# 1 Review

**Definition 1** *A vector space $V$ is called an inner product space if it is equipped with an inner product function that satisfies the following properties:*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

*i* *Linear in first entry:* $\langle v + \lambda u, w \rangle = \langle v, w \rangle + \lambda \langle u, w \rangle$

*ii* *Positivity:* $\langle v, v \rangle \geq 0 \quad \forall v \in V$

*iii* *Definiteness:* $\langle v, v \rangle = 0$ *if and only if* $v = 0$

*iv* *Conjugate Symmetry:* $\langle u, v \rangle = \overline{\langle v, u \rangle}$

**Definition 2** *Let $V$ be an inner product space. Vectors $u, v$ are called orthogonal if*

$$\langle u, v \rangle = 0$$

**Definition 3** *A norm on a vector space $V$ is a function such that:*

$$|| \cdot || : V \to \mathbb{R}$$

1. *Positivity:* $||v|| \geq 0 \quad \forall v \in V$
2. *Definiteness:* $||v|| = 0$ *if and only if* $v = 0$
3. *Absolute Homogeneity:* $||\lambda v|| = |\lambda| ||v||$
4. *Triangle Inequality:* $||u + v|| \leq ||u|| + ||v||$

**Theorem 1** *An inner product defines a norm on a vector space via:*

$$||v|| = \sqrt{\langle v, v \rangle}$$

**Definition 4** *A basis $v_1, ..., v_n$ of $V$ is called orthonormal if $||v_i|| = 1 \, \forall i$ and $\langle v_i, v_j \rangle = 0$ for any $i \neq j$*

**Definition 5** *A partial derivative of a function $f(x_1, ..., x_n)$ of multiple variables $x_1, ..., x_n$ with respect to some variable $x_i$, to be the usual derivative that treats all $x_j$ $j \neq i$ as constants. It is denoted as:*

$$\frac{\partial}{\partial x_j} f$$

Let $f : \mathbb{R}^m \to \mathbb{R}^n$ a nice enough function. We can write $f$ component wise as:

$$f(x) = f(x_1, ..., x_m) = (f_1(x), ..., f_i(x), ..., f_n(x))$$

Let $e_1, ..., e_m$ be an orthonormal basis for $\mathbb{R}^m$
Let $\tilde{e}_1, ..., \tilde{e}_n$ be an orthonormal basis for $\mathbb{R}^n$

**Definition 6** *We define the Jacobian of $f$ at a point $p \in \mathbb{R}^m$ to be the linear map $(Df)(p)$ given via:*

$$\begin{aligned} ((Df)(p))\, e_j &= \frac{\partial}{\partial \varepsilon} f(p + \varepsilon e_j) \\ &= \left( \lim_{\varepsilon \to 0} \frac{f_1(p + \varepsilon e_j) - f_1(p)}{\varepsilon}, ..., \lim_{\varepsilon \to 0} \frac{f_i(p + \varepsilon e_j) - f_i(p)}{\varepsilon}, ..., \lim_{\varepsilon \to 0} \frac{f_n(p + \varepsilon e_j) - f_n(p)}{\varepsilon} \right) \end{aligned}$$

Recall that if $e_j$ and $\tilde{e}_i$ are the standard bases, than the jacobian takes the following form:

$$(Df)(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\big|_p & \cdots & \frac{\partial f_1}{\partial x_m}\big|_p \\ \vdots & \frac{\partial f_i}{\partial x_j}\big|_p & \vdots \\ \frac{\partial f_n}{\partial x_1}\big|_p & \cdots & \frac{\partial f_n}{\partial x_m}\big|_p \end{pmatrix}$$

**Theorem 2** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a nice enough invertible function that maps $A$ into $f(A)$, with $A$ a nice set. Let $g : f(A) \to \mathbb{R}$ a nice enough function. We then have:*

$$\int_{f(A)} g(y) = \int_A g(f(x)) |det((Df)(x))|$$

## 2 Suggested Exercises

I) **Change of Coordinates**. The following maps are common change of coordinates used in physics. Compute the associated Jacobians:

i) Spherical Coordinates

i. Compute the Jacobian of:

$$f : (0, \infty) \times [0, \pi) \times [0, 2\pi) \to \mathbb{R}^3$$
$$f(r, \theta, \phi) = \begin{pmatrix} r \sin\theta \cos\phi & r \sin\theta \sin\phi & r \cos\theta \end{pmatrix}$$

ii. Use the change of variables theorem to compute the volume of a unit ball in 3d.

