1 Review of Calculus

1.1 Preliminary Definitions

Definition 1 *$U \subseteq \mathbb{R}$ is called open neighbourhood of $a \in \mathbb{R}$, if for any $p \in U \exists \varepsilon$ such that $(p - \varepsilon, p + \varepsilon) \subseteq U$ and $a \in U$.*

Definition 2 *A point $p \in \mathbb{R}$ is called the limit of a function $f : A \to \mathbb{R}$ as x approaches a if for any open neighbourhood U around p, there exists an open neighbourhood $V \subseteq A$ of point a such that $f(V) \subseteq U$. (i.e. if x keeps getting "closer" to a then $f(x)$ keeps getting closer to p)*

Definition 3 *A map $T : X \to Y$ is called linear if it satisfies the following property for all $u, v \in X$ and $\lambda \in R$:*

$$T(u + \lambda v) = T(u) + \lambda f(v) \tag{1}$$

1.2 The Derivative

Definition 4 *A linear function $Df : \mathbb{R} \to \mathbb{R}$ is called Jacobian of a function $f : [a, b] \to \mathbb{R}$ at some point x_0 if it is the best possible linear approximation of f at that point. Particularly:*

$$f(x_0 + u) = f(x_0) + Df(u) + R(u) \tag{2}$$

Where R is of order u^2, meaning: $\lim_{u \to 0} \frac{R(u)}{|u|} = 0$.

Corollary 1 *For a function $f : \mathbb{R} \to \mathbb{R}$, the Jacobian is multiplication by a single number. For a Jacobian of a function f at point x_0, we call this function the derivative of f and denote it via:*

$$f'(x_0) = \frac{df}{dx}\Big|_{x=x_0} = \frac{d}{dx} f\Big|_{x=x_0 a}$$

For derivatives with respect to time, we will often use the notation: $\frac{d}{dt} f = \dot{f}$

Corollary 2 *If the derivative of some function $f : \mathbb{R} \to \mathbb{R}$ exists, it is given via the following formula:*

$$\frac{d}{dx} f\Big|_{x_0} = \lim_{u \to x_0} \frac{f(x_0 + u) - f(x_0)}{u} \tag{3}$$

Derivative Suggested Exercises I

I) **Understanding the Definition**

   i) Explain intuitively why the limit formula for the derivative in equation (3) is correct? (i.e. gives you the best linear approximation to the function at the point).

   ii) Give a geometric interpretation of the derivative of a function. (*Hint: Draw a graph of a function, pick a point and draw something that helps clarify what the derivative is. You should think about what the Jacobian is: can you identify it's domain and target in the picture?.*).

II) **Computation Warmup**

   i) Compute the derivative of $f(x) = kx$ for k any constant at some point $x_0 \in \mathbb{R}$

   ii) Compute the derivative of $f(x) = ax^2$ for a any constant, at some point $x_0 \in \mathbb{R}$

III) **Bonus:** Suppose $f : \mathbb{R} \to \mathbb{R}^n$ with $f(t) = (f_1(t), \dots, f_n(t))$. How should the Jacobian of f look like?

1.3 Properties of Differentiation

1.3.1 Product Rule

Theorem 1 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions differentiable at x. Then the derivative of $f \cdot g$ at x is given via:*

$$\left.\frac{dfg}{dx}\right|_{x_0} = \left.\frac{df}{dx}\right|_{x_0} \cdot g(x) + f(x) \cdot \left.\frac{dg}{dx}\right|_{x_0}$$

Intuitively the rule above is clear. We want the derivative to give us the best linear approximation around x_0. Fix a point x_0. Then to linear order:

$$f(x_0 + \delta x) = f(x_0) + f'(x_0)\delta x \quad g(x_0 + \delta x) = g(x_0) + g'(x_0)\delta x$$

We can then use this to get a linear approximation for $f \cdot g$:

$$f(x_0+\delta x) \cdot g(x_0+\delta x) = (f(x_0)+f'(x_0)\delta x)(g(x_0)+g'(x_0)\delta x) = f(x_0)g(x_0)+(f'(x)g(x)+f(x)g'(x))\delta x+f'(x_0)g'(x_0)(\delta x)^2$$

The first term is a constant term, the last term is second order in δx and the second term is in fact of linear order in δx. The coefficient in front of δx must then be the desired derivative[1].

