Energy and Conservation of Mechanical Energy

Definitions

Recall our definition for work to be:

$$A = |\vec{F}| \cdot \cos\alpha \cdot |\vec{d}| \tag{1}$$

Where:

A - work done on object
$|\vec{F}|$ - magnitude of force the work of which is calculated
$|\vec{d}|$ - magnitude of displacement of object the work is done on
α - the angle between the displacement of the object and the vector of considered force

*Note that $|\vec{F}| \cdot \cos\alpha$ is the component of force $\vec{F}$ that is pushing in the same direction the object is moving.

Definition 1. *The change in energy of an object as it changes from one state to another state, is the work that was done to that object during this change.*

Potential Energy

Definition 2. *The change in potential energy is the work that must be applied to an object to get it into a specific position relative to some other object(s) from some initial position.*

Let's consider the potential energy an object of mass m has dues to its position relative to the earth.

The earth always applied some force of gravity F_g on objects near its surface pulling them directly towards its center. We can therefore only consider an objects position relative to its vertical distance away from the earth (opposite direction of pull of force).
This potential energy is therefore, the work required to apply to an object to raise it to some height h above the ground.

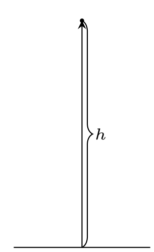

Figure 1: An object is lifted height h above the ground.

In order to lift an object, a force must be applied equal or greater than the earth's pull on the object.
Recall that:

$F_g = mg$

Therefore to raise an object a force equal to mg.

As the force is applied vertically upward (to counteract gravity), the angle between the displacement and applied force is $\alpha = 0$ and the displacement is equivalent to the height of the object above the ground.

Substitute this into the definition of work (1) to find potential energy:

$$\begin{aligned} A &= |\vec{F}| \cdot \cos\alpha \cdot |\vec{d}| \\ &= mg \cdot \cos 0 \cdot h \\ &= mgh \end{aligned}$$

Therefore:

$$E_p = mgh \tag{2}$$

Where:

E_p - potential energy of an object due to its position relative to earth
m - mass of object
h - position of object above earth

Kinetic Energy

Definition 3. *Kinetic Energy is the work that must be applied to an object to get it to its current state of motion.*

Let's look at kinetic energy gained by an object at constant acceleration.

Consider an object of mass m accelerating from rest due to some external force at a constant rate a. Let's look at its kinetic energy after some time t.

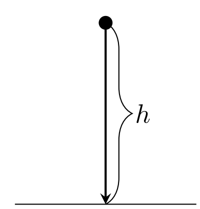

Figure 2: Ball is dropped from height h

To find this energy, we need to look at the work performed on the object. From its definition in equation (1), we must find its force and distance traveled.

From Newton's Second law get force directly in terms of acceleration and mass:

$F = ma$

From kinematic equations find the displacement of an object in terms of time t and a:

$|\vec{d}| = \frac{1}{2}at^2$

Substitute this and Newton's second law into definition of work (1) to find kinetic energy:

$$\begin{aligned} A &= |\vec{F}| \cdot |\vec{d}| \cdot \cos\alpha \\ &= ma \cdot \frac{1}{2}at^2 \cdot \cos\alpha \\ &= \frac{1}{2}m \cdot at \cdot at \cdot \cos\alpha \end{aligned}$$

Recall from kinematic equations, that assuming an object starts from rest $a \cdot t$ is its current velocity. Get:

$$A = \frac{1}{2} mv \cdot v \cos\alpha$$
$$= \frac{1}{2} mv^2 \cos\alpha$$

If we are considering the work that has been done to get an object moving from rest, then the force that achieved this must have been pointing in the same direction as the displacement and $\alpha = 0$ (assuming constant acceleration). Therefore:

$$E_k = \frac{1}{2}mv^2 \tag{3}$$

In the equation above we can directly see that kinetic energy of an object (unlike potential energy) does not depend on the force that "got the object moving".

Conservation of Energy

Let's investigate any relationships between potential and kinetic energy.

Consider some object of mass m raised to some height h. It is released and drops to the ground at some velocity v after some time t.

