1 Definitions & Review

Definition 1 *An alternating k tensor ω on a vector space V is an alternating multilinear map. Meaning that it has the following two properties:*

$$\omega(v_1, ..., v_i + \lambda u, ..., v_k) = \omega(v_1, ...v_i, ..., v_k) + \lambda\omega(v_1, ..., u, ..., v_k) \qquad \forall i$$

$$\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k) \qquad \forall i \neq j$$

Definition 2 *A differential k - form ω on smooth manifold M is a "k-form" at each point $p \in M$, which varies smoothly in p. We denote the set of $k-$forms on M via $\Omega^k(M)$.*

$$\omega(p, v) : T_pM \times ... \times T_pM \to \mathbb{R}$$

Definition 3 *We define a basis of $\Omega^1(\mathbb{R}^n)$ of forms: $\{dx^1, dx^2, ..., dx^n\}$, where $dx^i : \mathbb{R}^n \to \mathbb{R}$ is defined as:*

$$dx^i(a_1e_1 + ... + a_ne_n) = a_i$$

Definition 4 *Let $\omega^k \in \Omega^k(M)$ and $\omega^l \in \Omega^l(M)$ be differential forms on a smooth manifold M. We define their wedge product $\omega^k \wedge \omega^l \in \Omega^{k+l}(M)$ as:*

$$(\omega^k \wedge \omega^l)(v_1, ..., v_{k+l}) = \sum (-1)^\nu \omega^k(v_{i_1}, ..., v_{i_k})\omega^l(v_{i_{k+1}}, ..., v_{i_{k+l}})$$

Where ν is the sum of the permutation $i_1, ..., i_{k+l}$ and the sum is taken over all permutations such that $i_1 < i_2 < ... < i_k$ and $i_{k+1} < i_{k+2} < ... < i_{k+l}$ separately.

Theorem 1 *The set $\Omega^k(\mathbb{R}^n)$ is a vector space with a basis given by:*

$$\Omega^k(\mathbb{R}^n) = span(\{dx^{i_1} \wedge ... \wedge dx^{i_k}\})$$

Where the set considers all combinations $\{i_1, ..., i_k\} \subseteq \{1, ..., n\}$ such that $i_1 < i_2 < ... < i_k$

Corollary 1 *The set $\Omega^n(\mathbb{R}^n)$ is a 1 dimensional vector space:*

$$\Omega^n(\mathbb{R}^n) = span(dx^1 \wedge ... \wedge dx^n)$$

Note that $dx^1 \wedge ... \wedge dx^n$ is the usual determinant

Definition 5 *Let $\varphi : N \to M$, where N and M are smooth manifolds. Let $\omega \in \Omega^k(M)$. We define the pullback of φ as an operation on forms:*

$$\varphi^* : \Omega(M) \to \Omega(N)$$
$$(\varphi^*\omega)(x; v_1, ..., v_k) = \omega(\varphi(x); D\varphi v_1, ..., D\varphi v_n)$$

Where $x \in N$ and $v_i \in T_xN$

Definition 6 *Let $\omega \in \Omega^n(\mathbb{R}^n)$ be a top form on $\mathbb{R}^n$. From corollary 1, we know it must be of the form $\omega = f(x_1, ..., x_n)dx^1 \wedge ... \wedge dx^n$. We then define the integral of this form over some open set $U \subseteq \mathbb{R}^n$ to be the usual thing:*

$$\int_U \omega = \int_U f(x_1, ..., x_n)dx^1...dx^n$$

Definition 7 *Let M be a manifold, $U \subseteq M$ an open set. Let $\varphi : V \to U \subseteq M$ be a chart, such that $\varphi(V) = U$ and $V \subseteq \mathbb{R}^n$. Suppose $\omega \in \Omega^n(M)$. We then define:*

$$\int_U \omega = \int_V \varphi^*\omega$$

It can be shown that this integral does not depend on the choice of coordinate chart φ.