1.3.2 Chain Rule

Theorem 2 *Chain Rule. Let $g : \mathbb{R}^m \to R^n$ be differentiable at x and $f : \mathbb{R}^n \to R^l$ be differentiable at $g(x)$, then the Jacobian of $(f \circ g)(x)$ is given by:*

$$(Df)\big|_{g(x)} \circ (Df)\big|_{x}$$

The idea behind this equation is intuitively clear. If you want to compute the linear approximation of a composition of two functions, you linearly approximate the first function and then substitute that into the linear approximation of the second. The composition of two linear functions, is still a linear function so we can expect this to give a sensible result.

Corollary 3 *Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable at x and $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $g(x)$, then the derivative is given by:*

$$\left.\frac{d}{dx}(f \circ g)\right|_{x_0} = \left(\left.\frac{df}{dx}\right|_{g(x_0)}\right) \cdot \left(\left.\frac{dg}{dx}\right|_{x_0}\right)$$

The intuitition from the previous theorem can be informally encapsulated in the following computation [2]:

$$f(g(x_0 + \delta x)) = f(g(x_0) + g'(x_0)\delta x) = f(g(x_0)) + f'(g_x 0) \cdot (g'(x_0)\delta x)$$

1.3.3 List of Common Derivatives

$$\begin{gathered} \frac{d}{dx}x^n = nx^{n-1} \quad \forall n \geq 1 \\ \frac{d\frac{1}{x}}{dx} = -\frac{1}{x^2} \quad \frac{d\log x}{dx} = \frac{1}{x} \\ \frac{d}{dx}\sin(x) = \cos(x) \quad \frac{d}{dx}\cos(x) = -\sin(x) \\ \frac{d\exp(x)}{dx} = \exp(x) \end{gathered} \tag{4}$$

[1] This seemingly informal method of computing linear approximations can very much be rigorously justified. What we are really doing here, is expanding the function as in equation 2 and then never writing any of the terms with the $R(u)$ factor, because such things will not be of linear order.

[2] Note that this is once again essentially a calculation where we substitute everything using the form of equation (2) and "ignore" all of the $R(u)$ terms.

Derivative Suggested Exercises II

I) **Computational Practice:** Compute the following derivatives:

$$\frac{d}{dt}\sin(3t) \qquad \frac{d}{dt}(\cos(t))^2$$
$$\frac{d}{dt}\cos(t)\sin(t) \qquad \frac{d}{dt}\exp(\cos(t))$$
$$\frac{d}{dt}\frac{1}{t^n} \qquad \frac{d}{dt}\frac{1}{\sin(t)}$$

II) **Understanding Division:**

i) Let f be a differentiable function and $f \neq 0$.Compute the following derivative:

$$\frac{d}{dt}\frac{1}{f(t)}$$

ii) Use the result above to verify that:

$$\frac{d}{dt}\frac{f(t)}{g(t)} = \frac{f'(t)g(t) - f(t)g'(t)}{(g(t))^2}$$

III) **Inverse of Differentiation**
Find a function such f that the following are its derivative:

$$\frac{d}{dt}f = t^2 \qquad \frac{d}{dt}f = \sin(t)\cos(t)$$

1.4 Integration

Definition 5 *The area under the graph of the function $f : \mathbb{R} \to \mathbb{R}$ over the region $x \in [a, b]$ is called the integral of f. It is denoted as $\int_a^b f(x)dx$*

Theorem 3 *The Fundamental Theorem of Calculus I:*

$$\frac{d}{dt}\int_{t_0}^{t} f(\tau)d\tau = f(t)$$

Theorem 4 *Fundamental Theorem of Calculus II:*

$$\int_a^b \frac{dF}{dt}(t)dt = F(b) - F(a) = F\Big|_{t=a}^{t=b}$$

Theorem 5 *Change of Variables (one-dimensional)* [3]*:*

$$\int_a^b (\frac{d}{dt}g\Big|_t)f(g(t))dt = \int_{g(a)}^{g(b)} f(u)du$$

Theorem 6 *The following identity is often called integration by parts. It follows directly from product rule (check this yourself)*

$$\int_a^b u(x)\frac{dv(x)}{dx}dx = u(x)v(x)\Big|_a^b - \int_a^b \frac{dv(x)}{dx}u(x)dx \tag{5}$$

