0.1 Uncertainties Practice Problems

Calculate z and its uncertainty δz. Assume all variables are measured values.

0.1.1 Addition & Subtraction

1.

$z \pm \delta z = 3 \pm 1 + 2 \pm 2 + 10 \pm 1$

2.

$z \pm \delta z = 21 \pm 3 + 2 \pm 1 - 11 \pm 3$

3.

$z \pm \delta z = 12.51 \pm 0.04 - 11.04 \pm 0.01 - 0.10 \pm 0.09$

4.

$z = a + b + c$

5.
$z = a + b - c$

6. Solve for z and find its uncertainty δz:

$10 \pm 5 = 5 \pm 1 + z \pm \delta z + 10 \pm 3$

7. Solve for z and find its uncertainty δz:

$a = z - b + c$

0.1.2 Multiplication & Division

1.

$z \pm \delta z = \frac{1}{2} \cdot (12 \pm 1)$

2.

$z \pm \delta z = (0.45 \pm 0.02) \cdot (0.20 \pm 0.03)$
3.

$z \pm \delta z = \frac{45 \pm 3}{0.9 \pm 0.1}$

4.

$z \pm \delta z = \frac{49 \pm 7}{7 \pm 1} \cdot (3 \pm 1)$

5.

$z = a \cdot b$

6.

$z = 2 \cdot \frac{a}{b}$

7.

$$z = \frac{a-b}{c} \cdot d$$

8.

$$z = x^n$$

Where n is not a measured value and x is.

9. Solve for z and find its uncertainty δz:

$$a = \frac{z-b}{c} \cdot d$$

0.1.3 Transcendental Functions

*These questions require knowledge of calculus

1.

$$z = \sin x$$

2.

$$z = \tan x \cdot \cos \alpha$$

3.

$$z = \frac{\sin x + \cos x}{x^2}$$

4.

$$z = e^x$$

Where e - Euler's Number

5. Solve for z and find its uncertainty δz.

$$x = z^2 + 1$$

0.2 Solutions

0.2.1 Addition & Subtraction

Remember that when adding/subtracting measured numbers, we add their absolute uncertainty.

1.

$$\begin{aligned} z &= 3 + 2 + 10 \\ &= 15 \end{aligned}$$

$$\begin{aligned} \delta z &= \pm 1 \pm 2 \pm 1 \\ &= \pm(1 + 2 + 1) \\ &= \pm 4 \end{aligned}$$

2.

$$\begin{aligned} z &= 21 + 2 - 11 \\ &= 12 \end{aligned}$$

$$
\begin{aligned}
\delta z &= \pm 3 \pm 1 \pm 3 \\
&= \pm(3+1+3)
\end{aligned}
$$

3.

$$
\begin{aligned}
z &= 12.51 + 11.04 - 0.10 \\
&= 23.45 \\
\delta z &= \pm 0.04 \pm 0.01 \pm 0.09 \\
&= 0.14
\end{aligned}
$$

In Highschool, we assume that uncertainty can ONLY have 1 sig fig (as we assume we do not have equipment accurate enough to have uncertainty of two sig figs). We will therefore round the value above up so that it has only one significant figure:

$\delta z = 0.2$

4.

$z = a + b + c$

$$
\begin{aligned}
\delta z &= \pm \delta a \pm \delta b \pm \delta c \\
&= \pm(\delta a + \delta b + \delta c)
\end{aligned}
$$

5.

$z = a + b - c$

$$
\begin{aligned}
\delta z &= \pm \delta a \pm \delta b \pm \delta c \\
&= \pm(\delta a + \delta b + \delta c)
\end{aligned}
$$

6.

$10 \pm 5 = 5 \pm 1 + z \pm \delta z + 10 \pm 3$

$z \pm \delta z = 10 \pm 5 - 5 \pm 1 - 10 \pm 3$

$$
\begin{aligned}
z &= 10 - 5 - 10 \\
&= -5 \\
\delta z &= \pm 5 \pm 1 \pm 3 \\
&= \pm(5+1+3) \\
&= \pm 9
\end{aligned}
$$

7.

