Derivation of Formulas for Cross Product

Let's define cross product of two vectors $\vec{v}$ and $\vec{u}$ as some vector $\vec{m}$ which is perpendicular to these two vectors and has a magnitude equal to the area of the parallelogram formed by $\vec{v}$ and $\vec{u}$.

Consider:
$\vec{v} : \big(x_1; \quad y_1; \quad z_1\big)$
$\vec{u} : \big(x_2; \quad y_2; z_2\big)$
$\vec{m} : \big(a; \quad b; \quad c\big)$

Since $\vec{v} \perp \vec{m}$ then $\vec{v} \cdot \vec{m} = 0$.
Since $\vec{u} \perp \vec{m}$ then $\vec{u} \cdot \vec{m} = 0$.

Recall that the area of the parallelogram is: $A = |\vec{v}| \cdot |\vec{u}| \cdot \sin\alpha$. Where α is the angle between $\vec{v}$ and $\vec{u}$.

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha} \\ &= \sqrt{1-\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}\right)^2}\end{aligned}$$

Hence we get 3 equations:

$$\begin{cases} ax_1 + by_1 + cz_1 = 0 & (1) \\ ax_2 + by_2 + cz_2 = 0 & (2) \\ \sqrt{a^2+b^2+c^2} = \sqrt{x_1^2+y_1^2+z_1^2} \cdot \sqrt{x_2^2+y_2^2+z_2^2} \cdot \sqrt{1-\frac{(x_1x_2+y_1y_2+z_1z_2)^2}{(x_1^2+y_1^2+z_1^2)(x_2^2+y_2^2+z_2^2)}} & (3) \end{cases}$$

Using (1) we get:

$$a = -\frac{by_1+cz_1}{x_1}$$

Sub the line above into (2) and express b in terms of c.

$$-\frac{bx_2y_1+cx_2z_1}{x_1} + by_2 + cz_2 = 0$$
$$b \cdot (y_2 - \frac{x_2y_1}{x_1}) + c \cdot (z_2 - \frac{x_2z_1}{x_1}) = 0$$
$$b \cdot (y_2 - \frac{x_2y_1}{x_1}) = c \cdot (\frac{x_2z_1}{x_1} - z_2)$$
$$b = \frac{c \cdot (\frac{x_2z_1}{x_1} - z_2)}{y_2 - \frac{x_2y_1}{x_1}}$$
$$b \cdot (x_1y_2 - x_2y_1) = c \cdot (x_2z_1 - x_1z_2)$$
$$b = \frac{c \cdot (x_2z_1 - x_1z_2)}{x_1y_2 - x_2y_1}$$

Sub the line above into the expression which shows we got in the very first step to express a in terms of c.

$$
\begin{aligned}
a &= -\frac{c \cdot (y_1 \cdot \frac{x_2z_1 - x_1z_2}{x_1y_2 - x_2y_1} + z_1)}{x_1} \\
&= -\frac{c \cdot (x_2y_1z_1 - x_1y_1z_2 + x_1y_2z_1 - x_2y_1z_1)}{x_1 \cdot (x_1y_2 - x_2y_1)} \\
&= \frac{c \cdot (x_1y_1z_2 - x_1y_2z_1)}{x_1(x_1y_2 - x_2y_1)} \\
&= \frac{c \cdot (y_1z_2 - y_2z_1)}{x_1y_2 - x_2y_1}
\end{aligned}
$$

Now let's consider the third question equation. Square both sides and simplify the $\sin \alpha$ term.

$$
\begin{aligned}
&a^2 + b^2 + c^2 = (x_1^2 + y_1^2 + z_1^2) \cdot (x_2^2 + y_2^2 + z_2^2) \cdot \left(\frac{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2}{(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)} \right) \\
&a^2 + b^2 + c^2 = (x_1^2 + y_1^2 + z_1^2) \cdot (x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2 \\
&a^2 + b^2 + c^2 = x_1^2x_2^2 + x_1^2y_2^2 + x_1^2z_2^2 + x_2^2y_1^2 + y_1^2y_2^2 + y_1^2z_2^2 + x_2^2z_1^2 + y_2^2z_1^2 + z_1^2z_2^2 - x_1^2x_2^2 - y_1^2y_2^2 - z_1^2z_2^2 \\
&- 2x_1x_2y_1y_2 - 2x_1x_2z_1z_2 - 2y_1y_2z_1z_2 \\
&a^2 + b^2 + c^2 = x_1^2z_2^2 - 2x_1x_2z_1z_2 + x_2^2z_1^2 + x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 + y_1^2z_2^2 - 2y_1y_2z_1z_2 + y_2^2z_1^2 \\
&a^2 + b^2 + c^2 = (x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2
\end{aligned}
$$

Now let's substitute our equations for a and b in terms of c into the equation above.

$$
\begin{aligned}
&\frac{c^2 \cdot (y_1z_2 - y_2z_1)^2}{(x_1y_2 - x_2y_1)^2} + \frac{c^2 \cdot (x_2z_1 - x_1z_2)^2}{(x_1y_2 - x_2y_1)^2} + c^2 = (x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2 \\
&\frac{c^2}{(x_1y_2 - x_2y_1)^2} \cdot ((y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2) = (x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 \\
&+ (y_1z_2 - y_2z_1)^2 \\
&c^2 = \frac{((x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2) \cdot (x_1y_2 - x_2y_1)^2}{(x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2} \\
&c^2 = (x_1y_2 - x_2y_1)^2 \\
&c = \pm(x_1y_2 - x_2y_1)
\end{aligned}
$$

Here we notice something we did not mention in the definition. We get two possible solutions. This makes sense since a vector that is perpendicular can be going “up” from the plane or “down”. We need to choose one of em to have some kind of certainty. For reasons that we will not mention here, we will choose the positive solution. This will now be a part of our definition.

$$c = x_1y_2 - x_2y_1$$

Subbing this into the expression into the expressions for b and a we easily get that:

$$
\begin{aligned}
&a = y_1z_2 - y_2z_1 \\
&b = x_2z_1 - x_1z_2
\end{aligned}
$$

Hence we get that the cross product: $\vec{m} = \vec{v} \times \vec{u}$ is equal to:
$\vec{m} : \big(y_1 z_2 - y_2 z_1; \quad x_2 z_1 - x_1 z_2; \quad x_1 y_2 - y_2 z_1\big)$

Where:
$\vec{v} : \big(x_1; \quad y_1; \quad z_1\big)$
$\vec{u} : \big(x_2; \quad y_2; \quad z_2\big)$

We can note that the cross product is equal to the following determinant:

$$\vec{m} = \vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$