1 Relativistic Strings

1.1 The Nambu-Goto Action

Assume that space-time (configuration space), is a D-dimensional Euclidean space ($\mathbb{R}^D$). For convention we will index the coordinates starting with 0, where the x^0 coordinate is the time dimension.
This space is then equipped with a flat-space Minkowski metric. We use the -+++ metric.

$$\eta^{\nu\mu} = \begin{cases} -1 & \mu = \nu = 0 \\ 1 & \mu = \nu \neq 0 \\ 0 & \text{else} \end{cases}$$

We will now describe the trajectory of a string in this space. A classical point particle (0-dimensional object) traces out a 1-dimensional trajectory in the configuration space. We would then expect a string's trajectory to trace out a 2-dimensional surface called the **world-sheet**.
It can be parametrized via two coordinates: τ, σ. Let $X^\mu(\tau, \sigma)$ be a function the μth coordinate of points on the world-sheet. We will assume that $\sigma \in [0, \sigma_1]$ i.e. the string has finite length. Denote derivatives in τ with dots and derivatives with respect to σ with primes.

The world sheet can then be described by the following action:

$$S = \frac{1}{2\pi\alpha'} \int_{t_i}^{t_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{\left(\dot{X} \cdot X'\right)^2 - (\dot{X})^2 \left(X'\right)^2} \tag{1}$$

Note that in the equation above, norms and dot-products are taken with respect to the metric $\eta_{\nu\mu}$. The equation above is the area form of the surface. The order of terms is chosen to ensure content of square roots is positive, based on time-like separation arguments.
The constant α' is a dimension-full parameter. It corresponds to the inverse of string tensions T_0.

Varying action gives the following equations get:

$$\frac{\partial \mathcal{P}^\tau_\mu}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma_\mu}{\partial \sigma} = 0 \tag{2}$$

Where:

$$\mathcal{P}^\tau_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad \mathcal{P}^\sigma_\mu \equiv \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} \tag{3}$$

Note that these equations can be paired with boundary conditions:

$$\text{Dirichlet boundary condition:} \quad \frac{\partial X^\mu}{\partial \tau}(\tau, \sigma_*) = 0, \quad \mu \neq 0 \tag{4}$$

$$\text{Free endpoint condition:} \quad \mathcal{P}^\sigma_\mu(\tau, \sigma_*) = 0 \tag{5}$$

Where σ_* are the end points of the interval, σ is defined on.

We consider two types of string: open and closed. Open strings are parametrized by $\sigma \in [0, \pi]$ and closed strings are parametrized by $\sigma \in [0, 2\pi]$. Note that closed strings have the condition that:

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \tag{6}$$

1.2 Conserved Quantities

Note that the Euler Lagrange equations for an action with a density are given by:

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} = \frac{\partial \mathcal{L}}{\partial X^\mu} \tag{7}$$

In the case of the Nambu-Goto action in (1), see that α ranges over τ and σ, while μ ranges over the coordinates of the configuration space. Since the action does not depend on X^ν, get that $\mathcal{P}^\tau_\mu$ and $\mathcal{P}^\sigma_\mu$ form conserved world sheet currents, where the conservation law is given by equation in (2).

On this basis, we will refer to $\mathcal{P}^\tau$ as the world-sheet momentum current. We can then define the spacetime momentum of the string:

$$p_\mu(\tau) = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau,\sigma) d\sigma \tag{8}$$

Using the conservation laws for $\mathcal{P}$ and boundary conditions[1], get that p_μ is conserved with τ.
By performing contour integration over some path γ that connects $\sigma = 0$ and $\sigma = \sigma_1$ boundaries of the world sheet, can rewrite the momentum more generally as:

$$p_\mu = \int_\gamma \left(\mathcal{P}^\tau_\mu d\sigma - \mathcal{P}^\sigma_\mu d\tau\right) \tag{9}$$

Note that these conservation laws, correspond to invariance under spatial translations. The Nambu-Goto action however, is by construction Lorentz invariant. The conserved quantities for this symmetry and associated conservation law are given by:

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu \tag{10}$$

$$\frac{\partial \mathcal{M}^\tau_{\mu\nu}}{\partial \tau} + \frac{\partial \mathcal{M}^\sigma_{\mu\nu}}{\partial \sigma} = 0 \tag{11}$$

