1 Review

Definition 1 *Suppose we have outcomes λ which occur with probability $\mathcal{P}(\lambda)$. Then the expectation value of the process X of measuring these outcomes is defined as:*

$$\mathbf{E}(X) = \sum_{\lambda} \lambda \mathcal{P}(\lambda) \tag{1}$$

Theorem 1 *Suppose we are able to prepare a system to be in state ψ each time. The expectation value of making a measurement associated with observable A is then given by*[1]*:*

$$\langle A \rangle_{\psi} \equiv \mathbf{E}(A) = \langle \psi, A\psi \rangle \tag{2}$$

Definition 2 *Suppose we have a process X of measuring some outcomes with expectation value $\mathbf{E}(X)$. The standard deviation is defined as:*

$$\sigma_X^2 \cong E((X - \mathbf{E}(X))^2) = \sum_{\lambda} (\lambda - E(X))^2 \mathcal{P}(\lambda) \tag{3}$$

Theorem 2 *Suppose we prepare a system to be in a state ψeach time. We then consider the measurement associated with observable A. We call the standard deviation of this measurement the uncertainty. It is then computed via:*

$$\sigma_X^2 = \langle (A - \langle A \rangle_{\psi})^2 \rangle_{\psi} \tag{4}$$

Definition 3 *Let A, B be two linear operators. We define the commutator to be:*

Postulate 1 *The dynamics of a quantum state over time is given via the solution to the equation:*

$$i\hbar \partial_t |\psi(t)\rangle = H|\psi(t)\rangle$$

2 Suggested Exercises

I) **Warmup**. Suppose we have some two-level spin system. This means that we have some observable associated to a spin measurement, which in the $|\uparrow\rangle, |\downarrow\rangle$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i) For each of the following states, what is the probability to measure $\uparrow$? What about $\downarrow$?

$$\begin{gathered} |\psi_1\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \quad |\psi_2\rangle = \frac{3}{5}|\uparrow\rangle + \frac{4}{5}|\downarrow\rangle \\ |\psi_3\rangle = |\uparrow\rangle \quad |\psi_4\rangle = \frac{8}{10}|\uparrow\rangle + \frac{6}{10}|\downarrow\rangle \end{gathered} \tag{5}$$

ii) For each of the states above, compute the expectation value of S_z.

iii) Now let's define this basis [2]

$$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \, |\downarrow\rangle_x = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$$

Suppose that we have some other observable that in this basis is:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i. What is the expectation value of S_x in state $|\psi_1\rangle$? What about $|\psi_2\rangle$?

ii. Suppose we are in state $|\psi_2\rangle$. A measurement associated to S_x is made, and we get $\frac{\hbar}{2}$. What is the probability that if we did the measurement associated with S_z we also got $\frac{\hbar}{2}$?

[1] We now change conventions for the inner product to be linear in the second entry

[2] It is indeed a basis.

II) **Uncertainty Principle**.

i) During our session we showed theorems (1) and (2) in the case where all eigenvalues are distinct (non-degenerate case). Show that the theorems still hold if there are different eigenvectors with the same eigenvalue (degenerate case).

ii) We also showed that if you have observables p and x with $[x,p] = i\hbar$ then the following inequality is satisfied:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

Now suppose we had any two observables A, B with $[A, B] = k$. Derive an uncertainty principle for this case? What if we set $k = 0$? Why does this make sense?

III) **Particle in a Box Revisited**. Recall we considered the "particle in a box example". For a box of size L we had the following Hilbert Space.

$$\mathcal{H} = \{ f : [0, L] \to \mathbb{C} \mid \int_0^L |f|^2 < \infty \; f(0) = f(L) = 0 \; \}$$

Where the inner product is give via:

$$\langle f, g \rangle = \int_0^L \bar{g} f$$

We also defined operators:

$$p = -i\partial_x \quad x = x \cdot$$

Where x is just multiplication by x: $\psi(x) \to x \cdot \psi(x)$.
We then define the Hamiltonian (observable associated with energy).

$$H = \frac{p^2}{2m}$$

One thing I have not previously told you (and we will discuss in more detail later), is that the probability of the particle to be found in the region $[a, b] \subseteq [0, L]$ is given via:

i) Find the eigenstates of the Hamiltonian and their eigenvalues. What is their physical significance?

ii) Compute the following commutators:

$$[x, p] \quad [x, p^2] \quad [x, p^n]$$

Hint: Use Induction for the last one

iii) Let $|E_n\rangle$ be the state associated to the nth energy level. Let $\psi(t = 0, x) = \frac{1}{\sqrt{2}}(|E_0\rangle + |E_1\rangle)$. What is $\psi(t, x)$?.

iv) Compute the expectation value of the operator x as a function of time t.

v) Setting all constants ($\hbar, m, L$ etc.) to 1, write code in python that plots $|\psi(t, x)|^2$ on the range of $[0, L]$. Make a plots over a range of time of your choice and save them as .pngs. Then combine that in a gif and see how the particle "moves" around. How does this relate to your answer in the previous part?

vi) Write down equations of motion for $\rangle x \langle_\psi$ and $\rangle p \langle_p si$ for arbitrary ψ (*Hint: Use Schrödinger equation and maybe assume a product rule for inner products*)