0.1 Quotient of Upper Half-plane by Congruence Subgroups of $\mathbf{SL}_2(\mathbb{Z})$

0.1.1 Introduction

This project focused on examining the quotient spaces generated by the group action of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ on the upper half of the complex plane. We begin by defining required group theoretical structures in the upper half-plane and $\mathrm{SL}_2(\mathbb{Z})$. Then the appropriate quotient space along with a visual representation (Fundamental Domains) are introduced. From this, a general algorithm by H.A.Verrill for computing and drawing such a representation is introduced and intuitively motivated. Using a mathematica package by Kainberger, these Fundamental domains are drawn and endowed with simple triangulations, then used to identify the resulting space as either a sphere, n-torus or n-projective space. Finally, the relation to modular forms is motivated.

0.1.2 Definitions

The following definitions are modified from (Diamond and Shurman, 2005).

0.1.2.1 General Group Theory

Definition 1 *Let G be a group and H a subgroup. Given a $g \in G$ define the right coset Hg to be the following set:*

$$Hg = \{a \mid a \in \mathrm{SL}_2(\mathbb{Z}) \ : \ a = hg, \ h \in H\}$$

Definition 2 *Given a group G and a subgroup H, using cosets we can now define an equivalence relation $\sim_G$ on the on G where $x \sim_G y$ iff $x \in Hy$*

By applying inverses from H to the right of x and/or y it is easy to verify that this is a well defined equivalence relation.

Definition 3 *Given a coset Hg, any element y in Hg is called a coset representative and can be used to generate Hg as $Hg = Hy$*

Definition 4 *Given a group G and its subgroup H, define the index $[G : H]$ of H in G, as the number of distinct right cosets.*

0.1.2.2 Special linear group of degree 2

Definition 5 *Let the special linear group of degree 2 over the integers, $SL_2(\mathbb{Z})$ be the set of 2×2 matrices with integer entries, determinant 1 with the matrix multiplication operation.*

Notice that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the following two matrices T and S:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Definition 6 *Let the Upper half-plane $\mathcal{H}$ be the following subset of the complex plane with subspace topology*

$$\mathcal{H} = \{z \mid \mathrm{Im}(z) > 0, \ \ z \in \mathbb{C}\}$$

Definition 7 *Define a left group action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ (fractional linear transformation) for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$ as:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

0.1.2.3 Congruence Subgroups

Definition 8 *For any $N \in \mathbb{N}$ the principal congruence subgroup $\Gamma(N)$ of level N, is a subgroup of $SL_2(\mathbb{Z})$ such that:*

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \pmod N \right\}$$

Definition 9 *A congruence subgroup Γ of level N is any subgroup such that $\exists N \in \mathbb{N}$ with $\Gamma\,(N) \subseteq \Gamma$*

Definition 10 *An important congruence subgroup is $\Gamma_0\,(N)$ with:*

$$\Gamma_0\,(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \pmod N \right\}$$

Remark:

Principal congruence subgroups $\Gamma\,(N)$ have finite index $\Rightarrow$ there are finitely many right cosets $\Rightarrow$ $\exists$ finitely many matrices (coset representatives) $M_1, ..., M_k$ with:

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma(N)M_1 \sqcup ... \sqcup \Gamma(N)M_k$$

Since by definition for any congruence subgroup Γ have that $\exists N \in \mathbb{N}$: $\Gamma(N) \subset \Gamma$, get that all congruence subgroups are also of finite index.

Note that this choice of M_i is not unique.

0.1.3 Quotient of Upper Half-plane by Congruence Subgroup

0.1.3.1 Quotient by Congruence Group

Given a subgroup $H \leqslant \mathrm{SL}_2(\mathbb{Z})$, we can use fractional linear transformations to define an equivalence relation on $\mathcal{H}$. Using this equivalence relation, then we will construct a quotient space of $\mathcal{H}$ (Diamond and Shurman, 2005).

Definition 11 *For any $x, y \in SL_2(\mathbb{Z})$ $x \sim_\Gamma y$ iff $\exists\, \gamma \in \Gamma$ such that: $x = \gamma y$, where γ acts on y by the fractional linear transformation.*

This gives rise to a natural quotient map:

$$q_\Gamma : \mathcal{H} \to \mathcal{H} \big/ \sim_\Gamma$$

It is useful to have a visual representation of a quotient (for example the representation of a torus as a square with opposite identified sides). For this purpose, we give the following definition:

Definition 12 *The Fundamental Domain of a subgroup Γ of $SL_2(\mathbb{Z})$, is a connected subset $\mathcal{D} \subset \mathcal{H}$ with the following two properties*

1. For any $x \in \mathcal{H}$, $\exists\, z \in \mathcal{D}$ with $x \sim_\Gamma z$
2. For any $x, y \in \mathcal{D}$, x is not similar to y under the relation $\sim_\Gamma$

*We will later modify the Fundamental Domain of Γ to include a point at infinity and continue to study the topological space described by that region. It is however more important to first find a way to explicitly write down a Fundamental domain given some subgroup Γ, and we will return to the idea of modifying the space later.