Note that this current is anti-symmetric.
Similarly as with momentum, define the following conserved charges:

$$M_{\mu\nu} = \int_\gamma \left(\mathcal{M}^\tau_{\mu\nu} d\sigma - \mathcal{M}^\sigma_{\mu\nu} d\tau\right). \tag{12}$$

1.3 Choice of Parametrization

Earlier we noted that the Nambu-Goto action is reparametrization invariant. We can exploit this, to select convenient parametrizations in which the equations of motion are very simple. Since momentum is conserved, can choose

$$n \cdot X(\tau,\sigma) = \beta\alpha'(n \cdot p)\tau \tag{13}$$

Where $\beta = 1$ for open and $\beta = 2$ for closed strings and n is some D-dimensional vector.
It is also possible to choose a parametrization for σ such that:

$$n \cdot p = \frac{2\pi}{\beta} n \cdot \mathcal{P}^\tau \tag{14}$$

$$n \cdot \mathcal{P}^\sigma = 0 \tag{15}$$

Under these choices give us the following set of constraints and transforms equations of motions to be:

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} X'^\mu, \quad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu \tag{16a}$$

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0 \tag{16b}$$

1.4 Solutions to Wave Equation

We will now write down some solutions for the wave equations above for open and closed strings. Assume a space-filling D-brane (i.e. satisfy free boundary conditions as in (5)).

[1] We get p_μ is conserved, only for directions that have dirichlet boundary conditions

1.4.1 Open Strings

A general solution to the wave equation can be written as:

$$X^\mu(\tau,\sigma) = \frac{1}{2}\left(f^\mu(\tau+\sigma) + g^\mu(\tau-\sigma)\right)$$

Due to free-boundary conditions in all coordinates, get that f and g can differ by atmost a constant. The solution can then be rewritten as

$$X^\mu = \frac{1}{2}\left(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\right)$$

Looking at the derivative of the function above with respect to σ, notice that $f^{\mu\prime}$ is 2π periodic. Can write solution as fourier expansion. By integrating and rewriting as complex coefficients get the following form for the solution:

$$X^\mu(\tau,\sigma) = x_0^\mu + 2\alpha' p^\mu \tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^{\mu *}e^{in\tau} - a_n^\mu e^{-in\tau}\right)\frac{\cos n\sigma}{\sqrt{n}} \tag{17}$$

Also note that:

$$\dot{X}^\mu \pm X^{\mu'} = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau\pm\sigma)} \tag{18}$$

1.4.2 Closed Strings

Since closed strings are 2π periodic in sigma, can write down solution as:

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma) \tag{19}$$

After change of variables and taking derivatives of both sides, get that derivatives of X_L and X_R are also 2π periodic. Writing down fourier expansions of each, integrating and adding; after some rearrangement get:

$$X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu e^{in\sigma} + \bar{\alpha}_n^\mu e^{-in\sigma}\right) \tag{20}$$

From the periodicity condition and the fact that X^μ must be real get that:

$$\bar{\alpha}_0^\mu = \alpha_0^\mu \quad \alpha_{-n}^\mu = (\alpha_n^\mu)^* \tag{21}$$

Recalling the definition of momentum and simplification in (16a) have:

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \tag{22}$$

Using the expansion above, also make the following computations:

$$\dot{X}^\mu + X^{\mu\prime} = 2X_L^{\mu'}(\tau+\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \tag{23}$$

$$\dot{X}^\mu - X^{\mu\prime} = 2X_R^{\mu'}(\tau-\sigma) = \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in(\tau-\sigma)} \tag{24}$$

1.5 Light cone coordinates and gauge

Light cone gauge features a change in coordinates which is in general not Lorentz invariant:

$$X^+ = \frac{X^0 + X^1}{\sqrt{2}} \quad X^- = \frac{X^0 + X^1}{\sqrt{2}}$$

Additionally, we choose the following gauge:

$$n = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0\right) \tag{25}$$

Note that also:

$$\dot{X}^- \pm X^{-\prime} = \frac{1}{\beta\alpha'}\frac{1}{2p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2 \tag{26}$$

