Problem 1. Bonus Quantum Mechanics
The theory of quantum mechanics, is usually described by 5-6 postulates. Below is a slightly simplified version[1] of 2 of them.

(I) The set of states a system can be in is a special kind of vector space called the Hilbert Space. Here, we denote it by $\mathcal{H}$. For this exercise, take:

$$\mathcal{H} = \mathrm{L}^2 = \left\{ f(x) : \mathbb{R} \to \mathbb{C} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|^2 < \infty \right\}$$

Physically, if we say that a system is in state v, we mean that somehow [2] a vector $\psi \in \mathcal{H}$ encodes all of the information about what we can measure about that state.

(II) Every physically measurable quantity, corresponds to a hermitian[3] linear operator $A : \mathcal{H} \to \mathcal{H}$ called an observable. Do not worry about what Hermitian means. What matters is that a Hermitian linear operator over a hilbert space $\mathcal{H}$ has a Schauder [4] basis of eigenvectors:

$$\psi \in \mathcal{H} \Rightarrow \psi = \sum_{j=1}^{\infty} a_j \psi_j \quad \text{Where } A\psi_j = \lambda_j \psi_j \tag{1}$$

The eigenvalues correspond to the possible outcomes of the physical measurements.

Before we get to the actual problem, we also define the physicist's hermite polynomials $H_n(x)$. We define them recursively:

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= 2x \\ H_{n+1}(x) &= 2xH_n(x) - \frac{\partial}{\partial x} H_n(x) \end{aligned} \tag{2}$$

In this problem, we will consider the harmonic oscillator. This is a system that appears very often in quantum mechanics. Particularly, we will examine the observable associated to measuring the energy of the system called the Hamiltonian.

$$\begin{aligned} H &: L^2 \to L^2 \\ H &= -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2 \\ H(\psi(x)) &= -\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x) + \frac{1}{2}x^2\psi(x) \end{aligned} \tag{3}$$

Assume that this is a Hermitian operator and has a basis of eigenvectors that spans L^2 (in the sense of equation (1)). Let's define the following set of states (functions) ψ_n

$$\psi_n(x) = N_n e^{-\frac{x^2}{2}} H_n(x) \tag{4}$$

Where N_n is some constant, H_n is the nth Hermite polynomial defined in (2). (Please do not confuse H_n with H. H_n is a polynomial and H is an observable. The notation can be unclear).

(I) Energy of the Harmonic Oscillator

(i) Check that the hamiltonian H is linear

(ii) Compute $H(\psi_0(x))$ and $H(\psi_1(x))$. Are these eigenvectors? If so, what are their corresponding eigenvalues?

(iii) If you have a lot of free time verify (not during tutorial) that

$$H\psi_n = E_n\psi_n \qquad \text{Where: } E_n = \left(n + \frac{1}{2}\right)$$

Hint: Use equation (2) to express the derivative of H_n in terms of H_n and H_{n+1} without derivatives. You do not need to actually compute H_n. Also maybe use induction.

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[1] Once you learn what an inner product is in MAT247 (or just skip to the appropriate chapter in the textbook), you can read and understand their "full" version. A possible source is Chapter III of Cohen-Tannoudji.
[2] Do not worry about what this means.
[3] You will learn about what this is in 247. Don't worry about it for now.
[4] Please don't ask your TAs too many questions about all of this. Just use the equalities given to you. They are what really matters

What do these eigenvalues E_n physically represent? (*Answer: Since H is an observable associated to an energy measurement, E_n are the allowed energies of the system. Notice that they are evenly separated.*) [5]

(II) Raising and Lowering Operators

(i) We introduce a new differential operator:

$$p = -i\partial_x \tag{5}$$

Express ∂_x^2 in terms of p. Substitute this into the Hamiltonian.

(ii) Notice that $p \circ x \neq x \circ p$. Compute $p \circ x$ in terms of $x \circ p$.

(iii) We now define the lowering operator a and raising operator $a^\dagger$:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(x + ip\right) \\ a^\dagger &= \frac{1}{\sqrt{2}}\left(x - ip\right) \end{aligned} \tag{6}$$

Compute $a\psi_0$, $a\psi_n$ and $a^\dagger\psi_n$. Explain why a and $a^\dagger$ are called lowering and raising operators.

(iv) Compute $a^\dagger a$. Use the relation in part (ii) to express $a^\dagger a$ in terms of p^2 and x^2.

(v) Rewrite the Hamiltonian in terms of $a^\dagger a$ and the identity operator. Why does this make sense?

(III) **BONUS BONUS question**:
So far we have only talked about "stationary" states. We did not consider time dependence. To deal with this, we introduce yet another postulate:
The time evolution of a state in quantum mechanics is given via the following equation [6]

$$i\hbar\partial_t\psi = H\psi \tag{7}$$

Setting $\hbar = 1$, confirm that $e^{-iE_n t}\psi_n(x)$ satisfies the differential equation above. *Hint: Use the fact that ψ_n is an eigenstate of H.*
From this example, we can see that the problem of finding eigenvectors is central to quantum mechanics. Knowing the eigenvectors, substantially simplifies computing how the state evolves in time. Instead of having some complicated operator differential equation, we can instead find the eigenvectors ψ_n and eigenvalues λ_n of H. By the fact that H is Hermitian, these form a basis. The general solution to (7) with $\hbar = 1$ is then given via:

$$\psi = \sum_{n=1}^{\infty} e^{-i\lambda_n t}\psi_n$$

Verify that this general expression solves (7).

[5] Note that in the Hamiltonian above, we set the mass of our particle $m = 1$, $\hbar = 1$ and the oscillator frequency $\omega = 1$. The correct Hamiltonian with units is then $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$. We then also appropriately define $p = -i\hbar\partial_x$. In this case, the energy then gets a prefactor of $\hbar\omega$.

[6] This is known as the Schrödinger equation