1 Useful Definitions

Definition 1 *The* ***Cartesian product*** *of two sets X and Y, denoted as $X \times Y$, is the set of all pairs of elements, where the first element in the pair is from X and the second element is from Y. We can write this down as:*

$$X \times Y = \{(x, y)|x \in X, y \in Y\} \tag{1}$$

2 Suggested Exercises

I) **Completing the Square**. The goal of this exercise is to get comfortable with the technique called "completing the square". The idea is given an expression, rewrite it as the square of a number plus another number.

   i) Expand $(\alpha + \beta)^2$

   ii) Rewrite the following as squares of a sum of two numbers *Hint:Compare the expression with your expansion in part i)*:

$$\begin{gathered} 9 + 6 + 1 \\ 9 - 6 + 1 \\ 4x^2 + 4xy + y^2 \\ d^2 - 8d + 16 \end{gathered} \tag{2}$$

   iii) Now you hopefully got practice with recognizing when something is the square of the sum of two things. It is of course not always the case that something is a square. You can however usually rewrite an expression as a sum of something square and something else. Let's look at an example: Consider $y^2 + 4y + 2$. You can see that this is not quite something square, since $(y + 2)^2 = y^2 + 4y + 4$. This is almos like the first expression, we are just off by 2. Well, we can always change the original expression by 0 (i.e. not change it at all). So let's add and subtract the bit that we are missing.

$$y^2 + 4y + 2 + 2 - 2 = y^2 + 4y + 4 - 2 = (y + 2)^2 - 2$$

   We have therefore rewritten our initial expression as the sum of two things squared, and an extra bit. This procedure is called completing the square.
   Complete the square in this way for the following expressions:

$$\begin{gathered} x^2 + 6x - 2 \\ 9x^2 + 6x - 3 \\ 4x^2 + 5x + 1 \\ x^2 + kx \\ x^2 + kx + d \end{gathered} \tag{3}$$

   iv) Let's define the following polynomial function

$$p(x) = ax^2 + bx + c$$

   Where a, b, c are some known constants and $a \neq 0$. Solve the equation $p(x) = 0$ *Hint: Factor out a from the expression above, complete the square, rearrange take some square roots and rearrange again. At the end you should just get the quadratic formula.*

   v) In the formula you got before, you should have a part that looks like:

$$D = b^2 - 4ac$$

   (If you don't then you either made a mistake, or should add some fractions together.). This is often called the "Discriminant".

II) **Geometry of the Parabola**. In this exercise we will study the geometric properties of the graph of $p(x)$, which we will call a parabola. First let's be precise with what we mean by grah. We then have the following definition: The graph of a function $f(x)$, denoted by Γ_f is a subset of the plane (i.e. $\mathbb{R} \times \mathbb{R}$) given via:

$$\Gamma_f = \{(x, f(x))|x \in \mathbb{R}\}$$

In this exercise, we will aim to show that the graph of this function always has a minimum or maximum and is symmetric about some horizontal axis. We give these concepts rigorous definitions later in the question.

i) We begin by putting $p(x)$ into a nicer form. Assuming $a \neq 0$, complete the square for

$$p(x) = ax^2 + bx + c$$

Hint:factor out a as before.

ii) Assume $a > 0$. Let's show that the parabola has a minimum. First let's define what a minimum is:

Definition 2 *We say a function f has a minimum at x_0 if $f(x_0) \leq f(x)$ for all x.*

Using the form you rewrote $p(x)$ in, in previous part show that the function $p(x)$ has a minimum, i.e. find an x_0 such that $f(x_0) > f(x)$ for any oher x. *Hint: If you square something it is always positive.*

iii) How does this argument change for $a < 0$?

iv) From Q I, you found that if a quadratic equation has 2 roots, they are always centered around $-\frac{b}{2a}$. In the preious equation, you hopefully showed that at $x_0 = -\frac{b}{2a}$, the function $p(x)$ has a minimum/maximum depending on the sign of a. Now let's show that $p(x)$ should b symmetric around this point.

Definition 3 *We say that a function $f(x)$ is symmetric around a point x_0 if $f(x_0 - k) = f(x_0 + k)$ for any number k*

Draw a picture to describe this definition (this is open ended. Draw whatever you feel best helps visualize this definition). Then using the form you rewrote $p(x)$ in, in the first part of II, prove that $p(x)$ is symmetri around $x_0 = -\frac{b}{2a}$