SGN – Assignment #2

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Disclaimer: The story plot contained in the following three exercises is entirely fictional.

Exercise 1: Uncertainty propagation

The Prototype Research Instruments and Space Mission Technology Advancement (PRISMA) is a technology in-orbit test-bed mission for demonstrating Formation Flying (FF) and rendezvous technologies, as well as flight testing of new sensors and actuator equipment. It was launched on June 15, 2010, and it involves two satellites: Mango (Satellite 1, ID 36599), the chaser, and Tango (Satellite 2, ID 36827), the target.

You have been provided with an estimate of the states of Satellites 1 and 2 at the separation epoch t_{sep} = `2010-08-12T05:27:39.114` (UTC) in terms of mean and covariance, as reported in Table 1. Assume Keplerian motion can be used to model the spacecraft dynamics.

1. Propagate the initial mean and covariance for both satellites within a time grid going from t_{sep} to $t_{sep} + N\,T_1$, with a step equal to T_1, where T_1 is the orbital period of satellite 1 and $N = 10$, using both a Linearized Approach (LinCov) and the Unscented Transform (UT). We suggest to use $\alpha = 0.1$ and $\beta = 2$ for tuning the UT in this case.

2. Considering that the two satellites are in close formation, you have to guarantee a sufficient accuracy about the knowledge of their state over time to monitor potential risky situations. For this reason, at each revolution, you shall compute:

   - the norm of the relative position (Δr), and
   - the sum of the two covariances associated to the position elements of the states of the two satellites (P_{sum})

   The critical conditions which triggers a collision warning is defined by the following relationship:
   $$\Delta r < 3\sqrt{\max(\lambda_i(P_{\text{sum}}))}$$
   where $\lambda_i(P_{\text{sum}})$ are the eigenvalues of P_{sum}. Identify the revolution N_c at which this condition occurs and elaborate on the results and the differences between the two approaches (UT and LinCov).

3. Perform the same uncertainty propagation process on the same time grid using a Monte Carlo (MC) simulation *. Compute the sample mean and sample covariance and compare them with the estimates obtained at Point 1). Provide the plots of:

   - the time evolution for all three approaches (MC, LinCov, and UT) of $3\sqrt{\max(\lambda_i(P_{r,\text{i}}))}$ and $3\sqrt{\max(\lambda_i(P_{v,\text{i}}))}$, where $i = 1, 2$ is the satellite number and P_r and P_v are the 3x3 position and velocity covariance submatrices.
   - the propagated samples of the MC simulation, together with the mean and covariance obtained with all methods, projected on the orbital plane.

   Compare the results and discuss on the validity of the linear and Guassian assumption for uncertainty propagation.

*Use at least 100 samples drawn from the initial covariance

Table 1: Estimate of Satellite 1 and Satellite 2 states at t_0 provided in ECI J2000.

Parameter	Value
Ref. epoch t_{sep} [UTC]	2010-08-12T05:27:39.114
Mean state $\hat{\boldsymbol{x}}_{0,\text{sat1}}$ [km, km/s]	$\hat{\boldsymbol{r}}_{0,\text{sat1}} = [4622.232026629, 5399.3369588058, -0.0212138165769957]$ $\hat{\boldsymbol{v}}_{0,\text{sat1}} = [0.812221125483763, -0.721512914578826, 7.42665302729053]$
Mean state $\hat{\boldsymbol{x}}_{0,\text{sat2}}$ [km, km/s]	$\hat{\boldsymbol{r}}_{0,\text{sat2}} = [4621.69343340281, 5399.26386352847, -3.09039248714313]$ $\hat{\boldsymbol{v}}_{0,\text{sat2}} = [0.813960847513811, -0.719449862738607, 7.42706066911294]$
Covariance P_0 [km^2, km^2/s, km^2/s^2]	$\begin{bmatrix} +5.6e-7 & +3.5e-7 & -7.1e-8 & 0 & 0 & 0 \\ +3.5e-7 & +9.7e-7 & +7.6e-8 & 0 & 0 & 0 \\ -7.1e-8 & +7.6e-8 & +8.1e-8 & 0 & 0 & 0 \\ 0 & 0 & 0 & +2.8e-11 & 0 & 0 \\ 0 & 0 & 0 & 0 & +2.7e-11 & 0 \\ 0 & 0 & 0 & 0 & 0 & +9.6e-12 \end{bmatrix}$

Given the initial conditions in terms of mean and covariance for both satellite at the separation time t_{sep} we propagate them for 10 orbital periods of Mango. First, we retrieve the semi major axis as:

$$a = \frac{1}{\dfrac{2}{r} - \dfrac{v}{\mu}} \tag{1}$$

The compute the orbital period of Satellite Mango as follow:

$$T_1 = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{2}$$

With the Linearized Approach we want to study the displaced trajectories. Therefore we propagate the initial mean assuming a Keplerian motion. In this particular case the propagated mean coincides with the mean of the final distribution. For what concerns the covariance we can compute it with the STM:

$$P = \Phi(t_0, tf)P_0\Phi(t_0, t_f)^T \tag{3}$$

The linearized method fails to estimate correctly final mean and covariance for long-term propagation or highly nonlinear dynamics.
Another method is the uncertainty propagation using Uncented Transform. UT doesn't represent correctly the entire final Probability Density Function but compute only mean and covariance of the final distribution. This achieved sampling the initial PDF using a small number of points called Sigma Points.

