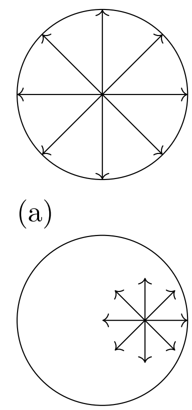
Pressure in Liquids and Archimedes' Principle

Prerequisite information

The following derivation will use concepts from the "pre-requisite information" document found in the physics section at `tinyurl.com/NFFCP` .

Pascal's Law. *Pressure applied on a liquid or gas is transmitted in all directions without a change in magnitude.*

Let's look at a balloon as an example. The gas in the center of the balloon (Figure 1 (a)), attempts to expand (applies the a force over an area) in all directions equally. It does not favor moving/pushing up over down as there is no difference between the two and therefore transmits the same pressure in all directions. The same is true for any part of gas inside the balloon (Figure 1 (b)).



(a)

(b)

Figure 1

Pressure inside a Liquid

Let's derive a formula for the pressure in a liquid of density ρ_l, h distance away from the surface. We will consider a case where the liquid is placed inside a rectangular container, with height h_0, width l and depth w.

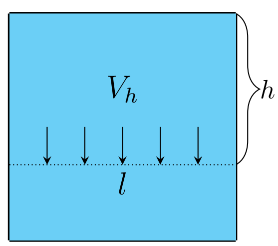


Figure 2: Side view of rectangular prism filled with a liquid of density ρ_l

Recall the definition of pressure:

$$P = \frac{F}{S} \tag{1}$$

Where:

P - pressure applied on object
F - force applied to object
S - area of surface of object

First let's identify what force is applied to the liquid h distance away from the surface (the dotted line in Figure 2). Only the weight of the column of liquid (shown with arrows on Figure 2) is pushing down on that line. We can recall that on Earth, the weight of the liquid will be:

$$F_g = m_l g$$

Where:

F_g - weight of liquid
m_l - mass of liquid
g - gravitational acceleration constant

* For the rest of the proof the l subscript is used to refer to the liquid.

The mass of the liquid can be expressed through its density and volume:

$$\rho_l = \frac{m_l}{V}$$

$$m_l = \rho_l V$$

In our case, denote the volume of liquid above the dotted line as V_h (Figure 2) and substitute the equation above into the equation for weight to get:

$$F_g = \rho_l g V_h$$

Substitute this into the original equation for pressure to get:

$$P = \frac{\rho_l g V_h}{S}$$

The weight of the liquid is applied on the cross-section of the container, with an area of $w \cdot l$ (Recall that Figure 2 is a side view, and w is the depth of the liquid column). Substitute this into the equation above:

$$P = \frac{\rho_l g V_h}{wl}$$

The volume of the column of liquid can be written as $V_h = lwh$. Substitute this into the equation above and simplify to get:

$$P = \frac{\rho_l\, g \cancel{lw} h}{\cancel{lw}}$$
$$P = \rho_l g h \tag{2}$$

This is the equation for pressure in a liquid h distance away from its surface. Note that pressure does not depend on the volume or shape of the container, only the height of the column of liquid. This formula is therefore true for any vessel containing the liquid.

Archimedes' Principle

Let's now determine an equation for the buoyant force acting on a cube with side length a and density ρ_c, placed in a liquid of density ρ_l.

* A c subscript will be used to refer to the cube in all further equations.

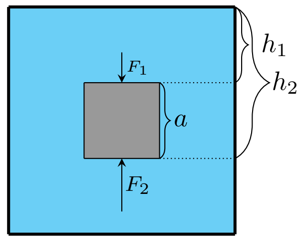


Figure 3: Side view of cube with density ρ_c submerged in liquid of density ρ_l

From Figure 3, it is evident that some pressure P_1 is applied to the top surface of the cube by the weight of the liquid. Using Pascal's principle we can notice that the liquid exerts a pressure of P_2 on the bottom face of the cube and therefore pushes it up with some force F_2. Both faces have the same area S.