A key example of definition (12) is $\mathcal{F}$, the following fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$.

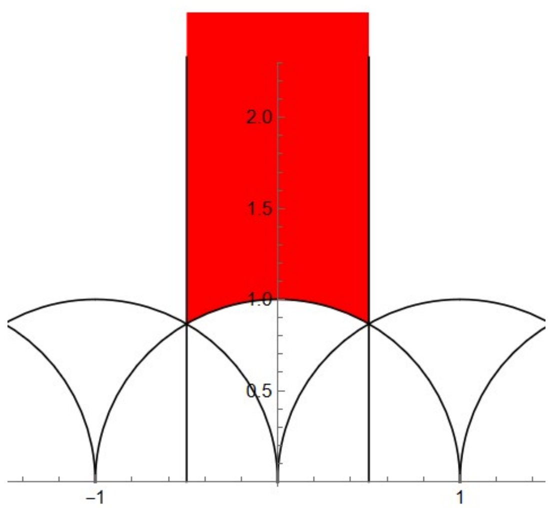

Figure 1: $\mathcal{F}$ - region shaded in red. Image generated by the "DrawFunDoms" mathematica package (Kainberger, 2021)

This Fundomental domain is given by (Kainberger, 2021):

$$\mathcal{F} = \left\{ z \in \mathcal{H} \mid (\mathrm{Re}(z), \mathrm{Im}(z)) \in \left[-\frac{1}{2}, 0\right] \times \mathbb{R}^2 - \{p \in \mathbb{R}^2 \mid |p| \leq 1\} \right\} \bigcup$$
$$\bigcup \left\{ z \in \mathcal{H} \mid (\mathrm{Re}(z), \mathrm{Im}(z)) \in \left(0, \frac{1}{2}\right) \times \mathbb{R}^2 - \{p \in \mathbb{R}^2 \mid |p| < 1\} \right\}$$

Note that the above depiction of the fundamental domain, is intuitively well explained by the fact that matrices T and S generate $\mathrm{SL}_2(\mathbb{Z})$.

Matrix T acts on $\tau \in \mathcal{H}$, by shifting point τ by 1 to the right. Due to this, lines $\mathrm{Re}(z) = -\dfrac{1}{2}$ and $\mathrm{Re}(z) = \dfrac{1}{2}$ are identified (hence the line at $\frac{1}{2}$ is left out of the definition of $\mathcal{F}$).

Matrix S acts on $\tau \in \mathcal{H}$ by sending it in/out of the unit disk centered at (0,0) (can be verified by computing relative magnitude of vector before and after transformation). Note that for a point on the unit circle $z = a + bi$, $z \cdot \overline{z} = 1 \Rightarrow \overline{z} = \dfrac{1}{z} \Rightarrow$ opposite halves of the circle are identified (Figure (1)). The definition of $\mathcal{F}$ therefore removes one of the halves of the circle.

Hence, the unit disk is not part of the fundamental domain, as all of its points are sent "out" of it by S and then can be shifted back into $\mathcal{F}$ by some iterations of T.

0.1.4 Determining the Fundamental Domain for a given group Γ

Now that we know $\mathcal{F}$, using the finite index property of congruence subgroups, we can find a way to express the fundamental domain of a given congruence subgroup Γ through some transformations on $\mathcal{F}$ (Kainberger, 2021).

By definition of the fundamental domain, recall that:

$$\mathcal{H} = \mathrm{SL}_2(\mathbb{Z})\mathcal{F} \tag{1}$$

Where $\mathrm{SL}_2(\mathbb{Z})\mathcal{F}$ denotes the union of orbits of all $x \in \mathcal{F}$.

At the same time since Γ is a finite index group $\exists$ $M_1, ..., M_k$ with:

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma M_1 \sqcup ... \sqcup \Gamma M_k \tag{2}$$

By substituting equation (2) into (1) get that:

$$\mathcal{H} = (\Gamma M_1 \sqcup ... \sqcup \Gamma M_k)\mathcal{F} \tag{3}$$

By double inclusion, it is easy to show that from the above statement it follows:

$$\mathcal{H} = \Gamma(M_1\mathcal{F} \sqcup ... \sqcup M_k\mathcal{F}) \tag{4}$$

Notice that the sums must remain disjoint, as if:

$M_i\mathcal{F} \cap M_j\mathcal{F} \neq 0 \Rightarrow$ $\exists$ $x, y \in \mathcal{F}$ with:

$M_1x = M_2y \Rightarrow M_2^{-1}M_1x = y$. Note that $M = M_2^{-1}M_1 \in \mathrm{SL}_2(\mathbb{Z}) \Rightarrow Mx = y$.

