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Topology

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Abstract

1 Russell's Paradox

Simplest Case: The set of all sequences of 0s and 1s is uncountable.
Suppose we could count such sequences

$$A_0 = (a_{00}, a_{01}, a_{02}, ...)$$
$$A_1 = (a_{10}, a_{11}, a_{12}, ...)$$
$$A_2 = (a_{20}, a_{21}, a_{22}, ...)$$

Now, define $B = (b_0, b_1, b_2, ...)$, where

$$b_0 = \begin{cases} 1 \text{ if } a_{00} = 0 \\ 0 \text{ if } a_{00} = 1 \end{cases} \quad b_1 = \begin{cases} 1 \text{ if } a_{11} = 0 \\ 0 \text{ if } a_{11} = 1 \end{cases} \quad etc. \tag{1}$$

Now, $B \neq A_0$ because $b_0 \neq a_{00}$, $B \neq A_1$ because $b_1 \neq a_{11}$, etc.
So B is not in the list $A_0, A_1, A_2, ...$ and so the set of all sequences of 0s and 1s is uncountable. COME BACK ADD NOTATIONS PART??

More verbosely, according to the unrestricted comprehension principle, for any sufficiently well-define property, there is the set of all and only the objects that have that property.

Let R be the set of all sets that are not members of themselves. If R is not a member of itself, then its definition entails that it is a member of itself; if it is a member of itself, then it is not a member of itself, since it is the set of all sets that are not members of themselves. The resulting contradiction is Russell's Paradox. In symbols:

$$\text{Let } R \coloneqq \{x | x \notin x\}, \text{ then } R \in R \iff R \notin R$$

1.1 Formal Presentation

The term ”naive set theory” is used in various ways. In one usage, naive set theory is a formal theory, that is formulated in a first order language with a binary non-logical predicate $\in$, and that includes the Axiom of extensionality:

$$\forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y)$$

and the axiom schema of unrestricted comprehension:

$$\exists y \forall x (x \in y \iff \varphi(x))$$

for any formula φ with the variable x as a free variable inside φ. Substitute $x \notin x$ for $\varphi(x)$. Then by existential instantiation (reusing the symbol y) and universal instantiation we have

$$y \in y \iff y \notin y$$

a contradiction. Therefore, this naive set theory is inconsistent.

2 Cantor's Diagonalization Argument

Cantor's Diagonalization Argument shows the set of Real numbers is not countable. That is, it is impossible to construct a bijection between $\mathbb{N}$ and $\mathbb{R}$.
Also impossible to construct a bijection between $\mathbb{N}$ and the interval $[0, 1]$ (whose cardinality is the same as that of $\mathbb{R}$).

2.1 Cantor's Proof

Suppose that $f : \mathbb{N} \to [0, 1]$ is any function. Make a table of values of f, where the first row contains the decimal expansion of $f(1)$, second row contains the decimal expansion of $f(2)$, the nth row contains the decimal expansion of $f(n)$, and so on.
Perhaps $f(1) = \frac{\pi}{10}, f(2) = \frac{37}{99}, f(3) = \frac{1}{7}, f(4) = \frac{\sqrt{2}}{2}, f(5) = \frac{3}{8}$, which gives us this infinite table:

n	$f(n)$
1	0.3141592653...
2	0.3737373737...
3	0.1428571428...
4	0.7071067811...
5	0.3750000000...
.	.
.	.
.	.

Now, can f possibly be onto? That is, can every number in $[0, 1]$ appear somewhere in the table?
No! Many cannot appear. For example we can take the digits of the main diagonal:

n	$f(n)$
1	0.**3**141592653...
2	0.3**7**37373737...
3	0.14**2**8571428...
4	0.707**1**067811...
5	0.3750**0**00000...
.	.
.	.
.	.

This gives us: 0.37210, now for funsies, add 1 to each digit. This gives us 0.48321.... This number cannot be in the table. Why?

The reason is because

1. It differs from $f(1)$ in its first digit
2. It differs from $f(2)$ in its second digit
3. It differs from $f(n)$ in its nth digit

It cannot equal $f(n)$ for any n- that is, it can't appear in the table.

Other examples can include subtracting 1 from each of the diagonal digits (0s become 9s), subtract 3 from the odds and add 4 to the evens, highlight a different set of digits, etc. As long as we follow the rule of choosing at least one digit per row and at most one per column, we can modify each digit and get another number not in the table. There's a real number that does not equal $f(n)$ for any positive integer n.
Why do this? Precisely that the function f can't possibly be onto- there will always be (infinitely many!) missing values. Therefore, there does not exist a bijection between $\mathbb{N}$ and $[0, 1]$.

