1 Definitions

1.1 Manifold

Definition 1 *A set M is an m-dimensional manifold if it is equipped with a collection $\{(U_i, \varphi_i)\}_{i \in \Lambda}$ where $U_i \subseteq M$ and $\varphi_i : R^m \supseteq \varphi_i^{-1}(U_i) \to U_i$.that satisfies the following properties*

i) Any point in M is covered by some U_i in the collection (i.e. $\forall p \in M \exists U_i s.t. p \in U_i$

ii) The functions φ_i is a bijection between $\varphi^{-1}(U_i) \subseteq R^m$ and U_i and $\varphi_i^{-1}(U_i)$ is an open subset of $\mathbb{R}^m$

iii) For any i, j the following map is a diffeomorphism $\varphi_i^{-1} \circ \varphi_j : \varphi_j^{-1}(U_i \cap U_j) \to \varphi_i^{-1}(U_i \cap U_j)$

We call φ_i "coordinate charts" and $\varphi_i^{-1} \circ \varphi_j$ transition maps.

The idea behind this definition is as follows. We have some set and we want to put some coordinates on it. Condition i), just requires that we can put coordinates on my whole space. Condition ii) requires that our coordinates are well defined (i.e. each coordinate refers only to one point). iii) We demand that if we have two different coordinate systems for the same part of my manifold, then these coordinate systems should be the same up-to diffeomorphism (my physics should not depend on the coordinate system I chose).

1.2 Tangent Space at a Point

Suppose we are given a manifold M. We now want to define a notion of "direction" on it. We will later need this, to define define vector fields on our manifolds (a vector that tells you where to go at each point) and define "flows" on them using differential equations.
Our manifold, has these "coordinates" that come from $\mathbb{R}^m$. In $\mathbb{R}^m$ we have a natural notion of direction, so we will try and use that to define tangent vectors on M.
It is easy to realize, that directions in $\mathbb{R}^m$, can be identified with derivatives of functions defined on $\mathbb{R}^m$. We therefore have the following definition:

Definition 2 *A vector X_{x_0} at a point $x_0 \in \mathbb{R}^m$, is a derivative in some direction $\vec{v}$ at point x. Note that this can be expressed in the basis of $\frac{\partial}{\partial x_i}$ with:*

$$X_{x_0} = \left.\frac{\partial}{\partial \vec{v}}\right|_x = a_1 \frac{\partial}{\partial x_i} + ... + a_m \frac{\partial}{\partial x_m}$$

Note that a derivative at a point is some linear map, that takes in a function and outputs a number. It also satisfies the Leibnitz (product) rule. In fact this fully characterizes derivatives.
We will therefore use this, as inspiration to define tangent vectors (directions at a point), on our manifold and consider things that look like derivatives at a point, on my manifold. Before we do this, we first need to prepare the functions we will be differentiating.

Definition 3 *Given a manifold M, we define*

$$C_p^\infty = \{f : U \to \mathbb{R} | U \textit{ an open set of } M \textit{ and for any } \varphi_i\ f(\varphi_i|_U) \textit{ is infinitely differentiable}\}$$

We can then define the tangent vectors at some point p in my Manifold:

Definition 4 *The following set, is called the tangent space to point $p \in M$. Elements of this set are called tangent vectors.*

$$T_pM = \left\{ X_p : C_p^\infty \to \mathbb{R} \;\middle|\; X_p - \textit{ linear and } X_p(f \cdot g) = X_p(f)g(p) + f(p)X_p(g) \right\}$$

It should be clear, that since T_pM is a collection of linear maps with some nice condition, it should itself be a vector space (which is a good idea for a tangent space. Directions make sense to be vectors). This definition is fairly intuitive, as we define the directions at a point p of our manifold to be things like taking derivatives in coordinate charts. In fact, we can show that the two are completely the same.
Let φ be some coordinate chart of M. Consider the following map:

$$\begin{aligned} \Psi :\ & \text{vectors at } x \text{ in } \mathbb{R}^m \to T_\varphi(x)M \\ & \left.\frac{\partial}{\partial \vec{v}}\right|_x \to \left(f \to \left.\frac{\partial}{\partial \vec{v}}\right|_x f \circ \varphi \right) \end{aligned} \tag{1}$$

It should be not too hard to check, that the above defines an isomorphism of vector spaces (i.e. the map above is linear, injective and surjective).

1.3 Tangent Bundles

In the previous section we defined the tangent space to a point of our manifold. We can now consider at taking the tangent spaces at every point of my manifold, and uniting them in one single object, called the tangent bundle.

Definition 5 *Let M be a manifold of dimension m with charts (U_i, φ_i). The tangent bundle is the set:*

$$TM = \sqcup_{p \in M} T_p M$$

With charts:

$$\psi_i : \varphi_i^{-1}(U_i) \times \mathbb{R}^m \to V_i \subseteq TM$$

Defined via:

$$\psi_i(x, v) = \left(f \to \frac{\partial}{\partial \vec{v}} \bigg|_x f \circ \varphi \right)$$

The claim is that with the structure above, TM is a manifold of dimension $2m$ (first coordinate tells me the point I am at, second coordinate tells me the vector at that point)

2 Suggested Exercises

I) The Sphere as a manifold.
We define the 2-dimensional sphere in $\mathbb{R}^3$ to be the following set:

$$S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 = 1 \right\}$$

We also define the north and south pole points to be:

$$N = (0, 0, 1) \quad S = (0, 0, -1)$$

Let:

$$U_N = S^2 \backslash N \quad U_S = S^2 \backslash S$$

We then consider the following construction:

$$\varphi_N : \mathbb{R}^2 \to U_N$$
$$(x, y) \to \text{ point where the line connecting } (x, y, 0) \text{ to } N \text{ intersects } S^2$$

$$\varphi_S : \mathbb{R}^2 \to U_N$$
$$(x, y) \to \text{ point where the line connecting } (x, y, 0) \text{ to } S \text{ intersects } S^2$$

(a) Write down explicitely what φ_N and φ_S are (i.e. give me a formula for their output)

(b) Compute the inverse of φ_S.

(c) Compute $\varphi_S^{-1} \circ \varphi_N$ and show that it is a diffeomorphism of $\mathbb{R}^2 \backslash (0, 0)$ and $\mathbb{R}^2 \backslash (0, 0)$

II) Consider the circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \,\Big|\, x^2 + y^2 = 1\}$$

We can give it a chart:

$$\varphi_1 : (-0.5, 0.5) \to \S^1$$
$$t \to (\cos(2\pi t), \sin(2\pi t))$$

i) Come up with another chart, to fully cover S^1 and make above into a manifold (convince yourself that these two charts would be compatible).

ii) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$. Write down the restriction of this function $f|_{\S^1}$in local coordinates.

iii) Compute the derivative of this function in local coordinates.

iv) Show that if a function is the restriction of some function $f(x, y)$ to the circle, then any $X \in T_{(0,0)}S^1$ acts on f at (0,0) as a multiple of $\frac{\partial}{\partial y}$. In other words, that:

$$X_{(0,0)} \left(f\big|_{S^1} \right) = c \frac{\partial f}{\partial y}\bigg|_{(0,0)}$$

III) Let M be an m dimensional manifold. Let φ be some chart of M. Show that Ψ as defined in (1) is indeed an isomorphism of vector spaces. This implies that the vector space