- Step 1:
  Compute Sigma points:

$$\begin{cases} \boldsymbol{\chi_0} = \hat{\boldsymbol{x}} \\ \boldsymbol{\chi_i} = \hat{\boldsymbol{x}} + \sqrt{(n+\lambda)P_{x_i}} \quad & i = 1, ..., n \\ \boldsymbol{\chi_i} = \hat{\boldsymbol{x}} - \sqrt{(n+\lambda)P_{x_i}} \quad & i = n+1, ..., 2n \end{cases} \tag{4}$$

where $\lambda = \alpha^2(n+k) - n$ is a scaling parameter.
Compute weights:

$$\begin{cases} W_0^{(m)} = \dfrac{\lambda}{(n+\lambda)} \\ W_0^{(c)} = \dfrac{\lambda}{(n+\lambda)} + (1-\alpha^2+\beta) \\ W_i^{(m)} = W_i^{(c)} = \dfrac{1}{2(n+\lambda)} \qquad i = 1,..,2n \end{cases} \tag{5}$$

- Step 2:
  Propagate Sigma Points with the nonlinear dynamic of Keplerian motion:

$$\boldsymbol{\varphi}_i = \boldsymbol{f}(\boldsymbol{\chi}_i) \qquad i = 0, ..., 2n \tag{6}$$

- Step3:
  Compute the weighted sample mean and covariance:

$$\boldsymbol{x} = \sum_{i=0}^{2n} W_i^{(m)} \boldsymbol{\varphi_i} \tag{7}$$

$$P_y = \sum_{i=0}^{2n} W_i^{(c)} [\boldsymbol{\varphi}_i - \boldsymbol{x}][\boldsymbol{\varphi}_i - \boldsymbol{x}]^T \tag{8}$$

The critical condition which triggers a collision warning happens from revolution 4 to 9 for both methods as we can see in the figure.

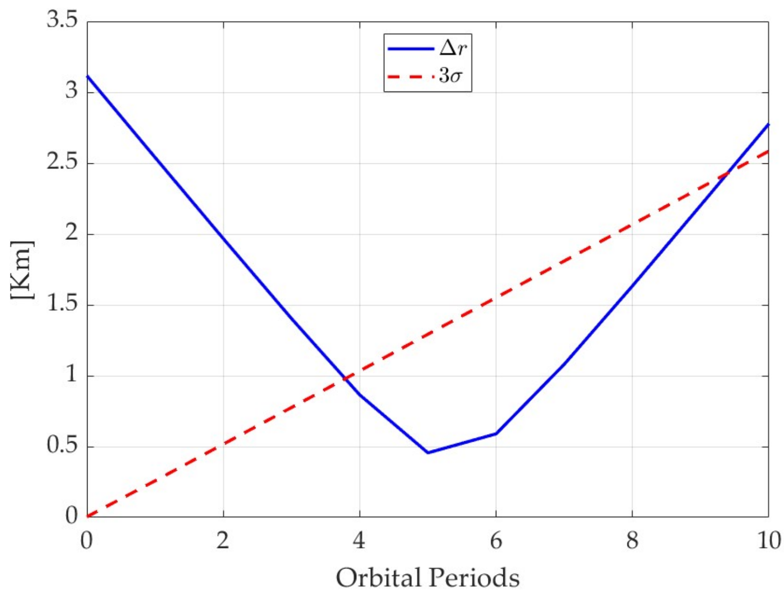

Figure 1: Critical condition analysis

To perform the uncertainty propagation with the Monte Carlo method we need to generate N samples(500 in our case), propagate them and estimate the final sample mean and covariance.

