1 Review of Session

Definition 1 *The span of a finite list of vectors in a vector space V over a field $\mathbb{F}$ is defined as:*

$$span(v_1, ..., v_k) = \{a_1v_1 + ... + a_kv_k \big| a_i \in \mathbb{F}\}$$

Definition 2 *A finite list of vectors $v_1, ..., v_k \in V$ is called linearly independent if for any i*

$$v_i \notin span(v_1, ..., v_{i-1}, v_{i+1}, ..., v_k)$$

We then gave an equivalent definition

Definition 3 *A finite list of vectors if linearly independent of the only set of scalars a_i such that*

$$a_1v_1 + ... + a_kv_k = 0$$

is $a_i = 0 \forall i \in \{1, ..., k\}$

Note that from the first definition, we see that a list of linearly dependent vectors (defined as a list of vectors that are not linearly independent), is a list from which you can always remove a vector, without changing the span.

Theorem 1 *From the definition above we claimed and proved that given a list of linearly dependent vectors can also be reduced to a list of linearly independent vectors that have the same span.*

Definition 4 *A vector space V is called finite dimensional if exist $v_1, ..., v_p \in V$ such that $V = span(v_1, ..., v_p)$*

Definition 5 *A basis of a finite dimensional vector space V, is a set of linearly independent vectors $v_1, ..., v_n$ such that $V = span(v_1, ..., v_n)$*

Definition 6 *The dimension of a finite dimensional vector space is the length of its basis.*

2 Suggested Exercises

I) **Linearly Dependent Vectors** Let a finite list of vectors be linearly dependent if it is not linearly independent

- Show that the two definitions of linear independence above are equivalent.
- Explain the "redundance theorem" 1

II) **Dimension of Vector Space** The goal of this exercise is to show that the dimension of a finite dimensional vector space is well defined. We will proceed by contradiction. Suppose a finite dimensional vector space has 2 bases of different length. Denote them as $v_1, ..., v_p$ and $u_1, ..., u_m$.

(a) What are the conditions on v_i and u_j (i.e. write down the definition of the basis for each of them).

(b) Consider the list of vectors $v_1, ..., v_p, u_1$. Show that it has the same span as the list $v_1, ..., v_p$.

(c) Prove that you can remove one of the vectors v_i from the list $v_1, ..., v_p, u_1$ without changing the span. Prove that after the removal, this list will be linearly independent.

(d) Apply induction and show that $u_1, ..., u_p$ must be linearly independent

(e) Conclude the proof.

(f) See how the proof above shows that any list longer than the basis, has to be linearly dependent.

3 Question III

3.1 Useful Definitions

Definition 7 *A linear map from a vector space to itself is called a linear operator.*

Definition 8 *A basis $\beta = \{v_1, ..., v_n\}$ of a finite dimensional vector space V is said to be upper triangular with respect to some linear operator $T : V \to V$ if $Tv_i \in span(Tv_1, ..., Tv_i)$. (i.e. the matrix ${}_\beta\mathcal{M}_\beta(T)$ is upper triangular.*

Definition 9 *A vector $v \in V$ is an eigenvector of some linear operator T if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$. We then call λ the eigenvalue of v.*

Theorem 2 *Any polynomial with complex coefficients has at least one complex root.*

This has the following consequence. Any polynomial of degree n with complex coefficients can be written as a product of n monomials as follows:

$$a_0 + a_1x + ... + a_nx^n = (x - \lambda_0) \cdots (x - \lambda_n)$$

3.2 Existence of Upper Triangular basis

The goal of this exercise, is to prove that any operator $T : V \to V$ over a finite dimensional vector space V over $\mathbb{C}$ has a basis in which it is upper triangular.

(I) Prove that the corollary above, follows from the theorem

(II) Let V be a vector space over $\mathbb{C}$ with $\dim(V) = n$ and $T \in \mathcal{L}(V)$

- (i) Let v be any non-zero vector. Consider the set of vectors:

  $$v, Tv, T^2 v, ..., T^n v$$

  Argue that these have to be linearly dependent. Write down some non-trivial linear combination that gives 0.

- (ii) Rewrite the result above, as some polynomial in T applied to v giving 0. Use the corollary above and conclude there must be at least one eigenvalue and eigenvector.

(III) We will now use the fact above, to prove that any operator $T \in \mathcal{L}(V)$ over a finite-dimensional complex vector space has a basis β, in which its matrix is upper triangular. Recall that a matrix is upper triangular in a basis $\beta = \{v_1, ..., v_n\}$ if

$$Tv_i \in \text{span}(v_1, ..., v_i) \quad \forall i$$

- (i) Prove that above is true if $\dim(V)$=1.
- (ii) We now proceed by induction. Suppose the result holds for any operator over a space with $\dim(V)$ = n>1. Let's show it for $\dim(V) = n + 1$

  By theorem in part II and III, know exists an eigenvalue λ. Let

  $$U = \text{Range}(T - \lambda I)$$

  Argue that $T - \lambda I$ is not surjective. Prove that U is invariant under T (i.e. $TU \subseteq U$)
- (iii) Use the induction hypothesis to argue that $T|_U$ has an upper-triangular basis $u_1, ..., u_m$.
- (iv) Extend $u_1, ..., u_j$ to a basis $\beta = \{u_1, ..., u_m, v_1, ..., v_k\}$ of V. Prove that $Tv_j \in \text{span }(u_1, ..., u_m, v_1, ..., v_j)$ *Hint: Consider the equation* $(T - \lambda I)v_k + \lambda v_k$
- (v) Conclude the result.