Kinetic Energy Formula

This is the derivation for the kinetic energy formula in a 1-dimensional case.

Consider an object with some mass m at rest. It has no kinetic energy. A force is applied making the object move and accelerate to some speed v. The object's kinetic energy will be equal to the work done by the net force applied to it.

First let's consider the case where this object accelerates uniformly at some rate a. Since acceleration is constant, the applied force is also constant.

Work is:
$W = E_k = F \cdot d$

Where d is the distance the object traveled while a force was applied to it.

Substitute $F = ma$ into the equation above to get:

$E_k = ma \cdot d \qquad (1)$

Recall the kinematics formula: $v^2 = v_1^2 + 2ad$

Where v_1 and v are initial and final speed respectively.

In our case, the object starts at rest, hence $v_1 = 0$.

$v^2 = 2ad$
$a = \frac{v^2}{2d}$

Substituting this into (1) get:

$E_k = m\frac{v^2}{2d} \cdot d = \frac{1}{2}mv^2$

We have derived the kinetic energy formula for constant acceleration. What if the object does not accelerate uniformly? Will the formula be the same?

Consider an object with a mass m at rest. It is then put into motion by some force F. Let

$s(t)$ be the function of the distance the object travels with respect to time.

Then (using the properties of the definite integral):

$W_k = E_k = \int_0^t F ds(t)$

Recall: $a = \frac{d^2 s(t)}{dt^2}$
Hence:

$F = ma = m \cdot \frac{d^2 s(t)}{dt^2}$

Substituting this into the equation above we get:

$E_k = \int_0^t m \cdot \frac{d^2 s(t)}{dt^2} ds(t) = m \int_0^t \frac{d^2 s(t)}{dt^2} \cdot \frac{ds(t)}{dt} dt = \frac{1}{2} m (\frac{ds(t)}{dt})^2$

But we know that $\frac{ds(t)}{dt} = v$. So:

$E_k = \frac{1}{2} m v^2$

Therefore the formula above is true for all types of motion.