Definition 1 *Let $A_1 \times ... \times A_n$ be some cartesian product of sets. We then define projection maps:*

$$pr_i : A_1 \times ... \times A_n \to A_i$$
$$pr_i(a_1, ..., a_n) = a_i$$

Definition 2 *Let E, F, B be manifolds [1] and $\pi : E \to B$ a surjective differentiable map. Then $(E, B, \pi; F)$ is called a fibre bundle if the following is true: For every $x \in B$ there exists an open neighbourhood $U \subseteq B$ containing x, and a diffeomorpshism*

$$\varphi_U : \pi^{-1}(U) \to U \times F$$

Such that

$$pr_1 \circ \varphi_U = \pi$$

In other words $\pi^{-1}U$ can be trivialized

The definition above implies that the following diagram commutes:

$$\pi^{-1}(U) \xrightarrow{\varphi_U} U \times F$$
$$\pi \searrow \qquad \swarrow pr_1$$
$$U$$

A fiber bundle is then denoted as $F \to E \xrightarrow{\pi} B$.

Definition 3 *Let $F \to E \xrightarrow{\pi} B$ and $F' \to E' \xrightarrow{\pi'} B$ be fibre bundles over the same manifold B. A bundle morphism is a smooth map $H : E \to E'$ such that:*

$$\pi' \circ H = \pi$$

Definition 4 *A fibre Bundle*

$$V \longrightarrow E \xrightarrow{\pi} M$$

Is a vector bundle of rank m if:
1) The fibre V is an $m-$dimensional vector space.
2) The charts (U, φ_U) are such that

$$pr_2 \circ \varphi_U : \pi^{-1}(x) \to V$$

Is a vector space isomorphism for any $x \in M$ with $x \in U$

Definition 5 *A vector bundle of rank 1 is called a line bundle*

Suggested Exercises

I) Give the following objects a fiber bundle structure (i.e. specify the maps π and collection (U, φ_U))

    i) Consider the cylinder of height 1. Express it as a fiber bundle over base S^1 with fiber $F = [0, 1]$

    ii) Let E be the Mobius band defined as follows:
    Let $I = [0, 1] \times [0, 1]$ be a square in $\mathbb{R}^2$. We define the following equivalence relation for the vertical edges of the square: $(0, y) \sim (1, 1 - y)$. The Mobius band is then the quotient

$$I/\sim$$

    (In other words, the points that are quotiented out, are made to be the same point). You should first think about why this indeed defines the Mobius band.
    Give this a fiber bundles structure, where the base is B and the fiber is $F = [0, 1]$

II) Prove that the tangent space of a manifold is a Vector Bundle.

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[1] Definition taken from "Mathematical Gauge Theory" by M.J.D. Hamilton