We will first solve for z, and then calculate our uncertainty based on our equation for z.

$$
\begin{aligned}
a &= z - b + c \\
z &= a + b - c \\
\delta z &= \pm \delta a \pm \delta b \pm \delta c \\
&= \pm(\delta a + \delta b + \delta c)
\end{aligned}
$$

0.2.2 Multiplication & Addition

Recall that when multiplying measured numbers, you add their relative uncertainties. So the absolute uncertainty of some measured number z where $z = a \cdot b$ can be calculated with the following formulas:

$$\delta z = |a| \cdot \delta b + |b| \cdot \delta a$$

$$\delta z = |ab| \cdot \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right)$$

For division relative uncertainties are also added. So, of some measured number $z = \frac{a}{b}$, the uncertainty is calculated using the following formulas:

$$\delta z = \frac{\delta a}{|b|} + \frac{|a|\delta b}{|b^2|}$$

$$\delta z = \left|\frac{a}{b}\right| \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right)$$

A solution will be presented using all possible formulas.

1.

$$\begin{aligned} z &= \frac{1}{2} \cdot 12 \\ &= 6 \end{aligned}$$

The $\frac{1}{2}$ has no assigned uncertainty and is therefore assumed to be an exact number (uncertainty of 0).

Non-Simplified Formula

$$\begin{aligned} \delta z &= |a| \cdot \delta b + |b| \cdot \delta a \\ &= |\frac{1}{2}| \cdot 1 + |12| \cdot 0 \\ &= 0.5 \end{aligned}$$

Relative Uncertainty

$$\begin{aligned} \delta z &= |ab| \cdot \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right) \\ &= \frac{1}{2} \cdot 12 \left(\frac{0}{\frac{1}{2}} + \frac{1}{12} \right) \\ &= 6 \cdot \frac{1}{12} \\ &= 0.5 \end{aligned}$$

Note that if you multiply a measured number by a constant, its uncertainty is effectively just multiplied by that constant.

2.

$$\begin{aligned} z &= 0.45 \cdot 0.20 \\ &= 0.090 \end{aligned}$$

Non-Simplified Formula

$$\begin{aligned}\delta z &= |a|\delta b + |b|\delta a \\ &= |0.45| \cdot 0.03 + |0.20| \cdot 0.02 \\ &= 0.0175 \\ &= 0.02\end{aligned}$$

Relative Uncertainty

$$\begin{aligned}\delta z &= |ab| \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right) \\ &= |0.45 \cdot 0.20| \left(\frac{0.02}{|0.45|} + \frac{0.03}{|0.20|} \right) \\ &= 0.0175 \\ &= 0.02\end{aligned}$$

Note that in this case, after rounding our uncertainty reaches up to the tenths decimal place, while the exact number z has sig figs up to the hundredth. We will therefore reduce the number of sig figs on the exact number to match the uncertainty.

$z \pm \delta z = 0.09 \pm 0.02$

3.

$$\begin{aligned}z &= \frac{45}{0.9} \\ &= 5 \cdot 10\end{aligned}$$

Non-Simplified Formula

$$\begin{aligned}\delta z &= \frac{\delta a}{|b|} + \frac{|a|\delta b}{|b^2|} \\ &= \frac{3}{|0.9|} + \frac{|3| \cdot 0.1}{|0.9^2|} \\ &= 8.88889 \\ &= 9\end{aligned}$$

Relative Uncertainty

$$\begin{aligned}\delta z &= |\frac{a}{b}| \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right) \\ &= \frac{45}{0.9} \left(\frac{3}{45} + \frac{0.1}{0.9} \right) \\ &= 8.88889 \\ &= 9\end{aligned}$$

4.

$$\begin{aligned}z &= \frac{49}{7} \cdot 3 \\ &= 21 \\ &= 2 \cdot 10\end{aligned}$$

Split this question into two parts. First consider the uncertainty of the fraction.

