Proof of Some Properties of Cross Product

Theorem 1 *The vector which is the cross product of two vectors is perpendicular to both of those vectors.*

Proof

Consider two vectors $\vec{a}$ and $\vec{b}$ where:
$\vec{a} : \left(x_1; \quad y_1; \quad z_1\right)$
$\vec{b} : \left(x_2; \quad y_2; \quad z_2\right)$

Then:

$$\begin{aligned}
\vec{m} &= \vec{a} \times \vec{b} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\
&= \vec{i} \cdot (y_1 z_2 - y_2 z_1) - \vec{j} \cdot (x_1 z_2 - x_2 z_1) + \vec{k} \cdot (x_1 y_2 - x_2 y_1)
\end{aligned}$$

Hence:

$\vec{m} : \left(y_1 z_2 - y_2 z_1; \quad x_1 z_2 - x_2 z_1; \quad x_1 y_2 - x_2 y_1\right)$

To check if $\vec{m} \perp \vec{a}$ we can calculate the dot product of both vectors. If it is equal to 0, then the vectors are perpendicular to each other. Hence:

$$\begin{aligned}
\vec{a} \cdot \vec{m} &= x_1 \cdot (y_1 z_2 - y_2 z_1) - y_1 \cdot (x_1 z_2 - x_2 z_1) + z_1 \cdot (x_1 y_2 - x_2 y_1) \\
&= x_1 y_1 z_2 - x_1 y_1 z_2 - x_1 y_2 z_1 + x_1 y_2 z_1 - x_2 y_1 z_1 + x_2 y_1 z_1 \\
&= 0
\end{aligned}$$

We can conduct a similar proof to show that $\vec{m} \perp \vec{b}$. This is left as an exercise for the reader.
$\therefore$ QED

Theorem 2 *The magnitude of the cross product is equal to the area of the parallelogram created by the two vectors the cross product is being taken of.*

Proof

Consider two vectors $\vec{a}$, $\vec{b}$ and their cross product: $\vec{m}$

$\vec{a} : \left(x_1; \quad y_1; \quad z_1\right)$
$\vec{b} : \left(x_2; \quad y_2; \quad z_2\right)$
$\vec{m} : \left(y_1 z_2 - y_2 z_1; \quad x_1 z_2 - x_2 z_1; \quad x_1 y_2 - x_2 y_1\right)$

First recall that the area of a parallelogram is:
$S = |\vec{a}| \cdot |\vec{b}| \cdot \sin\alpha$

Where α is the angle between $\vec{a}$ and $\vec{b}$.

$$
\begin{aligned}
\sin^2\alpha &= 1 - \cos^2\alpha \\
&= 1 - \left(\frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2}} \right)^2 \\
&= \frac{(x_1^2 + y_1^2 + z_1^2) \cdot (x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2})^2} \\
&= \frac{x_1^2x_2^2 + x_1^2y_2^2 + x_1^2z_2^2 + y_1^2x_2^2 + y_1^2y_2^2 + y_1^2z_2^2 + z_1^2x_2^2 + z_1^2y_2^2 + z_1^2z_2^2 - x_1^2x_2^2 + y_1^2y_2^2 - z_1^2z_2^2}{(\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2})^2} \\
&+ \frac{-2x_1x_2y_1y_2 - 2x_1x_2z_1z_2 - 2y_1y_2z_1z_2}{(\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2})^2} \\
&= \frac{x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 + y_1^2z_2^2 - 2y_1y_2z_1z_2 + y_2^2z_1^2 + x_1^2z_2^2 - 2x_1x_2z_1z_2 + x_2^2z_1^2}{(\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2})^2} \\
&= \frac{(x_1y_2 - x_2y_1)^2 + (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2}{(\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2})^2}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
S &= \sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2} \cdot \frac{\sqrt{(x_1y_2 - x_2y_1)^2 + (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2}}{\sqrt{x_1^2 + y_1^2 + z_1^2} \cdot \sqrt{x_2^2 + y_2^2 + z_2^2}} \\
&= \sqrt{(x_1y_2 - x_2y_1)^2 + (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2} \\
&= |\vec{m}|
\end{aligned}
$$

$\therefore$ QED