1 Review

Recall we discussed the following definitions:

Definition 1 *An equivalence relation $\sim$ on a set X, is a binary relation with the following properties $\forall a, b, c \in X$:*

i) $a \sim a$ (reflexive)

ii) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitive)

iii) $a \sim b \ \Rightarrow b \sim a$ (symmetric)

Note: you can give an equivalent definition by identifying $\sim$ with some subset $R \subseteq X \times X$, such that it has special properties as above.
Given this definition, it is natural to define the equivalence class of some element $x \in X$.

Definition 2 *The equivalence class of x is the set of all elements equivalent to it under some relation $\sim$. We will denote it using $[x]_\sim$*

$$[x]_\sim = \{y \in X \big| x \sim y\} \tag{1}$$

Suppose we have some vector space V and a subspace U. We can then define the following equivalence relation:

$$x \sim y \text{ if } x - y \in U \tag{2}$$

We will then define the quotient space as the set:

$$V/U = \{[x]_\sim \big| x \in V\} \tag{3}$$

Where $\sim$ is the equivalence relation from equation (2).

Also recall that if $U, W \subseteq V$ and $\lambda \in \mathbb{F}$ we can define the following

$$U + W = \{u + w \big| u \in U \, , \, w \in W\} \tag{4a}$$

$$\lambda U = \{\lambda u \big| u \in U\} \tag{4b}$$

2 Exercises

I) First let's familiarize ourselves with the properties of equivalence relations and equivalence classes

  i) Suppose $x \sim y$ for some equivalence relation $\sim$ on some set X. Show that $[x]_\sim = [y]_\sim$

  ii) Show that for any $x, y \in X$ either $[x]_\sim \cap [y]_\sim = \emptyset$ or $[x]_\sim = [y]_\sim$

II) In equation (2) we defined an equivalence relation. Show that it is indeed an equivalence relation.

III) In equation (3) we defined the quotient space as a set. For reasons we discussed in class, we want to give it a linear structure and turn it into a vector space. Note that the elements of V/U are sets! We can therefore equip them with operations in equation (4).

  i) We want to check compatibility between the operations of vector addition in the parent space V and set addition in the quotient space V/U. Show that:

$$[x + y]_\sim = [x]_\sim + [y]_\sim \tag{5a}$$

$$[\lambda x]_\sim = \lambda [x]_\sim \tag{5b}$$

  ii) Use above, to show that the set in equation (3), equipped with operations in (4) is a vector space over $\mathbb{F}$ (the same field as the one used for V) (i.e. show that it satisfies all the vector space axioms).

IV) Prove that if V is finite dimensional, then so is V/U

V) Compute the inverses of the following matrices. Do you notice anything interesting about their inverses?

$$\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -6 & 2 \\ 2 & 3 & -1 \\ 0 & 1 & 15 \end{pmatrix}$$