Basics of Trigonometry

The first sign of Similarity of Triangles states that the sides of two triangles with equal angles are proportionate to each other. This also means that for all triangles with the equal angles, the ratio between two respective sides within each triangle stay constant (prove this yourself).

But why do we choose to study right angled triangles over any other type of triangles? The answer is simple: right angles and right angled triangles are much more common and useful than other types of angles (E.G. Think about heights etc.) E.G. It is also always possible to split a triangle into two right angled triangles by drawing a a height from the biggest angle angle right angled triangles also have special properties like the Pythagorean theorem, making them easier to manipulate.

Definitions

The following are the three basic trigonometric ratios:

Sine of an angle

The sine of an angle is the ratio of the side opposite of the angle to the hypotenuse of the right angled triangle. It is denoted as $\sin\alpha$, where α is the angle in relation to which we define the sides we choose for the ratio.

$\sin\alpha = \frac{BC}{AB}$ (Figure 1)

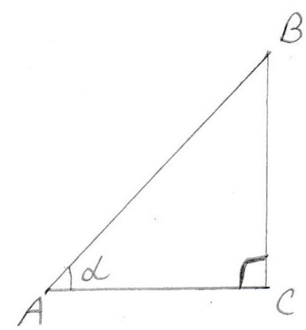

Figure 1

Cosine of an angle

The cosine of an angle is the ratio of the side adjacent to the angle to the hypotenuse of the right angled triangle. It is denoted as $\cos\alpha$, where α is the angle in relation to which we define the sides we choose for the ratio.

$\cos\alpha = \frac{AC}{AB}$ (Figure 1)

Tangent of an angle

The tangent of an angle is the ratio of the side opposite of the angle to the side adjacent to the angle in a right angled triangle. It is denoted as $\tan\alpha$, where α is the angle in relation to which we define the sides we choose for the ratio.

$\tan\alpha = \frac{BC}{AC}$ (Figure 1)

*Note that in some countries the tangent of an angle α is denoted as tg α instead of $\tan\alpha$.

Reciprocal Ratios

There are also names for reciprocals of the ratios:

Reciprocal of the sine of an angle : cosecant of an angle : $\csc\alpha = \frac{1}{\sin\alpha}$. (In some countries cosecant is denoted as "cosec α" instead of "csc α")

Reciprocal of the cosine of an angle: secant of an angle: $\sec\alpha = \frac{1}{\cos\alpha}$.

Reciprocal of the tangent of an angle: cotangent of an angle: $\cot\alpha = \frac{1}{\tan\alpha} = \frac{\cos\alpha}{\sin\alpha}$. (In some countries cotangent is denoted as "ctg α" instead of "$\cot\alpha$")

Unit Circle

To improve our understanding of the basic trigonometric ratios, lets introduce a set of new definitions of the trigonometric ratios based on the unit circle.Recall: the unit circle, is a circle with a radius of 1 unit.

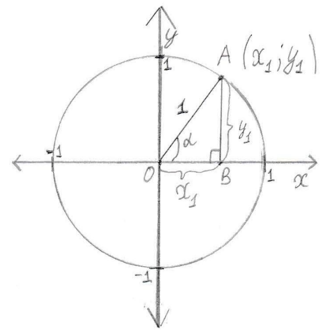

Figure 2

Definitions

Here are the definitions of the basic trigonometric ratios based on the unit circle.

Sine of an angle
The sine of an angle is the y-axis coordinate of point A. ($\sin\alpha = y_1$ on Figure 2)

Cosine of an angle
The cosine of an angle is the x-axis coordinate of point A. ($\cos\alpha = x_1$ on Figure 2)

Tangent of an angle
The tangent of an angle is the ratio of the y-axis coordinate of point A to the x-axis coordinate of point A. ($\tan\alpha = \frac{y_1}{x_1}$ on Figure 2)
*Note that from this definition we also get that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ (prove that yourself).

These definitions are equivalent to the definitions mentioned initially. According to them $\sin\alpha = \frac{AB}{OA}$. Recall that by the definition of the unit circle: $OA = 1$.
Hence: $\sin\alpha = \frac{AB}{OA} = \frac{AB}{1} = AB = y_1$. The same goes for cosine and tangent: $\cos\alpha = \frac{OB}{OA} = \frac{OB}{1} = x_1$; $\tan\alpha = \frac{AB}{OB} = \frac{y_1}{x_1}$

But Why?

