Preliminary knowledge

First let's look at the following expansion:

$$
\begin{aligned}
(z+y)^2 &= (z+y)(z+y) \\
&= z\cdot z + z\cdot y + y\cdot z + y\cdot y \\
&= z^2 + z\cdot y + z\cdot y + y^2 \\
&= z^2 + 2zy + y^2 \qquad (1)
\end{aligned}
$$

Now let's consider the expansion of $(z-y)^2$.

We will make the following substitution: $\widetilde{y} = -y$. Let's plug that into the equation above. We get:

$(z-y)^2 = (z+\widetilde{y}\,)^2$

This now looks like the expansion from above. Expand it using the equation (1) to get:

$$
\begin{aligned}
(z-y)^2 &= (z+\widetilde{y}\,)^2 \\
&= z^2 + 2z\widetilde{y} + \widetilde{y}^2
\end{aligned}
$$

Now let's substitute $\widetilde{y} = -y$ into the equation above to get:

$$
\begin{aligned}
(z-y)^2 &= z^2 + 2z(-y) + (-y)^2 \\
&= z^2 - 2zy + y^2
\end{aligned}
$$

It is important to note, that from the equations above it is evident that:

$\sqrt{z^2+y^2} \neq z+y$

Now let's consider the following expansion:

$$
\begin{aligned}
(x+y)(x-y) &= x^2 - xy + xy - y^2 \\
&= x^2 - y^2
\end{aligned}
$$

The left hand side $(x+y)(x-y)$ is factored form of the difference of squares (as evident from the statement above).

Completing the Square

Often times, it is useful to reorganize a mathematical statement in a way that would make it easy to manipulate.
One of such ways is making some statement look like $(z+y)^2$. This process is called completing the square.

Let's have a look at an example:

Let's turn part of $x^2 + 4x + 1$ into $(z+y)^2$

We can easily notice that: $(x+2)^2 = x^2 + 4x + 4 = (x^2 + 4x + 1) + 3$

Let's go back to our original statement, add 3 and subtract 3:

$x^2 + 4x + 1 + 3 - 3 = x^2 + 4x + 4 - 3 = (x + 2)^2 - 3$

This required us noticing what $(x + 2)^2$ expands into. But is there a more fool proof way of doing this?

Let's look at another example:

$4x^2 + 4x + 3$

To make a part of the statement above look like $(z + y)^2$, we need to make a part of it look like: $z^2 + 2zy + y^2$

First let's find something in our original statement that is already squared and choose it to be z^2. We can see that $4x^2 = (2x)^2$. Hence we will choose z as:
$z = 2x$

The only other part of the equation that has x is $4x$, therefore it must be the only other part that has a z in it. That is $2zy$. So let's find y:

$2zy = 4x$.

Let's substitute $z = 2x$ and solve for y:

$2(2x)y = 4x$

Divide both sides by $4x$

$y = 1$

Since y is just a number (has no xs), instead of searching for y^2 in the original statement, we could just add and subtract y^2 to the original statement to make things fit into $(z + y)^2$ to "complete" the square. We get:

$4x^2 + 4x + (1)^2 - (1)^2 + 3 = (2x + 1)^2 - 1 + 3 = (2x + 1)^2 + 2$

Let's have a look at one more example:

$3x^2 + x + 1$

We can immediately see that the thing with xs that is squared is: $3x^2$, so we select $z = \sqrt{3}x$.

$2zy = x$
$2 \cdot \sqrt{3}xy = x$
$y = \dfrac{1}{2\sqrt{3}}$

$3x^2 + x + (\dfrac{1}{2\sqrt{3}})^2 - (\dfrac{1}{2\sqrt{3}})^2 + 1$
$(\sqrt{3}x + \dfrac{1}{2\sqrt{3}})^2 + \dfrac{11}{12}$

Note, that with similar success we could have chosen $z = 1$ or even $z = 4x$, but that would mean that y would also contain an x, and that is not always useful. (You will see why later).

Solving the quadratic equation in a general way

Given the following equation:

$$ax^2 + bx + c = 0 \tag{2}$$

Where a, b, and c are some constants, let's find all values of x that make that equation true.

Our final goal is to get: $x =$ a number. Since we have x^2, we need to transform (2), so that we get x on one side and a number on the other. The issue is, we can't just do $ax^2 + bx = -c$, because in that case, we still can't do anything. We can't take a square root to get rid of the x^2. To do that, we must have something that look like: $(z + y)^2$ on one hand side, where either z or y have an x, and something without x on the other side.

