What is uncertainty

When conducting any experiment, there is always a question of how to accurately collect data? How can we know that the values provided by our equipment actually reflect what is happening in reality. How do we know that they are not entirely off? The concept of "uncertainty" is used to signify how inaccurate a measurement could be/is (its largest possible and smallest possible values). It is effectively a range within which, the "actual" value lies in for sure.

Let's consider a measurement taken with a house hold ruler as an example. It has markings at most 1mm apart from each other. It is therefore not possible to measure something smaller in size than 1mm. Any measurement with your ruler therefore has an uncertainty of ± 1mm, as what you recorded with it, can be 1mm more or less than the actual values.

All measured values have an uncertainty. We will use the following notation:

If we have some measured number x, then we will use:
x - to signify the number's "exact" magnitude (what was determined with equipment)
δx - to signify the number's uncertainty

We have just considered an example, where a number's uncertainty is attained by looking at the smallest possible unit that can seen with the tool. This is not always the case. Often times, scientific equipment has an associated uncertainty that has been determined by the manufacturer. The method of assuming the uncertainty to be the smallest unit of the provided scale, should be used only when that is not provided. In general, uncertainty can be a "situational" value (especially for high-school experiments).

Uncertainties are very very very easy

When analyzing data, we have to include uncertainty in our calculations. If the measured values have a range within which they could "actually" be in, then the final answer, after some operations were performed on this value will also have some range within which it can lie (and is not just an exact number). The calculation of these ranges is called "propagation of uncertainty". Let's see how this is done.

First we must find a way to mathematically express the uncertainty of a function.

Consider some measured value $a \pm \delta a$. It is used in some function $z(x)$. Let's look at the uncertainty $\delta z(a)$.

First construct a graph of $z(x)$:

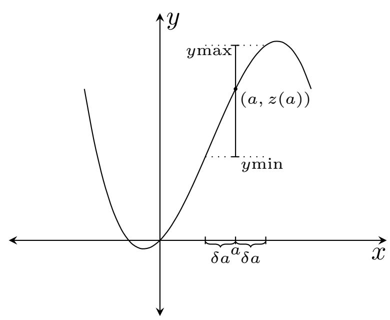

Figure 1: Graph of $z(x)$.

We know that the measured value $a \pm \delta a$, is the domain of $[a - \delta a \; ; \; a + \delta a]$. This means that $z(a \pm \delta a)$ is also some range of values, which is the function $z(x)$ on the domain of $[a - \delta a \; ; \; a + \delta a]$ (Figure 1). The uncertainty is used to define this range and is the "distance" from the exact vale $z(a)$ to maximum/minimum possible value.

The uncertainty $\delta z(a)$ is therefore the greatest possible distance between a value of $z(x)$ in the domain of $[a - \delta a \; ; \; a + \delta a]$ and $z(a)$ itself.

We can therefore define this uncertainty to be:

$$\delta z = \max\left(z\left(a \pm \delta a\right) - z(a)\right) \tag{1}$$

It is however rare that a measured number is used alone. We will therefore consider some multivariable function $z(a, b)$.
Where a and b are some measured numbers:
a with an absolute uncertainty of δa.
b with an absolute uncertainty of δb.
Please note that by definition: $\delta a > 0$ and $\delta b > 0$.

Analogically to equation(1) the absolute uncertainty of the operation will be defined as:

$\delta z = \max(z(a \pm \delta a, b \pm \delta b) - z(a, b))$

We will now derive the uncertainty formulas for the four basic operations: addition, subtraction, multiplication and division and show that uncertainties are very very very easy.

Addition

Consider the function:
$z = a + b$

Then using the definition of the absolute uncertainty:

$$\begin{aligned}
\delta z &= \max(z(a \pm \delta a, b \pm \delta b) - z(a, b)) \\
&= \max((a \pm \delta a) + (b \pm \delta b) - (a + b)) \\
&= \max(\pm \delta a \pm \delta b)
\end{aligned}$$

*In the max statement above we chose the largest value out of the following 4 possibilities:
1) $\delta a + \delta b$; 2) $-\delta a + \delta b$; 3) $\delta a - \delta b$; 4) $-\delta a - \delta b$.

Recall that $\delta a > 0$ and $\delta b > 0$. Hence:

$$\delta z = \delta a + \delta b$$

$\therefore$ When adding two measured numbers, the resultant uncertainty in the sum of the absolute uncertainties.

Subtraction

Consider the function:
$z = a - b$

Using the definition of the absolute uncertainty:

$$\begin{aligned}\delta z &= \max((a \pm \delta a) - (b \pm \delta b) - (a - b)) \\ &= \max(\delta a \pm \delta b) \\ &= \delta a + \delta b\end{aligned}$$

$\therefore$ When adding two measured numbers, the resultant uncertainty in the sum of the absolute uncertainties.

