Time Dilation

This the the derivation of the time dilation formula.

Consider a reference frame Σ and a reference frame Σ'. Σ' is moving relative to Σ with a constant velocity v. Let's say that both reference keep time using identical light clocks. In this clock, a beam of light bounces back and forth between two mirrors. The mirrors are placed parallel to the direction of motion (in the diagrams, it is to the right). When looking from Σ the clock in Σ will look like Figure 1, where h is the distance between mirrors. When looking from Σ the clock in Σ' will look like Figure 2 .

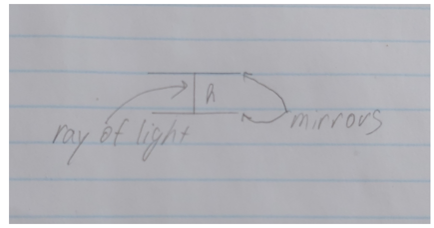

Figure 1: The ray of light just moves up and down.

Let t be the time it takes for the ray of light to travel from one mirror to the other. Recall postulate # 2: Speed of light is the same in all reference frames.

In Σ :

$$h = c \cdot t$$

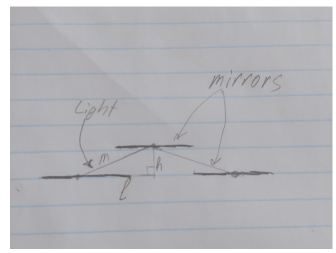

Figure 2: Since the reference frame Σ' is moving and we are an observer located in Σ, by the time the photon gets to the second mirror, it has already moved by some length l

In Σ':

$m = c \cdot t'$ (The light bouncing from one mirror to the other)
$l = v \cdot t'$ (The mirror moving along with the reference frame)

Using Pythagoras theorem:

$$h^2 = m^2 - l^2$$

Substitute the length of h from Σ and the lengths of m and l from Σ'

$$(ct)^2 = (ct')^2 - (vt')^2$$
$$(ct)^2 = t'^2(c^2 - v^2)$$

$$t'^2 = \frac{c^2t^2}{c^2-v^2}$$
$$t'^2 = \frac{c^2t^2}{c^2(1-\frac{v^2}{c^2})}$$
$$t'^2 = \frac{t^2}{1-\frac{v^2}{c^2}}$$

$$t' = \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$