1 Mathematical Review

Definition 1 *Let $T : V \to W$ be a linear map between V, W inner product spaces, with inner products $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_W$ respectively. We call a linear map $T^* : W \to V$ the adjoint of T and denote it with T^* if it has the following property:*

$$\langle w, Tv \rangle_W = \langle T^* w, v \rangle_V \tag{1}$$

Definition 2 *An operator $T : V \to$ is called self-adjoint (or Hermitian) if $T = T^*$*

Definition 3 *Let X be a set. A metric on X is a function with the following properties*

$$d : X \times X \to \mathbb{R} \tag{2}$$

i) $d(x, y) \geq 0 \quad \forall x, y \in X$

ii) *Distance from point to self:* $d(x, y) = 0$ *if and only if* $x = y$

iii) *Symmetry:* $d(x, y) = d(y, x)$

iv) *Triangle Inequality:* $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X$

Theorem 1 *Let V be an inner product space. Then we can define a metric using the norm induced by the inner product:*

$$d(x, y) = ||x - y|| = \sqrt{\langle x - y, x - y \rangle}$$

Definition 4 *Let X be a metric space and $x_1, ..., x_n, ...$ be some sequence in X. A sequence x_i is called a Cauchy sequence if for any $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that*

$$\forall m, n \geq N \quad d(x_m, x_n) < \varepsilon$$

Definition 5 *A vector space with a metric is called complete if any cauchy sequence has a limit.*

Definition 6 *An complete inner product space is called a Hilbert Space.*

Definition 7 *Suppose we have some numbers values λ_i that have probabilities $p(\lambda_i)$. Then the expectation value is defined via:*

$$E = \sum_i \lambda_i p(\lambda_i) \tag{3}$$

2 Postulates of Quantum Mechanics

In the most recent session we discussed the following postulates of Quantum Mechanics[1].

I) The state of an isolated system is fully described by a vector in a Hilbert space $\mathcal{H}$ over $\mathbb{C}$.

II) For any physical measurement that can be made on the system, there is a corresponding self-adjoint linear operator $A : \mathcal{H} \to \mathcal{H}$, which we call the observable associated with this measurement, with the following properties:

   i) The eigenvalues of the observable give all the possible measurements you can make.

   ii) If the system is in some normalized state $\psi \in \mathcal{H}$ (i.e. $||\psi|| = 1$) then the probability to get λ as a result of a measurement on the system is:

$$p(\lambda) = \sum_{v_i} |\langle v_i, \psi \rangle|^2 \tag{4}$$

   Where the sum is taken over all v_i eigenvectors of A with eigenvalue λ that are normalized to $||v_i|| = 1$.

III) Let the states of the system be described by a Hilbert space $\mathcal{H}$. Let A be some observable associated to some physical measurement. Let λ be an eigenvalue of A (i.e. a possible outcome of the physical measurement). Suppose the system is in some state $\psi \in \mathcal{H}$, when the physical measurement is performed and gives output λ. Then the state collapses into a new state following the equation [2]:

$$\psi \xrightarrow{\text{measuring } \lambda} \frac{\sum_{v_i} \langle \psi, v_i \rangle v_i}{|| \cdot ||} \tag{5}$$

Where the summation is once again taken over all normalized eigenvectors v_i of A with eigenvalue λ.

[1] There are more, but we look only at these for now

[2] Here the inner product is taken to be linear in the first entry

3 Suggested Exercises

1. **Mathematical Practice**. For the exercises below, take the fact that $\mathbb{R}$ is complete as given.

   i) Let $x_1, ..., x_n, ...$ be some sequence in a metric space X. Suppose this sequence has l and l' as a limit. Prove that $l = l'$.

   ii) Prove that any finite dimensional inner product space is complete.

2. **Made Up Quantum Mechanics**. Suppose our system is a light switch, which is completely isolated from the rest of the world. Suppose we have some device, that can measure the position of the light switch. We press a button and if the machine outputs 1 then that corresponds to the switch pointing up and if it outputs -1, then that corresponds to the switch pointing down.

   i) What are the eigenvalues of the observable A associated with this device? Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be the eigenvectors of this observable. Assume that this is all we can measure about this system. What is the dimension of the Hilbert Space? Write down the matrix of A in this $|\uparrow\rangle, |\downarrow\rangle$ basis.

   ii) Suppose the system is in state
   $$\psi = a|\uparrow\rangle + b|\downarrow\rangle$$
   Where a and b are some complex numbers. What is the probability that the light switch is measured to be looking up? down?

   iii) Suppose system is in state ψ. What is the expectation value [3] of the measurement above?

   iv) Suppose a measurement is indeed performed on ψ and we measure the switch to be pointing up. What is the state right after this measurement is performed?

3. Suppose we have some physical system described by some Hilbert Space. Let A be some observable, which has a countable basis of eigenvectors. Suppose the system is in some state ψ. Show that the expectation value $\langle A \rangle_\psi$ of the measurement associated with A will be given via:
   $$\langle A \rangle_\psi = \langle A\psi, \psi \rangle \tag{6}$$
   Note that if we take the inner product to be linear in the second entry (like all physicists do), then the formula above takes the nicer form of:
   $$\langle A \rangle_\psi = \langle \psi, A\psi \rangle$$

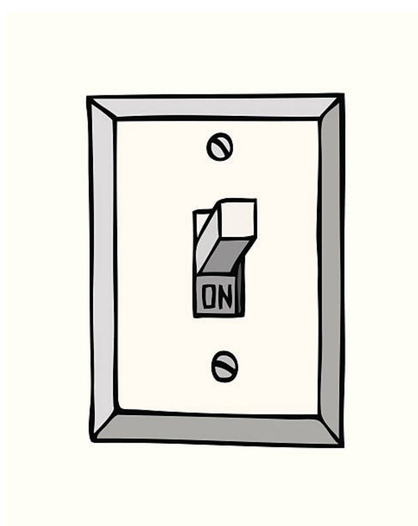

Figure 1: Our Quantum Mechanical System

[3] If you look very carefully at the definition of expectation value, you will see it tells you "what do you measure on average"