This is however a contradiction to the definition of the fundamental domain $\mathcal{F}$ as a set which does not contain two points equivalent to each other under $\sim_{\mathrm{SL}_2(\mathbb{Z})}$.

From equation (4), we can then claim that the following set $\mathcal{D}_\Gamma$ is a fundamental domain of Γ is:

$$\mathcal{D}_\Gamma = M_1\mathcal{F} \sqcup ... \sqcup M_k\mathcal{F} \tag{5}$$

Similarly as above, we can show that no two distinct points of $\mathcal{D}_\Gamma$ are equivalent due to $\sim_\Gamma$.

We now have that equation (5) reduces the problem of finding a fundomental domain of Γ, to finding a list $M_1, ..., M_k$ that ensure that $\mathcal{D}_\Gamma$ is connected. This is done with the following algorithm.

0.1.5 Algorithm for Choosing Valid Coset Representatives

Since any congruence subgroup has finite index (which is generally known for common congruence subgroups such as $\Gamma(N)$ or $\Gamma_0(N)$), we need to choose only finitely many coset-representatives. We will therefore construct our list $M_1, ..., M_k$ of matrices in $\mathrm{SL}_2(\mathbb{Z})$ for a given Γ inductively (Verrill, 2001).

0.1.5.1 Required Conditions

Recall that the resulting fundamental domain must have a point equivalent to any other point in $\mathcal{H}$ under $\sim_\Gamma$, must not contain any mutually equivalent points under $\sim_\Gamma$ and must be connected.

The first condition is already satisfied for $\mathcal{D}_\Gamma$ as shown above, so now determine conditions on M_i s for $\mathcal{D}_\Gamma$ to satisfy the above.

To make sure no two points in our fundamental domain are equivalent under $\sim_\Gamma$, as noted earlier in equation (3) we can simply choose the list such that ΓM_i form a disjoint union.

We therefore require that each coset representative generates a distinct right-coset (no two matrices are equivalent under relation from definition (2)) $\Rightarrow \forall i \neq j \; \Gamma M_i \neq \Gamma M_j \Rightarrow \; \nexists \, \gamma \in \Gamma$ with $\gamma M_i = M_j$

Note that the condition above can be rewritten as:

$$M_j M_i^{-1} \notin \Gamma \quad \forall i \neq j \tag{6}$$

Now we must guarantee that the union of all such domains is indeed connected. This condition can be motivated by considering the construction of $\mathcal{F}$ (Figure 1).

As mentioned earlier, the action of the matrix T and T^{-1} shifts the fundamental domain $\mathcal{F}$ left/right $\Rightarrow$ any $T^n\mathcal{F} \sqcup T^{n\pm1}\mathcal{F}$ is path-connected $\Rightarrow$ connected (see Figure (1)).

The same can be said about the action of S on $\mathcal{F}$, as S maps $\mathcal{F}$ to a circle, adjacent to $\mathcal{F}$ along a boundary at "the bottom" of $\mathcal{F}$ (see Figure 1) $\Rightarrow S^i\mathcal{F} \sqcup S_{i+1}\mathcal{F}$ is path connected $\Rightarrow$ connected.

We can therefore conclude, that given some initial connected region $M_1\mathcal{F}$, we can apply matrices S, T and T^{-1} to M_1, to generate a new coset representative M_2 such that $M_1\mathcal{F} \sqcup M_2\mathcal{F}$ is still connected (Verrill, 2001). We therefore get the restriction that:

$$\text{Each } M_i \text{is generated from some initial } M_1 \text{ by applying some sequence of } S, T, T^{-1} \tag{7}$$

0.1.5.2 Algorithm Outline

This algorithm is taken from (Verrill, 2001). Let Z_i be the list of already selected matrices at the beginning of step $i+1$ (resulting product of step i), with $Z_1 = \emptyset$.

Step 1 Select $M_1 = I$

Step i+1 Select any $M_p \in Z_i$. Multiply it by T and check if the resulting matrix TM_p is equivalent to any other matrix in Z_i under $\sim_\Gamma$ (check using condition (6)). If it is not equivalent to any other matrix, then add it to Z_i and begin the next step.

