Some review and terminology:

Lagrangian given by:

$$L = T - V \tag{1}$$

Where T - Kinetic Energy of entire system, V - Potential Energy of entire system.

From the least action principle we derived the following equations (Euler-Lagrange Equations):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{2}$$

Where q_i are generalized coordinates.

1. Consider some object of mass m moving in the $2d$ plane. Consider a cartesian coordinate system x, y. Now consider another coordinate system, where we denote the position of a particle with its distance from the origin $r \geq 0$ and the angle φ (in radians) that its radius vector makes with the positive x-axis (think unit circle angle).

   (a) Write down the x, y coordinates in terms of r and φ.

   (b) First of all, notice that it is fairly easy to write down $F = ma$ in the x, y coordinate system. It is however rather unclear how to write down $F = ma$ in terms of r, φ coordinates. What are the forces in the r and φ directions? What ARE the r and φ directions? Below is a convenient choice that is often made. Compute $F_{\hat{\varphi}}$ and $F_{\hat{r}}$ in terms of F_x and F_y.

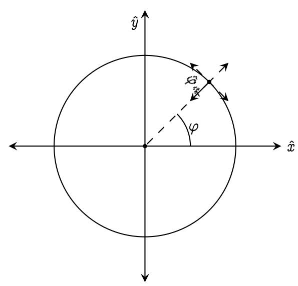

Figure 1: Consider the reference frame given by the $\hat{r}$ and $\hat{\varphi}$ axes.

   (c) Now suppose the particle travels some path $\gamma = (x(t), y(t))$ in the x, y coordinate system. From part 1), you have $x(t)$, $y(t)$ in terms of r and φ. Take a bunch of derivatives and express $\ddot{x}$ and $\ddot{y}$ in terms of derivatives of r and φ

   (d) Use the result above, the results for $F_{\hat{\varphi}}$ and $F_{\hat{r}}$ from part and Newton's second law to write down equations for $F_{\hat{r}}$ and $F_{\hat{\varphi}}$ in terms of r, φ and their derivatives. Recognize that this is very much not Newton's usual $F = ma$ when observed from this weird rotating reference frame. (hence we want Euler Lagrange equations to help us out).

   (e) Show that if the object is moving along a circle (it is not moving radially) at a constant speed, then we have $F_r = \frac{mv^2}{r}$

2. Consider a rigid pendulum of length l with mass m attached at the end. Introduce an x, y reference frame, with origin at center of circle the mass travels. Alternatively, introduce α as a coordinate (angle between position of mass and vertical).

   (a) Draw a free body diagram on the mass in the x, y reference frame. Write down Newton's Equations in the tangential reference frame from the previous question.

   (b) Now consider the Lagrangian for the system. Write down the Lagrangian in terms of α and $\dot{\alpha}$. Write down the Euler Lagrange equations.

   (c) Assuming that α is small (i.e. $\sin\alpha \approx \alpha$), show that the Euler Lagrange equation is identical to the equation for a spring ($m\ddot{x} = -kx$)

   (d) Solve the equation for α. Figure out what the period of oscillation is.

3. Consider the following system. You have a block of mass m_1 on a table. It is attached to a block of mass m_2, via string passing through a pulley. The second block is suspended on that string. See Figure (2). Assume that the pulley is massless and frictionless.

    (a) First assume that the table does not have friction. Draw a free body diagram for each block. Write down Newton's equations of motions. Choose appropriate coordinates for the masses, write down the Lagrangian and the Euler-Lagrange equations. Solve these equations of motion, assuming that the masses start at rest.

    (b) Now assume that the table has some static friction, with coefficient μ_k. Draw a new free body diagram. Write down and solve Newton's equations, assuming that the mass m_1 has some initial horizontal velocity v_1.

    (c) Suppose that the surface also has a coefficient of static friction μ_s. Suppose mass m_1 initially has a speed of v_0 moving the left in Figure (2). How light does the mass m_2 have to be, such that the system eventually ends at rest.

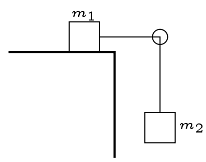

Figure 2: Two blocks connected by massless string. Block 1 is on a flat surface and block 2 is suspended on string.