Unit Circle

Let's consider a number line. A number line has some reference starting point O (where 0 is), a scale (how large 1 unit is) and a positive direction. How do we find some point M using its coordinate x on a number line? We start at 0 (which is point O). If $x > 0$ then we travel a distance x in the positive direction. If $x < 0$ then we travel a distance x in the negative direction. The end of this distance will be point M. To find x given point M, would mean finding the distance of OM, and giving it a positive or negative sign depending where M is located relative to O. In reality we don't always run in a straight line. An example of such would be running on a track in a circle. How would we express coordinate points on a circle?

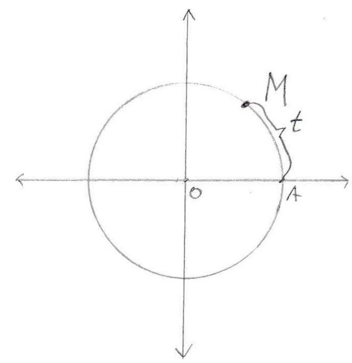

Figure 1

Definition

Given a unit circle (a unit circle is a circle with a radius of 1 unit) the starting point A - the right end of the horizontal diameter (Figure 1). Let's give every real number t a point on the circle using the following rule:
1) If $t > 0$, then "draw" some segment AM with a length t on the circumference of the circle starting at point A and going counter-clockwise (the positive direction). Point M will be the desired point.
2) If $t < 0$, the "draw" some segment AM with a length t on the circumference of the circle starting at point A and going clockwise (negative direction). M will be the desired point.
3) if $t = 0$ then A is the desired point.

For Example: consider point C (Figure 2). If we count counter-clockwise from A, point C is a quarter of a circle away from A, so its position away from A can be expressed as as a quarter of the total circumference of the circle. Since in a unit circle, the radius is 1, then its total circumference is $2\pi r = 2\pi$. A quarter of the total circumference is: $t = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$. C is located $\frac{\pi}{2}$ units away from A. At the same time if we count clockwise (in the negative direction), we need to go three quarters of the circle to get to point C. Hence C is also located at $-\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$ units away from A. But how do we relate these two values?

Consider the following statement:
If a point M on a unit circle corresponds to some value t, then it also corresponds to a value in the form of: $t + 2\pi k$ where k is any whole number ($k \in \mathbb{Z}$).

Indeed: 2π is the circumference of one circle. If you go one full circle either direction (positive or negative) you still end up in the same spot. k is then the number of full circles you end up going.

This is why point C is located at $t = \frac{\pi}{2} = \frac{\pi}{2} - 2\pi = -\frac{3\pi}{2}$

Radians

A point on the unit circle can also be used to express an angle. Given some point B its position on the unit circle represents the angle measure of $\angle AOB$ (Figure 2): where O is the center of the circle and A is the right end of the horizontal diameter. This means, that the size of any angle can be expressed by the position of a point on the unit circle. This unit of measurement is a radian (The symbol is: rad). For Example: Consider point D. It represents $\angle AOD$. Point D is located at π on the unit circle. Hence we can say that $\angle AOD = \pi$ rad.

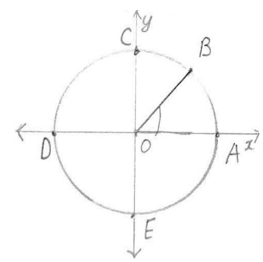

Figure 2

Let's now relate radians and degrees:
Let α be the measure of $\angle AOB$ in degrees and t be the measure of $\angle AOB$ in radians.

A full unit circle has a circumference 2π (since the radius of a unit circle is equal to 1 unit). The fraction of the circumference which is taken up by AB is also the fraction of the $360°$ taken up by $\angle AOB$.
AB to circumference is then: $\frac{t}{2\pi}$.
A full circle is $360°$. The fraction of the circle which is angle AOB is then also $\frac{\alpha}{360°}$. From here we get:
$\frac{t}{2\pi} = \frac{\alpha}{360°}$
Using this formula if we know the measurement of an angle in radians we can easily convert it to degrees and vice versa.

Let's consider a few examples (Figure 2):

$\angle AOC = 90°$. First rearrange the formula above and get:
$t = 2\pi \cdot \frac{\alpha}{360}$
If $\alpha = 90°$ then:
$t = 2\pi \cdot \frac{90°}{360°} = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$ rad
This also makes sense visually as $90°$ is a quarter of the circle. If the full circle is 2π then a quarter is: $\frac{1}{4} \cdot 2\pi$.
At the same time, $\angle AOC = -\frac{3\pi}{2}$.

Using the formula above we get that $\angle AOC = -\frac{3\pi}{2} = \frac{\frac{-3\pi}{2}}{2\pi} \cdot 360° = -270°$
On a unit circle we can now see that $-270°$ and $90°$ are the same angle. Explain why (Hint: use the same logic as shown on the previous page).

Exercises

1. What are the coordinates of points (write them down in two different ways): A, B,C,D. (Figure 3)(Eye ball it)
2. Express the following angles in both radians and degrees in two different ways for each unit : $\angle AOD$, $\angle AOE$, $\angle AOA$ (Figure 2).
E.G. $\angle AOC = -\frac{3\pi}{2}$ rad $= \frac{\pi}{2}$ rad $= 90° = -270°$
3. Express the following angles in radians in two ways: $45°; 60°; 300°; 330°$

Answers

1. A - $2\pi k$ B $\frac{\pi}{4}+2\pi$ C - $\frac{7\pi}{6}+2\pi k$ D - $\frac{11\pi}{6}+2\pi k \quad k\in\mathbb{Z}$
2. $\angle AOD=\pi+2\pi k$ rad $=180^\circ+360^\circ k$; $\angle AOE=\frac{3\pi}{2}+2\pi k$ rad $=270^\circ+360^\circ k$; $\angle AOA=2\pi k$ rad $=360^\circ k$ $k\in\mathbb{Z}$
3. $45^\circ=\frac{pi}{4}+2\pi k$ rad ; $60^\circ=\frac{\pi}{3}+2\pi k$ rad ; $300^\circ=\frac{5\pi}{3}+2\pi k$ rad; $330^\circ=\frac{11\pi}{6}+2\pi k$, $k\in\mathbb{Z}$

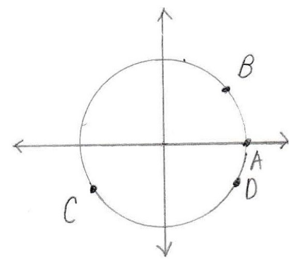

Figure 3