The Definition of a Limit for a Numerical Sequence

Numerical Sequences

Definition 1 *A numerical sequence is a function $a : \mathbb{N} \to \mathbb{R}$, that associates a real number to every integer. We will use the following notation: $a_i \equiv a(i)$.*

*We also often refer to the entire sequence as a_n (but that could also mean the nth element. This should be clear from context).

It is often convenient to write downdown a numerical sequences a_n as a list of real numbers, where the ith number in the list is a_i (an ordered list of numbers could be an alternative definition for a sequence).

Let's look at some examples:

1. Consider a function $a : \mathbb{N} \to \mathbb{R}$ where $a(i) = 1$
   - This function gives us the sequence where $\forall i \in \mathbb{N}$ (for all i in the natural numbers), $a_i = 1$.
   - We can write this sequence as the following list: $1, 1, 1, \ldots.$
2. Consider the sequence of natural numbers in increasing order. It is associated with a function $a : \mathbb{N} \to \mathbb{R}$ with $a(i) = i$
   - In this sequence, $\forall i \in \mathbb{N}$, $a_i = i$
   - We can write this sequence as the following list: 1,2,3,4,...
3. Another common sequence is the one give by the following function $a : \mathbb{N} \to \mathbb{R}$ with $a(i) = \frac{1}{i}$
   - Here, $\forall i \in \mathbb{N}$, $a_i = \frac{1}{i}$
   - The sequence can be also represented with list: $1, \frac{1}{2}, \frac{1}{3}, \ldots$
4. Consider the sequence given by the function $a : \mathbb{N} \to \mathbb{R}$ where: $a(i) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even} \end{cases}$
   - This sequence is associated with the list: 1,-1,1,-1,...

Motivating the limit

Suppose we are given some numerical sequence a_n. Let's look at what useful information we might want about the sequence.

Since we are given the sequence, we are always able to find its ith value. In other words we can always observe the sequence "locally" (at a given natural number). It is however often useful to look at more general trends, that let us study the sequence in its entirety, rather than at a specific point. We then get the following natural question:

How does the sequence behave as i keeps increasing (in other words, as i goes off to/approaches infinity)?

One such possibility is that there exists a number l, such that values of a_i generally approach/keep getting closer to l (get arbitrarily close to l).

Given this intuitive formulation, let's try to write the statement above in a more specific/rigorous fashion.

Defining the limit

Let's look at what it would mean mathematically, for values of a_i to approach a number l.

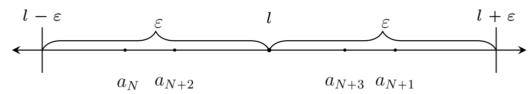

Figure 1: For an open neighbourhood $U(l, \varepsilon)$ around l with some radius ε there is an N, after which the entire sequence is contained in the neighbourhood.

First let's figure out what it means to approach/get closer to a number.

In real life, we know that something is approaching us if the distance between us is generally decreasing (it is overall getting closer rather than further away). So let's write down the distance between some ith element a_i and the number we are approaching l. From the pre-requisite information handout, recall that this can be written down as:

Distance between the numbers a_i and l is given by: $|a_i - l|$.

If our a_is get arbitrarily close to l, then we know that there must be a_is closer than any given distance away from l. In other words:

For any radius $\varepsilon > 0$ there must exist an i such that: $|a_i - l| < \varepsilon$.

Otherwise, that would imply that our sequence, doesn't get closer than a certain ε to l (and hence, does not actually approach it).

This however does not necessarily imply that the values of the sequence keep getting closer to l as i increases (what if the sequence keeps getting close to l, but then moving back?). We then need to demand, that at some point, the sequence does not move away further away from l than any given ε. In other words:

There exists some $N \in \mathbb{N}$, such that for all $i \geqslant N, i \in \mathbb{N}$, $|a_i - l| < \varepsilon$.

Combining this, with the idea that a_i must get arbitrarily close to l, we come up with the following definition:

Definition 2 *Given some numerical sequence a_n, the limit of a_n as n approaches infinity exists if there exists some number l such that:*

For all $\varepsilon > 0$ $\varepsilon \in \mathbb{R}$, there exists $N \in \mathbb{N}$, such that: for all $i \geqslant N, i \in N \Rightarrow |a_i - l| < \varepsilon$

Above definition is equivalent to the following:

For any open interval $U(l, \varepsilon)$ centered around l, there exists $N \in \mathbb{N}$ such that foor all $i \geqslant N, i \in N \Rightarrow a_i \in U(l, \varepsilon)$

We then call l the limit of a_n and write:

$$\lim_{n \to \infty} a_n = l$$

Take a moment to understand how the above definitions are equivalent (use Figure (1) as an aid).

Examples

Let's look at the limits of the same sequences as in the beginning of the handout.

1. Consider the sequence given by $a_i = 1$ (example 1 from above)

We can immediately guess that the limit of this sequence should be 1. Let's use our definition of the limit above, to check if that is indeed true.

Consider some distance away from l: $\varepsilon > 0$. We can always choose $N = 1$. We then know that $\forall i \geqslant N = 1$ we have:

$|a_i - l| = |a_i - 1| = |1 - 1| = 0 < \varepsilon \Rightarrow |a_i - 1| < \varepsilon$

This shows that $l = 1$, fits the definition of the limit and therefore $\lim_{n\to\infty} a_n = 1$.

