1 Review

Let $T : V \to W$, $S : W \to Y$ be a linear map, where V, W and Y are finite dimensional vector spaces.

Let the following be:

$$\begin{aligned} \alpha \ - \ & v_1, ..., v_n - \text{basis for } V \\ \beta \ - \ & w_1, .., w_m - \text{basis for } W \\ \gamma \ - \ & y_1, ..., y_p - \text{basis for } Y \end{aligned}$$

Note that, to fully describe T it is sufficient to describe how T acts on the basis vectors v_j (think about why this is again).

We then did the following calculation:

$$Tv_j = \sum_{i=1}^{m} b_{ij} w_i = b_{ij} w_i \;\; i \in \{1, ..., m\} \tag{1}$$

We can therefore see that the action of T on V is fully described by these number $b_{i,j}$.

We can therefore write these numbers in a matrix.

$${}_\alpha\mathcal{M}_\beta\,(T) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \tag{2}$$

We then considered the composition of two linear maps: $S \circ T$ and found that the numbers c_{ij} describing its action on V such that:

$$(S \circ T)\, v_j = c_{ij} w_i$$

Are given through the same numbers b_{ij} for T and numbers s_{ij} for S via:

$$c_{ij} = \sum_{i=1}^{m} s_{ik} b_{kj} \;\; i \in \{1, ..., m\} \tag{3}$$

We therefore use the formula above to define matrix multiplication such that:

$${}_\gamma\mathcal{M}_\alpha\,(S \circ t) = {}_\gamma\mathcal{M}_\beta(S) \cdot {}_\beta\mathcal{M}_\alpha(T) \tag{4}$$

So we in some sense multiply a row from ${}_\gamma\mathcal{M}_\beta(S)$ by a column of ${}_\beta\mathcal{M}_\alpha(T)$

2 Writing down a Vector and Dimension

In this vein, we want to also find a way to use matrix notation to write down the action of a matrix on a vector. For this we first need to figure out how to write down a vector as a list of numbers.
Given a basis α of V, we know we can specify any vector via a list of $n = dimV$ numbers a_j, where:

$$v = a_j v_j$$

Remember that in a matrix, the rows are like the output and the columns are like the input (recall the little arrows I drew on the matrix with v_j s and w_is. Since we want the vector to be an input into the matrix, it should look like its outputting these numbers into the matrix (if this makes any sense). We therefore choose to write down vectors as:

$$\mathcal{M}(v) = \begin{pmatrix} a_1 \\ \vdots \\ v_n \end{pmatrix}$$

Questions:
Given $T : V \to W$, what are the dimensions of ${}_\beta\mathcal{M}_\alpha(T)$ in terms of $\dim V$ and $\dim W$

Suppose you are multiplying an $m \times n$ matrix by a $p \times d$ matrix. For this operation to be valid, is there any restriction on how m, n, p, d have to be related? (Perhaps look at equation (3).)
Why does this restriction make sense, if we think of a matrix as a way to represent a linear map on a finite dimensional vector space? (*Hint: Think of vector space dimensions and what is going from where to where*).

3 Questions

0) Let V, W be 2 dimensional vector spaces over $\mathbb{R}$.
Let v_1, v_2, v_3 and w_1, w_2, w_3 be their respective bases. Suppose $T : V \to W$ such that:

$$\begin{aligned} Tv_1 &= 2w_1 + w_2 + 3_w 3 \\ Tv_2 &= 3w_2 + w_2 + w_3 \\ Tv_3 &= 4w_3 \end{aligned}$$

Write down $\mathcal{M}(T)$ in the bases above.

