1 Review

Definition 1 *A set M is called a differentiable manifold, if it is equipped with countably many charts*[1] (U_α, ψ_α) *where* $U_\alpha \subseteq \mathbb{R}^n$ *and* $\psi_\alpha : U_\alpha \to M$ *is injective and has the following properties:*

i) $\forall y \in M\ \exists \alpha$ *such that for some* $x \in U_\alpha\ \psi_\alpha(x) = y$

ii) Suppose $\psi_{U_\alpha} \cap \psi_{U_\beta} \neq \emptyset$ *then the following map is differentiable:*

$$\psi_\beta^{-1} \circ \psi_\alpha : \psi_\alpha^{-1}(\psi_\alpha(U_\alpha) \cap \psi_\beta(U_\beta)) \to U_\beta$$

Definition 2 *The $n-$dimensional sphere S^n, is the following subset of* $\mathbb{R}^{n+1}$

$$S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$$

2 Stereographic Projection

In the following exercise, you will give the 2-dimensional sphere a manifold structure, using the idea of a stereographic projection. We use N and S to refer to the points of the North and South pole.

i) Consider S^2 as placed sitting above the plane $\mathbb{R}^2$, such that the center is at $(0, 0, 1)$. Consider $(x, y, 0)$ in this plane. Draw a line connecting this point and the North Pole of the sphere at (0,0,2). This is connected by some line parametrized via:
$$l(t) = (0, 0, 2) + t \cdot ((x, y, 0) - (0, 0, 2))$$
Find the point of intersection of $l(t)$ with the sphere.

ii) Define a map $\psi_1 : \mathbb{R}^2 \to S^2\backslash\{N\}$ using above (*Hint: Recall that in the stereographic projection you shifted the sphere up by 1.*)

iii) Repeat the same procedure, but now doing a stereographic projection from the South Pole. (Rotate your sphere, so that the south pole is at the top and do the same stuff). Use this to define $\psi_2 : \mathbb{R}^2 \to S^2\backslash\{S\}$

iv) Compute ψ_2^{-1}

v) Compute $\psi_2^{-1} \circ \psi_1$ and confirm that this is a differentiable (or at least continuous) map.

vi) Conclude that this defines an atlas on S^2 and makes it a manifold.

[1] This collection of charts is called an Atlas.