

1 Definitions & Review

Definition 1 An alternating k tensor ω on a vector space V is an alternating multilinear map. Meaning that it has the following two properties:

$$\begin{aligned}\omega(v_1, \dots, v_i + \lambda u, \dots, v_k) &= \omega(v_1, \dots, v_i, \dots, v_k) + \lambda\omega(v_1, \dots, u, \dots, v_k) \quad \forall i \\ \omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) &= -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k) \quad \forall i \neq j\end{aligned}$$

Definition 2 A differential k -form ω on smooth manifold M is a “ k -form” at each point $p \in M$, which varies smoothly in p . We denote the set of k -forms on M via $\Omega^k(M)$.

$$\omega(p, v) : T_p M \times \dots \times T_p M \rightarrow \mathbb{R}$$

Definition 3 We define a basis of $\Omega^1(\mathbb{R}^n)$ of forms: $\{dx^1, dx^2, \dots, dx^n\}$, where $dx^i : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as:

$$dx^i(a_1 e_1 + \dots + a_n e_n) = a_i$$

Definition 4 Let $\omega^k \in \Omega^k(M)$ and $\omega^l \in \Omega^l(M)$ be differential forms on a smooth manifold M . We define their wedge product $\omega^k \wedge \omega^l \in \Omega^{k+l}(M)$ as:

$$(\omega^k \wedge \omega^l)(v_1, \dots, v_{k+l}) = \sum (-1)^\nu \omega^k(v_{i_1}, \dots, v_{i_k}) \omega^l(v_{i_{k+1}}, \dots, v_{i_{k+l}})$$

Where ν is the sum of the permutation i_1, \dots, i_{k+l} and the sum is taken over all permutations such that $i_1 < i_2 < \dots < i_k$ and $i_{k+1} < i_{k+2} < \dots < i_{k+l}$ separately.

Theorem 1 The set $\Omega^k(\mathbb{R}^n)$ is a vector space with a basis given by:

$$\Omega^k(\mathbb{R}^n) = \text{span}(\{dx^{i_1} \wedge \dots \wedge dx^{i_k}\})$$

Where the set considers all combinations $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ such that $i_1 < i_2 < \dots < i_k$

Corollary 1 The set $\Omega^n(\mathbb{R}^n)$ is a 1 dimensional vector space:

$$\Omega^n(\mathbb{R}^n) = \text{span}(dx^1 \wedge \dots \wedge dx^n)$$

Note that $dx^1 \wedge \dots \wedge dx^n$ is the usual determinant

Definition 5 Let $\varphi : N \rightarrow M$, where N and M are smooth manifolds. Let $\omega \in \Omega^k(M)$. We define the pullback of φ as an operation on forms:

$$\begin{aligned}\varphi^* : \Omega(M) &\rightarrow \Omega(N) \\ (\varphi^* \omega)(x; v_1, \dots, v_k) &= \omega(\varphi(x); D\varphi v_1, \dots, D\varphi v_n)\end{aligned}$$

Where $x \in N$ and $v_i \in T_x N$

Definition 6 Let $\omega \in \Omega^n(\mathbb{R}^n)$ be a top form on \mathbb{R}^n . From corollary 1, we know it must be of the form $\omega = f(x_1, \dots, x_n) dx^1 \wedge \dots \wedge dx^n$. We then define the integral of this form over some open set $U \subseteq \mathbb{R}^n$ to be the usual thing:

$$\int_U \omega = \int_U f(x_1, \dots, x_n) dx^1 \wedge \dots \wedge dx^n$$

Definition 7 Let M be a manifold, $U \subseteq M$ an open set. Let $\varphi : V \rightarrow U \subseteq M$ be a chart, such that $\varphi(V) = U$ and $V \subseteq \mathbb{R}^n$. Suppose $\omega \in \Omega^n(M)$. We then define:

$$\int_U \omega = \int_V \varphi^* \omega$$

It can be shown that this integral does not depend on the choice of coordinate chart φ .