Multiplication

Consider the function:
$z = a \cdot b$

Using the definition of the absolute uncertainty:

$$\begin{aligned}\delta z &= \max((a \pm \delta a) \cdot (b \pm \delta b) - a \cdot b) \\ &= \max(a \cdot b \pm a \cdot \delta b \pm \delta a \cdot b \pm \delta a \cdot \delta b - a \cdot b) \\ &= \max(\pm a \cdot \delta b \pm b \cdot \delta a \pm \delta a \cdot \delta b)\end{aligned}$$

It is assumed that the uncertainties are much smaller than the measure numbers itself. That means that $\delta a \cdot \delta b$ is much smaller than everything else in that statement (It is of the second order (a and b are the greatest, second biggest are their uncertainties and smallest are their uncertainties multiplied together)). We now drop this order (because it is small and insignificant $\Rightarrow$ we don't consider it).

$$\delta z = \max(\pm a \cdot \delta b \pm b \cdot \delta a)$$

In the statement above, a and b can be both negative and positive. To choose the maximum possible value of the statement, we will put absolute value brackets around a and b, to ensure they are the largest possible values.

$$\begin{aligned}
\delta z &= |a| \cdot \delta b + |b| \cdot \delta a \qquad (2) \\
&= |a| \cdot |b| \left(\frac{|a| \cdot \delta b}{|a| \cdot |b|} + \frac{|b| \cdot \delta a}{|a| \cdot |b|} \right) \\
&= |ab| \cdot \left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|} \right) \qquad (3)
\end{aligned}$$

Even though we could just calculate the absolute uncertainty using (2) directly, this is not always convenient. After some simple manipulation, in (2) we see that when multiplying two numbers, the resultant uncertainty is the sum of the original relative uncertainties ($\frac{\delta a}{|a|}$ and $\frac{\delta b}{|b|}$ are relative uncertainties, as they the ratio of the size of the uncertainty to the actual measured value). Often times using relative uncertainties is more convenient as they show you how large your error is on the scale of the measured process. Given this property, you can also state that when multiplying n numbers, then the resultant relative uncertainty is the relative uncertainties of all the numbers. This can be proven by grouping measured numbers and multiplying "two things at a time", until you have multiplied all the numbers.

Division

Consider the function:
$z = \frac{a}{b}$
Using the definition of the absolute uncertainty:

$$\delta z = \max \left(\frac{a \pm \delta a}{b \pm \delta b} - \frac{a}{b} \right)$$

Multiply top and bottom by the conjugate of the denominator.
* Note that the $\mp$ symbol shows the opposite operation of $\pm$ within the same equation.

$$\begin{aligned}
\delta z &= \max \left(\frac{(a \pm \delta a) \cdot (b \mp \delta b)}{(b \pm \delta b) \cdot (b \mp \delta b)} - \frac{a}{b} \right) \\
&= \max \left(\frac{ab \mp a \cdot \delta b \pm b \cdot \delta a \pm \delta a \cdot \delta b}{b^2 - (\delta b)^2} - \frac{a}{b} \right)
\end{aligned}$$

Once again: $\delta a \cdot \delta b$ and this time $(\delta b)^2$ are second order. Since they are so small, we drop these terms to get:

$$\begin{aligned}
\delta z &= \max \left(\frac{ab \mp a \cdot \delta b \pm b \cdot \delta a}{b^2} - \frac{a}{b} \right) \\
&= \max \left(\frac{a}{b} + \frac{\mp a \cdot \delta b \pm b \cdot \delta a}{b^2} - \frac{a}{b} \right) \\
&= \max \left(\mp \frac{a \cdot \delta b}{b^2} \pm \frac{\delta a}{b} \right)
\end{aligned}$$

Once again since a and b can be negative, we put absolute values around them to get the maximum, by making sure that all the values stay non negative). Hence:

$$\begin{aligned}
\delta z &= \frac{|a| \cdot \delta b}{|b|^2} + \frac{\delta a}{|b|} \\
&= \frac{|a|}{|b|}\left(\frac{|b|}{|a|} \cdot \frac{|a| \cdot \delta b}{|b|^2} + \frac{|b|}{|a|} \cdot \frac{\delta a}{|b|}\right) \\
&= \frac{|a|}{|b|}\left(\frac{\delta a}{|a|} + \frac{\delta b}{|b|}\right)
\end{aligned}$$

Using the statement above it is evident when dividing numbers, we add relative uncertainties. Using the same logic as in the multiplication section we can extend this rule to dividing n numbers.

