0.1 Kinematic Equations of Motion

The Kinematic Equations of motion are the basis of most physics courses, yet are often left unexplained and therefore pose a great challenge for high school students. Understanding their derivations is the best way to avoid any confusion when applying these formulas to a specific problem.

It considers the motion of a point particle, subject to constant acceleration. This is a common occurence in simple physical systems, where the force acting on some object is constant in magnitude and direction with time.

0.1.1 Derivations

First let's consider some particle Z, moving with some constant acceleration $\vec{a}$ (constant in magnitude and direction).
Recall the definition of acceleration as the change in speed over time (how much faster you get every second):

$$\vec{a} = \frac{\Delta\vec{v}}{t}$$

The change in velocity $\Delta\vec{v}$ is the difference between some final and initial velocities $\vec{v}_1$ and $\vec{v}_0$. Rewrite equation above to get:

$$\vec{a} = \frac{\vec{v}_1 - \vec{v}_0}{t}$$

Rearrange equation above to get:

$$\vec{a} = \frac{\vec{v}_1 - \vec{v}_0}{t}$$
$$\vec{a} \cdot t = \vec{v}_1 - \vec{v}_0$$

$$\vec{v}_1 = \vec{a}t + \vec{v}_0 \tag{1}$$

Using equation (1) we can now find an particle's velocity at any point in time, if it is constantly accelerating. Note that the equation above is a vector equation and can be split into two separate component ones (one for the x and one for the y directions). Let's consider the graph of speed in some direction n to time.

It will follow the equation:

$$v_n = a_n t + v_{0n}$$

Where:
The subscript n refers to the component in the n direction of each vector.
v_{0n} is the initial velocity is the nth direction.

From equation above get that:

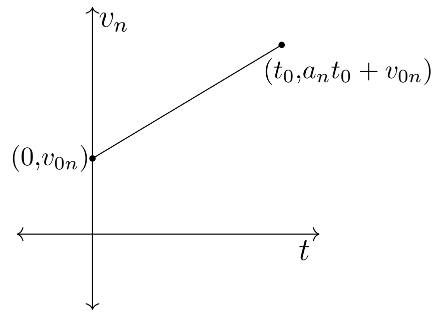

Figure 1: Speed of particle in the nth direction as a function of time.

At $t = 0$, $v_n = v_{0n}$

At $t = t_0$, $v_n = a_n t + v_{0n}$. (Where t_0 is just some time (a number).
Using the graph above let's try and find an equation for the position of the object after some time Δt.

First recall the definition of velocity as the change in position over time (displacement over time):

$$\vec{v} = \frac{\vec{d}}{t}$$

Like earlier, rewrite this in terms of some final and initial positions (coordinates) A_1 and A_0 and rearrange to get

$$A_1 = \vec{v} \cdot t + A_0 \tag{2}$$

Unlike acceleration, velocity is no longer constant ($\vec{v}$ is different for different values of t), so we cannot just use the equation above to calculate position A_1.

If we consider very small changes in time (look at the position of the object after) we can assume velocity to be constant, and calculate the change in position within that next small bit of time Δt.

Let's consider the graph of the velocity in the n th direction again:

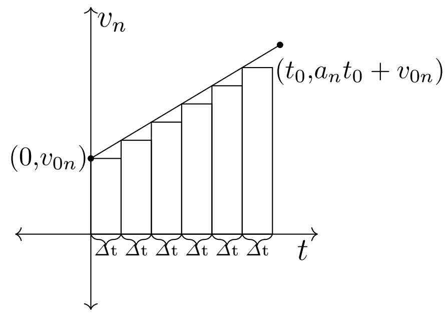

Figure 2: Assuming speed is constant at a small interval of time, we can approximate the distance travelled over that period of time as $v \cdot \Delta t$. This would give the area of one of the rectangles in the graph above. Summing up over these areas, would give an approximation for the distance traveled after some time t.

In Figure (2), we can notice that the area of the rectangle will be approximately the distance traveled by the object in the small increment of time Δt as the height of the rectangle is the current velocity v_n and therefore as $d = v \cdot \Delta t = S$. The total area traveled by the object will therefore be the sum of the areas of all the rectangles.

It is also easy to notice that as we make the time increment Δt infinitely small, the sum of the areas of all the rectangles, will simply be the total area between the line and the x axis. Hence the displacement in the direction nth will be the area under the velocity graph.

