Catalan Numbers

The Catalan Number series has many definitions. We will look at only one of them. First, let's consider the following problem.

The product abc can be interpreted in two ways: $(ab)c$ and $a(bc)$. While the order in which we multiply numbers a b and c does not matter, the intermediate steps do. The product $abcd$ can be calculated in in 5 ways. Instead of counting each separate case we can say that the number of ways we can distribute brackets for 4 numbers is:
number of ways we can multiply a times the number of ways we can multiply bcd, added to the number of ways we can multiply ab times the number of ways we can multiply cd and so on. We get:
$1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$
Now lets consider the product of $abcde$.
Let's look at the number of ways to distribute brackets in the following statements:

$a(bcde) : 1 \cdot 5 = 5$
$(ab)(cde) : 1 \cdot 2 = 2$
$(abc)(de) : 2 \cdot 1 = 2$
$(abcd)e : 5 \cdot 1$

Add all of these together to get 14 $\Rightarrow$ five numbers can be multiplied by grouping them in 14 different ways.

Definition

The Catalan numbers T_n is the number of ways to distribute brackets in the product of n numbers.

Recursive Formula

Using the examples we looked at above, let's devise a recursive formula for the Catalan Numbers. The multiplication of n numbers can be expressed as the product of the first few numbers and the the product of all the other numbers.

$$x_1 x_2 \cdots x_n = (x_1 \cdots x_r) \cdot (x_{r+1} \cdots x_n)$$

The first r numbers can be combined in T_r ways, while the rest of the numbers can be combined in T_{n-r} ways. Hence we get:

$$T_n = T_1 T_{n-1} + T_2 T_{n-2} \cdots T_{n-1} T_1 \tag{1}$$

Ordinary Generating Functions of the Catalan Numbers

First let's look at what an Ordinary Generating Function of a number sequence is:
$A(x) = \sum_{n=0}^{\infty} a_n x^n$
Where:
a_n is the n term in some number sequence a.

Let's find a formula for:

$f(x) = \sum_{n=0}^{\infty} T_n x^n$

Rewrite this as:

$f(x) = T_0 + T_1 x + T_3 x^2 + \cdots + T_n x^n + \cdots$

Let's square this to get:

$f^2(x) = T_0^2 + (T_0 T_1 + T_1 T_0)x + \cdots + (T_0 T_n + \cdots T_n T_0)x^n + \cdots$

Note that $T_0 = T_1 = 1 = T_0^2$.

Using the recursive formula we derived earlier, we can easily notice that: $T_0 T_1 + T_1 T_0 = T_2$. and in general:

$T_0 T_n + \cdots T_n T_0 = T_{n+1}$

From this, we can notice that $f^2(x)$ is essentially like $f(x)$ but all of the coefficients T_n are shifted by one to the right.

$f^2(x) = T_1 + T_2 x + T_3 x^2 + \cdots + T_n x^{n-1} + \cdots$

Multiply this by x to get:

$x f^2(x) = T_1 x + T_2 x^2 + T_3 x^3 + \cdots + T_n x^n + \cdots$

This looks identical to $f(x)$, but we subtracted T_0 from it, but $T_0 = 1$ Hence we get:

$x f^2(x) = f(x) - 1$

This is a quadratic equation, where $f(x)$ is the unknown.

$x f^2(x) - f(x) + 1 = 0$

Using the quadratic formula we get:

$f(x) = \dfrac{1 \pm \sqrt{1-4x}}{2x}$

As we can see, the quadratic formula gives us two possible solutions. But since this is a sum of specific numbers, there should be only one answer. How do we know which one is right?

First let's rewrite the equation above as:

$x f(x) = \dfrac{1 \pm \sqrt{1-4x}}{2}$.

If we substitute $x = 0$, then both the left hand side and the right hand side of the equation above must be equal to 0.

$\dfrac{1 + \sqrt{1 - 4 \cdot 0}}{2} = 1 \neq 0$

$\dfrac{1 - \sqrt{1 - 4 \cdot 0}}{2} = 0$

We can now choose the root with the negative sign and say:

$f(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$

Formula for the nth Catalan Number

Recall the formula from the section above:

$$xf(x) = \frac{1-\sqrt{1-4x}}{2}$$

First let's look at the square root of the equation: $\sqrt{1-4x}$ and consider a slightly simpler case at first:

$$\sqrt{1-x} = (1-x)^{\frac{1}{2}}$$

For those who know calculus and know about the taylor series, you can easily expand this as an infinite series.

Let's consider a more intuitive approach to expanding this into an infinite series. The following statement will be left without rigorous proof, but only an intuitive explanation.

