Orbiting the Earth

This handout contains instructions for a digital simulation of the trajectory of an object orbiting some planet.
In order to make a simulation, we must first come up with a a way to mathematically describe the trajectory of an object orbiting a planet.

To describe the trajectory of an object, we need to have a way to find out its position in time. We know from our kinematic equations, that we can find position from velocity, and velocity from acceleration.

Newton's Law of Gravitation 1. *The gravitational force of attraction acting on one object, due to another object is directed towards the center of that object. The magnitude of this force is modeled by the following equation*

$$F_g = \frac{Gm_1m_2}{r^2} \tag{1}$$

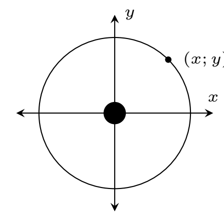

Figure 1: Object at (x, y) orbiting a planet at the origin of the reference frame.

Where:

m_1 and m_2 are the masses of the two objects
$G \approx 6.67 \cdot 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kgs}^2}$ - universal gravitational constant
r - distance between the centers of the two objects

Now let's introduce a way to write down the position of the object that is orbiting our planet mathematically.

We will use a Cartesian reference frame, where the origin is at the center of planet with mass m_2.
The position of orbiting object of mass m_1 will be denoted as a pair of Cartesian coordinates $(x; y)$.
Substitute equation (1) into Newton's second law for object 1 to get:

$$|\vec{a}| = \frac{Gm_2}{r^2} \tag{2}$$

From Newton's Law of Gravitation, we know that the acceleration vector of the orbiting object, will be directed towards the origin (as the gravitational force is directed towards the center of mass of the planet).

In equation (2), from Figure (1) and the Pythagorean theorem get that the distance between the center of the planet and orbiting object will be:

$$r = \sqrt{x^2 + y^2} \tag{3}$$

The vector of acceleration will be directed in the opposite direction of the radius vector, connecting the origin and the orbiting object and have the magnitude from equation (2). To get a vector pointing in this opposite direction, divide the radius vector by its magnitude (the radius) and multiply it by the magnitude of the acceleration from equation (2) to get:

$$\begin{aligned}\vec{a} &= \left(-\tfrac{x}{r}\cdot\tfrac{Gm_2}{r^2};\quad -\tfrac{y}{r}\cdot\tfrac{Gm_2}{r^2}\right)\\ &= \left(\frac{-x}{\sqrt{x^2+y^2}}\cdot\frac{Gm_2}{\left(\sqrt{x^2+y^2}\right)^2};\quad \frac{-y}{\sqrt{x^2+y^2}}\cdot\frac{Gm_2}{\left(\sqrt{x^2+y^2}\right)^2}\right)\\ &= \left(\frac{-xGm_2}{\left(\sqrt{x^2+y^2}\right)^3};\quad \frac{-yGm_2}{\left(\sqrt{x^2+y^2}\right)^3}\right) \end{aligned} \tag{4}$$

Now that we are able to calculate the acceleration of the orbiting object at any point in time, let's find a way to approximate the position and velocity of the orbiting object at very close points in time.

Recall the following kinematic equation:

$$\vec{v}_f = \vec{a}_i \Delta t + \vec{v}_i \tag{5}$$

Where:

$\vec{v}_f$ and $\vec{v}_i$ - initial and final velocities separated by time interval Δt
$\vec{a}_i$ - initial acceleration right before time interval Δt.

Even though this equation models a situation where acceleration is constant (in our case the direction of it changes, as the object moves), from small changes in time Δt, we can assume that acceleration does not change significantly (is essentially constant). This equation will therefore be a good approximation if we are trying to find the next velocity of the orbiting object from some previous initial value.

Analogically, if we can approximate the next position of the object after a small time interval, given the current position of the object with:

$$\vec{s}_f = \vec{v}\cdot\Delta t + \vec{s}_i \tag{6}$$

Where:

$\vec{s}_f$ and $\vec{s}_i$ - initial and final position vectors separated by some time interval Δt
$\vec{v}_i$ - initial velocity right before time interval Δt.

Given some initial position and velocity, the mass of the planet, using equations (4),(5) and (6), we can approximate consectuive positions separated by small time intervals Δt (and therefore the overall trajectory) of an orbiting object.

Instructions for Setting the Simulation up

Set up columns similar to the following image.

	A	B	C	D	E	F	G	H
1	delta t			m2	G			
2	0.005			5.972E+24	6.7E-11			
3								
4								
5	t	x	y	vx	vy	ax	ay	r
6	0	100000	1000	-631.08847	63108.8	-39827.266	-398.27266	100005
7	0.005	99996.8	1315.54	-830.2248	63106.9	-39825.414	-523.93747	100005

Figure 2: Set up columns for x and y directions of the position, velocity and acceleration vectors.

