1 Suggested Exercises I

1.1 Overview

First let's briefly review what we discussed today:

Definition 1 *We call the pair (X, τ) a topological space, where X is some set and $\tau \subseteq \mathcal{P}(X)$ (i.e. τ has subsets of X as elements) if τ satisfies the following axioms. Note that we call the elements of τ - open sets.*

1. $\varnothing \in \tau$ and $X \in \tau$
2. If $U_1, ..., U_n \in \tau \Rightarrow \bigcap_{j=1}^{n} U_j \in \tau$ (the intersection of finitely many open sets is still an open set)
3. If $U_i \in \tau$ for some $i \in \Lambda$ (where Λ is some indexing set) then $\bigcup_{i \in \Lambda} U_i \in \tau$ (the intersection of arbitrarily many open sets is an open set).

Mini exercises

1. Write down the definition of an arbitrary union of sets as in axiom 3 (Hint: I mean something of the form: A = { some elements | such that these elements satisfy some conditions }
2. Write down the definition of a finite intersection of sets $U_1, ..., U_n$ as in axiom 2

Recall we also talked about metric spaces!

Definition 2 *The pair (X, d) is a metric space, where X is a set and $d : X \times X \to \mathbb{R}^+$ a function, if d satisfies the following conditions. We also call d a distance function. (Note R^+ is the non-negative real numbers)*

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality)

When discussing metric space, it is very useful to discuss the notion of a ball. We give the definition as follows:

$$B_a(\varepsilon) = \{x \in X | d(a, x) < \varepsilon\}$$

Note: You read above as: Ball of radius ε centered at a.

The link between metric spaces and topologies.

One should imagine a metric as a ruler. It effectively allows you to measure the distance between two points. Therefore, one is able to tell whether two points are 'close' to one another or not. This idea of points being close to one another or not is the key concept in mathematics. Recall the $\delta - \varepsilon$ definition of continuity for functions f going from $\mathbb{R}$ to $\mathbb{R}$.

Definition 3 *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if $\forall a \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0$:*

$$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$$

Notice that from the previous discussion on metric spaces this definition could be rephrased not in terms of the metric but rather in terms of balls where the expression $|x - a| < \delta$ is replaced with $x \in B_a(\delta)$ and the expression $|f(x) - f(a)| < \varepsilon$ becomes $f(x) \in B_{f(a)}(\varepsilon)$. This definition does call upon the metric implicitly, yes, but there is something interesting here:

What if we defined 'closeness' not in terms of some metric but rather in terms of sets. This is the fundamental idea of topology.

But I hear your voices as I type up this line, 'but why, metrics are so cool and nice, don't take away my mathematical ruler'. And while I agree that metrics are very nice to work with and quite useful, there are times when they are too much.

Remember that the moment your space has a metric there is one fundamental fact you are imposing, and that is that two distinct points are never as close to each other as you like. For any such points you can find a positive distance between them. This is sometimes not something that you would want.

To give an analogy imagine you have a tightly knit friend group which we'll represent by the set $\{1, \ldots, n\}$ (the arbitrary n was chosen here so as to not assume the number of friends you have). If you want to measure the closeness of everyone here you quickly run into a problem because all of you are close to each other! Therefore a metric does really describe the situation we have. However the indiscrete topology would be perfect in this situation, in the indiscrete topology $\tau = \{\emptyset, \{1, \ldots, n\}\}$ every point is close to every other point. Therefore this topology perfectly describes your friend group! This and many more reasons are why considering topologies instead of just metric spaces is useful.

1.2 Exercises

I) Let (X, d) be a metric space. And consider the following set:

$$\tau = \{U \in \mathcal{P}(X) | \forall x \in U \exists \delta > 0 \text{ s.t } B_x(\delta) \subseteq U\}$$

   i) Show that (X, τ) satisfies the first axiom of a topological space
   ii) Show that (X, τ) satisfies the second axiom of a topological space
   iii) Show that (X, τ) satisfies the third axiom of a topological space

II) Topology of $\mathbb{R}^n$. Consider the topological space $(\mathbb{R}^n, \tau)$ with τ defined as in Q1, using the following norm:

$$d((x_1, ..., x_n), (y_1, ..., y_n)) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

   i) Suppose $n = 2$ and we are looking at the plane ($\mathbb{R}^2$) Draw a picture, of what it means for a set U to be an open set (i.e. in τ)
   ii) Check that d is indeed a metric. (Triangle inequality may be a little tricky. Hint: consider writing out what $d(x, y)^2$ is and maybe adding 0 somewhere).

Written by Ivan Ovchinnikov & Sabek Germame, Summer 2024.