$\therefore$ **UNCERTAINTIES ARE VERY VERY VERY EASY**

$\therefore$ QED

Uncertainties for a general function

Let's consider so general function $f(x)$. Suppose we have some measured value $a \pm \delta a$. Let's find $f(a \pm \delta a)$. First consider the following graph:

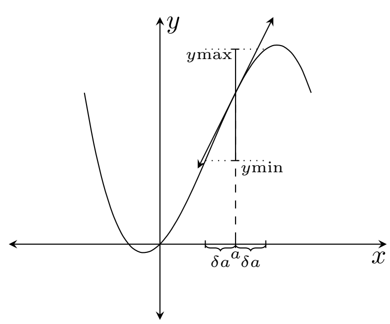

Figure 2: Graph of some arbitrary function $f(x)$.

The value a is found within some range $[a - \delta a \ ; \ a + \delta a]$. Therefore the value of $f(a)$ must lie within that domain. To find the range of $f(a)$, we must then find the maximum and minimum of $f(x)$ within this domain. This can be achieved, by finding the derivative of $f(x)$ and finding local minima and maxima. This can however take substantial time. It is much simpler to assume the uncertainty δa is small and "approximate" the function $f(x)$ as a straight line (its derivative). In this case, from the properties of a line, we directly get that the min and max are:

$$y_{\min} = f(a) - f'(a) \cdot \delta a$$

$$y_{\max} = f(a) + f'(a) \cdot \delta a$$

We then get the following range for $f(a)$: $[f(a) - f'(a) \cdot \delta a\ ; f(a) + f'(a) \cdot \delta a\ ]$

From this, it is clear that the uncertainty of $f(a)$ is:

$$\delta y = |f'(a)| \cdot \delta a \tag{4}$$

* Note that we add absolute value brackets to ensure our uncertainty is positive.

For the case of a multivariable function, we will use partial derivatives. (Partial derivatives with respect to a variable, are like normal derivatives. You just assume everything (other than the variable with respect to which the derivative is taken) is constant.

So for some function $f(x_1, x_2, ..., x_n)$ the uncertainty is:

$$\delta y = \left|\frac{\partial f}{\partial x_1}\right| \cdot \delta x_1 + \left|\frac{\partial f}{\partial x_2}\right| \cdot \delta x_2 + ... + \left|\frac{\partial f}{\partial x_n}\right| \cdot \delta x_n \tag{5}$$

Significant Figures

Uncertainty is not the only way to show the accuracy of a measurement. Sometimes, the uncertainty itself is not needed, but it is still useful to know the order of magnitude, up to which the measurement is accurate. This is done by writing digits only up to the order of magnitude which is the smallest measurable.

For example, if we measured something with a ruler that has an uncertainty of 0.1 cm, then any number we write with it, will only have digits up to the tenths place. These digits are called significant figures (sig figs for short).

For example, let's look at the following measurements made with a ruler accurate to ± 0.1 cm

l= 1.4 cm - this has two significant figures (1 and 4)
l= 1.0 cm (we write .0 to signify that the measurement is accurate to the tenths place). This still has two significant figures (1 and 0).

Let's look at another example. Let's say the accuracy of our measurement is ±10. How can we write proper significant figures of a measured value of let's say 100?

The number 100 has three significant figures, and would imply that the measurement is accurate to the ones digit, but want to show that it is accurate to the tens digit. For these cases we use scientific notation and write down:

$1.0 \cdot 10^2$

It is also important to note that leading zeroes do not count as sig figs.

For example:

In the number 0.0013, there are only two significant figures (1 and 3). The zeroes preceding them do not matter as they are not what is being measured!

The significant figures of a number are also sometimes called the mantissa. Let's look at a few examples:

1.45 - mantissa is 1.45

0.0034 - mantissa is the 34

0.0405 - mantissa is the 405

$1.45 \cdot 10^{-2}$ - mantissa is the 1.45

Just like with uncertainties, sig figs of measured numbers also change when they are operated on.

When adding or subtracting two measured numbers, the final sum will use the same number of decimal places, as the number with the least decimal places (the "worse accuracy is retained"). Here are a few examples:

$10.0 + 21 = 31$
$24.56 + 2.3 = 26.9$ (the number 2.3 has worse accuracy than 24.56, so the 0.06 is rounded to the nearest ten).

The idea behind this rule is in a sense based on propagation of uncertainty. We add absolute uncertainties when adding numbers, therefore the larger uncertainty will be more "dominant" over the smaller one.

E.G.
$1.0 \pm 0.1 + 1.45 \pm 0.01$

Here the 0.01 is much smaller than the 0.1. Overall in our new measurement (the sum) the 0.01 uncertainty is "less important" than the 0.1.

For multiplication and division, the product always has the lowest number of significant figures (out of all of the numbers used in the calculation).For example:

$3.00 \cdot 6.0 = 18$

3.00 has 3 sig figs, 6.0 has 2 sig figs. So the final answer has 2 sig figs.