

## Dual Space Discussion

The basis for the Dual Spaces session of the reading group, will be the section 3F in the 4th edition Linear Algebra done right by Sheldon Axler. I do have some modifications to how I would like our discussion to go relative to the book which I describe below (i.e. you do not need to read/understand everything in the chapter, but need to make sure to think about stuff below).

Particularly, I would like you to ditch Axler's notation of using primes for dual maps and dual spaces and use stars instead. Also everywhere in discussion below,  $V$  is finite dimensional.

- i) What is a dual space?
- ii) Given a basis of a vector space define a dual basis and argue that it is indeed a basis. (Given a basis  $v_1, \dots, v_n$  either use  $v_1^*, \dots, v_n^*$  as dual basis notation or  $v^1, \dots, v^n$ ).
- iii) Suppose we had a bilinear function  $\omega : V \times V \rightarrow \mathbb{F}$  with the property that  $\omega(v, v) \neq 0$  Consider the following map:

$$\begin{aligned} \omega(v, -) : V &\rightarrow V^* \\ v &\rightarrow \omega(v, -) \end{aligned}$$

Prove that this is a linear isomorphism of  $V$  and  $V^*$

Compare and contrast this isomorphism and the one we had above with the dual basis. Which one depends on choice of basis and which one doesn't?

- iv) Let  $T : V \rightarrow W$ . What is the dual map  $T^*$ .
- v) Suppose we pick bases for  $V$  and for  $W$ . What is the matrix of  $T^*$  in the respective dual bases.
- vi) In Axler's 4th edition, above block 3.125 there is a little comment on another way to prove 3.125. Prove it this way and draw a picture similar to one shown<sup>1</sup> in (1).
- vii) Prove that  $\dim \operatorname{Im}(T) = \dim \operatorname{Im}(T^*)$ . Draw a similar picture to one in (1) as part of your explanation.
- viii) Prove that row rank is equal to column rank.

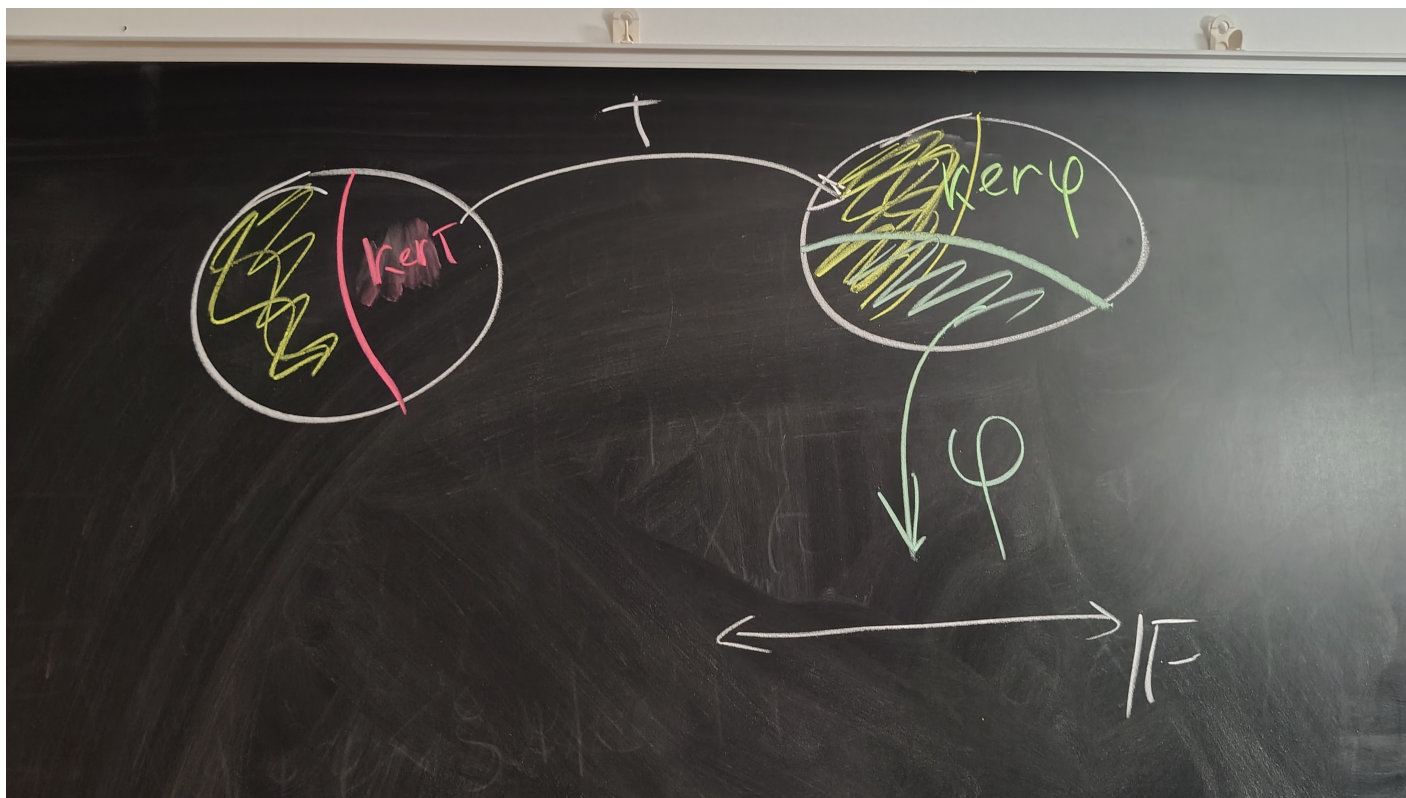


Figure 1: A picture that should be modified or improved to fit the context

<sup>1</sup>Make sure to draw with colours. Coloured chalk will be provided in the reading group.