1 Review

Definition 1 *The set of all possible states of a physical process is called phase space. As a particle moves in space, it traces out some path in phase space, which is parametrized by time.*

Definition 2 *Let X be a direction field. We call a path $\gamma(t)$ the integral curve of X of $\dot{\gamma} = X(\gamma(t))$*

Definition 3 *For the graph of a function φ to be an integral curve it is necessary and sufficient that for all t you have the relation:*

$$\frac{d\varphi}{dt} = v(t, \varphi(t))$$

We then say that φ is the solution to the differential equation:

$$\dot{x} = v(t, x)$$

Definition 4 *We call a vector that is drawn starting at a point p a vector attached at that point*

Definition 5 *Functions with attached vectors as input are called differential 1-forms, if they are linear when fixing the point at which we attach the vector. Any one form on $\mathbb{R}^2$ can be written as:*

$$\omega(p; v) = a(p)\ dx(v) + b(p)\ dy(v)$$

Where dx and dy are linear functionals that return the x and y coordinate of a vector respectively. If it is clear from context to which point we attach the vector, we omit this specification when writing inputs into $\omega(p, v)$ and just write $\omega(v)$

2 Suggested Exercises

I) **Phase Space**

- i) What is the phase space of a particle restricted to moving in 1 dimension?
- ii) What is the phase space of a particle restricted to moving in 2 dimensions?
- iii) What is the phase space of the process of picking two balls from two baskets (one from each) where the first basket has 3 red 1 green and the second has 2 red 1 green?

II) **Direction Fields**

- i) Draw by hand, the following direction fields:

$$X((x, y)) = (3, 0) \quad X((x, y)) = (x, y)$$
$$X((x, y)) = (y, x) \quad X((x, y)) = (-y, x)$$
$$X((x, y)) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}\right)$$

- ii) Now draw them using streamplot of matplotlib in python.
- iii) Plot the following vector field in Python

$$X((x, y)) = \left(x - \frac{1}{3}x^3 - y, x\right) \tag{1}$$

- iv) Find integral curves for the following direction fields:

$$X((x, y)) = (3, 0) \quad X((x, y)) = (x, y)$$
$$X((x, y)) = (x \cos x^2, y \log^2 y) \quad X((x, p)) = \left(\frac{p}{m}, -mg\right)$$

Hint: For the last one, realize it is a vector field in the phase space of a known system in classical mechanics.

III) **Differential 1-Forms**

- i) Find 1-forms ω such that $\omega(p; X(p)) = 0$ for all p, points in the space for the first 4 direction fields in the previous question.

ii) For each of the following 1-forms $\omega(p;v)$ find a direction field X such that $\omega(p;X(p))=0$

$$\omega = dx - dy \quad \omega = y\,dx - x\,dy$$

IV) **Integrating 1-Forms**

i) Let $\gamma : [0,1] \to \mathbb{R}^2$ be the path $\gamma(t) = (cos(2\pi t), sin(2\pi t))$. Integrate the 1-form $\omega = ydx - xdy$ over $\gamma([0,1])$. What does this integral represent?

ii) Let Γ be a straight line connecting $(1,0)$ and $(x,0)$ where $x > 1$. Integrate the 1-form $F = \frac{1}{(x^2+y^2)^{\frac{3}{2}}}(xdx+ydy)$.

iii) Now integrate the same form over the line connecting (0,1) and $(0,y)$ for $y > 1$. Can you recognize what physical system and quantity this form corresponds to? What is the significance of the integral of this form?

V) **Translation Invariant Vector Fields**

i) This was a theorem in Arnold that we forgot to discuss. Let $X(p)$ (where p is a point in my space), be a direction field. Suppose there exists a vector v such that $X(p+\lambda v) = X(p)$ for any $\lambda \in \mathbb{R}$. Also suppose that X is never parallel to v. Prove that finding the integral curves of this vector field, is equivalent to integrating some given function. (*Hint: The statement and proof of this theorem are in the section introducing integral curves.*

ii) Why is it significant that the vector field is never parallel to v?

iii) Why is it significant that the vector field is translation invariant for all λ?