(I) This question is about reproving the theorem we went over in class. I want you to write down everything explicitly. Like word for word every step. And write as legibly as possible. You can type if you want ($\LaTeX$only).

i) Write down the definition of an injective mapping

ii) Write down the definition of a surjective mapping

iii) Write down an alternative definition for injectivity that holds true specifically for linear maps.

iv) Write down the definition of a linear map T being invertible.

v) Think about it.

vi) Prove that if $T \in \mathcal{L}(V, W)$ is invertible that is equivalent to T being injective and surjective

(II) Now that we have talked about the inverse, it would be nice to figure out a practical way to compute the inverse of some linear map. We will use matrices to do this. This question is to motivate a method to find the inverse of a matrix (which I will show in class). Let $T \in \mathcal{L}(V, W)$ where V and W are finite dimensional.

i) Suppose we have some operators $A_1, ..., A_n \in \mathcal{L}(W)$, such that $A_n \circ \cdots \circ A_1 T = I$.

- What is the inverse of T?
- Are there any conditions on the A_i for the above equation to hold true?
- What do the A_i correspond to in terms of a basis of W.

ii) The point above, suggests that instead of trying to directly write down the inverse of T, we want to apply successive linear transformations to T and "build up" T^{-1} from those. So let's consider some simple transformations that may be helpful. Let $w_1, ..., w_m$ be a basis of W and $v_1, ..., v_n$ be a basis for V.

i. In the previous part, you may have recognized that that A_i correspond to some change of basis in W (or well that is what you had to say). Let's try and come up with a few ways to act on the basis without changing it. Let's think of the simplest change: swap w_i and $w_j \neq j$.

ii. Suppose W is finite dimensional with $\dim W = 2$. Let $A \in \mathcal{L}(W)$. Suppose A is the change of basis "swapping" w_1 and w_2. Write down a matrix for a linear operation swapping these two (in the w basis).

iii. Now suppose $\dim W = 3$. Write down a matrix for swapping w_1 with w_3.

iv. What would applying a matrix like this from the left or right, do to an arbitrary 3 by 3 matrix. What about n by n.

v. Now let's try something more complicated. Recall that if $w_1, ..., w_m$ is a basis, then $w_1 + w_2, w_2, ..., w_m$ is still a basis. Write down a matrix of operator A acting on a 2 dimensional vector space taking $w_1, w_2 \to w_1 + w_2, w_2$.

vi. Consider an arbitrary 2 by 2 matrix B. What does applying A to B on the right do? What does applying A to B on the left do?

vii. Write down the coefficients of a matrix related to a linear operator over some n dimensional space, which takes transforms the basis as:

$$w_1, ..., w_m \to w_1, ..., w_{i-1}, w_i + w_j, w_{i+1}, ..., w_j, ..., w_m$$

viii. Look at what you realized the 2 by 2 matrix does to an arbitrary matrix. Prove that the n dimensional thing, does the same to an n by n matrix.