Logarithms and their properties

Definition 1 *The logarithm of some number x base b is a function of x such that:*

$$
\begin{aligned}
f(x) &= \log_b(x) \\
b^{f(x)} &= b^{\log_b(x)} = x
\end{aligned} \tag{1}
$$

We will also define $f(x)$ only for $b > 0$.

Let's look at some examples:

1) Find $\log_2(4)$:

By definition (1), $\log_2(4)$ is a number such that:

$2^{\log_2(4)} = 4$

Notice that $4 = 2^2$

We then get :

$2^{\log_2(4)} = 2^2$

Hence:

$\log_2(4) = 2$

2) Find $\log_3(27)$

By definition (1) we get that:

$3^{\log_3(27)} = 27$

We can notice that:

$27 = 3^3$

We then get:

$3^{\log_3(27)} = 3^3$

Hence:

$\log_3(27) = 3$

So far, it may seem that logarithms are entirely useless (as we can always replace them with a number. This is however not always the case. Consider:

$\log_3(25)$

It is not obvious how we can express 25 as a power of 3.

It is however useful to be able to write down such a number without knowing its exact value (hence we need the log function).

Basic Properties

Property 1 *The logarithm of b to the power of x base b is equal to x*

$$\log_b(b^x) = x$$

Proof:

Recall that by definition (1) we get:

$b^{\log_b(b^x)} = b^x$

From this we can directly match powers and claim that:

$\log_b(b^x) = x$

Q.E.D.

Property 2 *The logarithm of the product of x and y base b is equal to the sum of logarithm of x base b and logarithm of y base b.*

$$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$$

Proof:

Substitute each of the logarithms from equation above into definition (1) to get:

$b^{\log_b(x)} = x$

$b^{\log_b(y)} = y$

$b^{\log_b(x \cdot y)} = x \cdot y$

Substitute the first two of these equations into the last equation to get:

$b^{\log_b(x \cdot y)} = b^{\log_b(x)} \cdot b^{\log_b(y)}$

Using exponent laws get that:

$b^{\log_b(x \cdot y)} = b^{\log_b(x) + \log_b(y)}$

We can once again match powers and get:

$\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

Q.E.D.

Property 3 *The logarithm of the quotient of x and y base b is equal to the difference of the logarithm x base b and logarithm y base b.*

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Proof:

Similarly to the proof for the product, let's substitute each of the logarithms from above into the definition equation (1).

$b^{\log_b(x)} = x$

$b^{\log_b(y)} = y$

$b^{\log_b\left(\frac{x}{y}\right)} = \dfrac{x}{y}$

Substitute the first two equations into the third equation to get:

$b^{\log_b\left(\frac{x}{y}\right)} = \dfrac{b^{\log_b(x)}}{b^{\log_b(y)}}$

Using power rules, rewrite equation above as:

$b^{\log_b\left(\frac{x}{y}\right)} = b^{\log_b(x)-\log_b(y)}$

Once again we can match powers to obtain the desired result:

$\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$

Q.E.D.

Practice Problems

Using the properties derived above, show that the following statements are true:

1. $\log_b\left(\dfrac{x}{y}z\right) = \log_b x - \log_b y + \log_b z$
2. $\log_b\left(\dfrac{x^2}{z}\cdot y\right) = 2\log_b x + \log_b y - \log_b z$
3. $\log_b(x^n) = n\log_b x$ Where n is some number.