From equation (2) we get that:

$$\begin{aligned} P_1 &= \rho_l g h_1 \qquad (3) \\ P_2 &= \rho_l g h_2 \\ &= \rho_l g (h_1 + a) \qquad (4) \end{aligned}$$

Rearrange equation (1) to express force through pressure:

$$F = PS$$

Substitute (3) and (4) into equation above to solve for forces F_1 and F_2.

$$F_1 = \rho_l g h_1 S$$

$$F_2 = \rho_l g (h_1 + a) S$$

These forces are acting in opposite directions. The net buoyant force pushing the cube upwards (also called the Archimedes force), is therefore equal to the difference $F_2 - F_1$:

$$\begin{aligned} F_A &= F_2 - F_1 \\ &= \rho_l g (h_1 + a) S - \rho_l g h_1 S \\ &= \rho_l g S h_1 + \rho_l g S a - \rho_l g S h_1 \\ &= \rho_l g S a \end{aligned}$$

Note that the buoyant force is always directed upwards. This can be seen from the equation above (3) as $\rho_l g S a > 0$.

Recall that pressure was originally applied on the top and bottom face of a cube with side length a. The area of each of these faces is then $S = a^2$. The volume of the entire cube $V_c = a^3$. Substitute this into the previous equation to get:

$$\begin{aligned} F_a &= \rho_l g S a \\ &= \rho_l g a^2 a \\ &= \rho_l g a^3 \\ &= \rho_l g V_c \end{aligned} \tag{5}$$

It is easy to notice that the buoyant force is equal to the weight of the liquid displaced by the cube ($\rho_l V_c = m_{lc} \Rightarrow F_A = \rho_l g V_c = m_{lc} g = F_{gl}$).

It is also important to note that equation (5) only depends on the volume of the cube submerged in the liquid, therefore the formula holds true for an object of any shape.

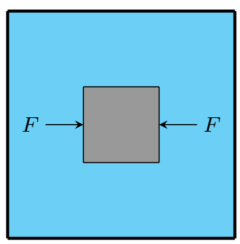


Figure 4: Side view of cube with density ρ_c submerged in liquid of density ρ_l

*In our derivation, we have ignored the forces due to liquid pressure applied to the sides of the cube. These can however be ignored as they will cancel out each other (Figure 4).

Why do things float

The direction in which an object moves is defined by the direction of the net force applied to it. In this case, the net force is the difference of the buoyant force and weight of the object: $F_{net} = F_A - F_g$ (Figure 5).
The weight of an object can be expressed through its density and volume: $F_g = m_o g = \rho_o V_o g$ (subscript o - refers to the submerged object).

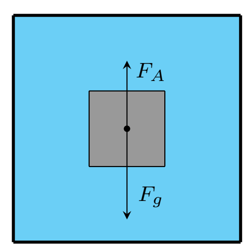


Figure 5: Side view of cube with density ρ_c submerged in liquid of density ρ_l

There are three possible cases:

$$\begin{cases} F_{net} > 0 \\ F_{net} < 0 \\ F_{net} = 0 \end{cases}$$

Let's consider each one separately.

$\mathbf{F_{net} > 0}$

If $F_{net} > 0$, then the buoyant force is greater than the weight of the submerged object. It will therefore float, as it is pushed up more than it is pulled down. In this case:

$\rho_l g V_o - \rho_o g V_o > 0$

$gV_o(\rho_l - \rho_o) > 0$

The inequality above is true only when the density of the liquid is greater than the density of the object ($\rho_l > \rho_o$). Therefore the object will float if placed in a liquid of higher density than itself. This is why despite being very heavy, large ships do not sink, as their density is in fact much less than that of water.

$\mathbf{F_{net} < 0}$

If $F_{net} < 0$, then the buoyant force is less than the weight of the submerged object. It will therefore float, as it is pulled down more than it is pushed up. In this case:

$gV_o(\rho_l - \rho_o) < 0$

The inequality above is true only when the density of the liquid is less than the density of the submerged object ($\rho_l < \rho_o$).

$\mathbf{F_{net} = 0}$

If $F_{net} = 0$, then the object will neither float nor sink. It will stay at rest/continue moving like it did before, until another force acts on it. In this case:

$gV_o(\rho_l - \rho_o) = 0$

This equation is true only when the density of the liquid is equal to the density of the submerged object ($\rho_l = \rho_o$).