2. Consider the sequence given by: $a_i = \frac{1}{i}$

Looking at this sequence, we can guess the limit to be $l = 0$. Once again let's verify that it is in fact the limit using the definition:

Let the distance from $l = 0$, $\varepsilon > 0$ be given.
Let's try and find an $N \in \mathbb{N}$ such that: $\forall i \geqslant N, |a_i - l| = |a_i - 0| = |a_i| = \left|\frac{1}{i}\right| < \varepsilon$.

Note that $0 < a_i \leqslant 1$. So if $\varepsilon \geqslant 1$, we can simply take $N = 1$ (and the entire sequence will be within ε of 0).

If $\varepsilon < 1$ then let N be the first natural number greater than $\dfrac{1}{\varepsilon}$. So we have:

$\frac{1}{\varepsilon} < N$ and $N \in \mathbb{N}$.

Divide both sides by N and multiply both sides by ε to get:

$\frac{1}{N} < \varepsilon$

If we take any $i \geqslant N$ we have:

$N \leqslant i \Rightarrow \frac{1}{i} = |\frac{1}{i}| \leqslant \frac{1}{N} < \varepsilon$

We have therefore found for any distance ε from 0, a N past which the entire sequence is contained in $(-\varepsilon, \varepsilon)$ and 0 fits the definition of the limit for the sequence $a_i = \frac{1}{i}$.

Infinite limit

Consider the sequence of the natural numbers, where $a_i = i$ (Example 3 from above).

Looking at Figure (2), we can see that this values of this sequence, get progressively larger and do not really approach any specific value and simply keep on getting larger. As n approaches infinity, a_i can get arbitrarily large. So there is no limit, in the sense of definition (2).

Let's try to define this improper limit, that would tell us that the sequence a_n gets arbitrarily large.

If a sequence a_n gets arbitrarily large, then we know that for any real number C, there must be an i such that: $a_i > C$. (As otherwise would imply that the sequence doesn't get any larger than C).

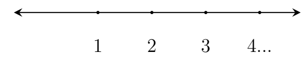

Figure 2: Sequence of natural numbers. Pick any point on this numberline and think of an open neighbourhood around that point that would not make this point the limit.

Like last time, this does not necessarily imply that the sequence "keeps on" getting larger forever, so we will also demand, that after some point N, the sequence does not "dip" below C. Let's now write the following definition:

Definition 3 *A sequence a_n has the limit infinity, if:*

For all $C \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $i \geqslant N \Rightarrow a_i > C$.

We then write that:
$\lim_{n\to\infty} a_n = \infty$

Similarly, we will define a limit for a sequence that goes to negative infinity. Check yourself that this definition follows the same logic as the one above:

Definition 4 *A sequence a_n has the limit negative infinity if:*

For all $C \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that for all $i \geqslant N \Rightarrow a_i < C$

Then: $\lim_{n\to\infty} a_n = -\infty$

Now let's show that the sequence $a_i = i$ does in fact have a limit of infinity, according to definition 3.

Let some real number C, be given. Set N to the first natural number greater than C. Since $a_i = i$, we get $a_N = N > C$.

Then if $i \geqslant N$, $a(i) = i \geqslant N > C$.

We have therefore shown that for this sequence $a_n = n$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = \infty$

These kinds of infinite limits are also often called "improper limits".

Sequence with no limit

Now let's consider the sequence a_n (example 4 from above) with $a_i = \begin{cases} 1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even} \end{cases}$

This sequence keeps alternating between the numbers 1 and -1. We can intuitively see that it neither approaches a specific number, neither does it go off to infinity (negative infinity). So it seems like there is no limit! Let's use definition (2) to define what "not having a limit" means.

First let's rewrite definition (2) and then negate it.

A sequence a_n has limit l if for all $\varepsilon > 0$ $\varepsilon \in \mathbb{R}$, there exists $N \in \mathbb{N}$, such that: for all $i \geqslant N, i \in N \Rightarrow |a_i - l| < \varepsilon$.

If a sequence has no limit, than the above is not true for any number $l \in \mathbb{R}$. For any number l, there must be some open interval of radius ε around l, that the sequence keeps "escaping". So we get:

Definition 5 *The limit of a sequence a_n does not exist if:*

For all $l \in \mathbb{R}$, there exists an $\varepsilon > 0$, such that for all $N \in \mathbb{N}$, exists an $i \geqslant N$ with $|a_i - l| \geqslant \varepsilon$.

Now let's use this definition, to show that the sequence a_n from above does not indeed have a limit.

Let's take any $l \in \mathbb{R}$. Suppose $l \neq -1$. Then we can always take $\varepsilon = \frac{|l-(-1)|}{2} = \frac{|l+1|}{2}$ (half the distance between l and -1). Then for any $N \in N$ take the first even number i greater than -1.

Since i is even have: $a_i = -1. \Rightarrow ||a_i - l| = |-1-l| = |l-(-1)| \geqslant \frac{|l-1|}{2} = \varepsilon \Rightarrow$ always exists an i with $|a_i - l| \geqslant \varepsilon$.

Find an ε for the case where $l = -1$ yourself, and complete the proof.