1 Review

Definition 1 *A functional Φ is a map which takes in a curve (function from $\mathbb{R}$ to some space) as input and outputs a number (real or complex).*

Definition 2 *Let Φ be a functional. We say that Φ is differentiable at γ if:*

$$\Phi(\gamma + h) - \Phi(\gamma) = F(h) + R$$

Where F depends on h linearly and $R(h,\gamma) = O(h^2)$ (i.e. if $|h| < \varepsilon$ and $|\frac{dh}{dt}| < \varepsilon \Rightarrow |R| < C\varepsilon^2$)
We call F the variation of Φ. It is common to denote F as $\delta\Phi$.

1.1 Action Principle

Suppose we have a physical system made of points. Let $q_1, ..., q_n$ be numbers, which can be used to specify the positions of all of these points. We call $q_1, ..., q_n$ generalized coordinates. We then call their derivatives $\dot{q}_1, ..., \dot{q}_n$, generalized velocities.

The Action Principle states that any mechanical system is characterized by some function $\mathcal{L}$ called the Lagrangian:

$$\mathcal{L}(q_1, ..., q_s, v_1, ..., v_s, t)$$

Usually, we write this in shortened form $\mathcal{L}(q, v, t)$.
Suppose at moments in time $t = t_1$ and $t = t_2$ the system takes positions characterized by q_1 and q_2. Then between these positions, the system moves along some path $q(t)$ such that the variation of the following functional at q is zero:

$$\begin{gathered} S(q(t)) = \int_{t_1}^{t_2} L(q_1(t), ..., q_n(t), \dot{q}_1, ..., \dot{q}_n, t) \\ \text{System takes path such that } (\delta S)|_q(t) = 0 \end{gathered} \tag{1}$$

2 Suggested Exercises

I) **Euler Lagrange Equations** From the Action principle above, we see that solving for the motion of a classical particle, it is essential to know how to compute δS and see what it means for it to be equal to 0.

i) Using the definition of the variation of S, convince yourself of the following:

$$\begin{aligned} \delta S|_{q(t)} &= S(q(t) + (\delta q)(t)) - S(q(t)) - R \\ &= \int\limits_{t_1}^{t_2} \left(\sum_{i=1}^{n} \frac{\partial \mathcal{L}(q,v,t)}{\partial q_i}\Big|_{(q(t),\dot{q}(t),t)} \cdot (\delta q_i)(t) + \sum_{i=1}^{n} \frac{\partial \mathcal{L}(q,v,t)}{\partial v_i}\Big|_{(q(t),\dot{q}(t),t)} \cdot (\dot{\delta q_i})(t) \right) dt \end{aligned} \tag{2}$$

Where $(\delta q_i)(t)$ are the curves you add to $q_i(t)$ as deviations to take the derivative (they are unrelated to q_i itself and $(\dot{\delta q_i})(t)$ is the time derivative of $(\delta q_i)(t)$. Note that this is like "taking a derivative in the direction" δq

ii) Show that above is equivalent to the following *Hint:Integrate something by parts.*

$$\delta S = \int\limits_{t_1}^{t_2} \left(\sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial q_i}\Big|_{(q(t),\dot{q}(t),t)} (\delta q_i)(t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_i}\Big|_{(q(t),\dot{q}(t),t)} \right) (\delta q_i)(t) \right) + \left(\frac{\partial \mathcal{L}}{\partial v_i}\Big|_{(q(t),\dot{q}(t),t)} (\delta q)(t) \right) \Bigg|_{(q_1,\dot{q}_1,t_1)}^{(q_2,\dot{q}_2,t_2)} \tag{3}$$

iii) Now by looking at the assumptions in the action principle, argue that the boundary term in the expression above must be zero. Then set $\delta S = 0$ and argue that since $(\delta q_i)(t)$ is arbitrary get that $\delta S = 0$ implies the following equations, called the Euler-Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial v_i}\Big|_{(q(t),\dot{q}(t),t)} \right) - \frac{\partial \mathcal{L}}{\partial q_i}\Big|_{(q(t),\dot{q}(t),t)} = 0 \qquad \forall i \tag{4}$$

Note that $\mathcal{L}$ is a function of variables $q_1, ..., q_n, v_1, ..., v_n, t$. Usually we abuse notation and write $\mathcal{L}(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t)$

instead. Here we use $\dot{q}$ as a symbol representing v, NOT the derivative of some path $q(t)$. Usually, the Euler Lagrange equations are then written:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\Big|_{(q(t),\dot{q}(t),t)}\right) - \frac{\partial \mathcal{L}}{\partial q_i}\Big|_{(q(t),\dot{q}(t),t)} = 0$$

Here the $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is a usual derivative of $\mathcal{L}$ with respect to the entry into which we plug in $\dot{q}_i(t)$; i.e. we treat $\dot{q}_i$ as a symbol when taking this derivative. Only then, do we plug in $q(t)$ into the expression as an actual path.

II) **Relation to Newtonian Mechanics** Suppose we have a particle, described by cartesian coordinates x_1, x_2, x_3, moving under the influence of a potential force F with potential $V(x_1, x_2, x_3)$.

i) Recall how F and V are related. Write it down.

ii) Let $x(t) = (x_1(t), x_2(t), x_3(t))$ be that according to the action principle gives us the motion of the system. Take the following lagrangian:

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2}m|\dot{x}|^2 - V(x)$$

Write down the Euler-Lagrange equations that $x(t)$ must satisfy. Show that these are in fact the same as Newton's equations. From this we can expect that for a physical system, the Lagrangian should be of the form:

$$\mathcal{L} = T - V$$

Where T is the kinetic energy of the system and V is the potential energy.

iii) Now what's the advantage of the Lagrangian formalism over Newtonian if they give the same equations!? Well note that the equations are the same only if we chose cartesian coordinates as our coordinate system. The key idea, is that the principle does not depend on the fact that we write everything in an x, y, z coordinate system (unlike Newton's equations). We could just as easily choose any generalized coordinates $q_1, .., q_n$ and as long as we rewrite $\mathcal{L}$ in terms of them, we expect to get the same answer. One of the important bits, is to know how to rewrite the kinetic energy in terms of other variables. Write down the kinetic energy $\frac{1}{2}m|\dot{x}|^2$ in terms of the following coordinates and their generalized velocities:

i. Rewrite kinetic energy in 3D in terms of spherical coordinates $r, \theta, \varphi, \dot{r}, \dot{\theta}, \dot{\varphi}$ where:

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned}$$

ii. Rewrite the kinetic energy in 2D in terms of polar coordinates $r, \dot{r}, \varphi, \dot{\varphi}$:

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}$$

iii. Rewrite the kinetic energy in 3D in terms of cylindrical coordinates $\rho, \dot{\rho}, \varphi, \dot{\varphi}, z, \dot{z}$:

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}$$