iii. Prove that the volume of a ball of radius $R$ is proportional to $R^3$ with coefficient $\frac{4}{3}\pi$

iv. Show that the measure $dx\, dy\, dz$ in polar coordinates becomes $r^2 \sin\theta\, dr\, d\theta\, d\phi$

ii) Cylindrical Coordinates

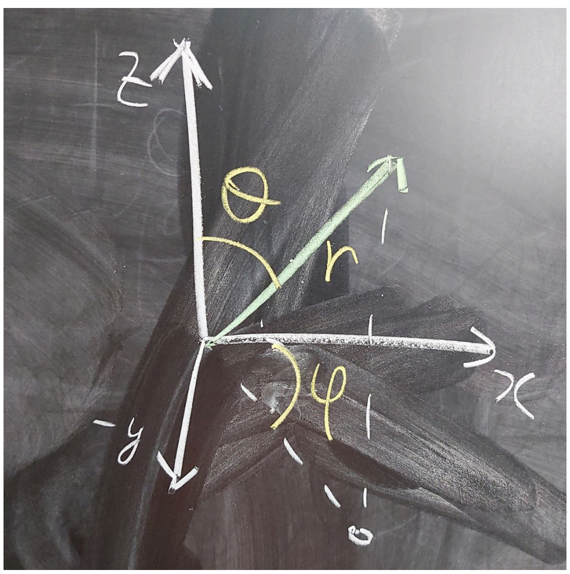
i. Compute the Jacobian of:

$$f : (0, \infty) \times [0, 2\pi) \times (-\infty, \infty) \to \mathbb{R}^3$$
$$f(\rho, \phi, z) = \begin{pmatrix} \rho \cos\phi & \rho \sin\phi & z \end{pmatrix}$$

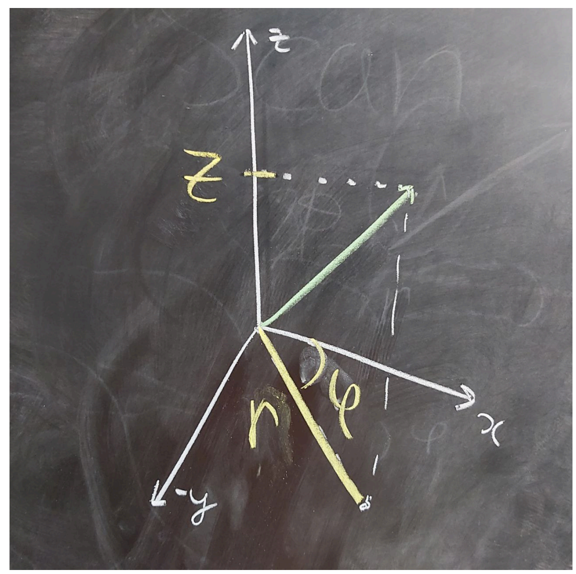
ii. Compute the volume of a cylinder of height $h$ and radius $R$ and confirm the change of variables theorem gives you what you expect

iii. Show that the measure $dx\, dy\, dz$ in cylindrical coordinates becomes $\rho\, d\rho\, d\phi\, dz$

iv. Compute the volume of a cylinder of height $h$ and radius $R$ with hollow core of radius $a$ (remove a cylinder of radius $a$ on the inside.



(a) Spherical Coordinates



(b) Spherical Coordinates

Figure 1: Diagrams for coordinate mappings

II) **More Practice**. Compute the Jacobian of the following functions:

- $$f(x, y, z) = x^2 - 3yz + z^3$$
- $$f(x, y) = \begin{pmatrix} -yx & x^2 \end{pmatrix}$$
- $$f(x, y) = \begin{pmatrix} yz & xz & xy \end{pmatrix}$$

III) **Inverse Function Theorem** Below we will prove a part of the inverse function theorem. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a nice function. Suppose the Jacobian $E = (Df)(p)$ at point $p$ is invertible. Let's show that in some small neighbourhood around $p$, the function $f$ is invertible. First some setup

We define the following:

$$|A| = \max(|E_{ij}|)$$

$$|x| = \max(|x_j|)$$

i) Show that $|x_0 - x_1| \leq 2\alpha|E \cdot x_0 - E \cdot x_1|$ where $\alpha$ is some constant that depends on $n$. (*Hint: Use the fact that* $E^{-1}E = I$

ii) Define $H(x) = f(x) - E \cdot x$. Show that $(DH)(p) = 0$

iii) Argue that exists a cube of size $\varepsilon$ such that $|(DH)(x)| \leq \frac{\alpha}{n}$. What condition should $DH$ satisfy for this to work

iv) Apply the mean value theorem to each component of $H_i$ and bound $|H_i(x_0) - H_i(x_1)| \leq \alpha|x_0 - x_1$

v) Use the fact above and the very first point, to conclude that $|f(x_1) - f(x_0)| \geq \alpha|x_1 - x_0|$

vi) Why does the above imply that $f$ is bijective into some open cube around $f(p)$