First let's look at $(1+x)^{\frac{1}{2}}$. Recall that using binomial expansion:

$$(1+x)^{\frac{1}{2}} = 1 + C_{\frac{1}{2}}^{1}x + C_{\frac{1}{2}}^{2}x^2 + \cdots + C_{\frac{1}{2}}^{n}x^n + \cdots$$

Note that this is an infinite series, unlike a binomial expansion where n is a whole number.
Recall that:

$$C_a^b = \frac{a!}{b!(a-b)!}$$

We are obviously unable to to find a factorial of $\frac{1}{2}$, so we will use the following trick: We cancel out the $(a-b)!$ with the $a!$ to get:

$$C_a^b = \frac{a\cdot(a-1)\cdots(a-b+1)}{b!}$$

Hence:

$$C_{\frac{1}{2}}^{n} = \frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!}$$

Technically you can prove that the statement above squared will give you $1+x$ on your own, but we will not do that within this document.
Now let's return to the following formula:

$$xf(x) = \frac{1-\sqrt{1-4x}}{2}$$

If we use the infinite binomial expansion we talked about above, substitute $-4x$ as x we will get:

$$\begin{aligned} xf(x) &= \frac{1-(1+C_{\frac{1}{2}}^{1}(-4x)+C_{\frac{1}{2}}^{2}(-4x)^2+\cdots+C_{\frac{1}{2}}^{n}(-4x)^n+\cdots)}{2} \\ &= \frac{-(C_{\frac{1}{2}}^{1}(-4x)+C_{\frac{1}{2}}^{2}(-4x)^2+\cdots+C_{\frac{1}{2}}^{n}(-4x)^n+\cdots)}{2} \end{aligned}$$

Now let's find the coefficient that we will have in front of x^n for n in the numerator (forget about the 2 in the denominator for now). * Note that the -4 is part of our coefficient.

$$\begin{aligned} C_{\frac{1}{2}}^{n}\cdot(-4)^n &= (-1)^n\cdot 2^{2n}\cdot\frac{\frac{1}{2}\cdot(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} \\ &= \frac{(-1)^n\cdot 2^{2n}}{n^1}\cdot\left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-3}{2}\right) \end{aligned}$$

For that last term we simply put $(\frac{1}{2} - n + 1)$ under a common denominator and factored out the negative.

We can notice that $1 \cdot 3 \cdot 5 \cdots 2n - 3$ is the multiplication of all odd numbers upto $2n - 3$.
We can also notice that the resultant product will have $n - 1$ minuses. This is true as there is a total of n numbers being multiplied, of which only the first one is positive. Hence we get:

$$\begin{aligned}
C^n_{\frac{1}{2}} \cdot (-4)^n &= \frac{(-1)^{2n-1} \cdot 2^{2n}}{n!} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n} \\
&= \frac{(-1)^{2n-1} \cdot 2^n}{n!} \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)
\end{aligned}$$

Note that $2n - 1$ is always an odd number so: $(-1)^{2n-1} = -1$. We also need to somehow simplify the multiplication of all odd numbers. We will do that by multiplying and dividing by all of the even numbers up to $2n - 3$.

$$\begin{aligned}
C^n_{\frac{1}{2}} \cdot (-4)^n &= -\frac{2^n \cdot 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{n!} \cdot \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n-2)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n-2)} \\
&= -\frac{2^n \cdot (2n-2)!}{n! \cdot 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n-2)}
\end{aligned}$$

We now need to get rid of the multiplication of even numbers in the denominator. It is obvious that there are $n - 1$ even numbers that make up that product. If we factor out a 2 from each one of those numbers, then we will get that:

$2 \cdot 4 \cdot 6 \cdots (2n - 2) = 2^{n-1} \cdot (n - 1)!$

Sub that into the equation above to get:

$$\begin{aligned}
C^n_{\frac{1}{2}} \cdot (-4)^n &= -\frac{2^n \cdot (2n-2)!}{n! \cdot 2^{n-1} \cdot (n-1)!} \\
&= -\frac{2 \cdot (2n-2)!}{n! \cdot (n-1)!}
\end{aligned}$$

Now let's once again go back to the original formula for $xf(x)$ and substitute these coefficients in:

$$\begin{aligned}
xf(x) &= \frac{-\sum_{n=0}^{\infty} -\frac{2 \cdot (2n-2)!}{n! \cdot (n-1)!} \cdot x^n}{2} \\
&= \sum_{n=0}^{\infty} \frac{(2n-2)!}{n! \cdot (n-1)!} \cdot x^n
\end{aligned}$$

Recall that:

$x \cdot f(x) = T_0 x + T_1 x^2 + \cdots + T_{n-1} x^n + T_n^{n+1} + \cdots$

This means that the coefficient in front of x^n is T_{n-1} - the $n-1$ Catalan number. Hence the formula for the $n-1$ Catalan number, using the summation statement we found earlier is:

$$T_{n-1} = \frac{(2n-2)!}{n!(n-1)!}$$

To find the formula for the nth number, just sub in $n = n+1$.

$$\begin{aligned} T_n &= \frac{(2n+2-2)!}{(n+1)!(n+1-1)!} \\ &= \frac{(2n)!}{(n+1)!n!} \end{aligned}$$

Just to rewrite this one last time: the formula for the nth Catalan number is:

$$T_n = \frac{(2n)!}{(n+1)!\,n!}$$