Suggested Exercises II

Definition 1 *A collection of subsets of X, $\mathcal{B}$ is a basis for some topology if it satisfies the following properties*

1. For any $x \in X$ $\exists B \in \mathcal{B}$ s.t. $x \in B$
2. If $x \in B_1 \cap B_2$ then $\exists B_3$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$

We then declare that the topology (i.e. the open sets) this generates is:

$$\tau = U \in \mathcal{P}(X) \Big| \forall x \in U \exists B \subseteq U \text{ s.t. } x \in B \in \mathcal{B}$$

1. Prove that any open set, with the definition above, can be expressed as a union of some $B_i \in \mathcal{B}$
2. Prove that τ is indeed a topology.
3. Give an example of 2 bases that give 2 different topologies on $\mathbb{R}$
4. Give an example of 2 different bases that give the same topology on $\mathbb{R}$
5. Suppose you have some space X such that two topologies can be defined: τ_1 and τ_2. Suppose for any open set $U \in \tau_1 \forall x \in U \exists V \subseteq U$, s.t. $x \in V \in \tau_2$ and for any open set $V \in \tau_2$ for any $x \in V$ exists an open set $U \subseteq V$, $x \in U \in \tau_1$. Show that these two topologies are indeed the same (i.e. if an open set is in τ_1 it must also be in τ_2).
6. Munkres, Exercise 5. Show that if $\mathcal{A}$ is a basis for a topology on X, then the topology generated by $\mathcal{A}$ equals the intersection of all topologies on X that contain $\mathcal{A}$.

Written by Ivan Ovchinnikov & Sabek Germame, Summer 2024.