Derivation of Formula for Dot Product Expressed in Terms of Coordinates

By definition dot product is: $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha$
Where: α is the angle between vectors $\vec{a}$ and $\vec{b}$

We need to prove that:
$|\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha = x_1 \cdot y_1 + x_2 \cdot y_2$

Where:
- x_1 and y_1 are the coordinates of $\vec{a}$
- x_2 and y_2 are the coordinates of $\vec{b}$

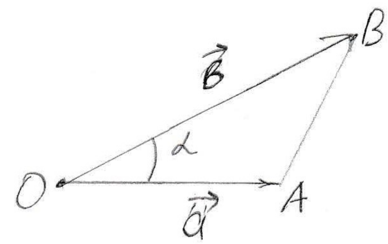

Figure 1

Proof # 1

If $\vec{a}$ and $\vec{b}$ are not colinear (Figure 1), then using cosine law:

$$AB^2 = OA^2 + OB^2 - 2OA \cdot OB \cdot \cos\alpha \qquad (1)$$

Since $\vec{AB} = \vec{b} - \vec{a}$, where $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, (1) can be written as:

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$$

From here we get:
$$\vec{a} \cdot \vec{b} = \frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 - |\vec{b} - \vec{a}|^2) \qquad (2)$$

Vectors $\vec{a}$, $\vec{b}$, and $\vec{b} - \vec{a}$ have coordinates of: $\{x_1; y_1\}, \{x_2; y_2\}$ and $\{x_2 - x_1; y_2 - y_1\}$.

$|\vec{a}|^2 = x_1^2 + y_1^2$
$|\vec{b}|^2 = x_2^2 + y_2^2$
$|\vec{b} - \vec{a}|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
Substituting this into (2) get:

$\vec{a} \cdot \vec{b} = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2)$
$\vec{a} \cdot \vec{b} = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2 - x_2^2 + 2x_1 \cdot x_2 - x_1^2 - y_2^2 + 2y_1 \cdot y_2 - y_1^2)$
$\vec{a} \cdot \vec{b} = \frac{1}{2}(2x_1 \cdot x_2 + 2y_1 \cdot y_2)$
$\vec{a} \cdot \vec{b} = x_1 \cdot x_2 + y_1 \cdot y_2$

$\therefore QED$

You can verify that cosine law would still be applicable if $\vec{a}$ and $\vec{b}$ are colinear ($\alpha = 0°$ or $\alpha = 180°$)

Proof # 2

Let α be the angle between $\vec{a}$ and $\vec{b}$.
Let β be the angle between $\vec{b}$ and the x axis.
Let c be the angle between $\vec{a}$ and the x axis.
x_1, y_1 and x_2, y_2 are the points $\vec{a}$ and $\vec{b}$ respectively.

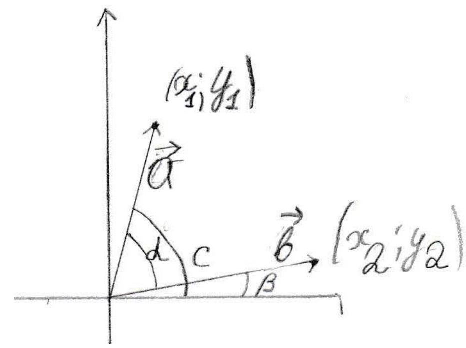

Figure 2

First we need to express as much as we can in terms of coordinate points:

Then:

$\tan\beta = \frac{y_2}{x_2}$
$\sin c = \frac{y_1}{\sqrt{x_1^2+y_1^2}}$
$\cos c = \frac{x_1}{\sqrt{x_1^2+y_1^2}}$
$$|\vec{a}| = \sqrt{x_1^2 + y_1^2} \tag{3}$$
$$|\vec{b}| = \sqrt{x_2^2 + y_2^2} \tag{4}$$

$\tan\alpha = \tan(c-\beta)$
$= \frac{\sin(c-\beta)}{\cos(c-\beta)}$

$= \frac{\sin c \cdot \cos\beta - sin\beta \cdot \cos c}{\cos c \cdot \cos\beta + \sin c \cdot \sin\beta}$

$= \frac{\sin c - \tan\beta \cdot \cos c}{\cos c + \sin c \cdot \tan\beta}$

$= \frac{\frac{y_1}{\sqrt{x_1^2+y_1^2}} - \frac{y_2}{x_2} \cdot \frac{x_1}{\sqrt{x_1^2+y_1^2}}}{\frac{x_1}{\sqrt{x_1^2+y_1^2}} + \frac{y_1}{\sqrt{x_1^2+y_1^2}} \cdot \frac{y_2}{x_2}}$

$= \frac{y_1 - \frac{y_2}{x_2} \cdot x_1}{x_1 + y_1 \cdot \frac{y_2}{x_2}}$

$$= \frac{y_1 \cdot x_2 - y_2 \cdot x_1}{x_1 \cdot x_2 + y_1 \cdot y_2} \tag{5}$$

Recall the identity:

$\tan^2\alpha + 1 = \frac{1}{\cos^2\alpha}$
$\cos\alpha = \sqrt{\frac{1}{\tan^2\alpha + 1}}$

Substitute (5) into the line above:

$= \sqrt{\frac{1}{(\frac{y_1 \cdot x_2 - y_2 \cdot x_1}{x_1 \cdot x_2 + y_1 \cdot y_2})^2 + 1}}$

$= \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{y_1^2 \cdot x_2^2 + y_2^2 \cdot x_1^2 + x_1^2 \cdot x_2^2 + y_1^2 \cdot y_2^2}}$

$= \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{y_1^2 \cdot (x_2^2 + y_2^2) + x_1^2 \cdot (x_2^2 + y_2^2)}}$
$= \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{(x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2)}}$
Substitute line above, (3) and (4) into the definition of dot product.
$\vec{a} \cdot \vec{b} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2} \cdot \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{(x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2)}}$
$= x_1 \cdot x_2 + y_1 \cdot y_2$

$\therefore QED$