Representing Linear Maps through Matrices

Definition 1 *Let V, W be finite dimensional vector spaces. Let $\alpha = \{v_1, ..., v_m\}$ be a basis for V and $\beta = \{w_1, ..., w_n\}$ be a basis for W. Let T be a linear map $T : V \to W$. We define a matrix representing T, to be a grid of numbers with n rows and m columns ${}_\beta\mathcal{M}_\alpha(T)$ where the entry in the ith row and jth column $c_{ij} = ({}_\beta\mathcal{M}_\alpha(T))_{ij}$ is defined via:*

$$Tv_j = c_{1j}w_1 + ... + c_{ij}w_i + ... + c_{nj}w_n = \sum_{i=1}^{n} c_{ij}w_i \tag{1}$$

We can now derive the rules for applying a matrix to a vector. Compute the expansion coefficients in the basis β of Tv for $v = a_1v_1 + ... + a_mv_m$ to be:

$$Tv = b_1w_1 + ... + b_nw_n$$

Where

$$b_i = \sum_{j=1}^{m} c_{ij}a_j$$

Now we let U be a vector space with basis $\gamma = \{u_1, ..., u_l\}$. Let $S : W \to U$ be a linear map. We want to define matrix multiplication such that:

$${}_\gamma\mathcal{M}_\alpha(S \circ T) = {}_\gamma\mathcal{M}_\beta(S) \cdot {}_\beta\mathcal{M}_\alpha(T) \tag{2}$$

Let $c_{ij} = ({}_\gamma\mathcal{M}_\alpha(S \circ T))_{ij}$
Let $a_{ij} = ({}_\beta\mathcal{M}_\alpha(T))_{ij}$
Let $b_{ij} = ({}_\gamma\mathcal{M}_\beta(S)_{ij}$

Compute that

$$c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} \tag{3}$$

It is easy to see that to change the input or output basis, you can multiply from left/right by identity martrix in appropriate basis:

$${}_{\beta'}\mathcal{M}_{\alpha'}(T) = {}_{\beta'}\mathcal{M}_\beta(Id) \cdot {}_\beta\mathcal{M}_\alpha(T) \cdot {}_\alpha\mathcal{M}_{\alpha'}(Id) \tag{4}$$