Note that I and J indices than run from 2 to $D-1$ (first two coordinates taken to be X^+ and X^-)

2 Quantization

Interpreting $\mathcal{P}^{\tau J}$ as momentum current, postulate the following commutation relations:

$$\left[X^{I}(\tau,\sigma),\mathcal{P}^{\tau J}\left(\tau,\sigma'\right)\right]=i\delta\left(\sigma-\sigma'\right)\eta^{IJ} \tag{27}$$

$$\left[x_0^I,p^J\right]=i\nu^{IJ} \tag{28}$$

2.0.1 Open Strings

Taking coefficients in Fourier expansion of solution in (17) to be operators, using commutation relations above conclude that:

$$\left[\alpha_m^I,\alpha_n^J\right]=m\eta^{IJ}\delta_{m+n,0}$$

Use this result, to define creation and annihilation operators that satisfy appropriate commutation relations. They will later be used to build the state space.

$$a_n^I=\frac{\alpha_n^I}{\sqrt{n}}\quad\text{ and }\quad a_n^{I\dagger}=\alpha_{-n}\frac{I}{\sqrt{n}},\quad n\geq 1 \tag{29}$$

These satisfy the expected commutation relations of:

$$\left[a_m^I,a_n^{J\dagger}\right]=\delta_{m,n}\eta^{IJ}$$

2.0.2 Critical Dimension

Define the following operator:

$$L_0^{\perp}\equiv\frac{1}{2}\alpha_0^I\alpha_0^I+\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I \tag{30}$$

Note that it can be used to express the following sum:

$$\frac{1}{2}\sum_{p\in\mathbb{Z}}\alpha_{-p}^I\alpha_p^I=\frac{1}{2}\alpha_0^I\alpha_0^I+\frac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I+a=L_0^{\perp}+a$$

Where a is the ordering constant.

Now turn our attention to operators of $M_{\mu\nu}$ as defined in 12. We expect the commutation relation:

$$\left[M^{-I},M^{-J}\right]=0$$

A non-trivial computation of this, gives us:

$$\left[M^{-I},M^{-J}\right]=-\frac{1}{\alpha' p^{+2}}\sum_{m=1}^{\infty}\left(\alpha_{-m}^I\alpha_m^J-\alpha_{-m}^J\alpha_m^I\right)\left\{m\left[1-\frac{1}{24}(D-2)\right]+\frac{1}{m}\left[\frac{1}{24}(D-2)+a\right]\right\}=0 \tag{31}$$

From this we can conclude that $D=26$ and $a=-1$

2.0.3 Closed Strings

For Fourier coefficients in solution for closed strings (20) compute:

$$\left[\bar{\alpha}_m^I,\bar{\alpha}_n^J\right]=m\delta_{m+n,0}\eta^{IJ} \tag{32}$$

$$\left[\alpha_m^I,\alpha_n^J\right]=m\delta_{m+n,0}\eta^{IJ} \tag{33}$$

$$\left[\alpha_m^I,\bar{\alpha}_n^J\right]=0. \tag{34}$$

Define the creation and annihilation operators as a rescaling of α s:

$$a_n^I = \frac{\alpha_n^I}{\sqrt{n}} \quad \text{and} \quad a_n^{I\dagger} = \frac{\alpha_{-n}^I}{\sqrt{n}}, \quad n \geq 1, \tag{35}$$

$$\bar{a}_n^I = \frac{\bar{\alpha}_n^I}{\sqrt{n}} \quad \text{and} \quad \bar{a}_n^{I\dagger} = \frac{\bar{\alpha}_{-n}^I}{\sqrt{n}} \quad n \geq 1. \tag{36}$$

We define Virasoro operators following the pattern in equation (9.79):

$$L_n^\perp = \frac{1}{2} \sum_{p \in Z} \alpha_p^I \alpha_{n-p}^I \tag{37}$$

$$\bar{L}_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{n-p}^I \tag{38}$$

