0.1 Definition of the Definite Integral

Let function $f(x)$ be defined on some segment (a, b). Split this segment into n segments with width $\Delta x_k = x_k - x_{k-1}$ where k is the order of the segment (first second third etc.). In each of these smaller line segments let's choose a random point ξ_k. Multiply this value of the function at that point and multiply it by the length of the segment it is in: $f(\xi_k) \cdot \Delta x_k$. This will be the area of that really badly shaded rectangle. Now let's add the area of all such rectangles:

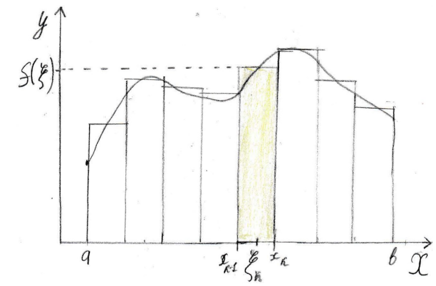

Figure 1: The graph of function $f(x)$. That squiggle is my attempt to write ξ.

$$S_n = \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k$$

This is called a Riemann sum. The symbol used for the Riemann sum in this guide will be: "S".

The definite integral of function $f(x)$ on the segment (a, b) is the limit of the Riemann sum where the number of line segments infinitely increases and their length approaches 0.

The definite integral from a to b is written using the following notation:

$$\int_a^b f(x)dx$$

Where:
a - lower bound
b - upper bound
$f(x)$ - function being integrated

Hence by definition:

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k$$

* Important to note that when Δx approaches 0, n approaches infinity.

0.2 Properties of Definite Integral

Property 1

For function $f(x)$ defined at $x = a$, the following statement is true:

$$\int_a^a f(x)dx = 0$$

Proof

This statement comes from the definition of the definite integral. Each Riemann sum for any division of (a, a) and for any choice of $(\xi_k$ is equal to zero since $\Delta x_i = x_i - x_{i-1} = 0$, $i = 1, 2, 3...n$. Hence, the limit of the Riemann sum is 0.

$\therefore$ QED

Property 2

For a function that can be integrated on (a, b) the following statement is true.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Proof

This property can also be derived from the definition from the definition of the Riemann integral. Now we start splitting the interval (a, b) from b to a.
The line segments are still the same and we choose the same points ξ_k, but we assign them different numbers (because we are now counting from the opposite side).
Let x_k and x_{k-1} be endpoints of an arbitrary segment in (a, b). If we number the line segments from b to a the same points will be represented with x_i and x_{i-1} then $x_k = x_{i-1}$ and $x_{k-1} = x_i$. (Once again, this is because we are counting from b to a and not a to b).
$\Delta x_{a\ to\ b} = x_k - x_{k-1}$
$\Delta x_{b\ to\ a} = x_i - x_{i-1} = x_{k-1} - x_k = -(x_k - x_{k-1}) = -\Delta x_{a\ to\ b}$

$$\begin{aligned}
\int_b^a f(x)dx &= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_{k_{b\ to\ a}} \\
&= \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot (-\Delta x_{k_{a\ to\ b}}) \\
&= -\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_{k_{a\ to\ b}} \\
&= -\int_a^b f(x)dx
\end{aligned}$$

Hence:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$\therefore$ QED

Property 3

For two functions $f(x)$ and $g(x)$ which can be integrated on (a, b) the following statement is true:

$$\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

Proof

Let's write down the Riemann sum for function $y = f(x) \pm g(x)$ for a given division of line segment (a, b) and choice of points ξ_k

$$\begin{aligned} S &= \sum_{k=1}^{n} (f(\xi_k) \pm g(\xi_k)) \cdot \Delta x \\ &= \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x \pm \sum_{k=1}^{n} g(\xi_k) \cdot \Delta x \\ &= S_f \pm S_g \end{aligned}$$