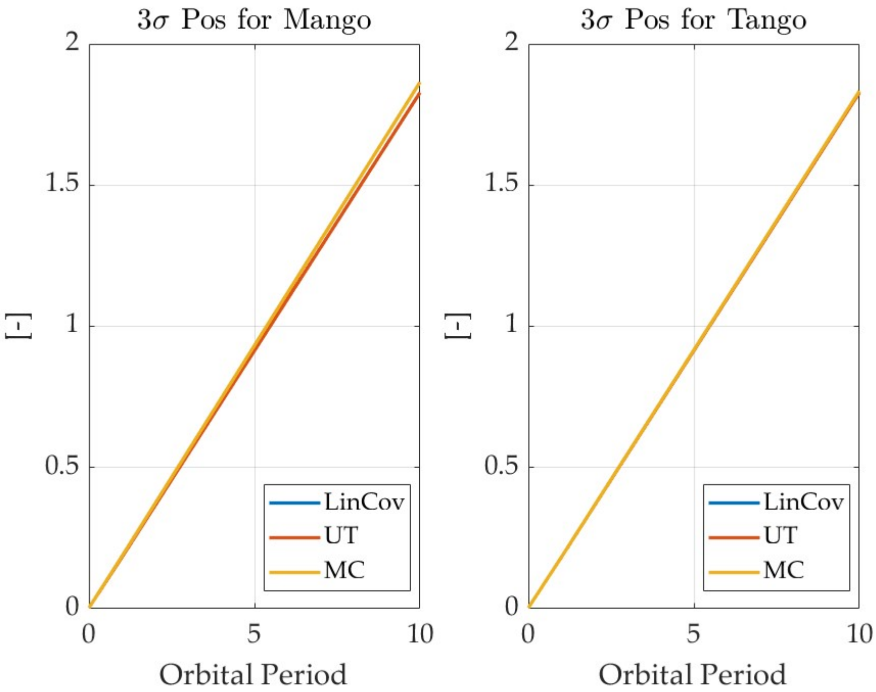

Figure 2: Evolution of 3σ for Position

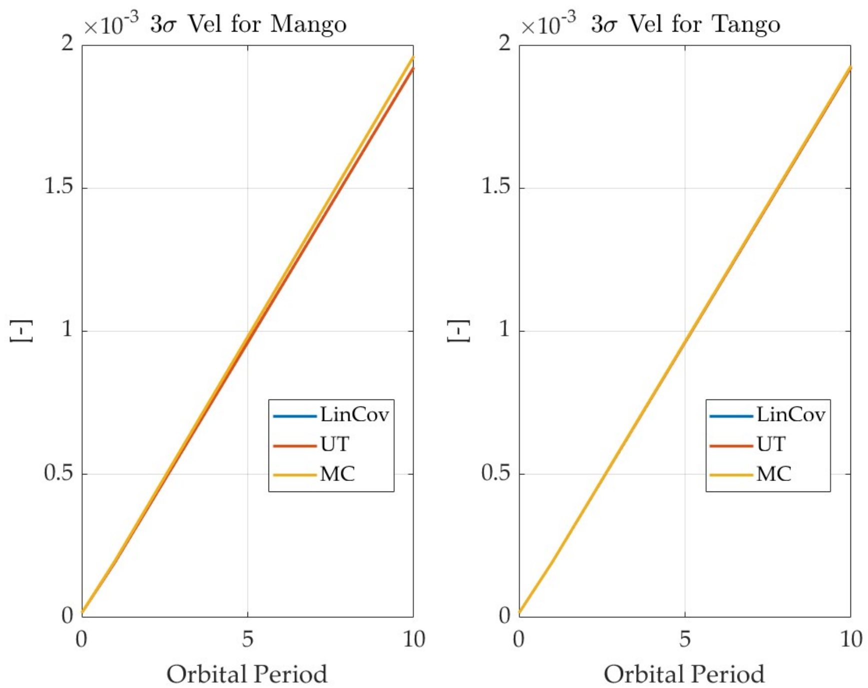

Figure 3: Evolution of 3σ for Velocity

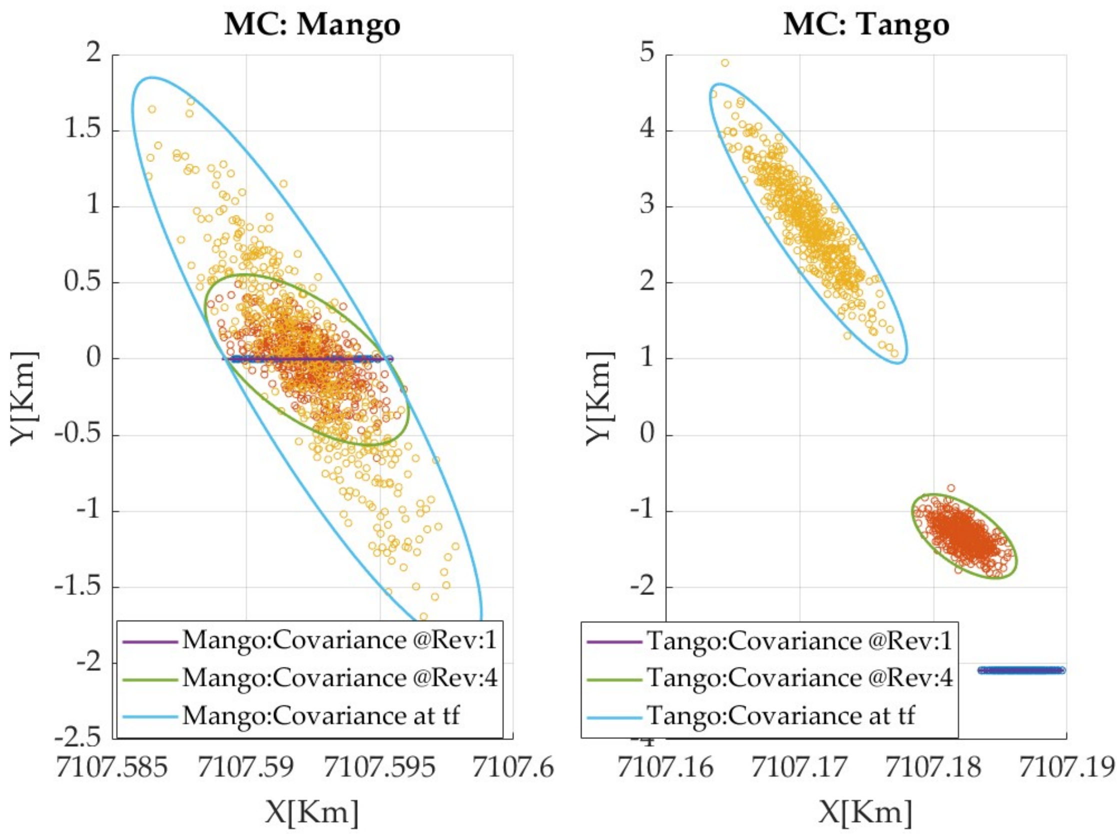

Figure 4: Monte Carlo propagation @Orbital Plane

Exercise 2: Batch filters

You have been asked to track Mango to improve the accuracy of its state estimate. To this aim, you shall schedule the observations from the two ground stations reported in Table 2.

1. *Compute visibility windows.* By using the mean state reported in Table 1 and by assuming Keplerian motion, predict the trajectory of the satellite over a uniform time grid (with a time step of 60 seconds) and compute all the visibility time windows from the available stations in the time interval from t_0 = `2010-08-12T05:30:00.000` (UTC) to t_f = `2010-08-12T11:00:00.000` (UTC). Plot the resulting predicted Azimuth and Elevation profiles in the visibility windows.
2. *Simulate measurements.* The Two-Line Elements (TLE) set of Mango are reported in Table 3 (and in WeBeep as 36599.3le). Use SGP4 and the provided TLEs to simulate the measurements acquired by the sensor network in Table 2 by:
   (a) Computing the spacecraft position over the visibility windows identified in Point 1 and deriving the associated expected measurements.
   (b) Simulating the measurements by adding a random error to the expected measurements (assume a Gaussian model to generate the random error, with noise provided in Table 2). Discard any measurements (i.e., after applying the noise) that does not fulfill the visibility condition for the considered station.
3. *Solve the navigation problem.* Using the measurements simulated at the previous point:
   (a) Find the least squares (minimum variance) solution to the navigation problem without a priori information using
      - the epoch t_0 as reference epoch;
      - the reference state as the state derived from the TLE set in Table 3 at the reference epoch;
      - the simulated measurements obtained for the KOROU ground station only;
      - pure Keplerian motion to model the spacecraft dynamics.

   (b) Repeat step 3a by using all simulated measurements from both ground stations.
   (c) Repeat step 3b by using J2-perturbed motion to model the spacecraft dynamics.
4. Provide the obtained navigation solutions and elaborate on the results, comparing the different solutions.
5. Select the best combination of dynamical model and ground stations and perform the orbit determination for the other satellite.