These come from:

$$\left(\dot{X}' + X''\right)^2 = 4\alpha' \sum_{n \in \mathbb{Z}} \left(\frac{1}{2} \sum_{p \in \mathbb{Z}} \bar{\alpha}_p' \bar{\alpha}_{n-p}' \right) e^{-in(\mathrm{r}+\sigma)} \equiv 4\alpha' \sum_{n \in \mathbb{Z}} \bar{L}_n^\perp e^{-in(\mathrm{r}+\sigma)} \tag{39}$$

$$\left(\dot{X}' - X'\right)^2 = 4\alpha' \sum_{n \in \mathbb{Z}} \left(\frac{1}{2} \sum_{p \in Z} \alpha_p' \alpha_{n-p}' \right) e^{-in(\tau-\sigma)} \equiv 4\alpha' \sum_{n \in Z} L_n^\perp e^{-in(\tau-\sigma)} \tag{40}$$

Writing the left hand side of using the expansion above, but also the familiar fourier expansion and matching coefficients get:

$$\sqrt{2\alpha'} \bar{\alpha}_n^- = \frac{2}{p^+} \bar{L}_n^\perp, \quad \sqrt{2\alpha'} \alpha_n^- = \frac{2}{p^+} L_n^\perp. \tag{41}$$

However recall that by periodicity, we concluded that $\alpha_0^- = \bar{\alpha}_0^-$ We therefore conclude that

$$L_0^\perp = \bar{L}_0^\perp \tag{42}$$

Using above, conclude:

$$\sum_{p \in \mathbb{Z}} \alpha_p^I \alpha_{-p}^I = \alpha_0^I \alpha_0^I + \sum_{p=1}^{\infty} 2 \cdot \alpha_p \alpha_{-p} - [\alpha_p, \alpha_{-p}] = \bar{\alpha}_0^I \bar{\alpha}_0^I + \sum_{p=1}^{\infty} 2 \cdot \bar{\alpha}_p \bar{\alpha}_{-p} - [\bar{\alpha}_p, \bar{\alpha}_{-p}] = \sum_{p \in \mathbb{Z}} \bar{\alpha}_p^I \bar{\alpha}_{-p}^I$$

We give these number operators names and rewrite them in terms of creation/annihilation operators:

$$\bar{N}^\perp \equiv \sum_{n=1}^{\infty} n \bar{a}_n^{I\dagger} \bar{a}_n^I \quad N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I\dagger} a_n^I \tag{43}$$

We therefore get the following level matching condition that:

$$N^\perp = \bar{N}^\perp \tag{44}$$

3 Constructing State Space

3.1 Open Strings

Define a number operator and compute the mass to be:

$$N^\perp \equiv \sum_{n=1}^{\infty} n a_n^{I^\dagger} a_n^I, \quad M^2 = \frac{1}{\alpha'} \left(-1 + N^\perp\right) \tag{45}$$

Take ground state:

$$\left| p^+, \vec{p}_T \right\rangle$$

Postulate that annihilation operator send ground states to 0. Build up state space with creation operators as:

$$|\lambda\rangle = \prod_{n=1}^{\infty}\prod_{I=2}^{25}\left(a_n^{I^\dagger}\right)^{\lambda_{n,I}}|p^+,\vec{p}_T\rangle$$

Where $\lambda_{n,I}$ is the number of times the creation operator with matching indices is applied to the ground state.

Consider the following sample computation of $N^\perp$ applied to a state of 1 creation operator:

$$\begin{aligned} N^\perp a_m^{I\dagger}|p^+,\vec{p}_T\rangle &= m a_m^{I\dagger}a_m^I a_m^{I\dagger}|p^+,\vec{p}_T\rangle \\ &= \left(m a_m^{I\dagger}a_m^{I\dagger}a_m^I + m a_m^{I\dagger}\left[a_m^I, a_m^{I\dagger}\right]\right)|p^+,\vec{p}_T\rangle \\ &= m a_m^{I\dagger}|p^+,\vec{p}_T\rangle \end{aligned} \tag{46}$$

Get that the $N^\perp$ operator, adds up the bottom "mode" indices of creation operators.