3 The cardinality of a set is less than the cardinality of its power set

If S is a set, then the power set $\mathcal{P}(S)$ is defined as the set of all subsets of S. For example, if $S = \{, 3, 4\}$ then

$$\mathcal{P}(S) = \{\{\}, \{1\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{3, 4\}, \{1, 3, 4\}\}$$

given finite S, easy to see that $|\mathcal{P}(S)| = 2^{|S|}$: (because to choose a subset R of S, you need to decide whether each element of S does or does not belong to R). In the above example, $|S| = 3$ and $|P(S)| = 8 = 2^3$ For infinite sets we can use a version of Cantor's argument to prove the following:

3.1 Theorem: For every set S, $|S| < |\mathcal{P}(S)|$

3.1.1 Proof:

Let $f : S \to \mathcal{P}(S)$ be any function and define

$$X \coloneqq \{s \in S | s \notin f(s)\}.$$

For example, if $S = \{1, 2, 3, 4\}$, then perhaps $f(1) = \{1, 3\}$, $f(2) = \{1, 3, 4\}$, $f(3) = \{\}$, and $f(4) = \{2, 4\}$. In this case X does not contain 1 (because $1 \in f(1)$), X does contain 2 (because $2 \notin f(2)$), X does contain 3 (because $3 \notin f(3)$), and X does not contain 4 (because $4 \in f(4)$), so $X = \{2, 3\}$.
Now, is it possible that $X = f(s)$ for some $s \in S$? If so, then either s belongs to X or it doesn't. But by the very definition of X, if s belongs to X then it doesn't belong to X, and if it doesn't then it does. This situation is impossible. So X cannot equal $f(s)$ for any s. But, just as in the original diagonal argument, this proves that f cannot be onto.

3.2 More on no Injection between A and its power set $\mathcal{P}(A)$

We have

$$f : A \to \mathcal{P}(A)$$

So the image of a point in A is a subset of A. Therefore, it makes sense to define the following set

$$B \coloneqq \{x \in A | x \notin f(x)\}$$

which contains all the points in A that belong to the set $f(x)$ so, in particular, by its definition, $B \subset A$.

Now we use the assumption that f is one-to-one, in particular it is surjective, so every subset of A has a pre-image. Let $a \in A$ be the preimage of B, that is

$$f(a) = B.$$

Now we try to reach a contradiction when we try to decide if a belongs, or not, to B :

Lets assume $a \in B = f(a)$, this is a contradiction as every element $b \in B$ does NOT belong to its image.
If $a \notin B = f(a)$, but B contains every element of A that does not belong to its image, that is $a \in B$. Another contradiction.

It is obvious that one of the previous assertions had to be true, since they are not, our assumption has to be wrong, and there is no such f.

Continuing with this, revisit B,

$$B \coloneqq \{x \in A : x \notin f(x)\}$$

This definition means B is a subset of A and so $B \in \mathcal{P}(A)$. So there is an element $a \in A$ such that $f(a) = B$. This give us two cases:

1. Case 1. $a \in f(a)$ implies $a \in B$ which is a contradiction
2. Case 2. $a \notin f(a)$ implies $a \in B$ which is a contradiction

Lets start with case 1:

Think about what "$a \in f(a)$" means. B is the set of all $x \in A$ which are not in $f(x)$. So if $a \in f(a)$, it cannot be the case that $a \in B$, by definition! But we do know that $f(x) = B$. So:

1. If $a \in f(a)$, then $a \in B$ (since $f(a) = B$)
2. But if $a \in B$, then $a \notin f(a)$ (since by definition B is the set of $a's$ which are not in $f(a)$).
3. So this is a contradiction! From "$a \in f(a)$" we've concluded "$a \notin f(a)$"- that means "$a \in f(a)$" can't be true! (If p implies not p, do you see why p can't be true?)

The other case is similar. If $a \notin f(a)$, then $a \in B$ (why? think about what B is.); but that means $a \in f(a)$ (why? think about what $f(a)$ is...). So, again, this leads to a contradiction.