If it is, then try multiplying by T^{-1} and S.

If all resulting matrices after multiplication by T, T^{-1}, S are equivalent to some other matrix $M_j \in Z_i$, then choose some other matrix $M_{p'} \in Z_i$ and repeat the process.

Algorithm Termination If all possible products by T, T^{-1}, S of all matrices are equivalent to some other matrix in Z_i or if $|Z_i|$ = index of Γ, then Z_i is the desired list of coset representatives $M_1, ..., M_k$.

0.1.5.3 Some remarks

We select $M_1 = I$ to get an image "centered" at y axis, that includes $\mathcal{F}$ (Verrill, 2001).

In order to preserve the "nice" location of $\mathcal{D}_\Gamma$, in each step $i+1$ it makes sense to select $M_p \in Z_i$ s such that M_p is a product of as few T, T^{-1}, S as possible, and is in some sense the "closest" to $I \Rightarrow TM_p\mathcal{F}$ is "closest" to $\mathcal{F} \Rightarrow$ closer to y-axis (Verrill, 2001).

The resulting fundamental domain $\mathcal{D}_\Gamma$ of Γ can then be drawn on the complex plane using the "DrawFunDoms" mathematica package (Kainberger, 2021).

0.1.6 Identifying the Resulting Quotient Space

Recall that the fundamental domain $\mathcal{D}_\Gamma$ was originally motivated to be the visual representation of the quotient space $\mathcal{H} / \sim_\Gamma$.

We will now modify the quotient by appending a point at infinity to the upper-half plane, making the resulting space compact. Using our image of $\mathcal{D}_\Gamma$, we would then be able to draw triangulations and identify the space it represents due to the following theorem (Lee, 2010):

Theorem 1 *The Euler characteristic of a standard surface presentation (compact) is equal to*

a) 2 for the sphere

b) 2-2n for the connected sum of n-tori

c) 2-n for connected sum of n projective spaces

Recall that (Lee, 2010)

Definition 13 *The Euler Charateristic $\chi(X)$ where X is a topological space represented by some 2-dimensional representation is given by*

$$\chi(X) = v - e + f$$

Where v, e, f are the numbers of vertices,edges and faces respectively of any valid triangulation of the representation (each triangle is made of 3 distinct points and vertices).

0.1.6.1 Modifying the Quotient

In order to guarantee finite triangulations of the resulting space, we will modify the fundamental domain to include a "point at infinity" (denoted ∞):

$$\mathcal{H}^* = \mathcal{H} \cup \{\infty\} \cup \mathbb{Q}$$

.

Where ∞ has the following properties (Diamond and Shurman, 2005):

For any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ have:

If $a = c = 0 \Rightarrow \gamma\infty = \frac{c}{d}$
If $a = 0$ $c \neq 0 \Rightarrow \gamma\infty = 0$
If $a \neq 0$ $c = 0 \Rightarrow \gamma\infty = \infty$ (note both c and d cannot be 0, as otherwise matrix non-invertible).
If $a \neq 0\, c \neq 0$ $\gamma\infty = \frac{a}{c}$

Note that from the above definition, all rational points in $\mathcal{H}^*$ are equivalent to $\infty \Rightarrow \infty \in \mathcal{F}^* \Rightarrow \infty \in \mathcal{D}^*_{\Gamma}$ and no rational point is in $\mathcal{D}^*_{\Gamma}$. Hence:

$$\mathcal{D}^*_{\Gamma} = \mathcal{D}_{\Gamma} \cup \{\infty\}$$

.

To accommodate for the added point, define the topology of $\mathcal{D}^*_{\Gamma}$ in the following way:

A subset $U \subseteq \mathcal{D}^*_{\Gamma}$ is open in $\mathcal{D}^*_{\Gamma}$ iff:

1. $U \subseteq \mathcal{D}_{\Gamma}$ is open in $\mathcal{D}_{\Gamma}$
2. $\infty \in U$ and $\mathcal{D}^*_{\Gamma} - U$ is compact in $\mathcal{D}_{\Gamma}$

Recall that $\mathcal{D}_{\Gamma}$ is also Hausdorff $\Rightarrow$ by problem 4 of the MAT327 Summer 2022 Midterm, get that $\mathcal{D}^*_{\Gamma}$ is compact, preserves the open sets of $\mathbb{R}^2$ (consider open set of all z with $\mathrm{Im}(z) > c$ for some $c \in \mathbb{R}$. The conjugate is compact (as always bounded in the $\mathrm{Re}(z)$ direction, due to translational properties of congruence subgroup matrices $\Rightarrow$ such a set is open) and is therefore a compact 2-dimensional representation of the quotient space.