Let $g = \dfrac{49 \pm 7}{7 \pm 1} = 7.$

Let $c = g \cdot (3 \pm 1) = 7 \cdot 3 = 21$

Then $z = g \cdot c$

Non-Simplified Formula

$$\begin{aligned}\delta g &= \frac{\delta a}{|b|} + \frac{|a|\delta b}{|b^2|} \\ &= \frac{7}{|7|} + \frac{|49| \cdot 1}{|7^2|} \\ &= 2\end{aligned}$$

$$\begin{aligned}\delta z &= |g|\delta c + |c|\delta g \\ &= |7| \cdot 1 + |3| \cdot 2 \\ &= 13 \\ &= 2 \cdot 10\end{aligned}$$

Relative Uncertainty

$$\begin{aligned}\delta g &= \left|\frac{a}{b}\right| \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|}\right) \\ &= \left|\frac{49}{7}\right| \left(\frac{7}{49} + \frac{1}{7}\right) \\ &= 2\end{aligned}$$

$$\begin{aligned}\delta z &= |gc| \left(\frac{\delta g}{|g|} + \frac{\delta c}{|c|}\right) \\ &= |7 \cdot 3| \left(\frac{1}{3} + \frac{2}{7}\right) \\ &= 13 \\ &= 2 \cdot 10\end{aligned}$$

5.

$z = a \cdot b$

Non-Simplified Formula

$$\delta z = |a|\delta b + |b|\delta a$$

Relative Uncertainty

$$\delta z = ab \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|}\right)$$

6.

$z = 2 \cdot \frac{a}{b}$

Non-Simplified Formula

$$\delta z = 2 \cdot \left(\frac{\delta a}{|b|} + \frac{|a|\delta b}{|b^2|}\right)$$

Relative Uncertainty

$$\delta z = 2 \cdot \frac{a}{b} \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|}\right)$$

7.

$z = \frac{a-b}{c} \cdot d$

We will come up with the following substitutions:

$g = a - b$

$f = \frac{a-b}{c} = \frac{g}{c}$

$z = \frac{a-b}{c} \cdot d = f \cdot d$

$$\delta g = \delta a + \delta b$$

Non-Simplified Formula

$$\delta f = \frac{\delta g}{|c|} + \frac{|g|\delta c}{|c^2|}$$

$$\begin{aligned}
\delta z &= |f|\delta d + |d|\delta f \\
&= \left|\frac{a-b}{c}\right| \cdot \delta d + |d| \cdot \left(\frac{\delta g}{|c|} + \frac{|g|\delta c}{|c^2|}\right) \\
&= \left|\frac{a-b}{c}\right| \cdot \delta d + |d| \cdot \left(\frac{\delta a + \delta b}{|c|} + \frac{|a-b|\delta c}{|c^2|}\right) \\
&= \left|\frac{a-b}{c}\right| \cdot \delta d + \left|\frac{a-b}{c}\right| |d| \cdot \left(\frac{\cancel{|c|}}{|a-b|} \cdot \frac{\delta a + \delta b}{\cancel{|c|}} + \right. \\
&\qquad \left. + \frac{\cancel{|c|}}{|a-b|} \cdot \frac{|a-b|\delta c}{|c^{\cancel{2}}|}\right) \\
&= \left|\frac{a-b}{c}\right| \cdot \delta d + \left|\frac{a-b}{c}\right| |d| \cdot \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \left(\left|\frac{\cancel{c}}{\cancel{(a-b)} \cdot d}\right| \cdot \left|\cancel{\frac{a-b}{c}}\right| \cdot \delta d\right) + \\
&\quad + \left|\frac{a-b}{c} \cdot d\right| \cdot \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \cdot \left(\frac{\delta d}{|d|}\right) + \left|\frac{a-b}{c} \cdot d\right| \cdot \\
&\qquad \cdot \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \cdot \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|} + \frac{\delta d}{|d|}\right)
\end{aligned}$$

Relative Uncertainty

$$\delta f = \left|\frac{g}{c}\right| \left(\frac{\delta g}{|g|} + \frac{\delta c}{|c|}\right)$$

$$\begin{aligned}
\delta z &= |fd| \left(\frac{\delta f}{|f|} + \frac{\delta d}{|d|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \left(\frac{\left|\frac{g}{c}\right| \left(\frac{\delta g}{|g|} + \frac{\delta c}{|c|}\right)}{|f|} + \frac{\delta d}{|d|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \left(\frac{\left|\frac{a-b}{c}\right| \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|}\right)}{\left|\frac{a-b}{c}\right|} + \frac{\delta d}{|d|}\right) \\
&= \left|\frac{a-b}{c} \cdot d\right| \left(\frac{\delta a + \delta b}{|a-b|} + \frac{\delta c}{|c|} + \frac{\delta d}{|d|}\right)
\end{aligned}$$