You might now ask: Well what is the difference between these two sets of definitions of trigonometric ratios and why would we want to complicate things with the unit circle?

The definitions that use the unit circle give several advantages. They allows us to consider trigonometric ratios for angles larger or equal to 90° (including concave angles) and negative angles. Using the first set of definitions we were unable to do that as you cannot have a right angled triangle with either 2 right angles, or an obtuse angle. You also cannot have a triangle with a concave angle. The unit circle is also a good tool for visualizing the trigonometric ratios making them easier to analyze.

Pythagorean Identity

Looking at Figure 2 we can see that:
$BA = y_1 = \sin\alpha$
$OB = x_1 = \cos\alpha$
$OA = 1$

Using Pythagorean theorem we get:
$y_1^2 + x_1^2 = 1$
$\sin^2\alpha + \cos^2\alpha = 1$ This is know as the Pythagorean identity.

Values for Basic Trigonometric Ratios for Certain Angles

0°=0 rad; 90°=$\frac{\pi}{2}$ rad; 180°=π rad; 270°= $\frac{3\pi}{2}$ rad

Consider point A at $t = 0$, it lies on the x-axis and is to the right of the y-axis (Figure 3). This means that the coordinates of point A are: $x = 1$, $y = 0$. Hence:
$\sin 0 = 0; \cos 0 = 1; \tan 0 = 0$

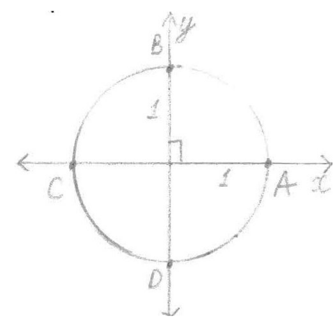

Figure 3

Consider point B at $t = \frac{\pi}{2}$, it lies on the y-axis and is above the x-axis (Figure 3). This means that the coordinates of point B are: $x = 0; y = 1$. Hence:
$\sin\frac{\pi}{2} = 1; \cos\frac{\pi}{2} = 0; \tan\frac{\pi}{2} = \text{undefined}$

Consider point C at $t = \pi$, it lies on the x-axis and is to the left of the y-axis (Figure 3). This means that the coordinates of point C are: $x =$ -1, $y = 0$. Hence:
$\sin\pi = 0; \cos\pi = -1; \tan\pi = 0$

Consider point D at $t = \frac{3\pi}{2}$, it lies on the y-axis and is below the x-axis (Figure 3). This means that the coordinates of point D are: $x = 0; y = -1$. Hence:
$\sin\frac{3\pi}{2} = -1; \cos\frac{3\pi}{2} = 0; \tan\frac{3\pi}{2} = \text{undefined}$

Recall that:
$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$
$\tan\frac{\pi}{4} = 1$

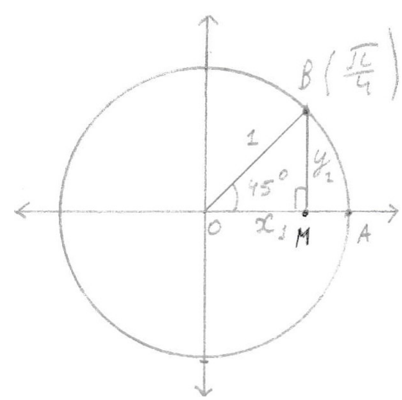

Figure 4

45°= $\frac{\pi}{4}$ rad

Consider a point B on a unit circle at $t = \frac{\pi}{4}$. This point makes an angle $AOB = 45°$ with the origin (Figure 4). If we draw a line perpendicular to the x-axis through point B we will get a right angled isosceles triangle BOM (Prove that on your own).
$MB = OM \Rightarrow x_1 = y_1 \Rightarrow \sin\frac{\pi}{4} = \cos\frac{\pi}{4}$ (1)

Recall the Pythagorean identity:
$\sin^2\alpha + \cos^2\alpha = 1$
Substitute (1) into the line above to get :
$\sin^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} = 1$
$2\sin^2\frac{\pi}{4} = 1$
$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
Please note that we take into account the positive root, since B "above" the x-axis. This means that the coordinate y_1 must be positive.
$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$
Once again, we say that cosine is positive, since B is to the right of the y-axis, hence the x-coordinate is positive.

