1 Review of Definitions

Definition 1 *A vector space V is called an inner product space if it is equipped with an inner product function that satisfies the following properties:*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$$

i *Linear in first entry:* $\langle v + \lambda u, w \rangle = \langle v, w \rangle + \lambda \langle u, w \rangle$

ii *Positivity:* $\langle v, v \rangle \geq 0 \quad \forall v \in V$

iii *Definiteness:* $\langle v, v \rangle = 0$ *if and only if* $v = 0$

iv *Conjugate Symmetry:* $\langle u, v \rangle = \overline{\langle v, u \rangle}$

Definition 2 *A norm on a vector space V is a function such that:*

$$|| \cdot || : V \to \mathbb{R}$$

1. *Positivity:* $||v|| \geq 0 \quad \forall v \in V$
2. *Definiteness:* $||v|| = 0$ *if and only if* $v = 0$
3. *Absolute Homogeneity:* $||\lambda v|| = |\lambda| ||v||$
4. *Triangle Inequality:* $||u + v|| \leq ||u|| + ||v||$

Definition 3 *Let V be an inner product space. Vectors u, v are called orthogonal if*

$$\langle u, v \rangle = 0$$

2 Additional Definitions

Definition 4 *A ball around point x_0 of radius ε is the set:*

$$B_\varepsilon(x_0) = \{y \quad | \quad ||y - x_0|| < \varepsilon\}$$

Definition 5 *A topological space is the pair (X, τ) where X is a set and τ is a topology. A topology τ is a collection of subsets of X (called open sets) with the following properties:*

i) *If $U_i \in \tau$ for some $i \in \Lambda$ then $\bigcup_{i \in \Lambda} U_i \in \tau$ (i.e. an arbitrary union of open sets is still open. Here Λ is some indexing set, i.e. a way to assign labels to the U_is.)*

ii) *If $U_1, ..., U_n \in \tau$ then $\bigcap_{i=1}^{n} U_i \in \tau$ (i.e. intersection of finitely many open sets is open)*

iii) $X \in \tau$ *and* $\emptyset \in \tau$

3 Suggested Exercises

I) **Topology.** In this exercise, you will gain some familiarity with the topology of metric spaces. A way to think about what a topology is (see definition 5), is defining the notion of points being "close" to each other without using a notion of distance. I.e., instead of saying how close points are together, just group points that are "close" into open sets.

   i) Let V be a vector space equipped with a norm. Prove that the following collection of subsets of V is indeed a topology on V:

   $$\tau = \{U \subseteq V \quad | \quad \forall u \in U \exists \varepsilon \text{ s.t. } B_\varepsilon(u) \subseteq U\} \tag{1}$$

   Hint: You can read the above statement as: set U such that around any point of that set we have a ball of finite size that fits inside U.

   ii) Let V,W be vector spaces with norms. Equip them with their own topologies τ_V, τ_W using the previous part. Prove that the two following definitions of continuity are equivalent:

Definition 6 *A function $f : V \to W$ is continuous if $\forall x \in V \ \forall \varepsilon > 0 \ \exists \delta$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$*

Definition 7 *A function $f : V \to W$ is called continuous if $U \in \tau_W \Rightarrow f^{-1}(U) \in \tau_V$*

Note that the second definition of continuity does not explicitly mention a notion of distance. It just uses open sets to define continuity. Suppose instead of defining open sets with a metric, I just told you what they are (say point by point). We see that the notion of a topology, is a generalization of "closeness" and continuity that does not explicitly talk about distance.

iii) Note that given a topology on a space, we can define the limit of a sequence. Let (X, τ) be a topological space. Let $\{x_i\}$ be a sequence of points. We then define the limit as follows:

Definition 8 *We call x a limit of the sequence x_i if for any open set U around x exists $N \in \mathbb{N}$ s.t. $\forall j > N \quad x_j \in U$*

Convince yourself, that this is exactly analogous to the definition of a limit you had in first year calculus.

II) **Gram Schmidt Decomposition** Let $v_1, ..., v_n$ be a set of linearly independent vectors.

i) Define vectors e_i via:

$$\begin{aligned} e_i &= \frac{v_i - \langle v_i, e_1 \rangle e_1 - ... - \langle v_i, e_{i-1} \rangle e_{i-1}}{||v_i - \langle v_i, e_1 \rangle e_1 - ... - \langle v_i, e_{i-1} \rangle e_{i-1}||} \\ &= \frac{v_i - \sum_{j=1}^{i-1} \langle v_i, e_j \rangle e_j}{||v_i - \sum_{j=1}^{i-1} \langle v_i, e_j \rangle e_j||} \end{aligned}$$

Argue that the denominator in the expression above is never 0. Argue that $\text{span}(v_1, ..., v_j) = \text{span}(e_1, ..., e_j)$

ii) Show that $\langle e_i, e_j \rangle = 0$ for $i \neq j$

III) **Cauchy-Schwarz-Bunyakovsky Inequality**. This inequality is incredibly important. It appears everywhere (including quantum mechanics). Please please please do this question. Let V be an inner product space. Let $u, v \in V$

i) Let

$$w = u - \frac{\langle u, v \rangle}{||v||^2} v$$

Show that $\langle w, v \rangle = 0$

ii) Note that

$$u = \frac{\langle u, v \rangle}{||v||^2} v + w$$

Show that

$$||u||^2 \geq \frac{|\langle u, v \rangle|^2}{||v||^2}$$

Rewrite above and conclude the following result known as the Cauchy-Schwarz-Bunyakovsky Inequality:

$$|\langle u, v \rangle| \leq ||u|| ||v|| \tag{2}$$