Initially it had a potential energy of (subscript i to indicate initial):

$E_{pi} = mgh$
When it hit the ground it was 0 m above the ground and therefore(subscript f to indicate final):

$E_{pf} = 0$

As the ball is released it has no initial velocity and therefore:

$E_{ki} = 0$

When it hits the ground it has some velocity v. Recall that this velocity can be calculated using the kinematic equations and get:

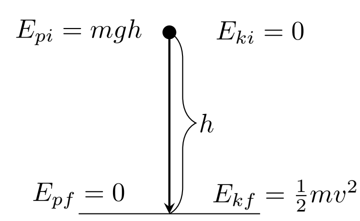

Figure 3: Ball is dropped from height h

$$E_{kf} = \frac{1}{2}mv^2$$
$$= \frac{1}{2}m(g \cdot t)^2$$
$$= mg\frac{1}{2} \cdot gt^2$$
$$= mgh$$
$$= E_{pi}$$

Therefore we see that:

$$E_{ki} + E_{pi} = E_{kf} + E_{pf} \tag{4}$$

In our case of zero initial velocity:

$$mgh = \frac{1}{2}mv^2 \tag{5}$$

Where:

m - mass of object
h - height object traveled from being released.
v - final velocity of object

Equation (4) is known as the law of conservation of energy. As we can see, the sum of potential and kinetic energies before and after the fall are conserved. In a sense potential energy can be "converted" into kinetic energy and vice versa.

Experimental verification

We will now verify this law on the special case considered above. An object is dropped from rest from some height h above the ground.

We expect relationship (5) to hold true. In our case of constant mass, we can simplify it as:

$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$
$gh = \frac{1}{2}v^2$

Rearrange this to get:

$$h = \frac{1}{2g}v^2 \tag{6}$$

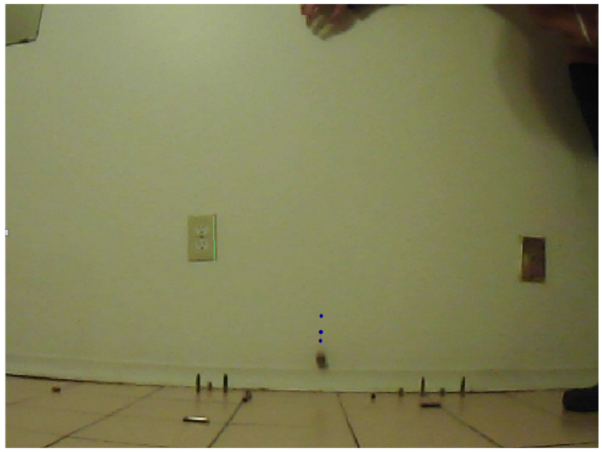

Figure 4: A bouncy ball was dropped from a height of 92 cm above the ground. This was video recorded to determine the ball's velocity.

We can now predict h "theoretically" through velocity and simultaneously measure it independently of our "theoretical" relationship.

In experiment we have measured:

$m = 0.02255$ kg
$h_{exp} = 0.92$ m (our experimental value)
Assume $g = 9.81\ \frac{\mathrm{m}}{\mathrm{s}^2}$

In the video recording, two frames right before the ball hits the ground are considered to determine its "impact" velocity:

At $t = 6.98633333333$ s, $y_{ball} = 0.335423524553$ m
At $t = 6.99466111111$ s, $y_{ball} = 0.297635030173$ m

Use this to calculate velocity as:

$$v = \frac{0.297635030173 - 0.335423524553}{6.99466111111 - 6.98633333333}$$
$$= -4.53764\ \frac{\mathrm{m}}{\mathrm{s}}$$

Using equation (6), we can now come up with a "theoretically" expected value for our height by substituting our impact velocity into that formula to get:

$$\begin{aligned} h_{theor} &= \frac{1}{2g}v^2 \\ &= \frac{1}{2 \cdot 9.81} \cdot (-4.53764)^2 \\ &= 1.04945\,\mathrm{m} \end{aligned}$$

To finally verify this law, compare h_{theor} and h_{exp}.

$$
\begin{aligned}
Er &= \left| \frac{h_{theor} - h_{exp}}{h_{theor}} \right| \\
&= \left| \frac{1.04945 - 0.92}{1.04945} \right| \\
&= 0.123351 \\
&= 12.3\%
\end{aligned}
$$

Our expected theoretical value differed from the experimental one by 12.3 percent. While this is not an ideal match, it still is a good indication that our initial proposition may be true.