Propagating the mean state of the satellite over the time grid we can retrieve the predicted measurements by the Stations of Kourou and Svalbard. Taking into account the minimum elevation angle for each Ground Station we can focus on the real measurements in the visibility windows.

The Batch filter is a method used for processing a set of measurements collectively, optimizing the estimation of the spacecraft's state over a given time span. As we can see in the figure Figure 11, using all the measurements available increase the precision of the filter minimizing the residuals. For a better solution we include the J2 perturbation caused by the Earth's oblateness, in fact in this way, we implemented a more accurate dynamical model. In the end we perform also the orbit determination for the satellite Tango. In the Table 5 are reported the results of the state estimation for each case.

Table 2: Sensor network to track Mango and Tango: list of stations, including their features.

Station name	KOUROU	SVALBARD
Coordinates	LAT = 5.25144° LON = -52.80466° ALT = -14.67 m	LAT = 78.229772° LON = 15.407786° ALT = 458 m
Type	Radar (monostatic)	Radar (monostatic)
Provided measurements	Az, El [deg] Range (one-way) [km]	Az, El [deg] Range (one-way) [km]
Measurements noise (diagonal noise matrix R)	$\sigma_{Az,El}$ = 100 mdeg σ_{range} = 0.01 km	$\sigma_{Az,El}$ = 125 mdeg σ_{range} = 0.01 km
Minimum elevation	10 deg	5 deg

Table 3: TLE of Mango.

```
1␣36599U␣10028B␣␣␣10224.22752732␣-.00000576␣␣00000-0␣-16475-3␣0␣␣9998
2␣36599␣098.2803␣049.5758␣0043871␣021.7908␣338.5082␣14.40871350␣␣8293
```

Table 4: TLE of Tango.

```
1␣36827U␣10028F␣␣␣10224.22753605␣␣.00278492␣␣00000-0␣␣82287-1␣0␣␣9996
2␣36827␣098.2797␣049.5751␣0044602␣022.4408␣337.8871␣14.40890217␣␣␣␣55
```