Where S_f and S_g are the Riemann sums of functions $y = f(x)$ and $y = g(x)$ for the given division of segment (a, b). Now if we consider the definite integral (the limit of the Riemann sum). We get:

$$\begin{aligned} \int_a^b (f(x) \pm g(x))dx &= \lim_{\Delta x \to 0} S \\ &= \lim_{\Delta x \to 0} (S_f \pm S_g) \\ &= \lim_{\Delta x \to 0} S_f \pm \lim_{\Delta x \to 0} S_g \\ &= \int_a^b f(x)dx \pm \int_a^b g(x)dx \end{aligned}$$

$\therefore$ QED

Property 4

A constant factor can be taken out of the integral sign. This means that for any function $f(x)$ that can be integrated on (a, b) the following statement is true:

$$\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$$

Proof

Once again, let's consider the Riemann sum of $y = k \cdot f(x)$

$$
\begin{aligned}
S &= \sum_{i=1}^{n} k \cdot f(\xi_i) \cdot \Delta x \\
&= k \cdot \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x \\
&= k \cdot S_f
\end{aligned}
$$

Where S_f is the Riemann sum of $y = f(x)$ on (a, b).
Now let's consider the definite integral (the limit of the Riemann sum).

$$
\begin{aligned}
\int_a^b k \cdot f(x)dx &= \lim_{\Delta x \to 0} S \\
&= \lim_{\Delta x \to 0} k \cdot S_f \\
&= k \cdot \lim_{\Delta x \to 0} S_f \\
&= k \cdot \int_a^b f(x)dx
\end{aligned}
$$

$\therefore$ QED

Property 5

If function $f(x)$ can be integrated on X and $a \in X$, $b \in X$, $c \in X$ then the following statement is true:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Proof

For this property we should consider several cases:
$a < b < c$
$a < c < b$
$b < a < c$
$b < c < a$
$c < a < b$
$c < b < a$

Here we will consider the proof for the first case (all other cases are proved in the exact same fashion).

If we split the interval (a, c) such that b is an end point of one of the segments. Since $a < b < c$ we can say that: $S_{ac} = S_{ab} + S_{bc}$. Where S_{ac}, S_{ab} and S_{bc} are Riemann sums for intervals $(a, c), (a, b)$ and (b, c) respectively. Hence (using the properties of the limit which we used in the proofs above (just too lazy to write things out in that much detail at this point)) by converting this expression in terms of Riemann sums into an expression in terms of definite integrals we get:

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
$$\int_a^b f(x)dx = \int_a^c f(x)dx - \int_b^c f(x)dx$$

From Property 2 we know:

$$\int_b^c f(x)dx = -\int_c^b f(x)dx$$

Now lets substitute the equation above into the previous equation.

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^c f(x)dx - \int_b^c f(x)dx \\ &= \int_a^c f(x)dx + \int_c^b f(x)dx\end{aligned}$$

$\therefore$QED

Other cases can be either proved using the same logic, or using the above statement and renaming points to fit all the other cases.

Property 6

If a function $y = f(x)$ can be integrated on $[a, b]$ and $f(x) \geq 0$ ($f(x) \leq 0$) then for any $x \in [a, b]$ the following statement is true. (* Note that the symbol that means "for all possible outcomes" is: $\forall$):

$$\int_a^b f(x)dx \geq 0 \qquad (\int_a^b f(x)dx \leq 0)$$

Proof

Since $a \leq b \Rightarrow \Delta x_k = x_k - x_{k-1} \geq 0$. Also, $f(x) \geq 0$ ($f(x) \leq 0$) on $[a, b]$ then no matter what ξ_k we choose the following is true: $f(\xi_k) \geq 0$ ($f(\xi_k) \leq 0$). Hence:

$$\sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k \geq 0 \qquad (\sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k) \leq 0)$$
$$\int_a^b f(x)dx \geq 0 \qquad (\int_a^b f(x)dx \leq 0)$$

$\therefore$ QED

Corollary

For functions $f(x)$ and $g(x)$ that can be integrated on $[a, b]$ The following statements are true:

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx, \qquad if \quad f(x) \geq g(x) \;\; \forall x \in [a, b]$$
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx, \qquad if \quad f(x) \leq g(x) \;\; \forall x \in [a, b]$$

