1 Overview

Definition 1 *A subset C of a topological space X is closed, if it's complement is open (i.e. $X \backslash C$ is open).*

Definition 2 *A point p is a limit point of a set $A \subseteq X$, where X is a topological space, if for any open set $U \in \tau_X$ such that $p \in U$, $A \cap U \backslash \{p\} \neq \emptyset$*

One way to colloquially describe a limit point is ”there is always something from A next to that point”. Note that a limit point of a set A need not be in A.

Definition 3 *Consider a sequence of points $x_i \in X$, where X is some topological space. A point $p \in X$ is the limit of the sequence X, if for any open set $U \in \tau_X$ with $p \in U$, $\exists N$ such that $\forall i > N$ $x_i \in U$. We denote the limit as $p = \lim\limits_{n\to\infty} x_n$*

2 Suggested Exercises IV

I) Properties of Closed Sets. During our session, we briefly discussed the properties of closed sets analogous to open sets. We stated that $X, \emptyset$ are also closed sets. We also noted that the arbitrary intersection of closed sets is a closed set. We have one statement left to prove. Finite union of closed sets, is still a closed set.

   i) Prove that $\bigcup\limits_{\alpha\in\Lambda}(X - U_\alpha) = X - \bigcap\limits_{\alpha\in\Lambda} U_\alpha$ (drawing a Venn Diagram of two intersecting circles helps.

   ii) Use the result above to prove that a finite union of closed sets is still a closed set.

II) Closed sets and Limit Points.

   i) Prove that a closed set contains all of its limit points (i.e. if p is a limit point of C then $p \in C$ for C closed)

   ii) Prove that if a set contains all of its limit points, then it must be a closed set.

III) Continuous Functions. Closed sets are an “alternative” definition of a topology (recall that if I tell you all the closed sets, then you know all the open sets and vice versa.). Prove that the following definitions of continuity are equivalent.

   i) A function $f : X \to Y$ is continuous, if $\forall V \in \tau_Y$ $f^{-1}(V) \in \tau_X$

   ii) A function $f : X \to Y$ is continuous, if for any closed set $C \subseteq Y$, $f^{-1}(C)$ is a closed set in X.

   iii) A function $f : x \to Y$ is continuous, if for any converging sequence x_i have that $\lim\limits_{n\to\infty} f(x_n) = f\left(\lim\limits_{n\to\infty} x_n\right)$ (*Hint: Use the definition in ii)*)