[3]Due to convention this method is also often called u-substitution

Suggested Integral Exercises

I) Compute the following integrals

$$\int_0^a dx \quad \int_0^a 3x^2 + 7$$
$$\int_0^a e^{kx} dx \quad \int_0^a xe^{x^2}$$
$$\int_0^a x\sin(x) \quad \int_0^a \sin(x)\cos(x)$$
$$\int_0^a xe^x \quad \int_0^a x^2e^x$$

II) Recall that Newton's equations of motion are given by:

$$F = ma = m\ddot{x}$$

i) Consider a particle restricted to 1 dimension. Suppose we attach a string to it. Then in some coordinate system, the force acting on the particle at position x is $F = -kx$.
We define the potential energy to be:

$$U(p) = -\int_0^p F(x)dx$$

Compute $U(x)$. Then show that the total energy:

$$E(x(t)) = \frac{1}{2}m(\dot{x})^2 + U(x(t))$$

Is constant with time. Show that if E is conserved for any F that does not depend on t.

1.5 Partial Differentiation

So far we have only concerned ourselves with functions that are defined on the real line (i.e. only have one parameter of input). In physics, we are however often conserved with quantities that can depend on multiple parameters (for example temperature can depend not only on time, but also where you are i.e. position). For this reason, we need to understand how to approximate functions $f : \mathbb{R}^n \to \mathbb{R}^m$. Since we know how to add vectors in $\mathbb{R}^m$, with a little thought it is easy to see that definition of the Jacobian remains completely unchanged.

It would however require some effort to understand what linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ look like [4]. This would take bit too much time (maybe 1-2 hours) and is not strictly necessary to in the discussion of analytical mechanics (though indispensible in physics in general). We therefore will consider a slightly simpler problem.

Let's restrict our attention to functions with a single output $f : \mathbb{R}^m \to \mathbb{R}$. In the discussion above, we got very comfortable with computing a linear approximation of functions that have only one input variable, in the form of a derivative. Our problem is that in the case of $f : \mathbb{R}^m to \mathbb{R}$, there are many "directions" in which we could be considering a linear approximation. We will then considering the simplest thing: fix a point x_0, chose some direction v and write down a linear approximation of the function in that direction.

Definition 6 *Let $f : \mathbb{R}^m \to \mathbb{R}$ be some function and $v \in \mathbb{R}^m$. Fix x_0. We can then consider $f_v(t) = f(x_0 + tv)$ and take a derivative with respect to t. This is called a partial derivative in the direction v.*

A common choice of directions, are those along unit vectors $e_i = (0, ..., 1, ..., 0)$. These partial derivatives have special notation:

$$\frac{\partial}{\partial x_i} f\bigg|_{x_0} = \lim_{u\to 0} \frac{f(x_0 + ue_i) - f(x_0)}{u} \tag{6}$$

[4] This is many ways the essence of standard first year linear algebra courses (assuming they are "Done Right"

1.5.1 Notational Point

We will now introduce a difference between $\frac{d}{dt}$ and $\dfrac{\partial}{\partial t}$ notations (which was not really indicated previously).
By definition, when we write $\frac{d}{dt}$, we assume that every variable could potentially be a function of t. This means that for example:

$$\frac{d}{dt}xy = \frac{d}{dt}x \cdot y + x \cdot \frac{d}{dt}y$$

When we write a partial derivative $\dfrac{\partial}{\partial t}$, we assume everything is a constant unless it is explicitly stated that it is a function of t. For example:

$$\frac{\partial}{\partial x}xy = y$$

Partial Derivative Suggested Exercises

I) Let $g : \mathbb{R} \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$ be both differentiable. Convince yourself that:

$$\left.\frac{\partial}{\partial t}f(g(t))\right|_{t_0} = \sum_{i=1}^{m} \left.\frac{\partial f}{\partial x_i}\right|_{g(t_0)} \cdot \left.\frac{\partial g_i}{\partial t}\right|_{t_0}$$

