Proportional Line Segments in Right Angled Triangles

Theorem 1 *The height of a right angled triangle that passes through the right angle, splits the triangle into two similar right angled triangles, each similar to the original triangle.*

Proof

Consider Triangle ABC where $\angle C$ is a right angle (Figure 1). CD is a height from point C to the hypotenuse AB. Let's prove that: $\triangle ABC \sim \triangle ACD, \triangle ABC \sim \triangle CBD, \triangle ACD \sim \triangle CBD$.

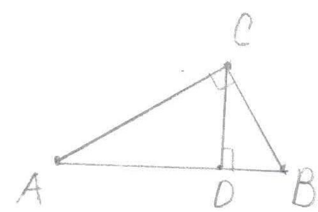

Figure 1

Triangles ABC and ACD are similar due to the first sign of similarity of triangles ($\angle A$ - common angle, $\angle ACB = \angle ADC = 90°$.) Hence: $\angle A = \angle BCD$.
In the same way, triangles ABC and CBD are similar ($\angle B$ - common angle, $\angle ACB = \angle BDC = 90°$).
Finally, triangles ACD and CBD are also similar due to the first sign of similarity of triangles: (Angle with vertex D are right angles and $\angle A = \angle BCD$).

$\therefore$ QED

Definition

A line segment XY is called the geometric mean of line segments AB and CD if:
$XY = \sqrt{AB \cdot CD}$

Now using the theorem above let's consider the following two statements.

Statement 1 *The height of a right angled triangle which passes through the right angle, is the geometric mean of line segments, into which the hypotenuse is split by this height.*

Proof

Consider triangle ABC, where $\angle C$ is the right angle (Figure 1). Let's draw a height CD from angle C to side AB. We need to prove that:
$CD = \sqrt{AD \cdot DB}$

Since $\triangle ADC \sim \triangle CBD$ then: $\frac{AD}{CD} = \frac{CD}{DB}$. Hence:
$CD^2 = AD \cdot DB$
$CD = \sqrt{AD \cdot DB}$
$\therefore$ QED

Statement 2 *The cathetus (a side adjacent to the right angle) of a right angled triangle is the geometric mean of the hypotenuse and the segment of the hypotenuse between that cathetus and the height passing through the right angle.*

Proof

Consider triangle ABC, where $\angle C$ is the right angle (Figure 1). Let's draw a height CD from angle C to side AB. We need to prove that:
$AC = \sqrt{AB \cdot AD}$ Since $\triangle ABC \sim \triangle ACD$ then: $\frac{AB}{AC} = \frac{AC}{AD}$. Hence:

$AC^2 = AB \cdot AD$
$AC = \sqrt{AB \cdot AD}$
$\therefore$ QED