To do this, we will need to make equation (2) be like equation (1).

$$ax^2 + bx + c = 0$$
$$ax^2 + bx + (\frac{b}{2\sqrt{a}})^2 - (\frac{b}{2\sqrt{a}})^2 + c = 0$$
$$ax^2 + bx + (\frac{b}{2\sqrt{a}})^2 = (\frac{b}{2\sqrt{a}})^2 - c \tag{3}$$

Why did we choose to add $(\frac{b}{2\sqrt{a}})^2$ to both sides?

Well we want things to look like: $(z + y)^2$.
Since we know that: $(z+y)^2 = z^2 + 2zy + y^2$, we need to find values for z and y, such that $(z+y)^2$ is part of $ax^2 + bx + c$

Let's choose $z = \sqrt{a}x$
That means that if we expand $(z + y)^2$ and plug in $z = \sqrt{a}x$ we will get:

$$(\sqrt{a}x + y)^2 = ax^2 + 2y\sqrt{a}x + y^2$$

Let's compare this to equation (2).

Since we want to put x into a square, we can't touch anything that has to do with x. That is why, we have to choose a y such that:

$$2y\sqrt{a}x = bx$$

From the equation above we can solve for y and get:

$$y = \frac{b}{2\sqrt{a}}$$

Now, to make equation (2) look like $z^2 + 2zy + y^2$, we just need to add the y term to both sides, which is exactly what we did to get to equation (3):

As mentioned earlier:

$$(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = ax^2 + bx + (\frac{b}{2\sqrt{a}})^2$$

Let's apply this statement to equation 3.

$$(\sqrt{a}x + \frac{b}{2\sqrt{a}})^2 = \frac{b^2}{4a} - c$$
$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\sqrt{\frac{b^2}{4a} - c}$$

$$
\begin{aligned}
&\sqrt{a}x = \frac{-b}{2\sqrt{a}} \pm \sqrt{\frac{b^2-4ac}{4a}} \\
&\sqrt{a}x = \frac{-b}{2\sqrt{a}} \pm \frac{\sqrt{b^2-4ac}}{2\sqrt{a}} \\
&\sqrt{a}x = \frac{-b \pm \sqrt{b^2-4ac}}{2\sqrt{a}} \\
&x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad (4)
\end{aligned}
$$

This is the general solution to equation (2).

A little bit of analysis

Let's look at equation (4). From the plus minus sign, we can see that there can be two possible solutions to equation (2) (two values of x, that when plugged into (2), make the equality true, $LS = RS$). But are there always two solutions?

Let's look at the following part of equation (4): $b^2 - 4ac$

This part is called the "discriminant" and is denoted by "D".

If $b^2 - 4ac > 0$, then as mentioned before there are two possible solutions (roots).

If $b^2 - 4ac = 0$, then there is only one solution. (Adding 0 is the same as subtracting 0, so the plus minus sign in that case doesn't change anything).

If $b^2 - 4ac < 0$, then there are no possible solutions. This is because in equation (4), we take the square root of the discriminant. If the discriminant is negative, then we can't take it's square root (You can't take a square root of a negative number (explain why)). Since we can't calculate the solution, then there is no solution.

Vieta's formulas

Let's consider some equation of the form of equation (2) which has at least 1 root ($D \geq 0$).
Let x_1 be the solution to (2) if we choose the positive sign in the plus minus of equation (4).
Let x_2 be the solution to (2) if we choose the positive sign in the plus minus of equation (4).

Let's look at the product of x_1 and x_2.

$$x_1 \cdot x_2 = \frac{-b + \sqrt{b^2-4ac}}{2a} \cdot \frac{-b - \sqrt{b^2-4ac}}{2a}$$

Let's make the following substitutions.