From the computation above and the formula for mass in (45), we see that the ground state has negative mass. This is named the **tachyon**.
There are also 24, massless basis states, corresponding to photons, given by:

$$a_1^{I^\dagger}|p^+,\vec{p}_T\rangle$$

3.2 Closed Strings

Calculate M^2

$$M^2 = -p^2 = 2p^+p^- - p^Ip' = \frac{2}{\alpha'}\left(L_0^\perp + \bar{L}_0^\perp - 2\right) - p^Ip^I. \tag{47}$$

$$m^2 = N^\perp + \bar{N}^\perp - 2 \tag{48}$$

Now we are finally ready to construct the state space using our creation operators.
The most general possible state we could think of would be given by:

$$|\,\lambda,\bar{\lambda}\rangle = \left[\prod_{n=1}^{\infty}\prod_{l=2}^{25}\left(a_n^{I\dagger}\right)^{\lambda_{n,I}}\right]\times\left[\prod_{m=1}^{\infty}\prod_{J=2}^{25}\left(\bar{a}_m^{J\dagger}\right)^{\bar{\lambda}_{m,j}}\right]|p^+,\vec{p}_T\rangle \tag{49}$$

We however, need to make sure we chose states that satisfy the condition:

$$\bar{N}^\perp = N^\perp \tag{50}$$

Operators count the number of barred and unbarred creation operators.
We consider states w. finite number of creation operators. Specifically interesting are cases with $N^\perp = \bar{N}^\perp = 1$. Have the form:

$$a_1^{I^\dagger}\bar{a}_1^{J\dagger}|p^+,\vec{p}_T\rangle$$

We can write down a general state:

$$\sum_{I,J} R_{IJ}a_1^{I^\dagger}\bar{a}_1^{J\dagger}|p^+,\vec{p}_T\rangle. \tag{51}$$

Now can decompose $R_{I,J}$ into 3 matrices:

$$\sum_{I,I}\hat{S}_{I,I}a_1^{\prime\dagger}\bar{a}_1^{\prime\dagger}|p^+,\vec{p}_T\rangle \tag{52}$$

$$\sum_{I,J}A_{IJ}a_1^{\prime\dagger}\bar{a}_1^{\prime\dagger}|p^+,\vec{p}_T\rangle \tag{53}$$

$$S'a_1^{\prime\dagger}\bar{a}_1^{\prime\dagger}|p^+,\vec{p}_T\rangle \tag{54}$$

Where $\hat{S}$ - traceless symmetric part, $A_{I,J}$ - antisymmetric part, S' - diagonal part.
In this case, the basis vectors given by:

- $\hat{S}$ in (52) correspond to graviton states;
- A_{IJ} in (53) correspond to the Kalb-Ramond field
- S' in (54) is the dilaton state

3.3 Closed Strings on Orbifolds

3.3.1 Closed Strings on $\mathbb{R}^1/\mathbb{Z}_2$ Orbifold

Consider a D- dimensional space, where the first $D-1$ coordinates are Euclidean and Dth coordinate is in the space:

$$\mathbb{R}^1/\sim$$

noindent Where $\sim$ is given by: $x \sim -x$
In this case, proceed similarly, but have new set of ground states: superpositions of states where momentum in last coordinate is positive and negative. i.e. let $\vec{p}$ denote momenutum in all I coordinates except last. Then the allowed basis vectors are given by:

$$\begin{array}{cc} \left|p^+, \vec{p}, p\right\rangle + \left|p^+, \vec{p}, -p\right\rangle & \left|p^+, \vec{p}, p\right\rangle - \left|p^+, \vec{p}, -p\right\rangle \\ \left|p^+, \vec{p}, p\right\rangle - \left|p^+, \vec{p}, -p\right\rangle & \left|p^+, \vec{p}, p\right\rangle + \left|p^+, \vec{p}, -p\right\rangle . \end{array}$$

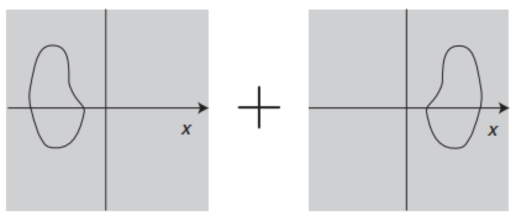

Figure 1: Identification $x \sim -x$ [1, p.297]

3.3.2 Twisted Sector

Note in new quotient space, have closed strings which are open in the original covering space (as x and $-x$ are now the same point; see figure below):