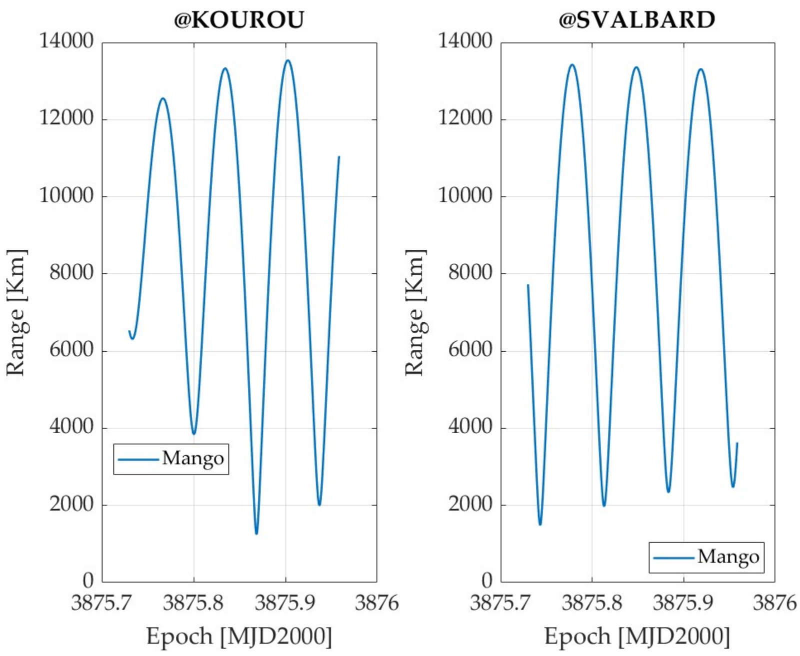

Figure 5: Predicted Range

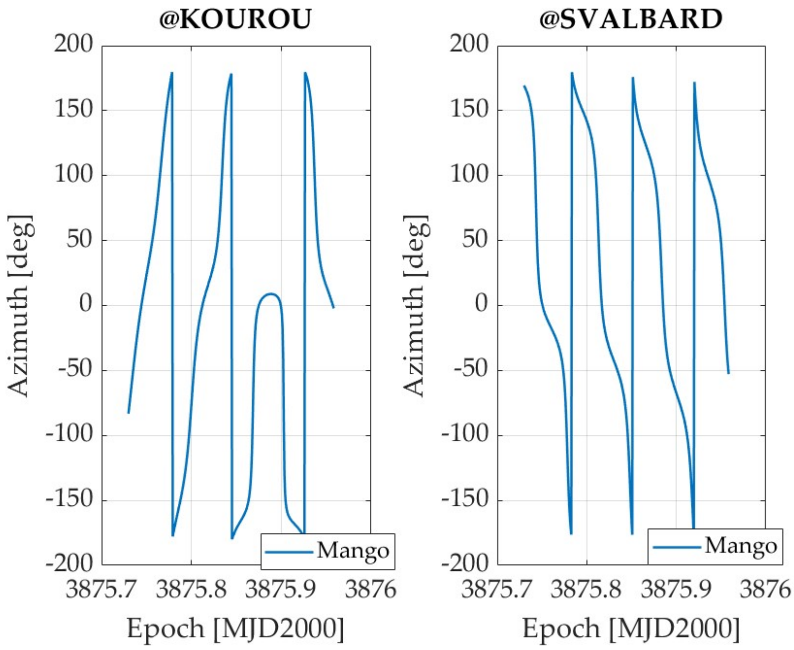

Figure 6: Predicted Azimuth

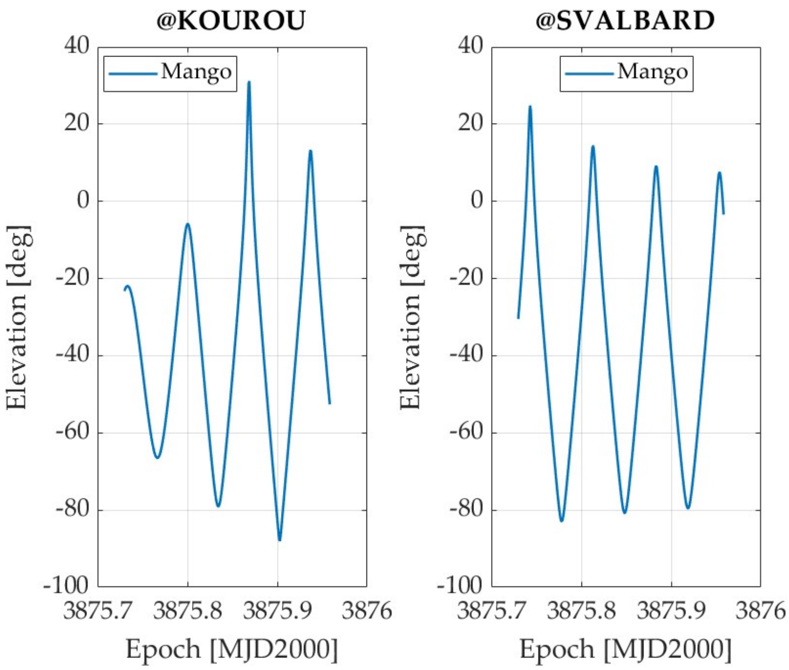

Figure 7: Predicted Elevation

Kourou	-14.907	14.286	-7.679	0.075	0.082	0.014
Kourou + Svalbard	-19.156	10.326	-2.899	0.010	0.008	0.005
Kourou + Svalbard + J2	$2.772e^{-3}$	$-1.125e^{-3}$	$9.3413e{-}4$	$-8.7175e^{-6}$	$-2.0576e^{-5}$	$-1.537e^{-6}$
Tango	$3.372e^{-1}$	$9.63e^{-1}$	-2.336	$2.073e^{-3}$	$2.308e^{-3}$	$-1.255e^{-4}$

