Relationship between the areas of two triangles with an equal angle

Theorem

If an angle of a triangle, is equal to the angle of another triangle, then the areas of these two triangles relate as the products of the sides that enclose the equal angles.

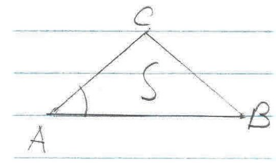

Figure 1

Proof

Let S and S_2 be the areas of triangles ABC and $A_1B_1C_1$, for which $\angle A = \angle A_1$

Let's prove that:

$\frac{S}{S_2} = \frac{AB \cdot AC}{A_1B_1 \cdot A_1C_1}$

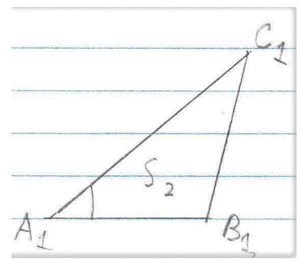

Figure 2

Let's put triangle ABC (Figure 1) on top of triangle $A_1B_1C_1$ (Figure 2) such that A and A_1 match, sides A_1B_1 and A_1C_1 align with rays AB and AC (Figure 3).

Triangles ABC and AB_1C have a common height CH. Therefore:

$\frac{S}{S_{AB_1C_1}} = \frac{AB}{AB_1}$

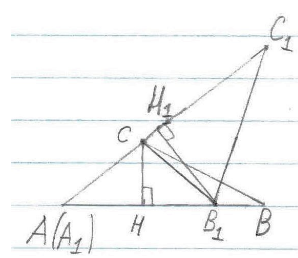

Figure 3

Triangles ABC and AB_1C have a common height B_1H_1. Therefore:

$\frac{S_{AB_1C}}{S_{AB_1C_1}} = \frac{AC}{AC_1}$

Now multiply the two to get:

$\frac{S}{S_{AB_1C}} \cdot \frac{S_{AB_1C}}{S_{AB_1C_1}} = \frac{AB \cdot AC}{AB_1 \cdot AC_1}$

$\frac{S}{S_2} = \frac{AB \cdot AC}{AB_1 \cdot AC_1}$

$\therefore$ QED