$z = -b$
$y = \sqrt{b^2 - 4ac}$

$$
\begin{aligned}
x_1 \cdot x_2 &= \frac{z+y}{2a} \cdot \frac{z-y}{2a} \\
&= \frac{(z+y)(z-y)}{4a^2}
\end{aligned}
$$

Here we can see that the numerator is the factored difference of squares. Let's write it as a difference of squares and sub the values of z and y back in.

$$\begin{aligned}
x_1 \cdot x_2 &= \frac{z^2 - y^2}{4a^2} \\
&= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} \\
&= \frac{b^2 - b^2 + 4ac}{4a^2} \\
&= \frac{4ac}{4a^2} \\
&= \frac{c}{a}
\end{aligned}$$

So we get:
$x_1 \cdot x_2 = \frac{c}{a}$

Now let's consider the sum of x_1 and x_2:

$$\begin{aligned}
x_1 + x_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2b}{2a} \\
&= \frac{-b}{a}
\end{aligned}$$

So we get:

$x_1 + x_2 = \frac{-b}{a}$

You might be asking, why are we doing all of this?

Well, now we have two equations with two unknowns (x_1 and x_2 are unknown, and a,b and c are given)! We can represent the solutions to the quadratic equation as solutions to the following system of equations!

$$\begin{cases} x_1 + x_2 = -\dfrac{b}{a} & (5) \\ x_1 \cdot x_2 = \dfrac{c}{a} & (6) \end{cases}$$

These equations are called Vieta's formula.

Using these two equations, we can express b and c in terms of x_1, x_2 and a.

$b = -a(x_1 + x_2)$
$c = a(x_1 \cdot x_2)$

Let's sub that back into equation (2).

$$
\begin{aligned}
ax^2 + bx + c &= ax^2 - a(x_1 + x_2)x + a(x_1 \cdot x_2) \\
&= a(x^2 - (x_1 + x_2)x + x_1 \cdot x_2) \\
&= a(x^2 - x_1 x - x_2 x + x_1 \cdot x_2)
\end{aligned}
$$

Now let's try and factor the statement above further!

$$
\begin{aligned}
a(x^2 - x_1 x - x_2 x + x_1 \cdot x_2) &= a(\, x(x - x_1) + x_2(x_1 - x)\,) \\
&= a(\, (x(x - x_1) - x_2(x - x_1)\,) \\
&= a(x - x_1)(x - x_2)
\end{aligned}
$$

So what does this mean?

We just proved that any quadratic equation that has at least 1 real root, can be factored in the form:

$ax^2 + bx + c = a(x - x_1)(x - x_2)$

Where x_1 and x_2 are the roots of equation (2). *Note, the roots here are interchangeable (you can switch their places and nothing changes) because of commutative properties of multiplication $(a \cdot b = b \cdot a)$.

Factoring

Often times, factoring is used to find the roots by guessing them! We use the system of equations mentioned above to help us guess the roots of an equation. Then we write it in factored form to double check if they are valid (expand into the original statement).

For example let's solve the following equation:

$x^2 - 3x + 2 = 0$

Here we have:

$a = 1$
$b = -3$
$c = 2$
Using Vieta's laws (5 and 6) we get:

$$
\begin{cases}
x_1 + x_2 = -3 \\
x_1 \cdot x_2 = 2
\end{cases}
$$

Now using this, we can guess and check to solve the system of equations above!
Since the sum of roots in the first equation is negative, we know that at least one of the roots must be negative.
Since the product of the roots is positive, and we know that one of the numbers is negative, that means that both numbers must be negative.

The numbers -3 and 2 are relatively small, so we will guess and check "small" negative numbers.

Let's try $x_1 = -1$

From the first equation we get that $x_2 = -2$

Let's try and sub that into the second equation!

$(-1) \cdot (-2) = 2$

It fits! So we guessed our roots, without having to sub stuff into the big and weird equation (4).

At the same time, this can be made even a little bit simpler. When we factor, instead of finding the roots of the equation x_1 and x_2, we can find the negatives of these roots: $\widetilde{x}_1 = -x_1$ and $\widetilde{x}_2 = -x_2$.

But how is this in any way helpful?

Let's first look back at Vieta's laws . If we substitute $x_1 = -\widetilde{x}_1$ and $x_2 = -\widetilde{x}_2$ into Vieta's equations (5 and 6) we get:

$$\begin{cases} \widetilde{x}_1 + \widetilde{x}_2 = \dfrac{b}{a} & (7) \\ \widetilde{x}_1\widetilde{x}_2 = \dfrac{c}{a} & (8) \end{cases}$$

If we also substitute these into the factored form of the original equation we get:

$ax^2 + bx + c = a(x + \widetilde{x}_1)(x + \widetilde{x}_2)$

So what's the difference?

