1 Pre-requisite Definitions

Definition 1 *Let V be a vector space and T an operator on V. Let λ be an eigenvalue of T. We define the λ eigen-subspace of V to be:*

$$Eig_T(\lambda) = \{v \in V \mid \}$$

Definition 2 *Let V be an inner product space. An operator T is called self-adjoint if:*

$$\langle Tu, v\rangle = \langle u, Tv\rangle \quad \forall u, v \in V$$

Theorem 1 *If V has countable dimensions and T is self-adjoint, then there exists an orthonormal basis of eigenvectors of T.*

2 Bloch's Theorem

I) A useful theorem/lemma

   i) Let S, T be operators on some vector space V. Suppose that $ST = TS$. Let λ be an eigenvalue of T. Prove that S restricts to $\mathrm{Eig}_T(\lambda)$. (In other words $S(\mathrm{Eig}_T(\lambda)) \subseteq \mathrm{Eig}_T(\lambda)$)

   ii) Prove that if T is not degenerate (i.e. if two eigenvectors have the same eigenvalue, they they are scalar multiples of each-other), then there is a basis in which both S, T are diagonal

   iii) Now suppose just that S and T are both self-adjoint and V is countable dimensional. Prove that we can find an orthonormal basis in which both S and T are diagonal. *Hint:Use part I)*

II) Read the proof "Using Operators" of Bloch's theorem on the Wikipidea page. Fill in the "skipped steps" in this proof.

III) Suppose the setup is as in the proof above (i.e. the Hamiltonian H is invariant under integer translations along some vectors a_1, a_2, a_3). Employing the labelling convention of eigenvalues of T_i with k where the eigenvalue associated to k is given via $\exp(ik \cdot a_i)$, show that some k must be equivalent. What is this condition? Google the Brillouin zone, and understand why we then want to restrict our crystall momentum k to the first Brillouin zone.

IV) If you have time, read the first 6 pages of " Fundamentals of the Theory of Metals" by Abrikosov.