Conservation of Mechanical Energy

Let's consider a similar situation to the example before.

A ball of mass m is moving vertically with some initial velocity $\vec{v}_0$. It then enters free fall initially being some height h_0 above the ground. Let's show that the sum of kinetic and potential energy will be conserved before and after some time interval t ("prove" equation (4)).

Using equations (2) and (3) directly write down the initial kinetic and potential energies:

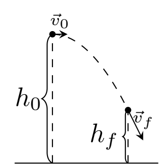

Figure 5: Ball falls with some initial velocity $\vec{v}_0$ from height h_0. After time t it has a new velocity $\vec{v}_f$ and is at some new height h_f.

$$
\begin{aligned}
E_{pi} &= mgh_0 \\
E_{ki} &= \frac{1}{2} m |\vec{v}_0|^2
\end{aligned}
$$

We need to find and express $\vec{v}_f$ and h_f in terms of $\vec{v}_0$ and $\vec{h}_0$ and some time interval t.

Introduce a reference frame with the origin on the ground below initial position of the ball. Let "up" and right be the positive directions.
First let's find v_f.
We will consider the velocity vector in the x and y, directions.

Since the object is in free fall, there is no acceleration in the x direction and $a_x = 0$. In the y direction the acceleration is $a_y = -g$. Write down the following kinematic equations:

$$
\begin{aligned}
\vec{v}_{fx} &= v_{0x} \\
\vec{v}_{fy} &= -gt + v_{0y}
\end{aligned}
$$

Calculate the magnitude of the final velocity with Pythagorean theorem as:
$|\vec{v}_f| = \sqrt{v_{0x}^2 + (-gt + v_{0y})^2}$

Substitute velocity above into equation (3) can calculate the final kinetic energy to be:

$$
\begin{aligned}
E_{kf} &= \frac{1}{2}m|\vec{v}_f|^2 \\
&= \frac{1}{2}m\left(\sqrt{v_{0x}^2 + (-gt + v_{0y})^2}\right)^2 \\
&= \frac{1}{2}m(v_{0x}^2 + (-gt + v_{0y})^2) \\
&= \frac{1}{2}m(v_{0x}^2 + v_{0y}^2 + g^2t^2 - 2v_{0y}gt) \\
&= \frac{1}{2}m(|\vec{v}_0|^2 + g^2t^2 - 2v_{0y}gt)
\end{aligned}
$$

Now find the new height h_f using the position kinematic equation for the y direction.

$$h_f = -\frac{1}{2}gt^2 + v_{0y}t + h_0$$

From equation (2) write down the potential energy to be:

$$E_{pf} = mg \cdot (-\frac{1}{2}gt^2 + v_{0y}t + h_0)$$

Recall that we have to prove the statement (4) to be true. Let's consider the sums of final potential and kinetic energies, and show that they are equal to the sum of initial kinetic and potential energies:

$$
\begin{aligned}
E_{pf} + E_{kf} &= mg \cdot (-\frac{1}{2}gt^2 + v_{0y}t + h_0) + \frac{1}{2}m(|\vec{v}_0|^2 + g^2t^2 - 2v_{0y}gt) \\
&= -\frac{1}{2}mg^2t^2 + mgv_{0y}t + mgh_0 + \frac{1}{2}m|\vec{v}_0|^2 + \frac{1}{2}mg^2t^2 - mv_{0y}gt \\
&= \cancel{-\frac{1}{2}mg^2t^2} + \cancel{\frac{1}{2}mg^2t^2} + \cancel{mgv_{0y}t} - \cancel{mv_{0y}gt} + mgh_0 + \frac{1}{2}m|\vec{v}_0|^2 \\
&= mgh_0 + \frac{1}{2}m|\vec{v}_0|^2 \\
&= E_{pi} + E_{ki}
\end{aligned}
$$

We have therefore shown that potential and kinetic energy is conserved for free fall motion.