Bead on Spinning Ring

Consider the following question from the Knight textbook:

69. ||| The 10 mg bead in FIGURE CP8.69 is free to slide on a frictionless wire loop. The loop rotates about a vertical axis with angular velocity ω. If ω is less than some critical value ω_c, the bead sits at the bottom of the spinning loop. When $\omega > \omega_c$, the bead moves out to some angle θ.

a. What is ω_c in rpm for the loop shown in the figure?

b. At what value of ω, in rpm, is $\theta = 30^\circ$?

FIGURE CP8.69

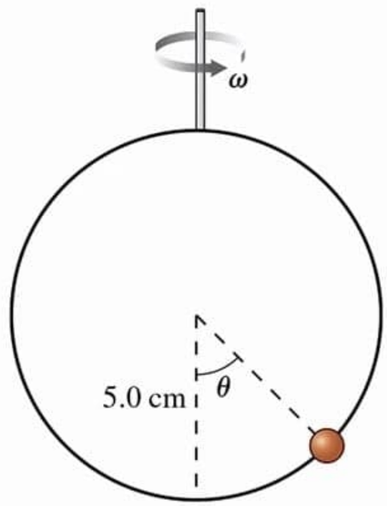

Figure 1: This is Q69 from the 4th edition of the textbook

Solution

Suppose that the bead is being spun on a ring of some radius l at some angular velocity ω. It will then move out to some angle θ. Consider the following free body diagrams:

(a) View from Side. Introduce a tangential reference frame with r positive pointing towards centre of ring and z positive pointing upwards.

(b) From top view, the bead undergoes uniform circular motion along a circle of radius $l\sin\theta$ (as evident from Figure (2a)). Introduce same tangential frame of reference from Figure (2a).

Figure 2

From the question we know that the bead stays at some fixed angle θ (Figure (2a)). We can then assume that there is no acceleration in the z direction

$\Rightarrow F_{znet} = 0$

Since the object is undergoing uniform circular motion, from Figure (2b) we get that:

$a_{tnet} = 0 \Rightarrow F_{tnet} = 0$

$$F_{rnet} = m\omega^2 \cdot (l\sin\theta) \tag{1}$$

Also note that:

$F_g = mg$

Consider the forces in Figure (2a):
$F_{znet} = N \cdot \cos\theta - F_g = 0$

From equation above get that:

$N = \dfrac{F_g}{\cos\theta} = \dfrac{mg}{\cos\theta}$

Now write down the equation in the r direction:

$F_{rnet} = N \cdot \sin\theta$

Substitute equation for N from above, equation (1) and solve for ω:

$\cancel{m}\omega^2 \cdot l \cdot \cancel{\sin\theta} = \dfrac{\cancel{m}g}{\cos\theta} \cdot \cancel{\sin\theta}$

$$\omega = \sqrt{\frac{g}{l \cdot \cos\theta}} \tag{2}$$

Now suppose we want to find some minimum ω_c for the bead to move away by some minimal θ. We then get that:

$\theta > 0 \Rightarrow \cos\theta < 1 \Rightarrow 1 < \dfrac{1}{\cos\theta}$

Substitute this inequality into equation (2):

$$\omega < \sqrt{\frac{g}{l}} \tag{3}$$

a) From inequality (3) we get that the minimum angular rotation required to move is:

$\omega_c = \sqrt{\dfrac{g}{l}}$

Using conditions from the question in Figure (1) and equation above get that:

$l = 5 \text{ cm} = 0.05 \text{ m}$
$g = 9.81 \ \frac{1}{\text{ms}^2}$

$w_c = \sqrt{\dfrac{g}{l}} = \sqrt{\dfrac{9.81}{0.05}} \dfrac{\text{rad}}{\text{s}} \cdot \dfrac{1\,\text{r}}{2\,\pi\text{rad}} \cdot \dfrac{60\,\text{s}}{1\,\text{min}} = 133.758 \text{ rpm}$

b) Solve for ω at $l = 0.05$ m and $\theta = 30°$ using equation (2):

$\omega = \sqrt{\dfrac{g}{l \cdot \cos\theta}} = \sqrt{\dfrac{9.81}{0.05 \cdot \frac{\sqrt{3}}{2}}} \dfrac{\text{rad}}{\text{s}} \cdot \dfrac{1\,\text{r}}{2\,\pi\text{rad}} \cdot \dfrac{60\,\text{s}}{1\,\text{min}} = 143.659 \text{ rpm}$