Prerequisite Definitions and Theorems

Neighbourhood

Before we can approach the formal definitions of limits we need to introduce a new concept: a neighbourhood.

Conceptually, the neighbourhood around point a on a number line, with some "radius" ε (pronounced "epsilon"), denoted as $U(a, \varepsilon)$, is the set of all points within ε distance away from point a, excluding points exactly ε away from a.

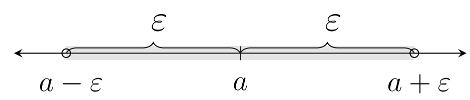


Figure 1: Neighbourhood $U(a, \varepsilon)$ around a, of radius ε shaded on number line.

A neighbourhood around $a \in \mathbb{R}$, is then formally defined as the following open interval (Figure 1).

$$U(a, \varepsilon) = (a - \varepsilon; a + \varepsilon) \tag{1}$$

Please note that ε is always defined as a positive value (If you set it as negative nothing really changes. Instead of being an interval from $a - \varepsilon$ to $a + \varepsilon$, it would be an interval from $a + \varepsilon$ to $a - \varepsilon$. For the sake of consistency, comfort, personal sanity and convenience we always define ε to be positive).

Punctured Neighbourhood

It is also very useful to introduce the concept of a punctured neighbourhood.

It is just a neighbourhood around some $a \in R$ excluding the point a.

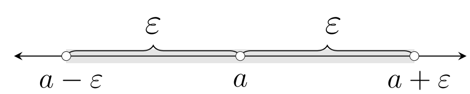


Figure 2: Punctured neighbourhood $\mathring{U}(a, \varepsilon)$ around a, of radius ε shaded on number line excluding point a.

It is often denoted with a ring above the symbol of a "normal" neighbourhood. E.G. if U is an un-punctured neighbourhood, then $\mathring{U}$ is a punctured neighbourhood. The definition is then as follows:

$$\mathring{U}(a, \varepsilon) = U(a, \varepsilon) - \{a\} = (a - \varepsilon; a + \varepsilon) \backslash \{a\} \tag{2}$$

This specific concept will be very important later.

*Note that both -$\{a\}$ and $\backslash\{a\}$ mean the same thing: to exclude element a from the set preceding it.

Rewriting the Definitions

Let's find a different way to write down definitions (1) and (2). Let's consider the "restrictions" on elements in the two types neighbourhoods (what must be true about a number, that would mean that it is part of some punctured/un-punctured neighbourhood).

Neighbourhood

Let U be a neighbourhood of radius ε around some $a \in \mathbb{R}$.

Take any point/element $x \in U$.

From definition (1) and Figure 1 we can clearly see that if $x \in U \Rightarrow a - \varepsilon < x < a + \varepsilon$

Subtract a from both sides of the inequality to get:

$$-\varepsilon < x - a < \varepsilon$$

It is possible to notice that the inequality above can be rewritten as:

$$|x - a| < \varepsilon$$

(This is true as $x - a$ is the distance between points, with the absolute value brackets simply converting the value to be positive).

We can then rewrite the definition of the neighbourhood (open interval) by expressing it as the following set:

$$U(a, \varepsilon) = \{x \,\big|\, |x - a| < \varepsilon\} \tag{3}$$

Punctured Neighbourhood

Let's try and use the logic above to come up with a new definition of a punctured neighbourhood.

Let's notice that the difference between a neighbourhood and a punctured neighbourhood is the absence of the point a.

We can then say that a punctured neighbourhood of radius ε around a, is the set of points (elements) that are less than ε distance away from a, but further than 0 distance away from a.

We can then write down the following inequalities:

$$-\varepsilon < x - a < 0 \text{ or } 0 < x - a < \varepsilon$$

Analogically to the previous argument (for the neighbourhood), we can notice that the inequality above is equivalent to the following:

$$0 < |x - a| < \varepsilon$$

We can then rewrite the definition of a punctured neighbourhood (punctured open interval), by expressing it as the following set:

$$\mathring{U}(a, \varepsilon) = \{x \,\big|\, 0 < |x - a| < \varepsilon\} \tag{4}$$

Triangle Inequality

The following triangle inequalities are true $\forall a, b \in \mathbb{R}$:

$$|a+b| \leq |a| + |b| \tag{5}$$

$$|a| - |b| \leq |a-b| \tag{6}$$

First consider statement (5):

We can prove this by examining 3 possible cases:

1. a and b are positive numbers
2. a and b are of opposite signs
3. a and b are both negative numbers

1) Suppose $a > 0$, $b > 0$. Then:

$a + b > 0 \Rightarrow a + b = |a+b| = |a| + |b|$ (by definition of absolute values)

The proof is identical if both a and b are negative.

2) Suppose $a > 0$, $b < 0$.

We will now consider 3 sub-cases:

a) $|a| > |b|$

b) $|a| < |b|$

c) $|a| = |b|$

a) Suppose $a > 0, b < 0, |a| > |b|$. Then:

$|a| + |b| = a + (-b) > a + b = |a+b| > 0$

b) Suppose $a > 0, b < 0, |a| < |b|$. Then:

$|a| + |b| = a + (-b) > (-b) - a = |a+b| > 0$

c) Suppose $a > 0, b < 0|a| = |b|$ Then:

$|a+b| = |a-a| = 0 \leq |a| + |b|$

Proof of the second inequality is left as an exercise.

Small Extension

This section will contain definitions that will be used in extension sections:

Open Set

For the real numbers $\mathbb{R}$, define an open set to be an arbitrary union of open intervals (could be around different points with different radii).

*Arbitrary Union - union of potentially infinitely many sets.

The empty set (set containing no elements denoted by $\varnothing$) as well as the set containing all real numbers, are also open sets.

Closed Set

A set A is called a closed set if the set $\mathbb{R} - A$ is an open set.

Practice problems

1. Which of the following are open sets:

   (a) $(1, 7) \cup (7, 20)$

   (b) $\{x \,\big|\, |x - 7| < 3\}$

   (c) $\{x \mid -0.1 < x - 2 < 0.1\}$

   (d) $\mathbb{R} - \{1\}$

   (e) $\mathbb{R}$

2. Draw the following sets on a number line (shade the open set)

   (a) $(1, 2)$

   (b) $(0, 1) \cup (-2, -1)$

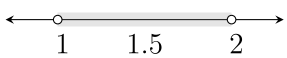
   (c) $\{x \mid |x| < 1\}$

Answers:

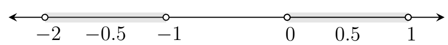
1. All of the above are open sets

    - It is a union of open intervals
    - This is an open interval around 7 with radius 3
    - This is an open interval around 2 with radius 0.1
    - Note that $\mathbb{R} - \{1\} = (-\infty, 1) \cup (1, \infty)$, and is therefore an open set. *Note that $(1, \infty)$ is the union of all intervals $(n, 2n)$ for $n \in \mathbb{N}$ $n \geq 1$
    - This is the union of all possible open intervals around all points. (or for example the union of intervals of all possible radii centered around some number a).

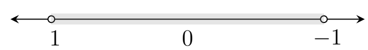
2. Drawings for Q2



(a)



(b)



(c)