Table 5: $x - x_0$ for each case

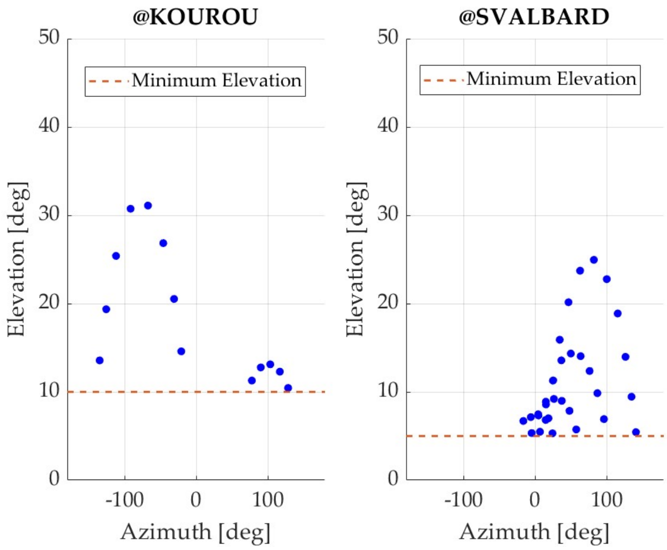

Figure 8: Azimuth and elevation profiles in the visibility windows

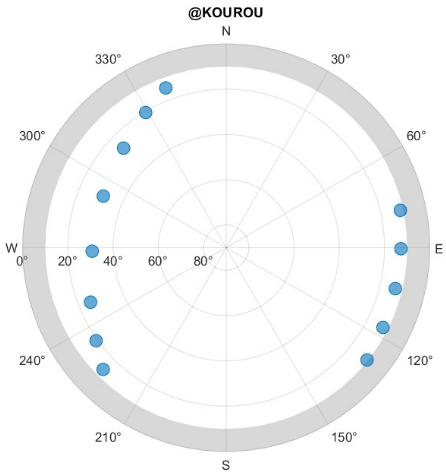

Figure 9: Kourou measurements

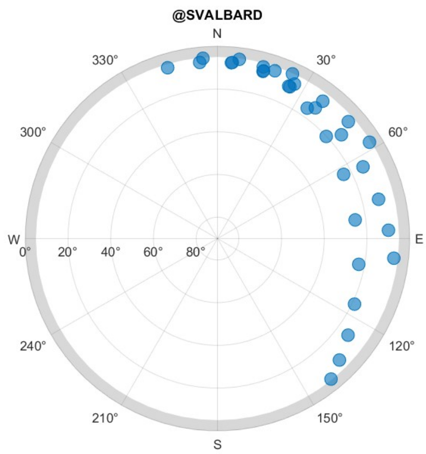

Figure 10: Svalbard measurements

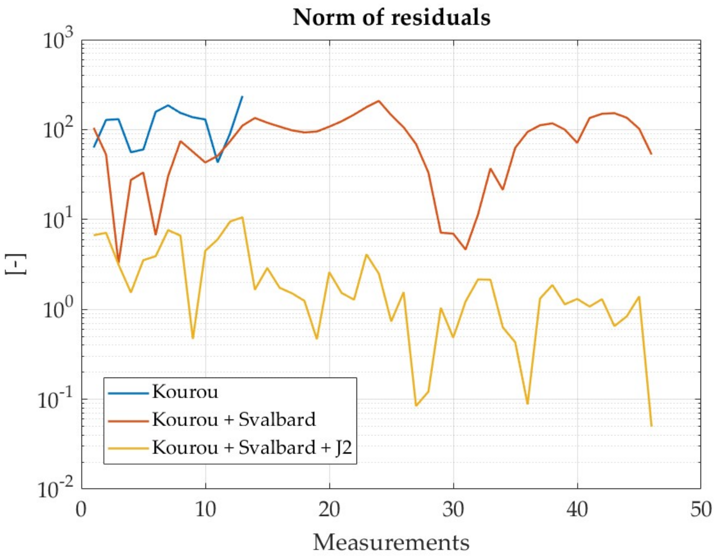

Figure 11: Norm of Residuals for the three cases

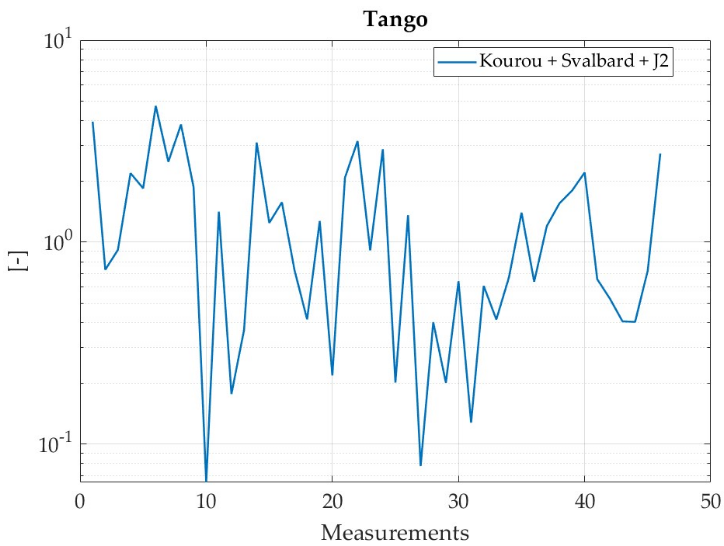

Figure 12: Norm of Residuals for Tango

Exercise 3: Sequential filters

According to the Formation Flying In Orbit Ranging Demonstration experiment (FFIORD), PRISMA's primary objectives include testing and validation of GNC hardware, software, and algorithms for autonomous formation flying, proximity operations, and final approach and recede operations. The cornerstone of FFIORD is a Formation Flying Radio Frequency (FFRF) metrology subsystem designed for future outer space formation flying missions.

FFRF subsystem is in charge of the relative positioning of 2 to 4 satellites flying in formation. Each spacecraft produces relative position, velocity and line-of-sight (LOS) of all its companions.

You have been asked to track Mango to improve the accuracy of the estimate of its absolute state and then, according to the objectives of the PRISMA mission, validate the autonomous formation flying navigation operations by estimating the relative state between Mango and Tango by exploiting the relative measurements acquired by the FFRF subsystem. The Two-Line Elements (TLE) set of Mango and Tango satellites are reported in Tables 3 and 4 (and in WeBeep as 36599.3le, and 36827.3le).

The relative motion between the two satellites can be modelled through the linear, Clohessy-Wiltshire (CW) equations[†]

$$\begin{aligned} \ddot{x} &= 3n^2x + 2n\dot{y} \\ \ddot{y} &= -2n\dot{x} \\ \ddot{z} &= -n^2 z \end{aligned} \tag{9}$$

where x, y,and z are the relative position components expressed in the LVLH frame, whereas n is the mean motion of Mango, which is assumed to be constant and equal to:

$$n = \sqrt{\frac{GM}{R^3}} \tag{10}$$

where R is the position of Mango at t_0.

The unit vectors of the LVLH reference frame are defined as follows:

$$\hat{\boldsymbol{i}} = \frac{\boldsymbol{r}}{r}, \quad \hat{\boldsymbol{j}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{i}}, \quad \hat{\boldsymbol{k}} = \frac{\boldsymbol{h}}{h} = \frac{\boldsymbol{r} \times \boldsymbol{v}}{\|\boldsymbol{r} \times \boldsymbol{v}\|} \tag{11}$$

To perform the requested tasks you should:

1. *Estimate Mango absolute state.* You are asked to develop a sequential filter to narrow down the uncertainty on the knowledge of Mango absolute state vector. To this aim, you shall schedule the observations from the SVALBARD ground station[‡] reported in Table 2, and then proceed with the state estimation procedure by following these steps:

   (a) By using the mean state reported in Table 1 and by assuming Keplerian motion, predict the trajectory of the satellite over a uniform time grid (with a time step of 5 seconds) and compute the first visibility time window from the SVALBARD station in the time interval from t_0 = `2010-08-12T05:30:00.000` (UTC) to t_f = `2010-08-12T06:30:00.000` (UTC).

   (b) Use SGP4 and the provided TLE to simulate the measurements acquired by the SVALBARD station for the Mango satellite only. For doing it, compute the spacecraft position over the visibility window using a time-step of 5 seconds, and derive the associated expected measurements. Finally, simulate the measurements by adding a random error (assume a Gaussian model to generate the random error, with noise provided in Table 2).

[†]Notice that the system is linear, therefore it has an analytic solution of the state transition matrix $\boldsymbol{\Phi}$

[‡]Note that these are the same ones computed in Exercise 2

(c) Using an Unscented Kalman Filter (UKF), provide an estimate of the spacecraft state (in terms of mean and covariance) by sequentially processing the acquired measurements in chronological order. Plot the time evolution of the error estimate together with the 3σ of the estimated covariance for both position and velocity.

2. *Estimate the relative state.* To validate the formation flying operations, you are also asked to develop a sequential filter to narrow down the uncertainty on the knowledge of the relative state vector. To this aim, you can exploit the relative azimuth, elevation, and range measurements obtained by the FFRF subsystem, whose features are reported in Table 6, and then proceed with the state estimation procedure by following these steps:

   (a) Use SGP4 and the provided TLEs to propagate the states of both satellites at epoch t_0 in order to compute the relative state in LVLH frame at that specific epoch.

   (b) Use the relative state as initial condition to integrate the CW equations over the time grid defined in Point 1a. Finally, simulate the relative measurements acquired by the Mango satellite through its FFRF subsystem by adding a random error to the expected measurements. Assume a Gaussian model to generate the random error, with noise provided in Table 6.

