Derivation of Centripetal Acceleration Formula (without Calculus)

Consider some object moving in a circle of radius r with a constant speed v. Let's try to find what magnitude force is required to keep this object moving in this fashion. We can first find the acceleration and find the magnitude of the force later with Newton's second law.

Specific Case

We must first find a way to mathematically describe the motion of an object along a circle. We can approximate it to be an equilateral triangle inscribed in that circle.

Let's find our acceleration.

Recall that:

$$|\vec{a}| = \frac{\vec{v}_2 - \vec{v}_1}{t_f} \tag{1}$$

Where:
$\vec{v}_1$ and $\vec{v}_2$ are initial and final velocities respectively
t_f - time object has been accelerating for.
In the case of a triangle, the object accelerates 3 times (at each vertex). Let's consider the acceleration at each one of them.

Introduce a reference frame with the origin at one of the vertices and the x-axis on one of the sides.
Recall that the object is moving at a constant velocity v. Its initial velocity before the turn is therefore:

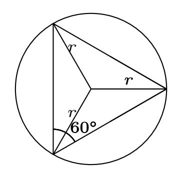

Figure 1

$\vec{v}_1 = \Big(-v; \quad 0\Big)$ As the angle of the triangle is $60°$, the final velocity will be:

$\vec{v}_2 = \Big(v\cos 60°; \quad v\sin 60°\Big)$

Now we can calculate the change in velocity as:

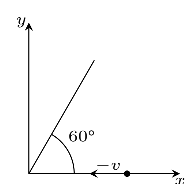

Figure 2: Reference frame at vertex of triangle

$$\begin{aligned}
\Delta\vec{v} &= \vec{v}_2 - \vec{v}_1 \\
&= \Big(v\cos 60° - (-v); \quad v\sin 60° - 0\Big) \\
&= \Big(v\cos 60° + v; \quad v\sin 60°\Big)
\end{aligned}$$

Substitute $\sin 60° = \frac{\sqrt{3}}{2}$ and $\cos 60° = \frac{1}{2}$ and calculate the magnitude of the change in velocity.

$$
\begin{aligned}
|\Delta\vec{v}| &= \sqrt{(v\cos 60^\circ + v)^2 + (v\sin 60^\circ)^2} \\
&= \sqrt{(\frac{1}{2}v + v)^2 + (\frac{\sqrt{3}}{2}v)^2} \\
&= \sqrt{(\frac{3}{2}v)^2 + (\frac{\sqrt{3}}{2}v)^2} \\
&= \sqrt{\frac{9}{4}v^2 + \frac{3}{4}v^2} \\
&= \sqrt{3v^2} \\
&= v\sqrt{3}
\end{aligned}
$$

As the object accelerates 3 times, we must multiply the magnitude in change in velocity above by 3, when we substitute it into equation (1) :

$$
\begin{aligned}
|\vec{a}| &= \frac{|\Delta\vec{v}|}{t_f} \\
&= \frac{3\sqrt{3}v}{t_f}
\end{aligned}
$$

We now only need to find the total time it takes the object to travel around our triangle.

It is easy to notice that:

$t_f = \dfrac{P}{v}$

Where:

P - perimeter of the triangle
v - velocity at which object travels along triangle

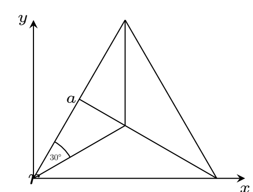

Figure 3: Find side length a.

The perimeter of the triangles is equal to:

$P = 3 \cdot a$

Where a is the side length of the triangle.