From both of the methods, we can directly see that regardless if you are multiplying or dividing 2 or 3 numbers, you can just directly add their relative uncertainties. (We also see that directly using the relative uncertainties approach can often times be easier (especially when dealing with division).

8.

$z = x^n$

This uncertainty is best derived using the relative uncertainty formula variant:

Note that:

$z = \overbrace{x \cdot x \cdot x \ldots x}^{n}$

Then:

$$\delta z = |x^n| \cdot \left(\frac{\frac{\left(\cdots \frac{\cancel{|x^3|} \cdot \left(\frac{\cancel{|x^2|}\left(\frac{\delta x}{|x|} + \frac{\delta x}{|x|} \right)}{\cancel{|x^2|}} + \frac{\delta x}{|x|} \right)}{\cancel{|x^3|} \ldots} + \ldots \right)}{\ldots}}{|x^{n-1}|} + \frac{\delta x}{|x|} \right)$$

$$= |x^n| \cdot \left(\overbrace{\frac{\delta x}{|x|} + \cdots + \frac{\delta x}{|x|}}^{n} \right)$$

$$= |x^n| \cdot \left(n \cdot \frac{\delta x}{|x|} \right)$$

For those of you who know calculus, you can easily notice that the equation above is the derivative of z with respect to x multiplied by its uncertainty:

$$\delta z = |x^n| \cdot \left(n \cdot \frac{\delta x}{|x|} \right)$$

$$= n \cdot \frac{|x^{\cancel{n}}|}{\cancel{|x|}} \cdot \delta x$$

$$= nx^{n-1} \cdot \delta x$$

$$= \frac{\partial z}{\partial x} \cdot \delta x$$

9.

$a = \frac{z - b}{c} \cdot d$

$\frac{ac}{d} = z - b$

$z = b + \frac{ac}{d}$

Make the following substitutions:

$$g = ac$$
$$f = \frac{ac}{d} = \frac{g}{d}$$
$$z = b + \frac{ac}{d} = b + f$$

Non-Simplified Formula	Relative Uncertainty
$\delta g = \lvert a\rvert \delta c + \lvert c\rvert \delta a$	$\delta g = \lvert ac\rvert \left(\frac{\delta a}{\lvert a\rvert} + \frac{\delta c}{\lvert c\rvert}\right)$
$\delta f = \frac{\delta g}{\lvert d\rvert} + \frac{\lvert g\rvert \delta d}{\lvert d^2\rvert}$	$\delta f = \left\lvert\frac{g}{d}\right\rvert \left(\frac{\delta g}{\lvert g\rvert} + \frac{\delta d}{\lvert d\rvert}\right)$
$\delta z = \delta b + \delta f$ $= \delta b + \frac{\delta g}{\lvert d\rvert} + \frac{\lvert g\rvert \delta d}{\lvert d^2\rvert}$ $= \delta b + \frac{\lvert a\rvert \delta c + \lvert c\rvert \delta a}{\lvert d\rvert} + \frac{\lvert ac\rvert \delta d}{\lvert d^2\rvert}$	$\delta z = \delta b + \delta f$ $= \delta b + \left\lvert\frac{g}{d}\right\rvert \left(\frac{\delta g}{\lvert g\rvert} + \frac{\delta d}{\lvert d\rvert}\right)$ $= \delta b + \left\lvert\frac{ac}{d}\right\rvert \left(\frac{\cancel{\lvert ac\rvert} \left(\frac{\delta a}{\lvert a\rvert} + \frac{\delta c}{\lvert c\rvert}\right)}{\cancel{\lvert ac\rvert}} + \frac{\delta d}{\lvert d\rvert}\right)$ $= \delta b + \left\lvert\frac{ac}{d}\right\rvert \left(\frac{\delta a}{\lvert a\rvert} + \frac{\delta c}{\lvert c\rvert} + \frac{\delta d}{\lvert d\rvert}\right)$