   (c) Consider a time interval of 20 minutes starting from the first epoch after the visibility window (always with a time step of 5 seconds). Use an UKF to provide an estimate of the spacecraft relative state in the LVLH reference frame (in terms of mean and covariance) by sequentially processing the measurements acquired during those time instants in chronological order. Plot the time evolution of the error estimate together with the 3σ of the estimated covariance for both relative position and velocity.

3. *Reconstruct Tango absolute covariance.* Starting from the knowledge of the estimated covariance of the absolute state of Mango, computed in Point 1, and the estimated covariance of the relative state in the LVLH frame, you are asked to provide an estimate of the covariance of the absolute state of Tango. You can perform this operation as follows:

   (a) Pick the estimated covariance of the absolute state of Mango at the last epoch of the visibility window, and propagate it within the time grid defined in Point 2c.

   (b) Rotate the estimated covariance of the relative state from the LVLH reference frame to the ECI one within the same time grid.

   (c) Sum the two to obtain an estimate of the covariance of the absolute state of Tango. Plot the time evolution of the 3σ for both position and velocity and elaborate on the results.

Table 6: Parameters of FFRF.

Parameter	Value
Measurements noise (diagonal noise matrix R)	$\sigma_{Az,El} = 1$ deg $\sigma_{range} = 1$ cm

The visibility window goes from `05:43:55(UTC)` to `05:54:30(UTC)`.We start evaluating the azimuth and elevation profiles during this time grid as shown in Figure 13

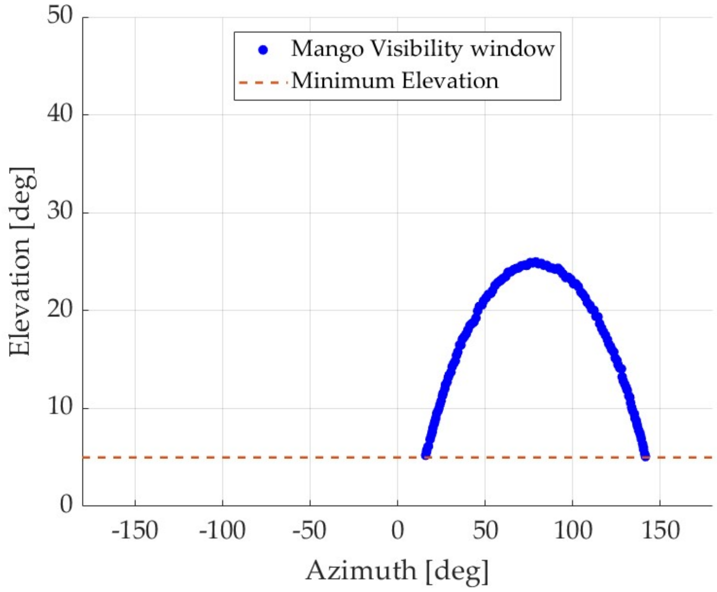

Figure 13: Visibility window

Using an UKF to estimate the spacecraft's state in terms of mean and covariance we retrieve the error in position and velocity in Figure 14

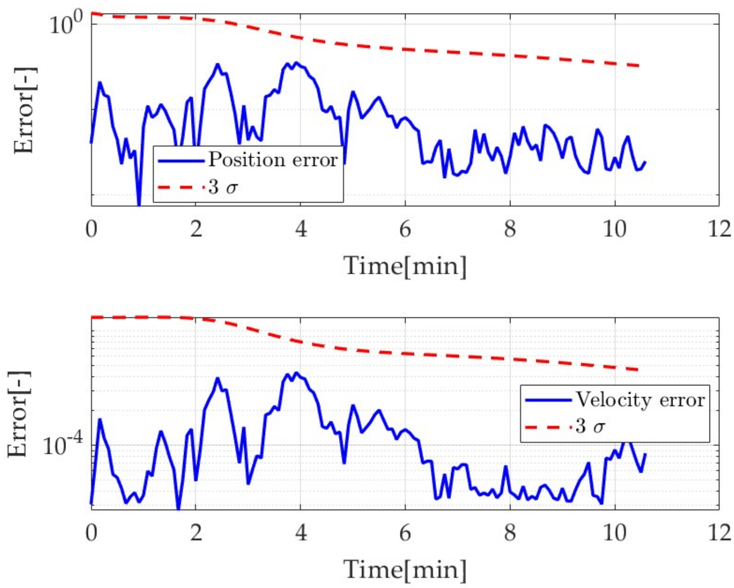

Figure 14: estimation errors

Then we need to assemble the rotation matrix from the ECI to LVLH Reference frame centered in Mango as:

$$\boldsymbol{r} = \begin{bmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{bmatrix} \qquad \dot{\boldsymbol{r}} = \begin{bmatrix} 0 & n & 0 \\ -n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \boldsymbol{r} \tag{12}$$

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r} & [0]_3 \\ \dot{\boldsymbol{r}} & \boldsymbol{r} \end{bmatrix} \tag{13}$$

where n is the mean motion of Mango, assumed to be constant.
At this point we can integrate the CW equation for the LVLH relative dynamics and with the FFRF system's measurements we can estimated relative mean and covariance. In Figure 15 and Figure 16 are reported the errors in position and velocity for this UKF application case.

