1 Quotient Spaces Overview

Today we finally understood why we care about Quotient Spaces. They are a simple description of the Image (or range) of any linear operator.

Recall that given some subspace U in a vector space V, we defined and equivalence relation $\sim$ such that:

$$x \sim y \text{ if } x - y \in U$$

We defined an equivalence class of an element $v \in V$ as:

$$[v]_\sim = \left\{x \in V \middle| x \sim v\right\}$$

We then defined a quotient space of V by U (V mod U) as the set:

$$V\!\diagup_{U} = \{[x]_\sim | x \in V\}$$

The noticed that if given operations of addition and multiplication by scalars for sets, the above is itself a vector space.

Now consider some linear map $T : V \to W$.
We recall that:

$$ker(T) = null(T) = \left\{v\ inV \middle| Tv = 0\right\}$$
$$Im(T) = \text{ Range}(T) = \left\{w \in W \middle| \exists v \in V \text{ s.t. } w = Tv\right\}$$

Recall that we previously proved that $ker(T)$ is a subspace of V and $Im(T)$ is a subspace of W.

We also define the following map:

$$\begin{aligned} \tilde{T} : V\!\diagup_{ker(T)} &\to W \\ \tilde{T}([v]_\sim) &= Tv \end{aligned} \tag{1}$$

As part of these exercises, you will show that the following diagram commutes (i.e. every map is a linear operator and regardless which composition of arrows you do, it is all the same map) and the following theorem

$$\begin{tikzcd} V \arrow[d, "\pi"] \arrow[dr, "T"] & \\ V\!\diagup_{ker(T)} \arrow[r, "\tilde{T}"] & W \end{tikzcd}$$

Theorem 1 *If $T : V \to W$ is a linear map then:*

$$V\!\diagup_{ker(T)} \cong Im(T)$$

2 Exercises

It is important to note, that in class I referred to a proof of all of this in the finite dimensional case, because I always chose a basis. These theorems however, all hold true in the infinite dimensional case too (why wouldn't they).

I) First we would like to show that $\tilde{T}$ is well defined and indeed a linear map. If we do that, then it should be obvious that the diagram above commutes.

   i) First note that it is possible that $[x]_\sim = [y]_\sim$ but $x \neq y$ (two different vectors, but in same equivalence class. By the definition of $\tilde{T}$ we would hope that $Tx = Ty$ (or otherwise the definition in equation (1) maps the same object to two different things, which makes no sense). Show that our hopes are correct

   ii) Show that $\tilde{T}$ is indeed linear (this should be like 1 line)

   iii) Convince yourself why $\tilde{T}$ would be surjective on the range of T

   iv) Prove that $\tilde{T}$ is injective

II) Since $\tilde{T}$ is injective and surjective and a linear map, we know it must have a linear inverse. Hence this is an isomorphism of ${}^{V}\!\diagup_{ker(T)}$ to $Im(T)$ (i.e. they are really the same space). This proves the theorem above. We can now use this theorem, to show that a map is an isomorphism, just by finding its kernel and image. First let's look at a completely trivial example:

- Suppose you have the identity operator $Id : V \to V$. Use the theorem, to prove that $V \cong {}^{V}\!\diagup_{0}$. (Compute the nullspace of Id and apply theorem)

III) Now Consider the following example (Axler 3.E Q): We want to show that

$$\mathcal{L}(V_1 \times ... \times V_n, W) \cong \mathcal{L}(V_1, W) \times ... \times mathcalL(V_n, W)$$

(a) This is fairly obvious. We can pretty much guess what the isomorphism should be. We just need to see that it is indeed an isomorphism. Write down the obvious isomorphism (call it say ϕ) (Hint: if $T \in \mathcal{L}(V_1 \times ... \times V_n, W)$, then you can define $\tilde{T}_i : V_i \to W$ in an easy way...) Note that:

$$\phi : \mathcal{L}(V_1 \times ... \times V_n, W)\mathcal{L}(V_1, W) \times ... \times mathcalL(V_n, W)$$

(b) Show that the operator must be surjective.

(c) Compute its kernel

(d) Use the theorem to show isomorphism. This is kind of a bad example, because we are just showing something is an isomorphism. That is usually not too bad.

IV) Use the isomorphism theorem, to immediately conclude Q.15

V) Q.18

VI) Draw the commutative diagram for Q20 and convince yourself of the fact that it actually commutes (don't write it out, just think about why it works)

VII) Remember the proof I talked about that involved choosing a basis, by first choosing the basis for the nullspace and then extending that to the basis of the entire space? Q.12