If we draw a height from the center of the circle to the sides of the triangle, we will get 3 isosceles triangles with two sides r and two 30° angles. The length of the big side is then:

$$
\begin{aligned}
a &= 2r \cdot \cos 30^\circ \\
&= 2r \cdot \frac{\sqrt{3}}{2} \\
&= \sqrt{3}r
\end{aligned}
$$

Substitute this into the equation for perimeter and then into the equation for time:

$$\begin{aligned} t_f &= \frac{P}{v} \\ &= \frac{3\sqrt{3}r}{v} \end{aligned}$$

Substitute this into the equation for acceleration from above to get:

$$\begin{aligned} |\vec{a}| &= \frac{3\sqrt{3}v}{t_f} \\ &= \frac{\cancel{3\sqrt{3}}v}{\frac{\cancel{3\sqrt{3}}r}{v}} \\ &= \frac{v \cdot v}{r} \\ &= \frac{v^2}{r} \end{aligned}$$

Substitute the equation above into Newton's second law to find the force:

$$|\vec{F}_c| = \frac{mv^2}{r} \tag{2}$$

This is often referred to as the centripetal force. It is the magnitude of force required to keep an object of mass m moving in a circle of radius r at constant velocity v.

General Case

This time, let's approximate the circle as an n sided regular polygon $A_1A_2A_3...A_n$ inscribed in that circle of radius r.
You can visually confirm that if we draw a polygon with a lot of sides, the shape will be close to that of a circle.

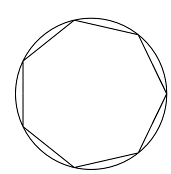

Figure 4

Consider an n sided regular polygon, with an inscribed circle of radius r. Let O be center of this circle.

Recall that acceleration is the change in velocity over some period of time:

$$\vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t}$$

Let's consider the change in velocity of a particle at one specific "corner" of the shape (Figure 2).

The particle comes in from the right with a velocity v_1 (We introduce a reference frame such that v_1 is traveling along the x axis and the origin is at some vertex A_k). We will assume at the corner, the particle instantly accelerates and starts traveling at velocity v_2.

Since we know that the magnitude of the velocity is constant and equal to v, we can find the coordinates of vectors v_1 and v_2.
Recall that A_k is the vertex of this regular polygon, and β is one of its angles.
In a regular convex polygon, all angles are equal and can be determined using the following formula :

$\beta = \frac{180°(n-2)}{n}$

So we get:

$\vec{v}_1 = \Big(-v; \quad 0\Big)$

$\vec{v}_2 = \Big(v \cdot \cos\beta; \quad v \cdot \sin\beta\Big)$

In this case, the change in velocity Δv_k at vertex A_k is:

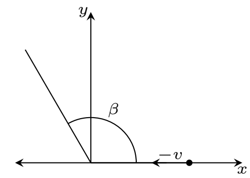

Figure 5: Reference frame at vertex of polygon

$$\begin{aligned}\Delta\vec{v}_k &= \vec{v}_2 - \vec{v}_1 \\ &= \Big(v \cdot \cos\beta - (-v); \quad v \cdot \sin\beta\Big) \\ &= \Big(v \cdot \cos\beta + v; \quad v \cdot \sin\beta\Big)\end{aligned}$$

The object makes n turns during one lap around the polygon. The overall change in velocity each lap is therefore:

$\Delta\vec{v}_f = n \cdot \Big(v \cdot \cos\beta + v; \quad v \cdot \sin\beta\Big)$

The magnitude of this change is therefore:

$$\begin{aligned}|\Delta\vec{v}_f| &= n \cdot \sqrt{(v \cdot \cos\beta + v)^2 + (v \cdot \sin\beta)^2} \\ &= n \cdot \sqrt{v^2\cos^2\beta + 2v^2\cos\beta + v^2 + v^2\sin^2\beta} \\ &= n \cdot \sqrt{v^2(\cos^2\beta + \sin^2\beta + 2\cos\beta + 1)} \\ &= nv \cdot \sqrt{1 + 2\cos\beta + 1} \\ &= nv \cdot \sqrt{2\cos\beta + 2} \\ &= \sqrt{2}nv\sqrt{1 + \cos\beta}\end{aligned}$$

The magnitude of the average acceleration of one full lap $\vec{a}$ is:

$$|\vec{a}| = \frac{\sqrt{2}nv}{t_f} \cdot \sqrt{1 + \cos\beta} \tag{3}$$

Where:
n - number of angles in the polygon the object is traveling around
v - constant velocity at which object is moving
β - angle of polygon
t_f - total time it takes object to make one lap

Now let's express t_f, through v, r, n and β.
We know that the total time it takes for the point to move across the entire polygon is:

$$t_f = \frac{P}{v} \tag{4}$$

Where P - is the perimeter of the shape.