Definition 8 *Let $\omega \in \Omega^k(M)$. Let $x \in M$. Pick some coordinate chart $\varphi : U \to M$ $U \subseteq \mathbb{R}^n$. Let $C(v_1, ..., v_n)$ be a parallelepiped with sides $v_1, ..., v_n$ starting at x. Define:*

$$A(v_1, .., v_n) = \int_{C_{v_1,...,v_n}} \varphi^* \omega$$

To lowest order, $A(v_1, ..., v_n)$ is approximated by a multilinear $k+1$ form, which depends smoothly on x. We call this from $d\omega$. This defines the exterior derivative operator:

$$d : \Omega^k(M) \to \Omega^{k+1} M$$

Theorem 2 *The exterior derivative $d : \Omega^k(M) \to \Omega k + 1(M)$ has the following properties, which are equivalent to its definition:*

i) $d(\omega_1 + \lambda\omega_2) = d\omega_1 + \lambda d\omega_2$

ii) $d\left(\omega^k \wedge \omega^l\right) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l$

iii) $d \circ d = 0$

iv) Let $f(x) \in \Omega^0(M)$ be a smooth function. Then $df = \frac{\partial f}{\partial x_1} dx^1 + ... + \frac{\partial f}{\partial x_n} dx^n$

Theorem 3 *Stokes Theorem: Let $\omega \in \Omega^{n-1}(M)$. Then:*

$$\int_{\partial M} \omega = \int_M d\omega$$

2 Suggested Exercises

I) Getting used to differential forms:

   i) Write the following explicitly in terms of dx^i and v_j

$$(dx^1 \wedge dx^2)(v_1, v_2) \qquad (dx^1 \wedge dx^2 \wedge dx^1)(v_1, v_2, v_3)$$

II) Integrating Differential Forms

   i) Let $\omega = ydx - xdy$. Let $S^1 \subseteq \mathbb{R}^2$ be the circle. Let $\varphi : (0,1) \to S^1$ be $\varphi(t) = (\cos(t), \sin(t))$. Compute $\int_{S^1} \omega$. Interpret your answer.

   ii) Let $\omega = xdy \wedge dzydx \wedge dz + zdx \wedge dy$ and $\varphi : (0,\pi) \times (0, 2\pi) \to S^2$ be $\varphi(\theta\,\varphi) = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$. Compute $\int_{S^2} \omega$. Interpret your answer.

III) Exterior Derivative. Compute the following:

   i) $d(\cos(y)dx - \sin(x)dy)$

   ii) Compute $d(f(x,y)dx + g(x,y)dy)$. Write down stokes theorem for these forms (this is called Green's theorem)

   iii) Compute $d(xdy \wedge dz + ydx \wedge dz)$

IV) Relation to grad, div and curl.

Consider the following set of maps:

$$\psi_1 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3)$$
$$\psi_1(f(x,y,z)e_1 + g(x,y,z)e_2 + h(x,y,z)e_3) = fdx + gdy + hdz$$

$$\psi_2 : \mathfrak{X}(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3)$$
$$\psi_2(f(x,y,z)e_1 + g(x,y,z)e_2 + h(x,y,z)e_3) = fdy \wedge dz - gdx \wedge dz + hdx \wedge dy$$

$$\psi_3 : C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$$
$$\psi_3(f(x,y,z)) = f(x,y,z)dx \wedge dy \wedge dz$$

Show that the following diagram commutes:

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "Id"] \arrow[d, "grad"] & \Omega^0(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \arrow[d, "d"] \\
\mathfrak{X}(\mathbb{R}^3) \arrow[r, "\psi_1"] \arrow[d, "curl"] & \Omega^1(\mathbb{R}^3) \arrow[d, "d"] \\
\mathfrak{X}(\mathbb{R}^3) \arrow[r, "\psi_2"] \arrow[d, "div"] & \Omega^2(\mathbb{R}^3) \arrow[d, "d"] \\
C^\infty(\mathbb{R}^3) \arrow[r, "\psi_3"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

In other words, show for example that doing curl on a vector field, is the same as applying d to some 1-form, where we identify forms and vector fields using ψ_1 and ψ_2. i.e. curl $= \psi_2^{-1} \circ d \circ psi_1$