3.1 Computation Practice

Compute the following applications of matrices to vectors:

$$1) \begin{pmatrix} 6 & 3 & 7 & 9 \\ 9 & 0 & 5 & 9 \\ 7 & 8 & 9 & 6 \\ 9 & 7 & 5 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \qquad 2) \begin{pmatrix} 4 & 2 & 3 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$3) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad 4) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Compute the following products of matrices:

$$5) \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 3 & 6 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad 6) \begin{pmatrix} 3 & 6 & 2 & 1 \\ 4 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 6 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$7) \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

8) Consider $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ - linear maps. Let e_1, e_2 be the standard basis (will refer to it as E) (e_1 - x direction, e_2 - y direction).
Let $v_1 = e_1 + e_2$ and $v_2 = e_1 - e_2$.
i) Draw v_1 and v_2 in plane (make sure this drawing is large, as you will add more stuff to it later).
Suppose T is such that in basis E you have:

$${}_E\mathcal{M}_E(T_1) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$${}_E\mathcal{M}_E(T_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

ii) Using the convenient matrix notation, compute, write down in terms of e_1 and e_2 and then draw in the plane (same drawing as before) the vectors:

$$T_1v_1 \quad T_2v_1 \quad T_1v_2 \quad T_2v_2$$
$$T_1T_1v_1 \quad T_2T_2v_2$$

iii) Look at your drawing, what do maps T_1 and T_2 "do" to the vectors?
iv) Sketch what the following matrix (in the standard basis) does to e_1 and e_2

$$\begin{pmatrix} 1.25 & -0.75 \\ -0.75 & 1.25 \end{pmatrix}$$

3.2 Choice of Basis

Definition 1 *A diagonal matrix, is a matrix such that* $a_{ij} = 0 \ \forall i \neq j$

Definition 2 *A linear operator, is a map from a vector space to itself. (i.e.* $T : V \to V$*)*

1. Why diagonal matrices are cool

   (a) Write down some examples of diagonal 3 by 3 matrices

   (b) Suppose a matrix of $T : V \ toV'$, where $dimV = dimV' = n$ in some bases $\alpha = v_1, ..., v_n \in V$ and $\alpha' = v_1', ..., v_n' \in V'$ is diagonal. (so ${}_{\alpha'}\mathcal{M}_\alpha(T)$ is diagonal.

   (c) Suppose T and S are linear maps such that $T : V \to V'$ and $S : V' \to V''$ where $\dim V = \dim V' \dim V'' = n$. Suppose given bases $\alpha, \alpha', \alpha''$ of V, V' and V'', the matrices for S and T in these bases are diagonal. Compute the product of $\mathcal{M}(S) \cdot \mathcal{M}(T)$ and simplify as much as possible.

   (d) Using your result in the question above, why are diagonal matrices so nice?

   (e) What does a matrix being diagonal, mean in terms of where it sends the basis vectors?

2. The power of change

   (a) Note that matrices depend on a choice of basis. This means if a matrix is diagonal in one basis, it may not necessarily be diagonal in another one. Let's check this. Let V and W be f. d. vsp. of the same dimension $n = 2$.
   Suppose e_1 and e_2 is some basis of V (denote by e) and w_1, w_2 is some basis of W (denote by w).
   Now suppose we have another basis of V given by
   $v_1 = e_1 + e_2$, $v_2 = e_2 - e_1$ (denote by v).
   Let $T : V \to W$ be a linear map such that

   $$Te_1 = \frac{1}{2}w_1 - \frac{1}{2}w_2$$
   $$Te_2 = \frac{1}{2}w_1 + \frac{1}{2}w_2$$

   Write down, ${}_w\mathcal{M}_e(T)$ and ${}_w\mathcal{M}_v(T)$. Is one of them diagonal?

   (b) This now poses an useful problem for us. How can we pick a basis, such that our matrix is diagonal in that basis. This problem is somewhat straightforward when we are mapping between two different spaces (say from V to W), because we can independently pick a basis for V and for W. This get's a lot harder when we start talking about linear operators (maps from a vector space to itself). Why is this? If you are mapping something from V to V, it would be very convenient to have the input and output of the matrix be in the same basis (think why? (think about compositions)).

   Let's look at an example. Suppose T is a linear operator on V, where $dimV = 2$
   Suppose in some basis α:

   $$ {}_\alpha\mathcal{M}_\alpha(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5}$$

   Find a basis β such that ${}_\beta\mathcal{M}_\beta(T)$ is diagonal. (*Hint: Look at computational questions 3 and 4, and maybe write that in terms of operators acting on vectors (like Tv, instead of matrices). Also think about 1 e) of this section.*)