Introduction to Vectors

Definition 1. *A vector quantity is a quantity that has both a magnitude and direction.*

Definition 2. *A scalar quantity is a quantity with only a magnitude.*

Examples

Let's look at some examples of vector quantities.

- Displacement:
  - Bill walked two meters to the right. There is both a direction (right) and a magnitude (two meters). It is important to note that: John walked two meters is a scalar (length) as it has now direction.
  - The term "displacement" is always used in reference to the vector quantity and the term "length"" is always used in reference to the scalar quantity.
- Velocity
  - A car is traveling north at a speed of 40 $\frac{\text{km}}{\text{h}}$. This quantity is has a direction (north) and a magnitude (40 $\frac{\text{km}}{\text{h}}$). However if we were to say: a peregrine falcon during a dive can reach a top speed of 390 $\frac{\text{km}}{\text{h}}$, this would be a scalar quantity as no direction of diving is provided.
  - The term "velocity" is always used in reference to the vector quantity and the term "speed"" is always used in reference to the scalar quantity.

*In general, to differentiate between the two quantities, a vector arrow is placed on top of vector quantities. E.G. $\vec{v}$.
To refer to the magnitude of a vector quantity, place absolute value brackets around its variable: $|\vec{v}|$

Representing Vectors

Since a vector has both a direction and a magnitude, it has to be represented by something that can express both a magnitude and a direction. Arrow have been chosen as the standard notation for vectors.

Definition 3. *A vector quantity can be represented by a vector arrow. The arrow's length is equal to the magnitude of the quantity. The direction its tip is pointing in, is the same as the direction of the vector quantity.*

Let's see if we can find a way to mathematically express this kind of arrow.

Consider some object O moving with some velocity $\vec{v}$. Let's introduce some arbitrary reference frame (Figure 1). In this reference frame the object is currently located at some point A. Let's draw an arrow of length $|\vec{v}|$ starting from point A, going into the direction of $\vec{v}$. The vector arrow representing $\vec{v}$ has some endpoint B. The length of line segment A to B

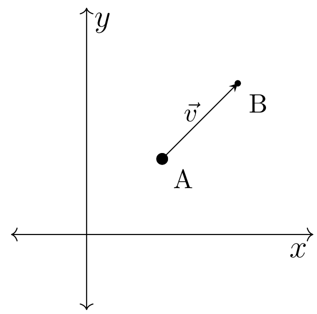

Figure 1

by definition is equal to $|\vec{v}|$ and is in the direction of $\vec{v}$. We have therefore established, that a vector can be expressed by through coordinates of its start point and end point. Note that it is important to specify which is the start point as going from A to B is not the same as going from B to A.

Let's look at another example to find out more about how vectors work. Consider two cars traveling parallel to each with the same speed of $|\vec{v}|$. The first car is currently at point A and the second car is at point B. Since both objects are traveling in the same direction with

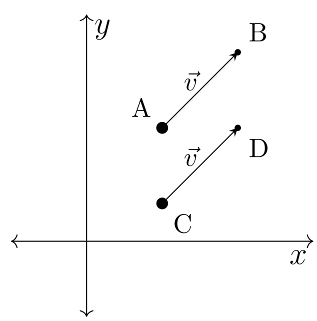

Figure 2

the same speed (same direction and magnitude), their velocities must be the same (equal to some $\vec{v}$). This implies that the vector arrow going from point A to B is equivalent to the vector arrow going from C to D. State the following property.

Vector Property 1. *Two parallel vectors of equal magnitude, with endpoints facing the same direction are equal.*

From the property above, it is clear that any given vector can be represented by an arrow starting at the origin, parallel to the given vector and equal in magnitude (Figure 3).
Now, to represent the same vector we need only one point (point Z), because by definition the origin is at (0,0) and does not need to be specified! You can therefore write: $\vec{a} = (x, y)$, where x and y are the coordinates of point Z. This is the general convention for mathematically expressing vectors.

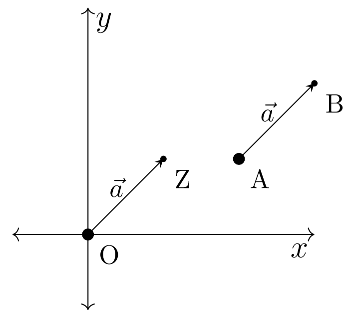

Figure 3

Let's figure out the coordinate of a vector arrow starting from the origin, which is equivalent(parallel and of equal length) to another given vector arrow going from points A (x_1, y_1) to B (x_2, y_2)(Figure 4). Draw line segments perpendicular to x axis from points Z,A and B.