Now, instead of searching for a sum of numbers equal to negative $\frac{b}{a}$, we search for a sum of numbers equal to something positive (which is arguably easier).
Let's look at an example:

Factor:

$x^2 + 4x - 21$

In this case:
$a = 1$
$b = 4$
$c = 21$

Using our "new" Vieta's laws we get that we are searching for two numbers that add up to 4, and when multiplied give us -21:

$$\begin{cases} \widetilde{x}_1 + \widetilde{x}_2 = 4 \\ \widetilde{x}_1 + \widetilde{x}_2 = -21 \end{cases}$$

7 and 3 are factors of 21. We can easily notice that 7-3=4. Hence, we guess our numbers to be: $\widetilde{x}_1 = 7$ and $\widetilde{x}_2 = -3$. We get:

$x^2 + 4x - 21 = (x + 7)(x - 3)$

In the case when $a \neq 1$, another simplification is required. Often times, $\frac{b}{a}$ and $\frac{c}{a}$ are not whole numbers. That would make life very difficult. So, if k_1 and k_2 are two factors of a such that: $k_1 \cdot k_2 = a$, then we can do the following:

$a(x+\widetilde{x}_1)(x+\widetilde{x}_2) = (k_1x + k_1\widetilde{x}_1)(k_2x + k_2\widetilde{x}_2)$

Let's rename our factoring once again:
$x_1^* = k_1\widetilde{x}_1$
$x_2^* = k_2\widetilde{x}_2$
So we are now trying to factor stuff in the following way:
$ax^2 + bx + c = (k_1x + x_1^*)(k_2x + x_2^*)$

Let's check out our Vieta's laws again and modify them one more time:
Modify equation (7):

$\widetilde{x}_1 + \widetilde{x}_2 = \dfrac{b}{a}$
$\widetilde{x}_1 + \widetilde{x}_2 = \dfrac{b}{k_1k_2}$
$k_1k_2\widetilde{x}_1 + k_1k_2\widetilde{x}_2 = b$
$k_2x_1^* + k_1x_2^* = b$

Modify Equation (8):
$\widetilde{x}_1\widetilde{x}_2 = \dfrac{c}{a}$
$\widetilde{x}_1\widetilde{x}_2 = \dfrac{c}{k_1k_2}$
$k_1k_2\widetilde{x}_1\widetilde{x}_2 = c$
$x_1^* \cdot x_2^* = c$

So we get:

$$\begin{cases} k_2x_1^* + k_1x_2^* = b & (9) \\ x_1^* \cdot x_2^* = c & (10) \end{cases}$$

Let's interpret this:
We are searching for two numbers that when multiplied give c. We also need to choose two factors of a. When the numbers we are searching for, are multiplied by the factors as: $k_1x_1^*$ and $k_2x_2^*$, they add up to b.

Let's have a look at a few examples:
Factor:

$2x^2 + 7x + 3$

In this case:
$a = 2$
$b = 7$
$c = 3$

a has only two factors: 2 and 1. Hence: $k_1 = 2$ $k_2 = 1$
c also has only two factors: 3 and 1.

From this we gain that we have two options of bracket combinations:

$(2x + 1)(x + 3)$
or
$(2x + 3)(x + 1)$

Then we remember that the following must be true:
$k_2 x_1^* + k_1 x_2^* = b$
$x_1^* + 2x_2^* = 7$

From here we can easily guess that:
$x_1^* = 1$ and $x_2^* = 3$

We get:
$2x^2 + 7x + 3 = (2x + 1)(x + 3)$

Let's look at another example:

$6x^2 + 7x + 2$

In this case:
$a = 6$
$b = 7$
$c = 2$

Factors of 6 are: 6,1 or 2,3.
Factors of 2 are: 2,1.
Let's first see of 6 and 1 work. In that case we would have two options of brackets:

$(6x + 1)(x + 2)$
or
$(6x + 2)(x + 1)$

Now we remember the second condition and check out the following:

$6 \cdot 2 + 1 \neq b = 7$
Also:
$6 \cdot 1 + 2 \neq b = 7$

This means, that using 6 and 1 as factors k_1 and k_2 is not correct. Let's now check out using $k_1 = 2$ and $k_2 = 3$. We then have two options:

$(2x + 1)(x + 3)$
or
$(2x + 3)(x + 1)$

Check out our second condition:

$2 \cdot 3 + 1 = b = 7$

This fits, so we get:
$6x^2 + 7x + 2 = (2x + 1)(x + 3)$