1 Review

We have discussed that the evolution of motion of a physical system, the position of which is somehow encoded in some generalized coordinates q (where q could stand for multiple numbers), is determined with the help of some lagrangian function:

$$\begin{aligned} \mathcal{L} : \text{ positions} \times \text{ velocities} \times \text{ time }^1 &\rightarrow \mathbb{R} \\ x, v, t \rightarrow \mathcal{L}(x, v, t) \end{aligned} \tag{1}$$

We then then define the action of the system to be:

$$\begin{aligned} S : \text{ paths in configuration space } &\rightarrow \mathbb{R} \\ S(q(t)) = \int_{t_1}^{t_2} \mathcal{L}(q(t), \dot{q}(t), t)\ dt \end{aligned} \tag{2}$$

The Action Principle, is then that the system travels along the path $q_(t)$ which extremizes the action, i.e. the functional derivative[2] at $q(t)$ is 0:

$$\delta S = 0 \tag{3}$$

We then showed, that $q(t)$ being such that it satisfied the action principle, is equivalent to $q(t)$ satisfying the following Euler-Lagrange equations:

$$\frac{d}{dt}\left(\left.\frac{\partial \mathcal{L}}{\partial v_i}\right|_{q(t)}\right) - \left.\frac{\partial \mathcal{L}}{\partial q_i}\right|_{q(t)} = 0 \tag{4}$$

Where everything in the above, are partial derivatives of $\mathcal{L}$ with respect to its entries. By abuse of notation, thinking of $\dot{q}_i = v_i$ as sometimes a coordinate and other times as the derivative of a path, the Euler-Lagrange Equations are more commonly written as:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{5}$$

Definition 1 *A quantity $Q(q, \dot{q}, t)$ is said to be conserved, if it does not change over time as the system evolves. i.e*

$$\frac{d}{dt}Q(q(t), \dot{q}(t), t) = 0 \tag{6}$$

2 Suggested Exercises

I) Show that for a system of n particles, with euclidean coordinates and the following lagrangian:

$$\mathcal{L} = T - V \tag{7}$$

Where T is the kinetic energy and V is the potential energy, the Euler-Lagrange equations give Newton's equationse.

II) **Conjugate Momentum** Suppose we have some physical system, with some lagrangian $\mathcal{L}$ and some coordinates q_i. We call the following quantity the conjugate momentum of q_i:

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

i) Compute the conjugate momentum of the following variables for the following lagrangian, describing the motion of a free particle in 2 dimensions:

$$\mathcal{L} = T_1 = (\dot{x})^2 + (\dot{y})^2$$

i. Compute the conjugate momentum of x and y.

ii. Rewrite above in polar coordinates, compute the conjugate momentum of φ. Does this resemble anything? (*Hint: Recall that φ is an angle. So this has to be* _________ *momentum.*

iii. Compute the conjugate momentum of r (in polar coordinates). Is it conserved?

ii) When is p_j a conserved quantity? (*Hint: Look at the Euler Lagrange equations, to deduce a condition on $\mathcal{L}$*)

[2] See Arnold Math Methods of Classical Mechanics or video for definition

iii) Suppose we introduced a second particle which did not interact with the one above (or anything else). The lagrangian would then become:

$$\mathcal{L} = T_1 + T_2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{x}')^2 + (\dot{y}')^2$$

Where T_2 is the kinetic energy of the second particle. Show that conjugate momenta are additive. Argue that when you have two particles, the quantity that is conserved is the sum of the two momenta.

iv) Consider the following Lagrangian, for a particle restricted to move in 1 d

$$\mathcal{L} = (\dot{x})^2 + (\dot{y})^2 - mgy \tag{8}$$

What common system does this describe? Which momentum is conserved and which one is not?

III) **Symmetry**

(a) Suppose we are given some Lagrangian $\mathcal{L}$. Suppose we then define a new lagrangian $\tilde{\mathcal{L}} = \mathcal{L} + \frac{d}{dt}\varPhi(q, \dot{q}, t)$, where $\varPhi$ is any function. Show that this does not affect the equations of motion. (*Hint: You can either just plug this into the equations of motion and realize they are unchanged. It is however better to think about how we derived the equations of motion and try to understand what this replacement does to the action.*

(b) We loosely define a symmetry as follows:

Definition 2 *A symmetry is a one-parameter transformation of the path $q(t) \to \tilde{q}(t, \lambda)$, such that the equations of motion are left invariant (i.e. the Lagrangian is changed by at most a time derivative of another function, meaning $\tilde{q}$ satisfies the same euler lagrange equations as q).*

Show that the following are symmetries of the following lagrangians:

- $(x, y) \to (x + \lambda, y)$ for $L = (\dot{x})^2 + (\dot{y})^2 - mgy$
- $q(t) \to q(t + \lambda)$ for $L = (\dot{x})^2 + (\dot{y})^2 - mgy$

(c) Now consider the following condition. Show that it must be equivalent to the definition above:

$$\left.\frac{\partial \mathcal{L}(\tilde{q}(t\lambda), \dot{\tilde{q}}(t, \lambda), t}{\partial \lambda}\right|_{\lambda=0} = \frac{d}{dt}\varPhi \tag{9}$$

IV) **Noether's Theorem** Noether's theorem says that every symmetry has an associated conserved quantity and vice versa. Below you will prove one direction of this theorem

(a) Suppose you are given some lagrangian and a symmetry transformation $q(t) \to \tilde{q}(t, \lambda)$. From the previous part of the question, you know that there exists some function $\varPhi$ such that equation (9) holds. Prove that then the following, is a conserved quantity *Hint: Plug in the Euler Lagrange equations somewhere when computing the derivative and it should all work out:*:

$$Q_\lambda = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \left.\frac{\partial \tilde{q}_i}{\partial \lambda}\right|_{\lambda=0} - \varPhi \tag{10}$$

(b) Suppose the following is a symmetry of your system: $q(t) \to q(t + \lambda)$. Write down the associated Q_λ. Suppose the lagrangian is of the form $\mathcal{L} = (\dot{x})^2 - V(x)$ where V does not explicitely depend on time. Compute Q_λ for this lagrangian? What is this quantity that you got[3]?

[3]You should get the energy and conclude that Energy is the conserved quantity associated to invariance under time translation