1 Exercises

Derivative Suggested Exercises I

I) **Understanding the Definition**

   i) Explain intuitively why the limit formula for the derivative in equation (**??**) is correct? (i.e. gives you the best linear approximation to the function at the point).

   ii) Give a geometric interpretation of the derivative of a function. (*Hint: Draw a graph of a function, pick a point and draw something that helps clarify what the derivative is. You should think about what the Jacobian is: can you identify it's domain and target in the picture?.*).

II) **Computation Warmup**

   i) Compute the derivative of $f(x) = kx$ for k any constant at some point $x_0 \in \mathbb{R}$

   ii) Compute the derivative of $f(x) = ax^2$ for a any constant, at some point $x_0 \in \mathbb{R}$

III) **Bonus:** Suppose $f : \mathbb{R} \to \mathbb{R}^n$ with $f(t) = (f_1(t), \dots, f_n(t))$. How should the Jacobian of f look like?

Derivative Suggested Exercises II

I) **Computational Practice:** Compute the following derivatives:

$$\frac{d}{dt}\sin(3t) \qquad \frac{d}{dt}(\cos(t))^2$$
$$\frac{d}{dt}\cos(t)\sin(t) \qquad \frac{d}{dt}\exp(\cos(t))$$
$$\frac{d}{dt}\frac{1}{t^n} \qquad \frac{d}{dt}\frac{1}{\sin(t)}$$

II) **Understanding Division:**

   i) Let f be a differentiable function and $f \neq 0$.Compute the following derivative:

$$\frac{d}{dt}\frac{1}{f(t)}$$

   ii) Use the result above to verify that:

$$\frac{d}{dt}\frac{f(t)}{g(t)} = \frac{f'(t)g(t) - f(t)g'(t)}{(g(t))^2}$$

III) **Inverse of Differentiation**
Find a function such f that the following are its derivative:

$$\frac{d}{dt}f = t^2 \qquad \frac{d}{dt}f = \sin(t)\cos(t)$$

Suggested Integral Exercises

I) Compute the following integrals

$$\int_0^a dx \quad \int_0^a 3x^2 + 7$$
$$\int_0^a e^{kx} dx \quad \int_0^a x e^{x^2}$$
$$\int_0^a x\sin(x) \quad \int_0^a \sin(x)\cos(x)$$
$$\int_0^a x e^{x} \quad \int_0^a x^2 e^{x}$$

II) Recall that Newton's equations of motion are given by:

$$F = ma = m\ddot{x}$$

i) Consider a particle restricted to 1 dimension. Suppose we attach a string to it. Then in some coordinate system, the force acting on the particle at position x is $F = -kx$.
We define the potential energy to be:

$$U(p) = -\int_0^p F(x)dx$$

Compute $U(x)$. Then show that the total energy:

$$E(x(t)) = \frac{1}{2}m(\dot{x})^2 + U(x(t))$$

Is constant with time. Show that if E is conserved for any F that does not depend on t.

Partial Derivative Suggested Exercises

I) Let $g : \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ be both differentiable. Convince yourself that:

$$\frac{\partial}{\partial t} f(g(t))\bigg|_{t_0} = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}\bigg|_{g(t_0)} \cdot \frac{\partial g_i}{\partial t}\bigg|_{t_0}$$

II) Compute the following derivatives:

$$\frac{\partial \sin(xy)y^2}{\partial x} \quad \frac{\partial e^{x^2\sin(y)}}{\partial y}$$
$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} e^{x^2\sin(y)} \quad \frac{\partial}{\partial y}\frac{\partial}{\partial x} e^{x^2\sin(y)}$$

III) Consider the function:

$$T(t) = \frac{1}{2}\left(\dot{x_1}^2 + \dot{x_2}^2\right)$$

i) Suppose you are given $r(t)$ and $\varphi(t)$ such that:

$$x = (x_1, x_2) = \left(r\cos\varphi\ ,\ r\sin\varphi\right)$$

Express $T(t)$ in terms of $r(t)$ and $\varphi(t)$

IV) Recall that acceleration is:

$$a = \ddot{x} = \big(\ddot{x}_1, \ddot{x}_2\big)$$

Suppose that the particle's motion is restricted to a circle of radius r. It's path $x(t)$ is then fully determined by some function $\varphi(t)$ via:

$$x(t) = r\big(\cos\varphi(t)\ ,\ \sin\varphi(t)\big)$$

Show that the acceleration, must be proportional to $x(t)$ (i.e. pointing along the same direction as the radius vector of x)

Suggested Exercises Functional Calculus

I) Let

$$F[f] = \int_0^1 \big(f(x)\big)^2 dx$$

i) Compute $DF_f[h]$ for arbitrary f and h

ii) Suppose that F is restricted only to functions f such that $f(a) = f(b)$. Suppose that some function g extremizes F. Find the condition that g must satisfy.

II) Let

$$F[x_1(t), x_2(t)] = \int_0^{t_1} \sqrt{\dot{x_1}^2 + \dot{x_2}^2} dt$$

.

Where $x_1(0) = x_2(0) = 0$ and $x_1(t_1) = x_2(t_1) = 1$

What does this functional compute?

Find $\tilde{x}(t)$ such that $DF[\tilde{x}] = 0$. Does this answer make sense?