Volume, Tensors and Alternating Forms

Let V be an n dimensional vector space over $\mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$).

Let $g : V \times V ... \times V \to \mathbb{F}$, be a map that takes k vectors as input and outputs a number in the field. We call g a k-multilinear map or a $k-$tensor iff

$$g(v_1, ..., v_i + \lambda u, ..., v_k) = g(v_1, ..., v_i, ..., v_k) + \lambda g(v_1, ..., u, ..., v_k) \quad \forall 1 \leq i \leq k$$

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(I) Practice with tensors. Let $V = \mathbb{R}^2$. We denote a vector in $v = (x, y)$. Which of the following are $k-$tensors?

(a) $g((x_1, y_1), (x_2, y_2)) = x_1 x_2 + y_1 y_2$
(b) $g((x_1, y_1), (x_2, y_2)) = x_1$
(c) $g((x_1, y_1), (x_2, y_2)) = x_1 y_1$
(d) $g((x_1, y_1), (x_2, y_2), (x_3, y_3)) = x_1 y_2 x_3$
(e) Prove that $1-$tensors are just linear functionals

(II) Volume. Consider the following problem. Let V be an n-dimensional vector space. Suppose we are given n vectors: $u_1, ..., u_n$. They define some parallepiped. How should we define the oriented volume of this shape? This is perhaps not an obvious problem in n- dimensions, so let's look at the 2-dimensional case, where $V = \mathbb{R}^2$.

(i) Draw two arbitrary vectors u_1, u_2 in $\mathbb{R}^2$. What is the "parallepiped" that they create. Shade it in.

(ii) We want to define the volume of this shape. This should be some function that takes in 2 vectors, and outputs the area of the parallelogram that it makes. Argue that this function has to be a $k-$tensor (i.e. it has to be linear). From now on we denote the k-multilinear function that gives the area of the parallelogram formed by two vectors u_1, u_2 as

$$\omega(u_1, u_2)$$

(iii) We want to consider the case when we make a parallelogram out of two collinear (parallel) vectors. Well then there is no parallelogram, it is just a line! So the volume should be 0! From this we demand the following condition:

$$\omega(v, v) = 0 \quad \forall v$$

Prove that this is equivalent to the following condition (*Hint: Consider the vector* $u - v$):

$$\omega(u_1, u_2) = -\omega(u_2, u_1) \quad \forall u_1, u_2 \in V \tag{1}$$

Tensors with this condition are called alternating tensors [1].
From now on we define ω to be an alternating 2-tensor.
Prove that if u_2 is a scalar multiple of u_1, then $\omega(u_1, u_2) = 0$. Why does this make sense as a definition of volume?

(iv) Now let e_1, e_2 be the standard basis of $\mathbb{R}^2$. Let u_1, u_2 be arbitrary vectors. Express u_1, u_2 in the standard basis, substitute into ω, expand and simplify.

(v) Using your expansion above, show what evaluation of ω needs to be specified, to fully determine ω on all of $\mathbb{R}^2 \times \mathbb{R}^2$.

(vi) Show that any alternating 2-tensor on $\mathbb{R}^2$, is a scalar multiple of an alternating tensor with $\omega(e_1, e_2) = 1$.

(vii) Show that if $\omega_(e_1, e_2) = 1$ then $\omega(u_1, u_2)$ is the determinant of a $2x2$ matrix with columns u_1 and u_2.

(III) Volume of n-dimensional parallelepiped. Let $\omega(u_1, ..., u_n)$ be an $k-$tensor on $\mathbb{R}^n$ (i.e $u_i \in \mathbb{R}^n$).
We define ω to be an alternating $k-$tensor, if it has the property that:

$$\omega(u_1, ..., u_i, ..., u_j, ..., u_k) = -\omega(u_1, ..., u_j, ..., u_i, ..., u_k)$$

(i.e. swapping two inputs, changes output by a minus sign).
Let $\det(u_1, ..., u_n)$ be an alternating $n-$tensor with $\det(e_1, ..., e_n) = 1$.

- Prove that ω being an alternating k-tensor is equivalent to omega being a k-tensor such that $\omega(u_1, ..., u_k) = 0$ if $u_i = u_j$ for any i, j.
- Prove that the condition above fully determine the function $\det(u_1, ..., u_n)$ (*Hint: Expand* u_1,...,u_n *in some basis and use property above*
- Show that any alternating$-n$ tensor over $\mathbb{R}^n$ is a scalar multiple of det

[1] You can maybe think of this condition as "sweeping the area" from the u_1 towards u_2 vs from u_2 towards u_1.