0.2.3 Transcendental Functions

Recall that the uncertainty for some arbitrary function $z(x_1, x_2, ..., x_n)$ is approximated using the following formula:

$$\delta z = \left\lvert\frac{\partial z}{\partial x_1}\right\rvert \cdot \delta x_1 + ... + \left\lvert\frac{\partial z}{\partial x_n}\right\rvert \cdot \delta x_n$$

1.

$$z = \sin x$$
$$\begin{aligned}\delta z &= \left\lvert\frac{\partial z}{\partial x}\right\rvert \cdot \delta x \\ &= \left\lvert\frac{\partial \sin x}{\partial x}\right\rvert \cdot \delta x \\ &= \lvert \cos x\rvert \cdot \delta x\end{aligned}$$

For example if

$$z = \pi \pm 0.1$$

Then:

$$\begin{aligned}z &= \sin \pi \\ &= 0\end{aligned}$$

$$
\begin{aligned}
\delta z &= |\cos x| \cdot \delta x \\
&= |\cos \pi| \cdot 0.1 \\
&= 0.1
\end{aligned}
$$

2.

$z = \tan x \cdot \cos \alpha$

$$
\begin{aligned}
\delta z &= \left|\frac{\partial z}{\partial x}\right| \cdot \delta x + \left|\frac{\partial z}{\partial \alpha}\right| \cdot \delta \alpha \\
&= \left|\frac{\partial(\tan x \cdot \cos \alpha)}{\partial x}\right| \cdot \delta x + \left|\frac{\partial(\tan x \cdot \cos \alpha)}{\partial \alpha}\right| \cdot \delta \alpha \\
&= \left|\cos \alpha \cdot \frac{1}{\cos^2 x}\right| \cdot \delta x + |-\tan x \cdot \sin \alpha| \cdot \delta \alpha \\
&= \left|\frac{\cos \alpha}{\cos^2 x} \cdot\right| \delta x| + |\tan x \cdot \sin \alpha| \cdot \delta \alpha
\end{aligned}
$$

3.

$z = \dfrac{\sin x + \cos x}{x^{2x}}$

$$
\begin{aligned}
\delta z &= \left|\frac{\partial z}{\partial x}\right| \cdot \delta x \\
&= \left|\frac{\partial\left(\frac{\sin x + \cos x}{x^2}\right)}{\partial x}\right| \cdot \delta x \\
&= \left|\frac{\partial \frac{\sin x}{x^2}}{\partial x} + \frac{\partial \frac{\cos x}{x^2}}{\partial x}\right| \cdot \delta x \\
&= \frac{\cos x \cdot x^{\cancel{2}} - 2\cancel{x} \cdot \sin x}{x^{\cancel{4}}} + \frac{-\sin x \cdot x^{\cancel{2}} - 2\cancel{x} \cdot \cos x}{x^{\cancel{4}}} \\
&= \frac{x \cos x - 2 \sin x - x \sin x - 2 \cos x}{x^3}
\end{aligned}
$$

4.

$z = e^x$

$$
\begin{aligned}
\delta z &= \left|\frac{\partial z}{\partial x}\right| \cdot \delta x \\
&= \left|\frac{\partial e^x}{\partial x}\right| \cdot \delta x \\
&= |e^x| \cdot \delta x
\end{aligned}
$$

5.

$x = z^2 + 1$

$z^2 = x - 1$

$z = \pm\sqrt{x-1}$

$$\begin{aligned}
\delta z &= \left|\frac{\partial z}{\partial x}\right| \cdot \delta x \\
&= \left|\pm\frac{\partial\left(\pm\sqrt{x-1}\right)}{\partial x}\right| \cdot \delta x \\
&= \left|\pm\frac{1}{2\sqrt{x-1}}\right| \cdot \delta x \\
&= \left|\frac{1}{2\sqrt{x-1}}\right| \cdot \delta x
\end{aligned}$$