II) Compute the following derivatives:

$$\frac{\partial \sin(xy)y^2}{\partial x} \qquad \frac{\partial e^{x^2 \sin(y)}}{\partial y}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}e^{x^2 \sin(y)} \qquad \frac{\partial}{\partial y}\frac{\partial}{\partial x}e^{x^2 \sin(y)}$$

III) Consider the function:

$$T(t) = \frac{1}{2}\left(\dot{x_1}^2 + \dot{x_2}^2\right)$$

i) Suppose you are given $r(t)$ and $\varphi(t)$ such that:

$$x = (x_1, x_2) = \big(r\cos\varphi \ , \ r\sin\varphi\big)$$

Express $T(t)$ in terms of $r(t)$ and $\varphi(t)$

IV) Recall that acceleration is:

$$a = \ddot{x} = \big(\ddot{x}_1, \ddot{x}_2\big)$$

Suppose that the particle's motion is restricted to a circle of radius r. It's path $x(t)$ is then fully determined by some function $\varphi(t)$ via:

$$x(t) = r\big(\cos\varphi(t) \ , \ \sin\varphi(t)\big)$$

Show that the acceleration, must be proportional to $x(t)$ (i.e. pointing along the same direction as the radius vector of x)

1.6 Elements of Functional Calculus

Definition 7 *A functional is a map that takes in functions and outputs a number.*

Examples

i) The evaluation functional: $\mathrm{ev}_x[f] = f(x)$

ii) Given some function g and $a, b \in \mathbb{R}$, we can define: $f \to \int_a^b f(t) \cdot g(t) dt$

iii) The following is a common function: $F[f] = \int_{-\infty}^{\infty} f^2(t) dt$

Similar to above, we are interested in the notion of a best linear approximation to a functional. The definition for this is identical to that of a derivative:

Definition 8 *Let $F[f]$ be some functional. We call DF : functions $\to$ numbers the Jacobian (or best linear approximation) of F at f, if it is linear and*

$$F[f + h] = DF_f[h] + R(f, h) \tag{7}$$

Where R is of order h^2, meaning: $\lim_{u \to 0} \frac{R(u)}{|u|} = 0$.

Note that like before, we have the problem of having "too many" directions, along which we can take this best linear approximation. In general, as compared to partial derivatives before, the space of functions is an infinite dimensional space. We can still use the same trick from before and simply pick a direction to differentiate in:

Theorem 7 *Let $F[f]$ be a functional. The best approximation of F at f in the direction g is given via:*

$$DF_f[h] = \lim_{\varepsilon \to 0} \frac{F[f + \varepsilon h] - F[f]}{\varepsilon} = \frac{\partial}{\partial \varepsilon} F[f + \varepsilon g]\Big|_{\varepsilon = 0} \tag{8}$$

Definition 9 *We say that f is a saddle point of F or that f extremizes F if:*

$$DF_f[h] = 0 \qquad \forall h$$

Suggested Exercises Functional Calculus

I) Let

$$F[f] = \int_0^1 \left(f(x)\right)^2 dx$$

i) Compute $DF_f[h]$ for arbitrary f and h

ii) Suppose that F is restricted only to functions f such that $f(a) = f(b)$. Suppose that some function g extremizes F. Find the condition that g must satisfy.

II) Let

$$F[x_1(t), x_2(t)] = \int_0^{t_1} \sqrt{\dot{x_1}^2 + \dot{x_2}^2} dt$$

.

Where $x_1(0) = x_2(0) = 0$ and $x_1(t_1) = x_2(t_1) = 1$

What does this functional compute?

Find $\tilde{x}(t)$ such that $DF[\tilde{x}] = 0$. Does this answer make sense?

2 Exercises

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I) **Understanding the Definition**

   i) Explain intuitively why the limit formula for the derivative in equation (3) is correct? (i.e. gives you the best linear approximation to the function at the point).

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