Hence theorem (1) can be used to identify the space given by $\mathcal{D}^*_{\Gamma}$.

0.1.7 Examples of Triangulations

Now let's consider a few examples of such spaces represented by $\mathcal{D}^*_{\Gamma}$, draw their triangulation and using theorem (1), identify them as either a sphere, n-torus or n-projective space.

0.1.7.1 Triangulation of the Fundamental Domain of $\mathbf{SL}_2(\mathbb{Z})$

Consider the following triangulation of $\mathcal{F}^*$ drawn over the Fundamental Domain generated with the 'DrawFunDoms" mathematica package written by Paul Kainberger:

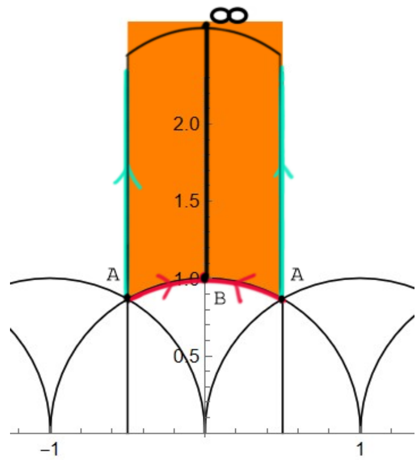

Figure 2: This triangulation is given by the two triangles made with points A, B and ∞. Note that this is a valid triangulation as A, B and ∞ are distinct points. There are however only 3 distinct edges, due to identifications discussed in section 0.1.3.1.

From the above triangulation image we get that $v = 3$ $e = 3$ $f = 2$, hence the Euler characteristic is:

$$\chi(\mathcal{F}) = v - e + f = 3 - 3 + 2 = 2$$

Using theorem (1), we conclude from the above triangulation that $\mathcal{F}^*$ a sphere.

0.1.7.2 Triangulation of Fundamental Domain of $\Gamma_0(11)$

Draw the fundamental domain of $\Gamma_0(11)$ using the "DrawFunDoms" mathematica package and add the following triangulation:

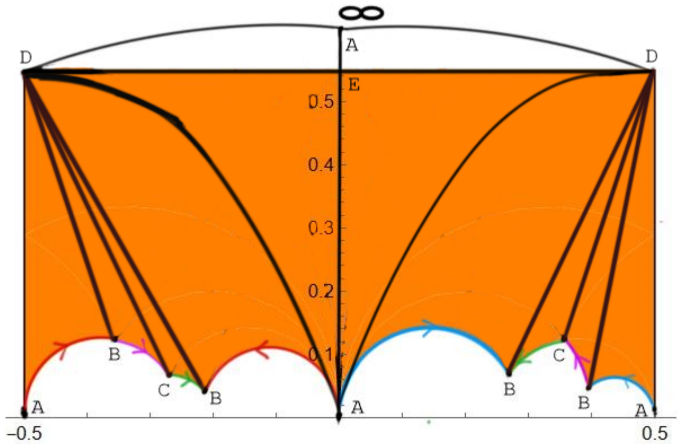

Figure 3: The identifications drawn above were determined numerically with the inverse point tool of the "DrawFunDoms" package.

From the Figure above determine that: $v = 5, e = 17, f = 12$. Now compute the Euler Characteristic to be:

$$\chi(\mathrm{D}_{\Gamma_0(11)}) = 5 - 17 + 12 = 0$$

By theorem (1), this is either a 1 torus or projective space of degree 2. However from Figure (3), it is evident that there are no twisted edge pairs $\Rightarrow$ this is a presentation of an orientable surface $\Rightarrow$ representation above gives a torus (Lee, 2010).

0.1.8 Conclusion

This project outlined a simple algorithm for specifying a valid fundamental domain on $\mathcal{H}$ of any given congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ in a way that allows for clear graphing. These images were then used to draw a representation of the quotient space $\mathcal{H} \big/ {\sim_\Gamma} \cup \{\infty\}$ and use the Euler characteristic to identify it as a sphere, torus or real-projective space.

The proposed method provides a simple way to visualize the action of a congruence subgroup Γ on $\mathcal{H}$, which is otherwise non-trivial when simply looking at the subgroup definition (see Figure (3)).

Note that the genus for orientable Riemann surfaces (as in examples above), can be directly computed from the Euler characteristic. Hence, being able to determine the type of space is very useful to studying modular forms, as the genus of the quotient $\mathcal{H} \big/ {\sim_\Gamma} \cup \{\infty\}$ is required to determine the dimension formulas for associated vector spaces (Diamond and Shurman, 2005).

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