Suggested Exercises III

0.1 Subspace Topology

Definition 1 *Let (X, τ_X) be a topological space. Let $A \subseteq X$ be any subset. We can then define a topology on A by:*

$$\tau_A = \{U_A \mid U_A = A \cap U_X \;\; U_X \in \tau_X\} \tag{1}$$

I) Prove that τ_A indeed defines a topology on A.

II) Consider $\mathbb{R}$ with the metric topology (topology given to it by open balls in it's metric). Let $A = [0, 1]$. Show that $[0, 0.5)$ is an open set in the subspace topology of A.

0.2 Continuous Function

Recall that the preimage of a set $V \subseteq Y$ of a function $f : X \to Y$ is defined as

$$f^{-1}(V) = \{x \in X \big| f(x) \in V\} \tag{2}$$

We then use this to define continuous functions.

Definition 2 *Let X, Y be topological spaces with topologies τ_X and τ_Y respectively. A function $f : X \to Y$ is continuous if*

$$\forall V \in \tau_Y \quad f^{-1}(V) \in \tau_X \tag{3}$$

You can rephrase above succinctly as: f is continuous if the preimage of any open set in Y is an open set in X.

I) Consider the step function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & , x \leq 0 \\ 1 & , x > 0 \end{cases}$$

  i) Thinking of f as a function between topological spaces $\mathbb{R}$ (give $\mathbb{R}$ the standard metric topology); show that f is not continuous

  ii) You can also think of f as a function $f : \mathbb{R} \to \{0, 1\}$. Where $\{0, 1\}$ has the subspace topology from $\mathbb{R}$. Modify the proof from the previous part,to show that the function is still not continuous in this case.

II) Let's think of why defining a continuous function in this way is at all natural. To begin with; what are we trying to do here? We are trying to define a function which somehow preserves the topological space structure. Let's think of why having pre-images of open sets be open, is a good way to go.

Let's carefully look at the definition. It tells us that if something is an open set after being mapped, that means it originally used to be an open set. Well this is exactly what it want. This means that whatever structure we have in the target space, was part of the structure in the original space.

  i) Consider the following space $Y = \{1, 2, 3\}$, with a topology $\tau_Y = \{\{1\}, \{2, 3\}, \{1, 2, 3\}, \varnothing\}$. Let $X = \{a, b, c, d\}$. Suppose $f(a) = f(b) = 1$ and $f(c) = 2$ $f(d) = 3$ Write down as many possible topologies on X that would make f a continuous function.

  ii) Prove that $f(x) = x$ is a continuous function (if you think of it as $f : \mathbb{R} \to \mathbb{R}$)

  iii) Suppose $g : X \to Y$ and $f : Y \to Z$ are continuous. Prove that $f \circ h$ is also continuous.

III) Note however, that a continuous map does not preserve all of the topological structure. i.e. For $f : X \to Y$, if U is open in X, then for continuous f, $f(U)$ does not need to be open! In general, this property has its own name

A map $f : X \to Y$ is open if for any $U \in \tau_X$, $f(U) \in \tau_Y$.

  (a) Consider the function $f(x) = x^2$. Draw it's image on the real line. Prove that this is not an open map.

  (b) Show that a map f is a homeomorphism if and only if it is bijective, continuous and open.