Cosine Law

Theorem 1 *The square of one side of a triangle is equal to the sum of squares of the other sides of the triangle minus the doubled product of those two sides with the cosine of the angle between them.*

Proof

In triangle ABC let: $BC = a$, $AC = b$, $AB = c$, $BH = h$ - height , $AH = m$, $HC = n$, $\angle A = \alpha$ (Figure 1).

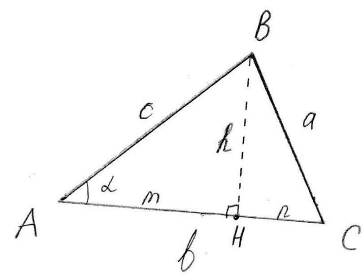

Figure 1

We need to prove that:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

Using the Pythagorean Theorem it is obvious that:

$a^2 = h^2 + n^2$

Now we just need to express h and n in terms of c, b, and α.

Recall that:

$\sin\alpha = \dfrac{h}{c}$

$h = c \cdot \sin\alpha$

Now we just need to express n through c, b and α.

$n = b - m$

Recall that:

$\cos\alpha = \dfrac{m}{c}$

$m = c \cdot \cos\alpha$

Therefore:

$n = b - c \cdot \cos\alpha$

Substitute this into the original Pythagorean Theorem statement and simplify:

$$\begin{aligned}
a^2 &= h^2 + n^2 \\
&= (c \cdot \sin\alpha)^2 + (b - c \cdot \cos\alpha)^2 \\
&= c^2 \cdot \sin^2\alpha + b^2 - 2bc\cos\alpha + c^2 \cdot \cos^2\alpha \\
&= c^2 \cdot (\sin^2\alpha + \cos^2\alpha) + b^2 - 2bc\cos\alpha \\
&= b^2 + c^2 - 2bc\cos\alpha
\end{aligned}$$

$\therefore$ QED