Expressing the Dot Product Formula in Terms of Vector Coordinates for two 3d Vectors

Suppose you have two vectors in 3D space: $\overrightarrow{OA} = \vec{a}$ with coordinates x_1, y_1, z_1 and $\overrightarrow{OB} = \vec{b}$ with coordinates x_2, y_2, z_2.

We will now prove that:
$\vec{a} \cdot \vec{b} = x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2$
Where α is the angle between $\vec{a}$ and $\vec{b}$

Proof

By definition:

$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha$

First, using Pythagorean theorem we can express the magnitudes of each vector through its coordinate points:

$|\vec{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$
$|\vec{b}| = \sqrt{x_2^2 + y_2^2 + z_2^2}$

We still need to express $\cos\alpha$ in terms of coordinates.

Consider the triangle OAB. The side AB can be represented with $\overrightarrow{AB} = \vec{c} = \vec{b} - \vec{a}$

$|\vec{c}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Using cosine law:

$$\begin{aligned}
|\vec{c}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cdot \cos\alpha \\
&= |\vec{a}|^2 + |\vec{b}|^2 - 2 \cdot \vec{a} \cdot \vec{b}
\end{aligned}$$

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= \frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 - |\vec{c}|^2) \\
&= \frac{1}{2}(x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - x_1^2 + 2x_1x_2 - x_2^2 - y_1^2 + 2y_1y_2 - y_2^2 - z_1^2 + 2z_1z_2 - z_2^2) \\
&= x_1x_2 + y_1y_2 + z_1z_2
\end{aligned}$$

$\therefore QED$