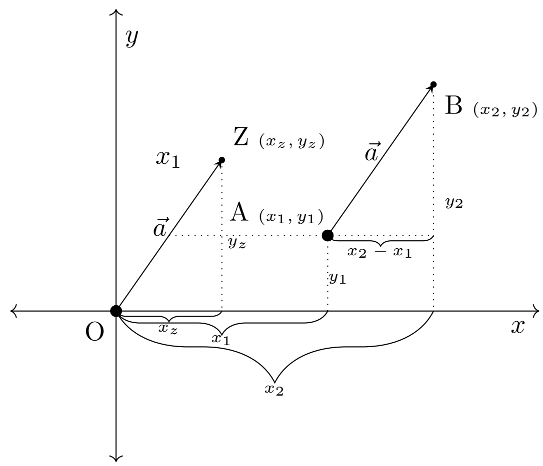

Figure 4

From Figure 4, it is easy to notice that the right angled triangles with hypotenuses OZ and AB are equal (angle side angle theorem). Therefore, from Figure 4 we get directly that:

$$\begin{aligned} \vec{v} &= (x_z, y_z) \\ &= (x_2 - x_1, y_2 - y_1) \end{aligned} \tag{1}$$

Let's now find the magnitude $|\vec{a}|$ of vector $\vec{a}$ using our arrow OZ.
Once again, we draw lines perpendicular to the x axis from point Z. In the right angled triangle with hypotenuse OZ, from Pythagorean theorem we get:

$$OZ^2 = x_z^2 + y_z^2$$

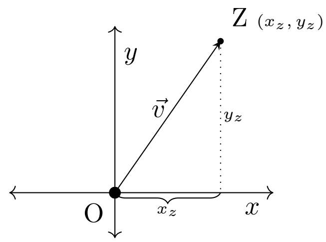

Figure 5

From the definition of the a vector arrow recall that $OZ = |\vec{a}|$ to get:

$$|\vec{a}| = \sqrt{x_z^2 + y_z^2} \tag{2}$$

Let's consider one more problem. An object is moved in time t. This displacement is represented by some vector $\overrightarrow{d}$ with coordinates (x_d, y_d). How can we find its velocity vector $\vec{a}$ and its coordinates (x_v, y_v)? If an object's displacement is equivalent to a displacement

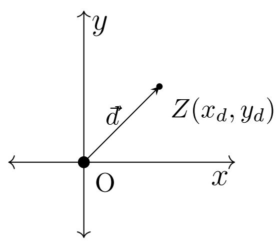

Figure 6

from point O to Z, then its velocity vector should also be pointing in the direction going from O to point Z (Figure 6). It should therefore be a scaled $\overrightarrow{d}$ (vector in the same direction but of different length). We know that the speed of the object should be equal to the magnitude of its displacement divided by time. Therefore we get:

$$|\vec{v}| = \frac{|\overrightarrow{d}|}{t}$$

Substitute equation (2) into formula above and square both sides to get:

$$\sqrt{x_v^2 + y_v^2} = \frac{\sqrt{x_d^2 + y_d^2}}{t}$$
$$x_v^2 + y_v^2 = \frac{x_d^2}{t^2} + \frac{y_d^2}{t^2}$$

In the equation above, we can "assume" that the x (y) coordinates of velocity will match the x (y) coordinates of displacement and get:

$$\begin{aligned}
x_v^2 &= \frac{x_d^2}{t^2} \\
x_v &= \frac{x_d}{t} \qquad (3) \\
y_v^2 &= \frac{y_d^2}{t^2} \\
y_v &= \frac{y_d}{t} \qquad (4)
\end{aligned}$$

Equations (3) and (4) will be the coordinates of the velocity vector $\vec{v}$ in terms of the coordinates of the object's displacement vector $\vec{d}$ and the change in time during the displacement t. It is important to note, that when taking the square root to get this equations, no $\pm$ sign was added. Since the velocity vector is pointing in the same direction as the displacement vector, the signs of their respective coordinates must be the same.