Notation
In lecture, we talked about kinetic energy E_k and potential energy E_p. It is useful to know that often times, we use the following letters: $V \equiv E_p$ and $K \equiv E_k$. (I will likely use V for potential and E_k for kinetic). Of course E or E_{tot} still refer to the total energy $V + E_k$.

1. Consider a particle restricted to 1 spatial dimension with mass m. Suppose there is a force acting on the particle given by: $F = -kx$, where x is the position of the particle. This kind of force, describes a mass on a spring. When the mass is at $x = 0$, then the spring is neither compressed or extended. If $x < 0$, then the spring is compressed. For $x > 0$, the spring is extended.

   (a) Draw the spring and the mass where spring is compressed, extended and at equilibrium. Show which way the force is pointing in each direction and understand why $F = -kx$ therefore makes sense as a force that models the force of a spring. [1]

   (b) Compute V for this particle. (Hint: To make calculations nicer, set the reference point at 0).

   (c) Write down the equation for the total energy.

   (d) Show that the total energy is conserved (Hint: Take the derivative with respect to time).

   (e) Now suppose the particle begins at rest at some point x_0. Write down the energy at this point (in space and time).

   (f) Look very closely at the equation and tell me what is the maximum speed the particle can achieve. (Hint: What is the maximum value of V?).

   (g) Plot V as a function of x. Draw a horizontal line at the maximum value of V (label its value) and think about which regions in space the particle can and cannot access (and why?).

   (h) Now write down newton's law and substitute the force in said law. You get a differential equation. Solve it (Hint: Got to your notes and look at the complex numbers bit. Don't forget that the particle initially starts at rest at x_0).

   (i) Hopefully by now you have installed jupyter notebook. Open a jupyter notebook file. Define a variable $x0$ (set it to say 1), a variable m (set it to 1) and a variable k and set that also to 1. Now define a lambda function `x = lambda t: your differential equation solution goes here`. In the first cell write `import numpy as np` and `import matplotlib.pyplot as plt`.
   Now define a numpy array of equally spaced points from 0 to 2π, with say 100 points. Call this array whatever you like (just not whatever you called your lambda function.
   Define another array by applying the lambda function to the numpy array above.
   Now do:
   ```
   mainFigure = plt.figure()
   mp = mainFigure.add_axes([0,0,1,1])
   mp.plot( t values, your computed x values, ls='--', color = your fav color)
   ```

   (j) Now define a separate figure and plot `mp.plot(xvalues ,tvalues, ls='', marker='o' )`

   (k) Make a plot that for the potential energy as a function of x. Plot the thing you did above on top of this plot for potential energy (ask me if you don't get what it means).

   (l) Notice that $F = -\frac{d}{dx}V$. So in some sense, it might actually be more convenient to think of energy rather than Forces (this is ideally what we would like to do in the long run, but not now).

2. We will now do a little math exercise (to fully answer the question with integrals along curves (specifically for finding energy). First let's define what a curve is.

   Definition 1 *A curve in* $\mathbb{R}^2$ *is a continuous function* $\gamma : [a, b] \to \mathbb{R}^2$ *(or* $\mathbb{R}^3$*).*

   (a) Let γ be a curve. This is a function that takes in a number and outputs a point. We can therefore write this function as $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ (So like two functions. One for each coordinate). Note that we said that γ has to be continuous. Without going into too much detail, we will define this to mean that γ_1 and γ_2 are continuous functions of (s). [2]
   Now recall how we defined E_p in multiple dimensions. Write it down.
   The formula involves $\dot{\gamma}$ - the tangent to the curve at some point. Let's try and come up with an equation for the tangent to the curve at some point $\gamma(s)$. Let's find inspiration to come up with this definition:

[1]You might imagine that for some springs, if you extend them by a lot, the material starts to break down a bit (or some other effect) and the force would be non-linear with space, or even change with time (material erodes over time). You would be correct. But this is usually a good approximations. Springs for which this is a good approximation (i.e. The force they exert is about $-k\Delta x$ where Δx is their displacement from some rest position) are called Hookean. The equation $F = -kx$ used to describe springs is called Hooke's law.

[2]In math, we often use t instead of s to parametrize curves (and indeed t does look nicer). Here I will use s, to avoid confusing it with time. This letter is also quite commonly used though.

(b) Suppose we had some continuous function $f(s)$ of one variable. Then we can consider its graph in $\mathbb{R}^2$ i.e. the set of points $\mathit{\Gamma} = \{(x, y)|y = f(x)\}$. Write down the parametrization for this graph (i.e. a function $\gamma : \mathbb{R} \to \mathbb{R}^2$, $\gamma(s) = (\gamma_1(s), \gamma_2(s))$.

(c) Now think about it really hard. You want a vector, that has the same slope as the derivative of the function. So you want a vector such that $\frac{y}{x} = f'(x)$. Now what if $x = 1$. What vector would you have? (write it down). Now think about, how can you get this vector from the equation of γ?[3]

(d) Now use this as inspiration to write down an equation for $\dot{\gamma}$ for a general curve γ (not just the graph of a function).

(e) Now write down the equation of $\overrightarrow{F}(\gamma(s)) \cdot \dot{\gamma}(\gamma(s))$. (So we are taking the dot product of the force at a point on the curve, and times its tangent vector.

(f) Now, when we write E_p as an integral, we can just integrate the formula we have above with respect to s. So we get that if $\gamma : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$), $E_p = \displaystyle\int_a^b \overrightarrow{F}(\gamma(t))\dot{\gamma}(\gamma(s))ds$. [4]

(g) As an example, consider the force of gravity $F_g = -mg\hat{y}$ (I am looking at 2 dimensions). Now consider the curve $\gamma : [0, a] \to \mathbb{R}^2$, given by $\gamma(s) = (s, 2s)$. Compute E_p of a particle of mass m going from $(0, 0)$ to $(a, 2a)$ along the curve γ [5]

(h) Try and show that in fact E_p is independent of the path taken (consider some arbitrary path γ and try and show that the integral always gives the same value). [6]

(i) The result above gives us exactly what we would like! Really potential energy should not depend on path we take. Now in the more general case, it turns out that potential energy does not depend on the path you take the integral with if the force does not depend on time (only on position) [7].

3. Figure out what units Forces are in (Hint: look at $F = ma$).[8]

[3] Does the notation for $\dot{\gamma}$ now make sense?

[4] Remember how I wrote something like $\int F_1 dx + \int F_2 dy + \int F_3 dz$. Well notice that $x = \gamma_1(s)$ and recall that $dx = d\gamma_1 = \dot{\gamma}_1 ds$. Look at the formula and see how this is the same thing.

[5] You already know the potential energy is $E_p = mgh = 2mga$. Make sure that once you compute the integral, that is what you get.

[6] This may be difficult to prove, we might do it together in class.

[7] Hey we have seen something like this before! Remember that energy is conserved for such forces!

[8] This unit is called the Newton.