Decomposition of a Vector into Given Vectors

Two Dimensional Case

Theorem 1 *Any vector in a plane can be decomposed into two given non-collinear vectors in the same plane. The coefficients of the decomposition are unique.*

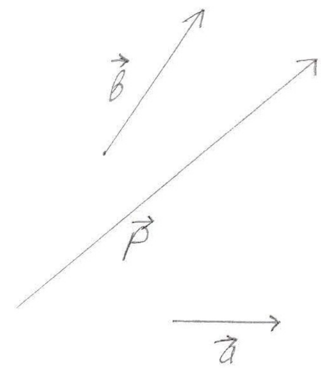

Figure 1

Proof

Let $\vec{a}$ and $\vec{b}$ be the given non-collinear vectors First lets prove that any vector $\vec{p}$ can be decomposed into vectors $\vec{a}$ and $\vec{b}$ (Figure 1). There are two possible cases:

1)
Vector $\vec{p}$ is collinear to either $\vec{a}$ and $\vec{b}$ (Let's take $\vec{b}$ as an example). In this case, we can express vector $\vec{p}$ in the form $\vec{p} = y\vec{b}$, where y is some number. Hence: $\vec{p} = 0 \cdot \vec{a} + y \cdot \vec{b}$. Vector p is decomposed into $\vec{a}$ and $\vec{b}$.

2) Vector $\vec{p}$ is not collinear to $\vec{a}$ or $\vec{b}$. Let's mark some point O and draw the following vectors from it: $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OP} = \vec{p}$ (Figure 2). Through point P draw a line parallel to line OB, and label the intersection of this line with line OA as A_1. From here we get: $\vec{p} = \overrightarrow{OA_1} + \overrightarrow{A_1P}$. Vectors $\overrightarrow{OA_1}$ and $\overrightarrow{A_1P}$ are respectively collinear to vectors $\vec{a}$ and $\vec{b}$. This means that there exist numbers x and y such that: $\overrightarrow{OA_1} = x\vec{a}$, $\overrightarrow{A_1P} = y\vec{b}$. Hence: $\vec{p} = x\vec{a} + y\vec{b}$. This means that $\vec{p}$ was decomposed into $\vec{a}$ and $\vec{b}$.

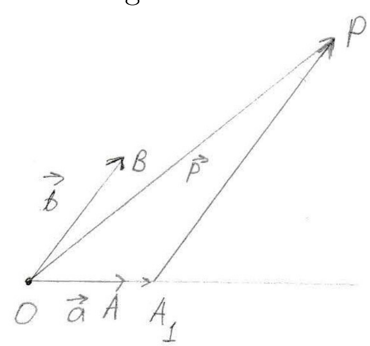

Figure 2

Now let's prove that coefficients x and y can be determined in a unique way.
Suppose that along with the decomposition $\vec{p} = x\vec{a} + y\vec{b}$ there is some other decomposition: $\vec{p} = x_1\vec{a} + y_1\vec{b}$. Subtracting one equation from the other we get:

$\vec{0} = (x - x_1)\vec{a} + (y - y_1)\vec{b}$

This is equality is true only in the case where coefficients : $x - x_1$ and $y - y_1$ are equal to zero.

If we suppose that $x - x_1 \neq 0$ then from the derived equality we get:

$\vec{a} = \vec{a} = -\frac{y-y_1}{x-x_1}\vec{b}$

This means that $\vec{a}$ is collinear to $\vec{b}$, which contradicts our initial conditions. Hence: $x - x_1 = 0$ and $y - y_1 = 0$. From here we get: $x = x_1$ and $y = y_1$. This means that the coefficients of decomposition of $\vec{p}$ are determined in a unique way.

$\therefore$ QED

Three Dimensional Case

Theorem 2 *Any vector can be decomposed into three given non-coplanar vectors. The coefficients of the decomposition are determined in a unique way.*

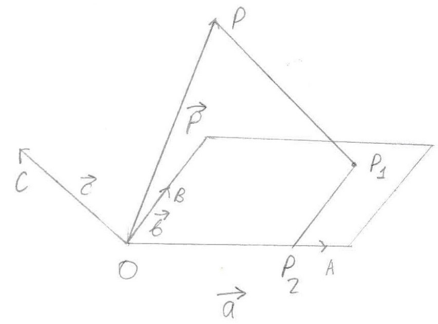

Figure 3

Proof

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the given non-coplanar vectors. First let's prove that any vector $\vec{p}$ can be expressed in the following form:

$\vec{p} = x\vec{a} + y\vec{b} + z\vec{c}$

Let's mark an arbitrary point O and draw the following vectors starting from it (Figure 3):

$\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c},\ \overrightarrow{OP} = \vec{p}.$

Through point P draw a line parallel to line OC and mark the intersection of this line with plan AOB as P_1. If $P \in OC$, then choose O as P_1). From P_1 draw a line parallel to line OB. Label the point of intersection of this line with line OA as P_2 (if $P_1 \in OB$ then choose point O as point P_2). We get:

$\overrightarrow{OP} = \overrightarrow{OP_2} + \overrightarrow{P_2P_1} + \overrightarrow{P_1P}$

Vectors: $\overrightarrow{OP_2}$ and $\overrightarrow{OA}$, $\overrightarrow{P_2P_1}$ and $\overrightarrow{OB}$, $\overrightarrow{P_1P}$ and $\overrightarrow{OC}$ are collinear which means that there exist numbers x, y, z such that: $\overrightarrow{OP_2} = x \cdot \overrightarrow{OA}$, $\overrightarrow{P_2P_1} = y \cdot \overrightarrow{OB} + z \cdot \overrightarrow{OC}$.

Recalling that $\overrightarrow{OA} = \vec{a},\ \overrightarrow{OB} = \vec{b},\ \overrightarrow{OC} = \vec{c},\ \overrightarrow{OP} = \vec{p}$, we get the above equation is of the required form:

$\vec{p} = x \cdot \vec{a} + y \cdot \vec{b} + z \cdot \vec{c}$

Now let's prove that the coefficients of this decomposition are unique.

Suppose that with the decomposition above there is some other decomposition:

$\vec{p} = x_1\vec{a} + y_1\vec{b} + z_1\vec{c}$

Subtracting these two decompositions we get: $\vec{0} = (x - x_1)\vec{a} + (y - y_1)\vec{b} + (z - z_1)\vec{c}$

This equality is true only if:

$x - x_1 = 0,\ y - y_1 = 0; z - z_1 = 0.$

If we suppose that for example: $z - z_1 \neq 0$ then from the above equality we get that:

$$(z - z_1)\vec{c} = -(x - x_1)\vec{a} - (y - y_1)\vec{b}$$
$$\vec{c} = -\frac{x - x_1}{z - z_1}\vec{a} - \frac{y - y_1}{z - z_1}\vec{b}$$

This would mean that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, but this contradicts our original suppositions. Meaning that the assumption that $z - z_1 \neq 0$ is incorrect. We get:

$x = x_1,\ y = y_1,\ z = z_1$

Hence the coefficients are unique.

$\therefore$ QED