Coplanar Vectors

Let's prove that if the triple scalar product of three vectors is equal to zero, then these vectors will be coplanar. Consider vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\vec{m}$. (Note that all of these vectors have a common starting point). Given that: $\vec{m} \perp \overrightarrow{OA}$, $\vec{m} \perp \overrightarrow{OB}$, $\vec{m} \perp \overrightarrow{OC}$. To prove that these vectors will be coplanar, it would be sufficient to prove that the lines OA, OB and OC are coplanar.

Theorem 1 *If three lines OA, OB and OC are perpendicular to some other line m and intersect at some point O, then these lines lie in the same plane.*

Proof

Consider lines OA, OB and OC which are all perpendicular to some line m and intersect at some point O (Figure 1). Since m is perpendicular to OA and OB, then it is also perpendicular to the plane α, which is defined by these two lines. (This statement will not be proven here, just assume it to be true). This means that m is perpendicular to any line $\in \alpha$.

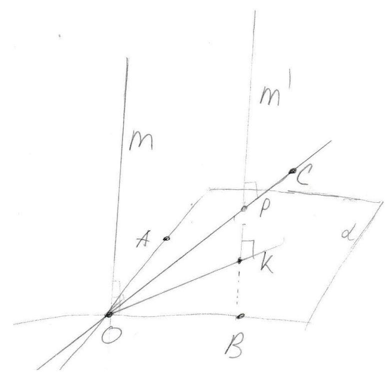

Figure 1

Let's suppose that $OC \notin \alpha$. In that case, we can draw a line m', parallel to m such that it intersects OC at some point P, which does not coincide with point O. Since $OC \notin \alpha$ and point $O \in alpha$, then point $P \notin \alpha$. This means that m' would intersect α at some other point k. Since m' is parallel to m it is also perpendicular to OC and α. This means that we would have a triangle OPK with two right angles: $\angle OKP$ and $\angle KPO$.
This is impossible, hence OA, OB and OC are coplanar.
$\therefore$ QED

Hence, $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ as described above, are coplanar.