So each possibility- $a \in f(a)$ and $a \notin f(a)$- lead to contradictions. So neither can hold- but one of them has to!
This means that, somewhere, we must have made an incorrect assumption. And the only assumption we've made is "there is a one-to-one correspondence between A and $\mathcal{P}(A)$". So that assumption must be false.

4 The SchrU+00F6der-Bernstein Theorem

There is a marvelous criterion for the existence of a one-to-one correspondence between two sets. NOTE TO SELF: COME BACK AND FIGURE OUT THEOREM NAMING

Theorem 1 (The Schroder-Bernstein Theorem). *If there are one-to-one mappings*

$$f : A \to B \text{ and } g : B \to A, \tag{2}$$

then there is a one-to-one correspondence between A and B.

Proof. In order to prove this theorem, we first prove the following preliminary result. □

Lemma 2 (Lemma 1.9 NAMING). *If $B \subset A$ and $f : A \to B$ is one-to-one, then there exists a function $h : A \to B$, which is a one-to-one correspondence.*

Proof. Take $B \subset A$ and suppose $B \neq A$. Recall that $A - B = \{a \in A | a \notin B\}$. Define

$$C = \bigcup_{n \geq 0} f^n(A - B),$$

where $f^0 = \mathrm{id}_A$ and $f^k(x) = f(f^{k-1}(x))$. Define the function $h : A \to B$ by

$$h(z) = \begin{cases} f(z), & \text{if } z \in C, \\ z, & \text{if } z \in A - C. \end{cases} \tag{3}$$

By definition, $A - B \subset C$ and $f(C) \subset C$. Suppose $n > m \geq 0$. Observe that

$$f^m(A - B) \cap f^n(A - B) = \emptyset.$$

To see this, suppose $f^m(x) = f^n(x')$, then $f^{n-m}(x') = x \in A - B$. But $f^{n-m}(x') \in B$ and so $x \in (A - B) \cap B = \emptyset$, a contradiction. This implies that h is one-one, since f is one-one.
We next show that h is onto: FIX ALIGNMENT

$$\begin{aligned} h(A) = f(C) \cup (A - C) \\ = f\big(\bigcup_{n \geq 0} f^n(A - B)\big) \cup \big(A - \bigcup_{n \geq 0} f^n(A - B)\big) \\ = \bigcup_{n \geq 1} f^n(A - B) \cup \big(A - \bigcup_{n \geq 0} f^n(A - B)\big) \\ = A - (A - B) = B \end{aligned}$$

So h is a one-to-one correspondence.

□

Now that we are able to prove the theorem, lets recall it:

Theorem 3 (Cantor-Schroder-Bernstein Theorem). *If there are one-one mappings*

$$f : A \to B \text{ and } g : B \to A, \tag{4}$$

then there is a one-one correspondence between A and B

Proof of the Schroder-Bernstein Theorem. Let $A_0 = g(B) \subset A$ and $B_0 = f(A) \subset B$. Then $g_0 : B \to A_0$ and $f_0 : A \to B_0$ are one-one correspondences, each induced by g and f respectively. Let $F = f_0{`}_0 : B \to B_0$ denote the one-one function. Lemma 1.9 applies to $(B, B_0, F)$, so there is a one-one correspondence $h : B_0 \to B$. The composition $h{`}_0 : A \to B_0 \to B$ is the desired equivalence of sets. □

4.1 More on CSB and Lemma 1.9

Cardinality: The cardinality of a finite set with n elements is n.

For infinite sets, we say that $|A| \leq |B|$ if there is a one-to-one $f : A \to B$ or an onto map $g : B \to A$. For this definition we'd like to have $|A| \leq |B|$ and $|B| \leq |A|$ imply $|A| = |B|$. We define $|A| = |B|$ to mean "there is a bijection between A and B" or "there is a one-to-one and onto $f : A \to B$.

Now the statement $|A| \leq |B|$ and $|B| \leq |A| \implies |A| = |B|$ is not obvious. This is the CSB Theorem in Chapter 1.

Question: How would you ever think of the construction in the proof of Lemma 1.9?
Answer: You would look at an example and try to construct the bijection in the example and then try to generalize.

Example 1 (For lemma 1.9 so $B \subset A$, and $f : A \to B$ is 1-1)**.**

$$A = \{0, 1, 2, 3, ...\} = \mathbb{N},$$

$$B = \{0, 2, 4, ...\} = 2\mathbb{N}$$

$$f : A \to B \text{ would be } f(x) = 6x$$

To understand lemma 1.9, try to follow the construction of the proof in an example like this.