Recall that:
$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$ we get:
$\tan\frac{\pi}{4} = 1$

30°= $\frac{\pi}{6}$ rad

Consider a point B on a unit circle at $t = \frac{\pi}{6}$. This point makes an angle $AOB = 30°$ with the x-axis (Figure 5). If we draw a line perpendicular to the x-axis through point B we will get a right angled triangle BOM with a $30°$ angle. From previous theorems (Search somewhere in the Geometry folder), we know that in a right angled triangle with a $30°$ angle, the side opposite to that angle is half the length of the hypotenuse. Hence:

$BM = \frac{1}{2}OB \Rightarrow y_1 = \frac{1}{2}\cdot 1 \Rightarrow \sin\frac{\pi}{6} = \frac{1}{2}$

B is above the x-axis, hence the y-coordinate must be positive $\Rightarrow$ $\sin\frac{\pi}{6}$ is positive.

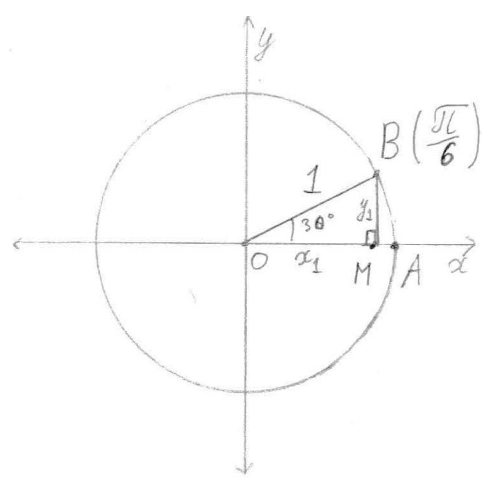

Figure 5

Now let's use the Pythagorean Identity:
$\sin^2\frac{\pi}{6} + \cos^2\frac{\pi}{6} = 1$
$(\frac{1}{2})^2 + \cos^2\frac{\pi}{6} = 1$
$\cos^2\frac{\pi}{6} = \frac{3}{4}$
$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
B is to the right of the y-axis, so the x-coordinate is positive $\Rightarrow \cos\frac{\pi}{6}$ is positive.

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$\tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

60°= $\frac{\pi}{3}$ rad

Consider a point B on a unit circle at $t = \frac{\pi}{3}$. This point makes an angle $AOB = 60°$ with the x-axis (Figure 6). If we draw a line through point B perpendicular to the x-axis we will get a right angled triangle BOM with a $30°$ angle. We know that in a right angled triangle with a $30°$ angle, the side opposite to that angle is half the length of the hypotenuse. Hence:
$OM = \frac{1}{2}OB \Rightarrow x_1 = \frac{1}{2} \Rightarrow \cos\frac{\pi}{3} = \frac{1}{2}$
Point B is to the right of the y-axis,so its x coordinate must be positive. This means that cosine must be positive too.

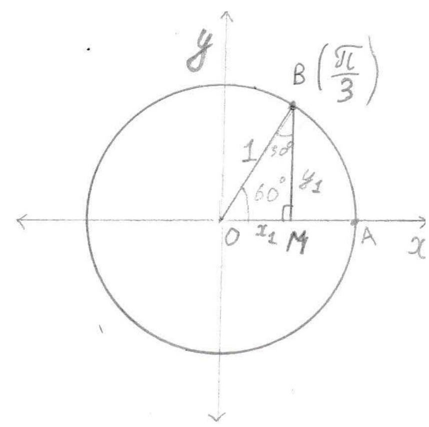

Figure 6

Using the Pythagorean identity:
$\sin^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} = 1$
$\sin^2\frac{\pi}{3} + (\frac{1}{4})^2 = 1$
$\sin^2\frac{\pi}{3} = \frac{3}{4}$
$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
Point B is above the x-axis, so its y coordinate must be positive. This means that sine must be positive too.

Finally:
$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

$\tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$