Proof

If $f(x) \geq g(x)$ (or $f(x) \leq g(x)$) $\Rightarrow f(x)-g(x) \geq 0$ ($f(x)-g(x) \leq 0$). Let $h(x) = f(x)-g(x)$. Since $h(x) \geq 0$ (or $h(x) \leq 0$) we can use property 6 and then apply property 3.

$$\int_a^b h(x)dx \geq 0 \quad\;\; f(x) \geq g(x) \quad \forall x \in [a, b]$$
$$\int_a^b (f(x) - g(x))dx \geq 0$$
$$\int_a^b ((x)fx - \int_a^b g(x)dx \geq 0$$
$$\int_a^b f(x) \geq \int_a^b g(x)dx$$

$$\int_a^b h(x)dx \leq 0 \quad\;\; f(x) \leq g(x) \quad \forall x \in [a, b]$$
$$\int_a^b (f(x) - g(x))dx \leq 0$$
$$\int_a^b ((x)fx - \int_a^b g(x)dx \leq 0$$
$$\int_a^b f(x) \leq \int_a^b g(x)dx$$

$\therefore$ QED

This property signifies that you are allowed to integrate inequalities (definite integrals).

Property 7

If function $y = f(x)$ can be integrated on $[a, b]$ then the following inequality is true:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Proof

It is trivial that: $-|f(x)| \leq f(x) \leq |f(x)|$. In the previous property we determined that it is possible to integrate an inequality. Integrating this inequality get:

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x) \leq \int_a^b |f(x)|dx$$

From this we get that:

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

This conclusion is correct since we have an inequality of the form:
$-a \leq b \leq a$ Where: $a \geq 0$. If we take the absolute value of b and consider all of its positive values, it is obvious that for all $b > 0$, $|b| \leq |a|$. Now let's consider negative values of b. Since b is greater than negative a, that means that the absolute value of $-a$ would be greater than the absolute value of any negative value b can take on. Hence from the original statement we get that: $|b| \leq a$

$\therefore$ QED

Property 8

If $y = f(x)$ and $y = g(x)$ can be integrated on $[a, b]$ and $g(x) \geq 0$ $\forall x \in [a, b]$ then the following statement is true:

$$m \cdot \int_a^b g(x)dx \leq \int_a^b f(x) \cdot g(x)dx \leq M \cdot \int_a^b g(x)dx$$

Where $m = \min\ f(x)$ and $M = \max\ f(x)$ on $[a, b]$

Proof

Since m and M are the min and max of the function on $[a, b]$ then the following statement must be true:
$m \leq f(x) \leq M$ $\forall x \in [a, b]$
Multiplying this inequality by the non-negative function $y = g(x)$ get:
$m \cdot g(x) \leq f(x) \cdot g(x) \leq M \cdot g(x)$
Now integrate this inequality to get the statement from above.

$\therefore$ QED

Corollary

If $g(x) = 1$ then the inequality will have the following form: (Note that a horizontal line makes a rectangle with the x axis and has the area $l \cdot w$. If $w = 1$, then the area is numerically equal to the length of the rectangle).

Hence : $\int_a^b g(x)dx = 1 \cdot (b-a) = (b-a)$:

Substituting that into the theorem above we get:

$$m \cdot (b-a) \leq \int_a^b f(x)dx \leq M \cdot (b-a)$$

Property 9

If the function $y = f(x)$ can be integrated on $[a, b]$, $m = \min f(x)$ and $M = \max f(x)$ on $[a, b]$, then there is a number μ is in the range of $f(x) \forall x \in [a, b]$ such that:

$$\int_a^b f(x)dx = \mu \cdot (b-a)$$

Proof

If we use in inequality in the corollary above and divide it by $(b-a)$. Then the following value will be between m and M, and equal to the definite integral of f(x) if multiplied by $(b-a)$:

$$\frac{\int_a^b f(x)dx}{b-a}$$

We can select this number to be μ and since it is less than the max of $f(x)$ and the min of $f(x)$ on $[a; b]$, then we know it will also be in the range of $f(x)$ on that same interval.

$\therefore$ QED