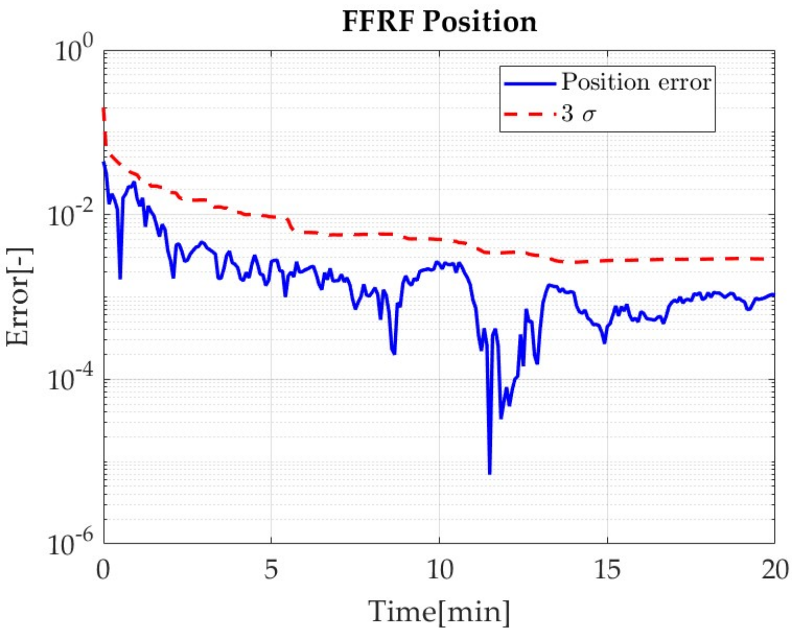

Figure 15: estimation error in position

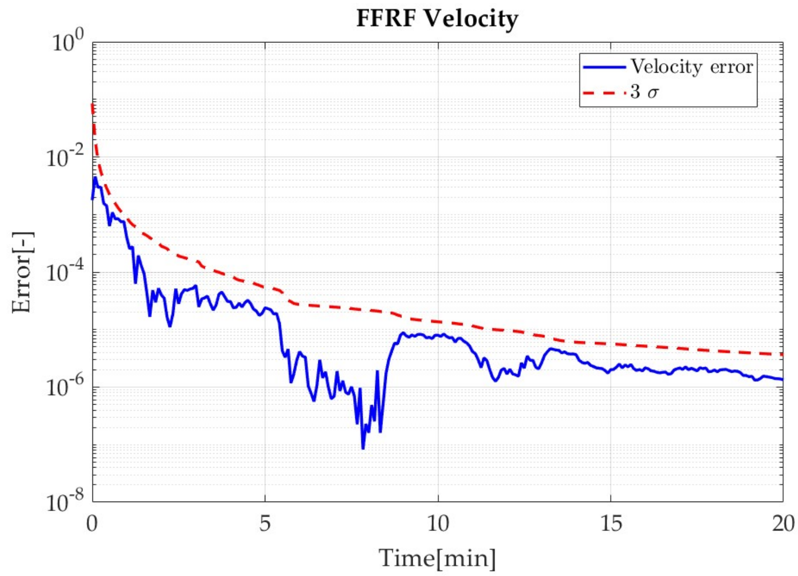

Figure 16: estimation error in velocity

In order to reconstruct Tango absolute covariance we have to go back to the ECI reference frame with the rotation matrix:

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{r}^T & [0]_3 \\ \dot{\boldsymbol{r}}^T & \boldsymbol{r}^T \end{bmatrix} \tag{14}$$

After propagating the absolute state and covariance of Mango after the visibility window we can sum the rotated relative covariance of Tango and see the evolution of 3σ for both position and velocity.

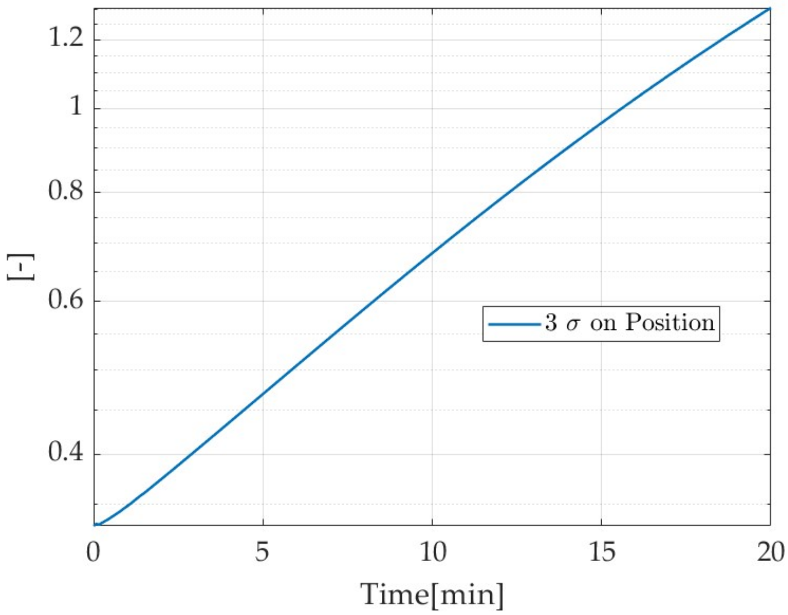

Figure 17: 3σ for position

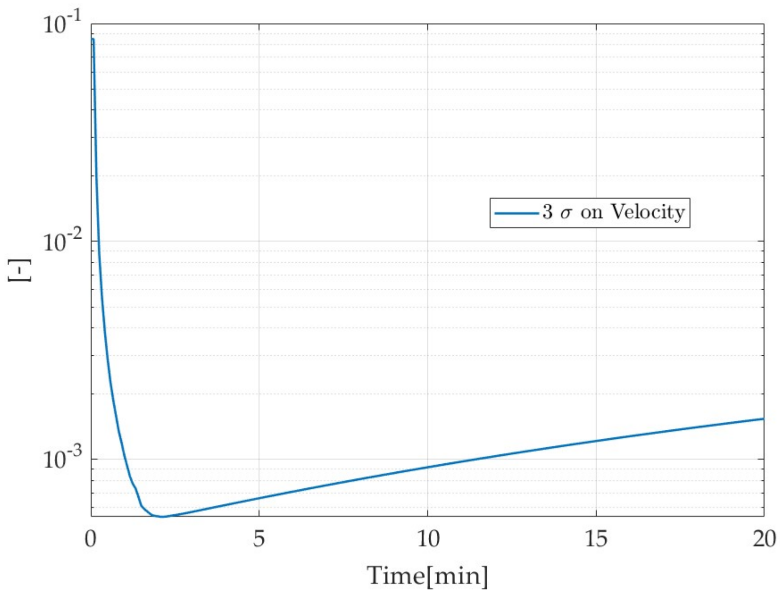

Figure 18: 3σ for velocity