0.1 Antiderivative and Indefinite Integral

Definition #1
Function $y = F(x)$ is called the antiderivative of a function $y = f(x)$ in a given domain X, if for all x in X the following is true: $F'(x) = f(x)$

Theorem 1 *If $y = F(x)$ is the antiderivative of a function $y = f(x)$ on an interval X, then the function $y = f(x)$ has infinitely many antiderivatives and all of them have the form $y = F(x) + C$*

Proof
Let $y = F(x)$ be the antiderivative of a function $y = f(x)$ on X. This means that $F'(x) = f(x)$. Let's find the derivative of $y = F(x) + C$

$(F(x) + C)' = F'(x) + C' = f(x) + 0 = f(x)$

This means that $y = F(x) + C$ is an antiderivative of function $y = f(x)$

Now we need to prove that this form encompasses the set of all possible antiderivatives.

Let $y = F_1(x)$ and $y = F(x)$ be two antiderivatives for function $y = f(x)$ on the domain X. This means that for all x on X the following is true:
$F_1'(x) = f(x)$ and $F'(x) = f(x)$

Consider a function $y = H(x)$ where $H(x) = F_1(x) - F(x)$. Let's find the derivative:

$H'(x) = (F_1'(x) - F(x))' = F_1'(x) - F'(x) = f(x) - f(x) = 0$

It is known that if the derivative of $y = H(x)$ on the domain X is equal to zero, then the function is constant on X.
This means that $H(x) = C$.
Hence: $F_1(x) - F(x) = C \Rightarrow F_1(x) = F(x) + C$

$\therefore QED$

Definition # 2
If a function $y = f(x)$ on some domain X has an antiderivative $y = F(x)$, then the set of all antiderivatives (the set of function in the form of $y = F(x) + C$) is called the indefinite integral. It is written as:

$\int f(x)dx$