*The r column was added just for convenience when calculating and using radial distance away from planet in other equations. It has the following equation:

=SQRT(B6^2+C6^2) Set up the first row of position,velocity vector coordinates as manual input of initial conditions. Use equation (4) to calculate the acceleration coordinates on this first row.

The acceleration in the x direction in this first row will have the following sample Excel formula, based on equation (4):

=-B6/H6*$E$2*$D$2/(H6^2)

*There are dollar signs around D and E in D2 and E2in order to prevent the row number of these values from changing when we drag them down. These are the mass of the mass of the planet and the universal gravitational constant, which do not change from row to row.

Use formulas (5) and (6) to calculate all velocity and position values in the next row using initial values from the previous row. For example, the cell B7 from Figure (2) (x coordinate in second row, will have the following Excel formula.

=B6+D6*$A$2

*Once again, there are dollar bill signs around A in A2, to prevent this value to be changing between rows.

The t ime column should have the following equation:

A6+$A$2

After you have set all of this up, select the second row and drag it down for as many time points as you would like. Remember you can always decrease the size of the time interval in cell A2 for greater accuracy of the simulation.

Using the points in column B and C, you can construct a scatter plot of the trajectory of the orbiting object.

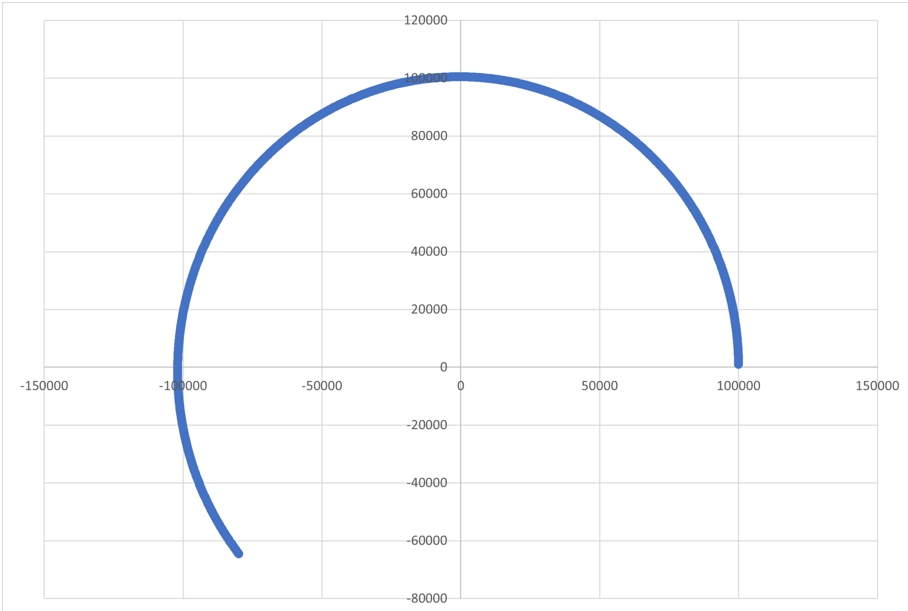

Figure 3: Trajectory of some object orbiting a planet of mass $m_2 = 5.972 \cdot 10^{24}$ kg, with an initial position at (100000;1000) and initial velocity of $\vec{v}\,(-631.0884705\frac{\mathrm{m}}{\mathrm{s}}; 63108.84705\frac{\mathrm{m}}{\mathrm{s}})$, as simulated in Excel using a time increment $\Delta t = 0.005\ s$.

Try out other initial conditions to look at other possible trajectory shapes.

Connection with Centripetal Motion

On this image, we can see that the simulated trajectory is almost nearly that of a perfect circle. Let's see how we can relate orbiting to circular motion.

Recall that the force that must be applied to an object to make it travel at a constant speed along a circle is:

$$F_c = \frac{mv^2}{r} \tag{7}$$

The velocity of this object, will always be tangent to the circle. This demonstration used this formula to select the initial velocity and position of the object, so that the required centripetal force is matched by the gravitational force:

$$\begin{aligned} \frac{Gm_1m_2}{r^2} &= \frac{m_1v^2}{r} \\ \frac{Gm_2}{r} &= v^2 \\ v &= \sqrt{\frac{Gm_2}{r}} \end{aligned}$$

From our simulation we can see, that the gravitational force can "produce" circular orbits.