10^{-3}\, Pa \cdot s, \quad \mu_a = 5 \times 10^{-4}\, Pa \cdot s.$$

The relative permeability for each phase and capillary pressure follow a Brooks-Corey model:

$$\kappa_\ell = (1 - s_a)^2(1 - s_a^{5/3}), \quad \kappa_a = s_a^{11/3}, \quad p_c = 5 \times 10^3 s_a^{-1/3}$$

We consider a heterogeneous medium where $\kappa = 5 \times 10^{-8}\, m^2$ everywhere, except for in the square $[25, 50] \times [25, 50]$ where κ is 1,000 times lower.

3.1.2 Numerical Results

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	64	2.20E-2	-	8.81E-3	-	3.58E-2	-
0.125	256	5.70E-3	1.95	4.41E-3	0.99	1.15E-2	1.64
0.0625	1024	1.31E-3	2.12	2.24E-3	0.98	5.55E-3	1.06
0.03125	4096	4.43E-4	1.56	1.12E-3	0.96	2.56E-3	1.11
0.015625	16384	1.67E-4	1.41	5.80E-4	0.99	1.26E-3	1.03

Table 3.1: rates of convergence for $\mathbb{Q}_1 - \mathbb{RT}_1-$ BDF1

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	64	2.11E-2	-	8.44E-3	-	4.08E-2	-
0.0625	256	5.55E-3	1.94	2.20E-3	1.94	5.08E-3	3.01
0.015625	1024	1.38E-3	1.99	5.55E-4	1.98	1.17E-3	2.12
0.003906	4096	3.44E-4	2.01	1.39E-4	2.00	2.89E-4	2.02
0.000976	16384	8.71E-5	1.98	3.47E-5	2.00	7.21E-5	2.00

Table 3.2: rates of convergence for $\mathbb{Q}_1 - \mathbb{RT}_1-$ BDF1 where $\Delta t = h^2$

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	144	3.94E-3	-	8.62E-3	-	3.43E-2	-
0.125	576	3.14E-3	0.32	4.41E-3	0.97	1.12E-2	1.55
0.0625	2304	6.22E-4	2.34	2.25E-3	0.97	5.54E-3	1.08
0.03125	9216	3.21E-4	0.96	1.15E-3	0.94	2.57E-3	1.11
0.015625	36864	1.49E-4	1.10	5.81E-5	0.99	1.26E-3	1.03

Table 3.3: rates of convergence for $\mathbb{Q}_2 - \mathbb{RT}_2-$ BDF1

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	144	3.61E-3	-	8.62E-3	-	3.43E-2	-
0.0625	576	7.33E-4	2.30	4.41E-3	1.83	1.12E-2	2.96
0.015625	2304	1.81E-4	2.02	2.25E-3	1.96	5.54E-3	2.11
0.00390625	9216	4.35E-5	2.06	1.15E-3	1.99	2.57E-3	2.02
0.000976	36384	1.08E-5	2.01	3.76E-5	2.00	7.16E-5	2.01

Table 3.4: rates of convergence for $\mathbb{Q}_2 - \mathbb{RT}_2-$ BDF1 where $\Delta t = h^2$

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	64	3.61E-3	-	8.62E-3	-	3.43E-2	-
0.0625	256	7.33E-4	2.30	4.41E-3	1.83	1.12E-2	2.96
0.015625	1024	1.81E-4	2.02	2.25E-3	1.96	5.54E-3	2.11
0.00390625	4096	4.35E-5	2.06	1.15E-3	1.99	2.57E-3	2.02
0.000976	16384	1.08E-5	2.01	3.76E-5	2.00	7.16E-5	2.01

Table 3.5: rates of convergence for $\mathbb{Q}_1 - \mathbb{RT}_1-$ BDF1 STAB METHOD

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	64	2.38E-2	-	9.23E-3	-	2.39E-2	-
0.0625	256	5.52E-3	2.11	2.37E-3	1.96	5.27E-3	2.18
0.015625	1024	1.39E-3	1.99	6.04E-4	1.97	1.19E-3	2.15
0.00390625	4096	3.50E-4	1.99	1.51E-4	2.00	2.91E-4	2.03
0.000976	16384	8.79E-5	2.00	3.78E-5	2.00	7.25E-5	2.00

Table 3.6: rates of convergence for $\mathbb{Q}_1 - \mathbb{RT}_1-$ BDF1 where $\Delta t = h^2$ STAB METHOD

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	144	4.72E-3	-	9.11E-3	-	2.22E-2	-
0.0625	576	1.88E-3	1.33	4.69E-3	0.96	1.27E-2	0.80
0.015625	2304	9.48E-4	0.99	2.43E-3	0.94	5.31E-3	1.26
0.00390625	9216	4.40E-4	1.11	1.24E-3	0.97	2.50E-3	1.09
0.000976	36384	2.08E-4	1.08	6.27E-4	0.99	1.12E-3	1.04

Table 3.7: rates of convergence for $\mathbb{Q}_2 - \mathbb{RT}_2-$ BDF1 STAB METHOD

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	144	3.61E-3	-	8.62E-3	-	3.43E-2	-
0.0625	576	7.33E-4	2.30	4.41E-3	1.83	1.12E-2	2.96
0.015625	2304	1.81E-4	2.02	2.25E-3	1.96	5.54E-3	2.11
0.00390625	9216	4.35E-5	2.06	1.15E-3	1.99	2.57E-3	2.02
0.000976	36384	1.08E-5	2.01	3.76E-5	2.00	7.16E-5	2.01

Table 3.8: rates of convergence for $\mathbb{Q}_2 - \mathbb{RT}_2-$ BDF1 where $\Delta t = h^2$ STAB METHOD

Δt	DOF's	p_ℓ		S_a		S_v	
		L^2 error	Rate	L^2 error	Rate	L^2 error	Rate
0.25	144	4.72E-3	-	9.11E-3	-	2.22E-2	-
0.0625	576	4.42E-4	3.42	1.12E-3	2.87	2.51E-3	3.15
0.015625	2304	5.03E-5	3.13	1.58E-4	2.98	2.97E-4	3.08
0.00390625	9216	6.29E-6	3.00	1.98E-5	3.00	3.68E-5	3.01

Table 3.9: rates of convergence for $\mathbb{Q}_2 - \mathbb{RT}_2-$ BDF1 where $\Delta t = h^3$ STAB METHOD

Chapter 4

Conclusion

4.1 Outcome

4.2 Outlook