Remark on the Tangential Frame of Reference

Consider some object undergoing circular motion (any kind, as long as radius is constant). We will then claim the following statement:

Remark 1 *For a body undergoing circular motion the radial component of its acceleration vector in the tangential frame of reference will always be equal to:*

$$a_r = (\dot{\gamma})^2 \cdot l \tag{1}$$

Where:

γ - angular position of the object (angle radius makes with some fixed diameter) $\Rightarrow$ $\dot{\gamma}$ - angular speed of the object (recall the dot notation to refer to a derivative of time. Also yes, I am using γ and not θ, because θ is a dumb letter. At the same time, please do not confuse γ with angular acceleration. In this handout we will denote angular acceleration as $\ddot{\gamma}$.)

l - radius of the circular trajection of object.

Let's try and prove this statement.

Consider the following diagram:

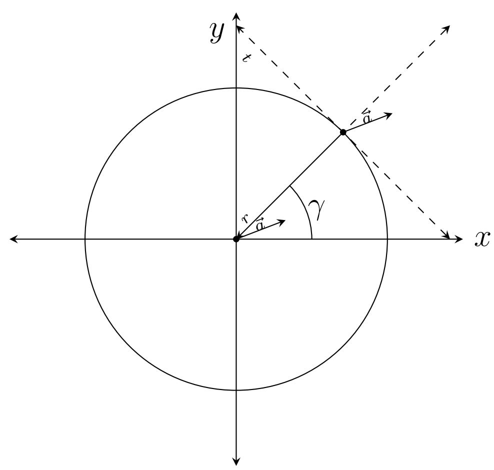

Figure 1: An object is moving around in a circle. At some instance it has an angular position γ. Introduce a tangential reference frame with positive direction of t going counter clockwise and positive direction of r towards the center of the circle.

Let $\vec{a}$ be its acceleration vector in the x, y reference frame.
Let's find these x, y coordinates of our acceleration vector. Suppose that the angular position is give as a function of time $\gamma(t)$ and the radius of the circle is l.

From here on, we will use the dot notation to denote time derivatives.

Let's consider the x and y position coordinates of the object in terms of γ.

$x = l \cos \gamma$
$y = l \sin \gamma$

Now let's find the x and y components of the velocity vector by differentiating the x and y coordinates.

$v_x = l\,(\cos\gamma)' = -\dot\gamma\sin\gamma$
$v_y = l\,(\sin\gamma)' = \dot\gamma\cos\gamma$

Now let's find the x and y components of the acceleration vector by differentiating v_x and v_y components.

$$a_x = l(-\dot\gamma\sin\gamma)' = -l\left(\ddot\gamma\sin\gamma + (\dot\gamma)^2\cos\gamma\right) \tag{2}$$

$$a_y = l(\dot\gamma\cos\gamma)' = l\ddot\gamma\cos\gamma - l(\dot\gamma)^2\sin\gamma \tag{3}$$

Now we have the x, y coordinates of $\vec{a}$.

We want to rewrite the x, y coordinates of $\vec{a}$ as r, t coordinates.

As seen in Figure 1, we can then "move" $\vec{a}$ to the origin of the tangential frame of reference and it will still be the same vector.

Let's consider this "moved" vector. Introduce an x, y reference frame with same origin as the tangential reference frame. Now consider the r and t components of $\vec{a}$ and express them in terms of the x and y components of $\vec{a}$

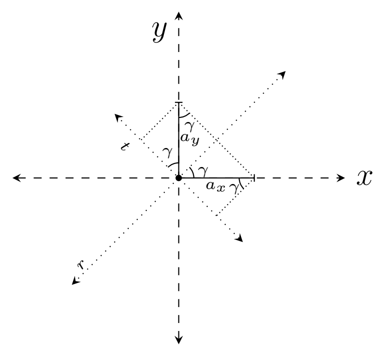

Figure 2: Draw perpendiculars from a_x and a_y to r and t axes. Then find which angles are equal to γ

From the diagram above we get that:

$a_r = -(a_y\sin\gamma + a_x\cos\gamma)$

$a_t = a_y\cos\gamma - a_x\sin\gamma$

Substitute equations (2) and (3) into equations above and simplify to get:

$$\begin{aligned}
a_r &= -(a_y\sin\gamma + a_x\cos\gamma)\\
&= -\left((\ddot\gamma\cos\gamma - (\dot\gamma)^2\sin\gamma)l\sin\gamma - \left(\ddot\gamma\sin\gamma + (\dot\gamma)^2\cos\gamma\right)l\cos\gamma\right)\\
&= \cancel{-l\ddot\gamma\sin\gamma\cos\gamma} + l(\dot\gamma)^2\sin^2\gamma + \cancel{l\ddot\gamma\sin\gamma\cos\gamma} - (-l(\dot\gamma)^2\cos^2\gamma)\\
&= l(\dot\gamma)^2(\sin^2\gamma + \cos^2\gamma)\\
&= l(\dot\gamma)^2
\end{aligned}$$

This is the statement that we wanted to prove. Q.E.D.

Since we are here anyways, let's find a_t:

$$
\begin{aligned}
a_t &= a_y \cos\gamma - a_x \sin\gamma \\
&= l(\ddot{\gamma}\cos\gamma - (\dot{\gamma})^2 \sin\gamma)\cos\gamma + l\left(\ddot{\gamma}\sin\gamma + (\dot{\gamma})^2\cos\gamma\right)\sin\gamma \\
&= l\ddot{\gamma}\cos^2\gamma - \cancel{l(\dot{\gamma})^2 \sin\gamma\cos\gamma} + l\ddot{\gamma}\sin^2\gamma + \cancel{l(\dot{\gamma})^2 \sin\gamma\cos\gamma} \\
&= l\ddot{\gamma}(\cos^2\gamma + \sin^2\gamma) \\
&= l\ddot{\gamma}
\end{aligned}
$$