Derivation of the Newton Leibniz Formula

Theorem 1 *If $y = f(x)$ is continuous on $[a, b]$ and $F(x)$ is one of the antiderivatives of $y = f(x)$ on this domain, then the Newton-Leibniz formula holds true:*

$$\int_a^b f(x)dx = F_b - F_a$$

Proof

First let's introduce the concept of an integral with a variable upper bound.
If $y = f(x)$ is continuous on $[a, b]$ then for $x \in [a, b]$ the integral of the following form is a function with a variable upper bound:

$$\Phi(x) = \int_a^x f(t)dt$$

Let's find the derivative of this function:

$$\Phi'(x) = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} \tag{1}$$

Let's first expand the numerator using the properties of the definite integral.

$$\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt$$

Using Property 9 we get:

$$\begin{aligned}\int_x^{x+\Delta x} f(t)dt &= f(\xi) \cdot (x + \Delta x - x) \\ &= f(\xi) \cdot \Delta x\end{aligned}$$

Now let's substitute this back into (1):

$$\begin{aligned}\Phi'(x) &= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(\xi) \cdot \Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} f(\xi)\end{aligned}$$

Since ξ is between x and $x + \Delta x$, then if $\delta x = 0$; $\xi = x$
Hence:

$$\begin{aligned}\Phi'(x) &= \lim_{\Delta x \to 0} f(\xi) \\ &= f(x)\end{aligned}$$

Since $\Phi'(x) = f(x)$ then by the definition of the antiderivative: $\Phi'(x)$ is the antiderivative of $f(x)$. We can write the set of antiderivatives in the following way:

$$F(x) = \Phi(x) + C = \int_a^x f(t)dt + C$$

Now let's use the First Property of the definite integral:

$$F(a) = \int_a^a f(t)dt + C = C \tag{2}$$

Now let's calculate $F(b)$

$$F(b) = \int_a^b f(t)dt + C$$

Substitute (2) into the line above to get:

$$F(b) = \int_a^b f(t)dt + F(a)$$
$$\int_a^b f(t)dt = F(b) - F(a)$$

$\therefore$ QED