Definition 8 Let $\omega \in \Omega^k(M)$. Let $x \in M$. Pick some coordinate chart $\varphi : U \rightarrow M$ $U \subseteq \mathbb{R}^n$. Let $C(v_1, \dots, v_n)$ be a parallelepiped with sides v_1, \dots, v_n starting at x . Define:

$$A(v_1, \dots, v_n) = \int_{C_{v_1, \dots, v_n}} \varphi^* \omega$$

To lowest order, $A(v_1, \dots, v_n)$ is approximated by a multilinear $k+1$ form, which depends smoothly on x . We call this from $d\omega$. This defines the exterior derivative operator:

$$d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$$

Theorem 2 The exterior derivative $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ has the following properties, which are equivalent to its definition:

- i) $d(\omega_1 + \lambda\omega_2) = d\omega_1 + \lambda d\omega_2$
- ii) $d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l$
- iii) $d \circ d = 0$
- iv) Let $f(x) \in \Omega^0(M)$ be a smooth function. Then $df = \frac{\partial f}{\partial x_1} dx^1 + \dots + \frac{\partial f}{\partial x_n} dx^n$

Theorem 3 Stokes Theorem: Let $\omega \in \Omega^{n-1}(M)$. Then:

$$\int_{\partial M} \omega = \int_M d\omega$$

2 Suggested Exercises

I) Getting used to differential forms:

- i) Write the following explicitly in terms of dx^i and v_j

$$(dx^1 \wedge dx^2)(v_1, v_2) \quad (dx^1 \wedge dx^2 \wedge dx^1)(v_1, v_2, v_3)$$

II) Integrating Differential Forms

- i) Let $\omega = ydx - xdy$. Let $S^1 \subseteq \mathbb{R}^2$ be the circle. Let $\varphi : (0, 1) \rightarrow S^1$ be $\varphi(t) = (\cos(t), \sin(t))$. Compute $\int_{S^1} \omega$. Interpret your answer.
- ii) Let $\omega = xdy \wedge dzydx \wedge dz + zdx \wedge dy$ and $\varphi : (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{S}^2$ be $\varphi(\theta, \varphi) = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$. Compute $\int_{S^2} \omega$. Interpret your answer.

III) Exterior Derivative. Compute the following:

- i) $d(\cos(y)dx - \sin(x)dy)$
- ii) Compute $d(f(x, y)dx + g(x, y)dy)$. Write down stokes theorem for these forms (this is called Green's theorem)
- iii) Compute $d(xdy \wedge dz + ydx \wedge dz)$

IV) Relation to grad, div and curl.

Consider the following set of maps:

$$\begin{aligned} \psi_1 : \mathfrak{X}(\mathbb{R}^3) &\rightarrow \Omega^1(\mathbb{R}^3) \\ \psi_1(f(x, y, z)e_1 + g(x, y, z)e_2 + h(x, y, z)e_3) &= f dx + g dy + h dz \\ \psi_2 : \mathfrak{X}(\mathbb{R}^3) &\rightarrow \Omega^2(\mathbb{R}^3) \\ \psi_2(f(x, y, z)e_1 + g(x, y, z)e_2 + h(x, y, z)e_3) &= f dy \wedge dz - g dx \wedge dz + h dx \wedge dy \\ \psi_3 : C^\infty(\mathbb{R}^3) &\rightarrow \Omega^3(\mathbb{R}^3) \\ \psi_3(f(x, y, z)) &= f(x, y, z)dx \wedge dy \wedge dz \end{aligned}$$

Show that the following diagram commutes:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^3) & \xrightarrow{Id} & \Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \\ \downarrow grad & & \downarrow d \\ \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\psi_1} & \Omega^1(\mathbb{R}^3) \\ \downarrow curl & & \downarrow d \\ \mathfrak{X}(\mathbb{R}^3) & \xrightarrow{\psi_2} & \Omega^2(\mathbb{R}^3) \\ \downarrow div & & \downarrow d \\ C^\infty(\mathbb{R}^3) & \xrightarrow{\psi_3} & \Omega^3(\mathbb{R}^3) \end{array}$$

In other words, show for example that doing curl on a vector field, is the same as applying d to some 1-form, where we identify forms and vector fields using ψ_1 and ψ_2 . i.e. $\text{curl} = \psi_2^{-1} \circ d \circ \psi_1$