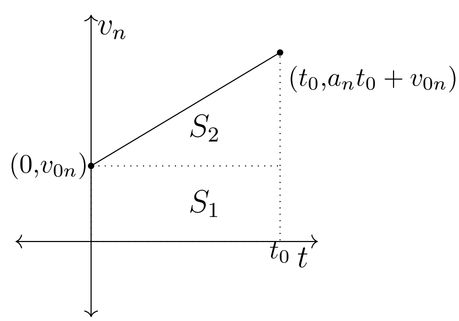

Figure 3: The area under the velocity graph can be split into two shapes with some physical interpretation. Rectangle with area S_1 (distance traveled due to initial velocity) and a triangle with area S_2 - distance traveled due to acceleration.

From Figure (3) get that:

$$A_{1n} = S_1 + S_2 \tag{3}$$

Where:

A_{1n} - nth coordinate of position of object.
S_1 - area of rectangle with width t_0 and height v_{0n}
S_2 - area of triangle with width t_0

From Figure (3) get that:

$S_1 = v_{0n} \cdot t_0$

$S_2 = \frac{1}{2} \cdot a_n t_0^2$

Using equation above and equation (2) get:

$$A_{1n} = \frac{1}{2} a_n t_0^2 + v_{0n} t_0 + A_{0n} \tag{4}$$

Using equation (4), we can now calculate the position of a point particle which moves at a constant acceleration at time t_0 given its initial position and speed.

0.1.2 Last Kinematic Equation

This equation poses an interesting relationship that can be derived from equations (1),(3).

Consider some initial and final velocities $\vec{v}_0$ and $\vec{v}_1$.
If the object moves at some constant acceleration $\vec{a}$, then from equation (2) get that $v_{1n} = a_n t_0 + v_{0n}$ (in the nth direction).

Lets find the difference of squares of v_{0n} and v_{1n}

$$\begin{aligned} v_{1n}^2 - v_{0n}^2 &= (a_n t_0 + v_{0n})^2 - v_{0n}^2 \\ &= (a_n t_0 + v_{0n})(a_n t_{0n} + v_{0n}) - v_{0n}^2 \\ &= a_n^2 t_0^2 + a_n t_0 v_{0n} + a_n t_0 v_{0n} + \cancel{v_{0n}^2} - \cancel{v_{0n}^2} \\ &= a_n^2 t_0^2 + 2 a_n t_0 v_{0n} \\ &= 2a_n \left(\frac{1}{2} a_n t_0^2 + v_{0n} t_0 \right) \end{aligned}$$

Substitute equation (3) into equation above and rearrange to get:

$v_{1n}^2 - v_{0n}^2 = 2a(A_{1n} - A_{0n})$
$v_{1n}^2 - v_{0n}^2 = 2a_n d_n$

$$v_{1n}^2 = v_{0n}^2 + 2a_n d_n \tag{5}$$

Note that while equation (4) uses exact position, equation above only uses displacement in the nth direction, while!

Remark for Later

It is now immediately apparent, what the physical interpretation of equation (5) is. The answer to this question becomes more clear once we study Newton's second law and the conservation of mechanical energy.

If we multiply equation (5) by $\frac{1}{2}m$ and rewrite d_n as $A_{1n} - A_{0n}$ we will get

$$\frac{1}{2}mv_{0n}^2 - ma_nA_{0n} = \frac{1}{2}mv_{1n}^2 - ma_nA_{1n} \tag{6}$$

Now consider the case where our particle is subjected to the force of earth's gravity with the usual coordinate system such that:

$a_x = 0$, $a_y = -g$

Substitute these accelerations into equation (6) for the appropriate direction and add them together to get:

$$\frac{1}{2}m(v_{0x}^2 + v_{0y}^2) + 0 + mgA_{0y} = \frac{1}{2}m(v_{1x}^2 + v_{1y}^2) + 0 + mgA_{1y} \tag{7}$$

It is easy to notice that $v_{0x}^2 + v_{0y}^2 = |\vec{v}_0|^2$. Also note that $A_{0y} = h_0$ is the height of the particle "above ground" (or any fixed level) and so we finally get

$$\frac{1}{2}mv_0^2 + mgh_0 = \frac{1}{2}mv_1^2 + mgh_1 \tag{8}$$

This is the equation of conservation of mechanical energy! Tracing back on our steps, we see that equation (5), is the same thing but along a specific component divided through by $\frac{1}{2}m$. From this, the physical significance of the equation becomes apparent.