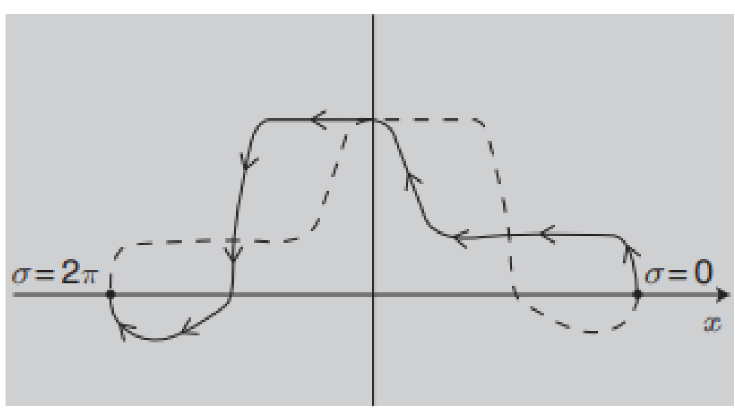

Figure 2: Twisted string drawn in Covering Space [1, p. 299]

This gives rise to a new set of states. To develop this, need to re-solve the wave equations for strings with condition on the last coordinate:

$$X(\tau, \sigma + 2\pi) = -X(\tau, \sigma)$$

In this case, get half-integer Fourier coefficients for the last coordinate. This changes the Number and Mass operators to be:

$$\bar{N}^{\perp} = \sum_{p=1}^{\infty} \bar{\alpha}^i_{-p} \bar{\alpha}^i_p + \sum_{k \in \mathbb{Z}^+_{\text{odd}}} \bar{\alpha}_{-\frac{k}{2}} \bar{\alpha}_{\frac{k}{2}} \quad N^{\perp} = \sum_{p=1}^{\infty} \alpha^i_{-p} \alpha^i_p + \sum_{k \in \mathbb{Z}^+_{\text{odd}}} \alpha_{-\frac{k}{2}} \alpha_{\frac{k}{2}} \tag{55}$$

$$\frac{1}{2}\alpha' M^2 = N^{\perp} + \bar{N}^{\perp} - \frac{15}{8} \tag{56}$$

Notice that once again have tachyonic ground state.

4 Field Theories

Now let's see how the string theory states derived above, are equivalent to those obtained from quantization of already known field theories.

4.1 Scalar Fields

Consider the following action for a scalar field:

$$S = \int d^D x \left(-\frac{1}{2} \eta^{\mu v} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{57}$$

Varying this action, yields the following equation:

$$\left(\partial^2 - m^2 \right) \phi = 0 \tag{58}$$

Where $\partial^2 \equiv \eta^{\mu v} \partial_\mu \partial_\nu$
Consider the classical field configuration form:

$$\phi_p(t, \vec{x}) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p}} \left(a(t) e^{i\vec{p}\cdot\vec{x}} + a^*(t) e^{-i\vec{p}\cdot\vec{x}} \right)$$

Quantizing the solution above and using light-cone coordinates, get only set of creation and annihilation operators, indexed by momenta. Build up the state space, by applying them to some vacuum state $|\Omega\rangle$. One particle states are given by:

$$a^{\dagger}_{p^+, p_T} |\Omega\rangle$$

Note that this has the same indices as the massless particles due to an open string and we get expected correspondence.

4.1.1 Weak Gravitational Fields

Assume weak gravitational fields are give via:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$

Where η and h are symmetric.
In Fourier space, get that the linearized equation of motion of this fluctuation is:

$$S^{\mu v}(p) \equiv p^2 h^{\mu v} - p_\alpha \left(p^\mu h^{\nu\alpha} + p^v h^{\mu\alpha} \right) + p^\mu p^v h = 0$$

Using this equation and lightcone gauge, get that h^{IJ} must be traceless.
From this conclude that the one particle basis states would be given by:

$$a^{IJ\dagger}_{p^+, p_T} |\Omega\rangle$$

This has again the same indices as the basis for graviton states in (52).

From these examples, see that the indices string theoretic states match those of the graviton state from field theory.

References

[1] B. Zwiebach. *A first course in string theory*. Cambridge university press, 2004.