1 Review

Definition 1 *Let $T : V \to W$ be a linear map between V, W inner product spaces, with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively. We call a linear map $T^* : W \to V$ the adjoint of T and denote it with T^* if it has the following property:*

$$\langle w, Tv \rangle_W = \langle T^* w, v \rangle_V \tag{1}$$

Theorem 1 *Let V, W be finite dimensional inner product spaces and $T : V \to W$. Then the adjoint of T always exists, is a unique linear map. Let $v_1, ..., v_m$ and $w_1, ..., w_m$ be orthonormal bases for V and W. Then T^* is defined via:*

$$\begin{aligned} T^* w_j &= \langle T v_1, w_j \rangle v_1 + \cdots + \langle T v_m, w_j \rangle v_m \\ &= \sum_{i=1}^{m} \langle T v_i, w_j \rangle v_i \end{aligned} \tag{2}$$

Definition 2 *An operator $T : V \to$ is called self adjoint (or Hermitian) if $T = T^*$*

Definition 3 *Let T be an operator over V. A vector $v \in V$ is called an eigenvector of T if $Tv = \lambda v$. The number λ is called the eigenvalue of v.*

Theorem 2 *The Spectral Theorem: Let V be a finite dimensional inner product space over $\mathbb{C}$. Let T be a self adjoint operator, then T has a basis of orthonormal eigenvectors with real eigenvalues.*

2 Review of things you have not yet learned

Definition 4 *Let X be a set. A metric on X is a function with the following properties*

$$d : X \times X \to \mathbb{R} \tag{3}$$

i) $d(x, y) \geq 0 \quad \forall x, y \in X$

ii) Distance from point to self: $d(x, y) = 0$ *if and only if* $x = y$

iii) Symmetry: $d(x, y) = d(y, x)$

iv) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Theorem 3 *Let V be an inner product space. Then we can define a metric using the norm induced by the inner product:*

$$d(x, y) = ||x - y||$$

Definition 5 *Let X be a metric space and $x_1, ..., x_n, ...$ be some sequence in X. A sequence x_i is called a Cauchy sequence if for any $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that*

$$\forall m, n \geq N \quad d(x_m, x_n) < \varepsilon$$

Definition 6 *A vector space with a metric is called complete if any cauchy sequence has a limit.*

Definition 7 *Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ with a metric. A set of linearly independent vectors $\{v_1, ..., v_i, ...\}$ is called a Schauder basis if for any $v \in V$ there exists a unique sequence α_j such that:*

$$\sum_{j=1}^{n} \alpha_j v_j \xrightarrow{n \to \infty} v$$

Where we take the limit in the topology induced by the metric from the inner product (usual definition from fist year, where we replace | | with || ||).

Definition 8 *An complete inner product space is called a Hilbert Space.*

3 Suggested Exercises

I) **(Understanding the Adjoint)** Let $T : V \to W$ be a linear map between two (potentially infinite dimensional) inner product spaces.

i) Suppose T^* exists. Prove that it must be unique. *Hint: Note that w and v in the defining property are arbitrary. Use properties of inner product.*

ii) Now suppose V and W are finite dimensional. Show that T^* as defined in equation (2) satisfies the property in (1).

iii)

II) **(Fun Facts)** Let $T : V \to V$ be a linear operator, with V over $\mathbb{C}$.

(a) Without using the spectrral theorem prove that the eigenvalues of a self adjoint operator are real.

(b) Without using the spectral theorem, prove that if T is self adjoint and u, v are its distinct eigenvectors, then u is orthogonal to v.

(c) Notice that

$$\langle Tu, w\rangle = \frac{\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle}{4} + \frac{\langle T(u+iw), u+iw\rangle - \langle T(u-iw), u-iw\rangle}{4}$$

Now use this to prove that if $\langle Tv, v\rangle = 0 \forall v$ then $T = 0$.

(d) Prove that T is self adjoint if and only if $\langle Tv, v\rangle \in \mathbb{R} \quad \forall v \in V$

III) **(Spectral Theorem)**

i) Review the proof of the fact that any operator on V over $\mathbb{C}$ with V finite dimensional, has a basis in which it is upper triangular

ii) Review the Gram-Schmidt procedure

iii) Prove that ${}_\alpha\mathcal{M}_\alpha(T^*) = {}_\alpha\mathcal{M}_\alpha(T)^\dagger$, where α is an orthonormal basis and $\dagger$ means the conjugate and transpose of a matrix (i.e. you take the transpose and take complex conjugate of each entry).

iv) Prove the Spectral Theorem

v) Prove by direct computation that the eigenvalues of a Hermitian 2×2 matrix are real.

IV) **(The Schauder Basis)**

i) Prove that any basis over finite dimensional V is also a Schauder basis [1].

ii) Let $v_1, ..., v_n, ..$ be an orthonormal Schauder basis. Then fix $v \in V$ with

$$v = \sum_{i=1}^{\infty} \alpha_i v_i$$

Prove that $\alpha_i = \langle v, v_i\rangle$

iii) Now define $S_n = \sum_{i=1}^{n} \alpha_i v_i$. Prove that this is the vector $v \in \text{span}(v_1, ..., v_n)$ that has the minimal distance to v *Hint: Compute the distance between v and S_n and then complete the square in an easy way somewhere. No need to expand v in terms of anything.*

V) **Bonus Exercise:(l^2 space)** We define l^2 to be the space of all square summable sequences:

$$l^2 = \{(x_1, ..., x_n, ...) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

This is an inner product vector space (where addition is defined componentwise and multiplication is defined by multiplying all entries. The inner product is defined via:

$$\langle (x_1, ..., x_n, ...)\ ,\ (y_1, ..., y_n, ...)\rangle = x_1\overline{y_1} + ... + x_n\overline{y_n} + ...$$

[1] You can read more about complete metric spaces and separable Hilbert spaces in "Elements of the Theory of Functions and Functional Analysis" by Kolmogorov and Fomin.

i) Write down an orthonormal Schauder basis for the space above. Prove that it is indeed a Schauder basis.

ii) Use the Cauchy-Schwarz inequality over $\mathbb{C}^n$ to show that the inner product above is well defined i.e. is finite (*Hint: Use the Cauchy-Schwarz on finite dimensional subspaces and then take a limit.*)

iii) Prove that this is indeed a Hilbert Space.

VI) (**Reading**)

- Read Introduction and Statement of Postulates in Chapter III of Cohen-Tannoudji Quantum Mechanics Volume I & II upto the time evolution section (p.215-p.222)