Lorentz Transformations

This is the derivation of the Lorentz Transformation equations for a 1-dimensional case.

Recall the 2 postulates :
1. All laws of physics have the same mathematical form in all inertial reference frames (reference frames that move in uniform motion with respect to one another)
2. The velocity of light in empty space is the same in all reference frames.

Consider two reference frames: Σ and Σ'. Assume that these reference frames have the same origins at $x' = 0$; $x = 0$ at time $t' = t = 0$. Let v be the speed of Σ' with respect to Σ
Let's start with two general linear transformation equations

$x' = ax - bt$ (1)
$t' = dx + et$ (2)

Consider the point x' $= 0$

Substituting that into equation (1):

$0 = ax - bt$
$ax = bt$
$\frac{x}{t} = \frac{b}{a}$

Then:

$v = \frac{b}{a}$

Now, let's substitute the point when $x = 0$ into equations (1) and (2) and divide equation (1) by equation (2)

$x' = 0 - bt$
$t' = 0 + et$
$\frac{x'}{t'} = -\frac{b}{e}$

Since we know that the relative velocity of reference frames is constant, we can say that the origin of Σ is moving with a velocity $-v$ with respect to Σ'. Since both reference frames have a common origin $x = 0$; $x' = 0$ at $t' = t = 0$ we can say that: $\frac{x'}{t'} = -v \Rightarrow v = \frac{b}{e}$

Since $b = e \cdot v$ but also: $b = a \cdot v$. We get that $a = e$
Let's rewrite the new forms of equations (1) and (2) .

$$x' = ax - avt \qquad (3)$$
$$t' = dx + at \qquad (4)$$

We need to make sure that postulate # 2 is incorporated in the equations (speed of light travels at the same speed c in both reference frames). Let's say a ray of light starts at the origin at time $t' = t = 0$ and moves along the x direction. Then: $x = c \cdot t$, but that would also mean that $x' = c \cdot t'$ (since speed of light has to be the same in all reference frames). Substitute this into (3) (4) (This is kind of like substituting a point into a function to find a coefficient)

$$ct' = act - avt \qquad (5)$$
$$t' = dct + at \qquad (6)$$

Divide (5) by (6) to get:

$$c = \frac{act-avt}{dct+at}$$
$$c(dct + at) = act - avt$$
$$dc^2 = -av$$
$$d = -\frac{av}{c^2}$$

Substitute this back into equations (3) and (4)

$$x' = a(x - vt) \qquad (7)$$
$$t' = a(t - \frac{vx}{c^2}) \qquad (8)$$

These equations describe a person looking from Σ into Σ'. We would expect for the same relation to be true if we are looking at Σ from Σ'. (Note the speed between frames will now

be negative, so we substitute -v as v)
Using equations (7) and (8) we get:

$x = a(x' + vt')$ (9)
$t = a(t' + \frac{vx'}{c^2})$ (10)

Substitute (7) and (8) into (9) and (10) to solve for a:

$x = a(a(x - vt) + v \cdot a(t - \frac{vx}{c^2}))$
$x = a^2(vt - vt + x - \frac{v^2 x}{c^2})$
$x = a^2 x(1 - \frac{v^2}{c^2})$
$a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

a is normally denoted as γ
$\frac{v}{c}$ is normally denoted as β
So:

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

So the final Lorentz transformation equations will be:

$x' = \gamma(x - vt)$
$y' = y$
$z' = z$
$t' = \gamma(t - \frac{vx}{c^2})$

Equations for the x direction and time can be re-written as:

$x' = \gamma(x - \beta ct)$
$ct' = \gamma(ct - \beta x)$