We need to find the perimeter of the polygon in terms of r, n and β. In a regular polygon the perimeter is equal to:

$P = n \cdot a$

Where:
a - side length
n - number of sides

We now just need to find the side length a.

Revisit the diagram of the n sided polygon inscribed in a circle of radius r. If we connect each vertex with the center of the circle O, we will get n isosceles triangles with sides of length r. The height towards side a will bisect the angle at vertex O.
Since all of these triangles are equal, we therefore get that in Figure 2:

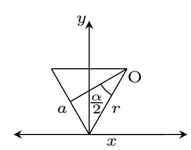

Figure 6: Vertex of n sided polygon connected with center of circle

$$\begin{aligned} \alpha &= \frac{360}{n} \cdot \frac{1}{2} \\ &= \frac{180}{n} \end{aligned}$$

With some simple trigonometry we get:

$a = 2r\sin(\frac{180}{n})$

Substitute this into the equation for perimeter and equation for t_f (4) to get:

$$\begin{aligned} t_f &= \frac{P}{v} \\ &= \frac{n \cdot a}{v} \\ &= \frac{2nr\sin(\frac{180}{n})}{v} \end{aligned}$$

Substitute this back into equation (3) to get:

$$
\begin{aligned}
|\vec{a}_f| &= \frac{\sqrt{2}nv}{t_f} \cdot \sqrt{1+\cos\beta} \\
&= \frac{\sqrt{2}nv}{\frac{2nr\sin(\frac{180}{n})}{v}} \sqrt{1+\cos\beta} \\
&= \frac{\sqrt{2}\cancel{n}v \cdot v}{2\cancel{n}r\sin(\frac{180}{n})} \cdot \sqrt{1+\cos\beta} \\
&= \frac{\sqrt{2}v^2}{2r\sin(\frac{180}{n})} \cdot \sqrt{1+\cos\beta} \\
&= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{1+\cos\beta}}{\sin(\frac{180}{n})}
\end{aligned}
$$

Recall that:

$$
\begin{aligned}
\beta &= \frac{180 \cdot (n-2)}{n} \\
&= 180 - \frac{360}{n}
\end{aligned}
$$

Using the following identities, simplify the formula for acceleration above:

$\cos(180 - x) = -\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$

$$
\begin{aligned}
|\vec{a}| &= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{1+\cos\beta}}{\sin(\frac{180}{n})} \\
&= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{1+\cos(180-\frac{360}{n})}}{\sin(\frac{180}{n})} \\
&= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{1-\cos(\frac{360}{n})}}{\sin(\frac{180}{n})} \\
&= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{1-(\cos^2(\frac{180°}{n}) - \sin^2\frac{180°}{n})}}{\sin(\frac{180°}{n})} \\
&= \frac{\sqrt{2}v^2}{2r} \cdot \frac{\sqrt{2\sin^2(\frac{180°}{n})}}{\sin(\frac{180°}{n})} \\
&= \frac{\cancel{\sqrt{2}}v^2}{\cancel{2}r} \cdot \frac{\cancel{\sqrt{2}}\cancel{\sin\frac{180°}{n}}}{\cancel{\sin\frac{180°}{n}}} \\
&= \frac{v^2}{r}
\end{aligned}
$$

This is the formula for the magnitude of centripetal acceleration for an object moving at constant velocity v. Pair this with Newton's second law to find the magnitude of the centripetal force that can be used to keep an object moving in a circle of radius r at some constant velocity v:

$$|\vec{F}_c| = \frac{mv^2}{r} \tag{5}$$