## 0.1 Definitions Review

**Definition 1** *A curve $\gamma$ in $\mathbb{R}^2$, is a set $\Gamma \subseteq \mathbb{R}^2$ equipped with an invertible function $\gamma : [c,d] \to \mathbb{R}^2$ such that $\gamma([c,d]) = \Gamma$ and such that $\gamma$ is differentiable in each entry with respect to $t$. i.e. write $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ where $\gamma_1$ and $\gamma_2$ are differentiable wrt. to $t$ for $t \in [c,d]$.*

We define the derivative of a function $\gamma : \mathbb{R} \to \mathbb{R}^2$ as the derivative of each entry[1]:

$$\dot{\gamma} = \frac{d\gamma}{dt} = \left( \frac{\mathrm{d}}{\mathrm{d}t}\gamma_1, \frac{\mathrm{d}}{\mathrm{d}t}\gamma_2 \right) = (\dot{\gamma_1}, \dot{\gamma_2}) \tag{1}$$

So given some function $f : \gamma \to \mathbb{R}$ we can integrate it along $\gamma$ using:

$$\int_\gamma f(x,y) = \int_{[c,d]} f(\gamma(t)) \sqrt{(\dot{\gamma_1})^2 + (\dot{\gamma_2})^2} \mathrm{d}t \tag{2}$$

Also recall that integrals have the following nice property

$$\int_a^b g(f(x)) f'(x) dx = \int_{f(a)}^{f(b)} g(u) du \tag{3}$$

## 0.2 Suggested Exercises

1. First let's see that equation (2) works as expected in a familiar setting of computing the length of a curve. Suppose I had a rectangle of height 1 and length $l$. Then the area of the rectangle, would be numerically equal to its length. Having a rectangle of height 1, corresponds to setting $f(x) = 1$ in equation (2). We therefore get that the length of a curve $\Gamma$ is given via:

$$l(\Gamma) = \int_\Gamma 1 = \int_{[c,d]} \sqrt{(\dot{\gamma_1})^2 + (\dot{\gamma_2})^2} \mathrm{d}t \tag{4}$$

   i) Now suppose you wanted to get the length of the graph of some function $g(x)$ on some interval $[c,d]$. Note that the graph of this function is the set:

$$\Gamma = \left\{ \big(x, g(x)\big) \middle| x \in [c,d] \right\}$$

   Use this to write down a parametrization of the curve that is the graph of $g(x)$. Recognize that then the equation (4), becomes the familiar equation you have seen in class.

   ii) Consider the line segment of the line $y = kx$ for $x \in [0,1]$. Write down its parametrization that you figured out in the previous part. Compute the length of this line segment using the given equation. Check that it coincides with what we would expect the length to be from Pythagorean theorem.

   iii) ***Defining Pi:*** Recall that the points on a unit circle obey the following equation:

$$x^2 + y^2 = 1$$

   Use this to parametrize the circle in the top right quadrant (i.e. $y > 0, x > 0$). Write down the integral for the length of this arc. Compute this integral using desmos (or approximate it via Riemann sums). Note that a quarter of the circumference of the unit circle should be $\frac{\pi}{2}$. Use this fact to check that your computation gives you what you expect.

2. Consider the function $g(x) = 4x^{\frac{3}{2}}$. This defines a curve for $x \in [0,1]$, in the $x,y$ plane. Using the parametrization you got in question 1, find the length of this curve.

3. Let $S$ be the unit circle in $\mathbb{R}^2$

   (a) Find a nicer parametrization of entire circle via $\phi : [0,1] \to S$. (Hint: use sines & cosines).

   (b) Consider the functions $f(x,y) = \frac{x}{x^2+y^2}$ and $g(x,y) = \frac{y}{x^2+y^2}$. Compute $\int_S f$ and $\int_S g$

   (c) Now consider $f - ig$ where $i$- imaginary root. What is that equal to as a function in terms of $z = x + iy$?. Suppose that $\int \frac{1}{z} \mathrm{d}z = \int_S f - i \int_S g$.

   (d) Recall, what is $\int \frac{1}{x} dx$ for real numbers?

[1] Note that this is a vector in $\mathbb{R}^2$

(e) Assuming that for complex numbers have $e^{\log z} = z$, conclude that $e^z$ is a periodic function of a single complex variable with period $2\pi i$.

4. Reparametrization invariance: Note that our equation (2) uses some choice of parametrization. What does this mean? Suppose you are given some curve $\Gamma$, which is just some set of points in $\mathbb{R}^2$. We want to give it a parametrization (a function $\gamma : [c, d] \to \Gamma$). Physically, you can think about $\gamma$ as the position of a car driving along the curve at some time $t$. You can drive through the same curve in many different ways (go fast the first half and slow the second half or vice versa). So does our integral depend on the parametrization we chose? The answer is no and you will show why.

   (a) Suppose you have two parametrizations $\gamma : [a, b] \to \Gamma$ and $\tilde{\gamma} : [c, d] \to \Gamma$. Recall that we assume that $\gamma, \tilde{\gamma}$ are invertible (denote the inverse via: $\gamma^{-1}$). We will also use the convention that $t \in [a, b]$ and $\tilde{t} \in [c, d]$ (i.e. we write $\gamma(t)$ and $\tilde{\gamma}(\tilde{t})$ ). We then can define $\varphi = \tilde{\gamma}^{-1} \circ \gamma$ (i.e. the following diagram commutes, meaning you can follow the arrows):

$$\begin{tikzcd} {[a,b]} \arrow[d, "\varphi"] \arrow[dr, "\gamma"] & \\ {[c,d]} \arrow[r, "\tilde{\gamma}"] & \Gamma \end{tikzcd}$$

   (b) Convince yourself that the diagram above does indeed commute and you have that: $\tilde{\gamma} \circ \varphi = \gamma$

   (c) Rewrite the equation (2) for the $\gamma$ parametrization. Now notice that instead of $\gamma$, you can instead write $\tilde{\gamma} \circ \varphi$. Use this to rewrite your integral in terms of $\tilde{\gamma}$ and $\varphi$. Note to write down your derivatives in Leibnitz notation, and not Newton notation (this will be important later).

   (d) Now, note that by chain rule, we have: $\frac{df(g(t))}{dt} = \left.\frac{df}{dg}\right|_{g(t)} \cdot \left.\frac{dg}{dt}\right|_{t}$. Using this fact, rewrite all the derivatives in the square root for your equations, to include some derivatives of that look like $\frac{d}{d\phi}$. You should be able to factor out something out of the square root.

   (e) Use equation (3), to do a $u$-substitution on your integral. Write out this new integral.

   (f) Note that now, if you write $\tilde{t} = u$, you get exactly equation 2) but for the parametrization $\tilde{\gamma}$. You have therefore shown, that the value of the integral does not depend on our choice of parametrization and depends entirely on the curve itself (as we expected).