Intuitionistic and Many-Valued Logics

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Abstract

When studying any logic we wish to give a full account of both its syntax (the symbols and formal expressions) and its semantics (the meaning we assign to the symbols and formal expressions). The fundamental differences between any two logics are the differences in how syntax and semantics are defined for these logics, and these differences make certain logics more appropriate for one kind of reasoning as opposed to another. Brouwer has conceived off and has been a strong proponent of Intuitionistic Logic, which compared to Classical Logic lets us correctly reason about a larger number of arguments in mathematics. Later Heyting developed rigorously the syntax and the Kripke semantics for Intuitionistic Logic. Whilst Kripke's semantic treatment of Intuitionistic Logic has been adopted as the standard way of giving semantics to this logic, in this paper we wish to investigate a truth-valued semantic treatment of Intuitionistic Logic. The main results we wish to discuss in this paper are due to Gödel and Jaśkowski which together imply that Intuitionistic Logic can be given a truth-valued semantics with countably infinitely many truth values.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

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Introduction

Logic is the study of valid arguments, if we cannot distinguish between valid arguments, those that preserve truth, and invalid arguments then we cannot reason well about the world about us. This is even more pertinent in the world of mathematics, as logic is what distinguishes a valid proof from an invalid proof, and without being able to tell this difference the whole discipline would dissolve into absurdity. One could also look at the way logic is used in the development of axiomatic set theories, in contrast to naive set theories based off natural language, which are used as a foundation to build the rest of mathematics as a motivation for the study of logic. In most day-to-day mathematics and conversations, we use Classical Logic as it broadly captures the truth-preservation/validity qualities we expect from valid arguments. However, there exist many scenarios where Classical Logic fail to capture certain quirks of reasoning, or we may wish to redefine the notion of validity from truth-preservation to provability for example. In such cases we leave Classical Logic and work with non-classical logics which do capture these quirks/new notions of validity and reason using these logics.

When Bertrand Russell was studying his undergraduate degree at Cambridge, he remarks in his autobiography [35] on the un-rigorous nature with which the subject was taught. Supposedly, after the conclusion of one of his real analysis lectures, Bertrand asked his professor about the definition of the term "infinitesimal", the professor replies with a circular definition (something of the form "The infinitesimal is something which is infinitely small"), and after Bertrand voices his displeasure of such a lack of rigour the professor scolds Bertrand for daring to challenge the contemporary knowledge. In Bertrand's own words, *"My mathematical tutors had never shown me any reason to suppose the Calculus anything but a tissue of fallacies."* It is this desire for rigour and certainty in the foundations of mathematics, which among many others Russell possessed, that motivated the study of mathematical logic. Since Logic is instrumental in the statement and development of the foundations of modern day mathematics, we now wish to present a brief timeline of key developments in the field of mathematical logic and its contributions to formalising mathematics.

We begin in the mid-19th century when Bernard Bolzano began his investigations into *collections* of mathematical objects which all share a common property, such as the collection of all even numbers [4]. These *collections* are the predecessor of what we today know as sets. Later the name set was given to such a collection and mathematicians became more interested in their formal development.

The second half of the 19th century sees serious work in the development of the notion and constructive power of sets [33] [34], which becomes known as the Theory of Sets (or simply Set Theory). Two important mathematicians conducting research into Set Theory at the time are Richard Dedekind and George Cantor. Dedekind's main contribution was to in showing how many important mathematical structures can be described in the language of Set Theory, notably Dedekind's construction of the algebraic numbers (see Chapter 3 of *Ferreirós* [14]) and his technique of Dedekind Cuts to build the set of real numbers $\mathbb{R}$ [9]. This work in showing the expressibility of mathematics through Set Theory helped to garner support and attract further interest in the subject.

It was with the work of George Cantor, however, that Set Theory became the contender for the language with which to formalize the foundations of mathematics. Throughout the late 19th century, George Cantor publishes his papers detailing how Set Theory can be used as a foundation for the study of mathematics. Cantor, among other things, presents his work on *Cardinal Arithmetic* and *Transfinite Arithmetic*, as well as discusses the famous and celebrated *Continuum Hypothesis* (see chapters 6,7 and 8 of *Ferreirós* [14]), further showcasing the descriptive power of Set Theory and firmly distinguishing it, alongside mathematical logic, as an important field of mathematical study. Cantor's work has attracted the favor of many important mathematicians of the time, including Frege, Hilbert and later Russell, with Hilbert cementing the importance of the study of Set Theory for the future of mathematics with his address to the International Congress of Mathematicians in Paris in the year 1900 [20]. In his address Hilbert praised Cantor's work in the development of Set Theory and lays out his vision for a fully rigorous and consistent axiomitisation of the foundations of mathematics (in other an axiomitisation that does not contain any inherit contradictions) using the language and results of Set Theory. Hilbert ends his speech by stressing how with such a formalisation mathematicians could turn the derivaiton of mathematical truths into an automated procedure, famously declaring that "in mathematics there is no *ignorabimus*."[1]

However in the early 20th century mathematicians discover a wide range of contradictions in Cantor's set theory (nowadays referred to as *Naive Set Theory* to reflect this), such as Russell's Paradox [10], Hilbert's Paradox [28], and Burali-Forti Paradox [7] to name a few, which casts serious doubt over the suitability of Set Theory as a formalisation of mathematics. In response mathematicians begin work on resolving these inherent paradoxes. In 1908, Ernst Zermelo provides a consistent axiomitisation of a version of Set Theory which, whilst strictly weaker than Cantor's theory, still allows for constructions of desired mathematical objects and the study of cardinal arithmetic [43]. In the 1920's Abraham Fraenkel would point out a shortcoming of Zermelo's axiomitisation, and offer a way to amend Zermelo's work to take account of this gap [19]. The amended system is known as ZF (for Zermelo-Fraenkel), and with the eventual adoption by

[1] "ignorabimus" is Latin for "we will not know."

the mathematical community of the Axiom of Choice into the ZF framework we obtain the ZFC (for Zermelo-Fraenkel with Choice) system which contemporary mathematics is based on.

Zermelo's contributions, as well as Russell's and Whitehead's answer to the contradictions of Naive Set Theory through their publication of *Principia Mathemaitca* [36] served to motivate work in the Hilbert Program despite the opposition of prominent mathematicians, notably Poincarè [29] and later on by Brouwer and Weyl [42]. Brouwer for example argued that the infinite arbitrary sets which were present in the work of Hilbert's Program where flawed and that the foundations of mathematics need to be based of constructivist principles.

Adjacent to all this, in 1920 Łukasiewicz conceives of his three valued logic $Ł_3$ which marks the beginning of a formal study of Many-Valued Logics [23]. Work on $Ł_3$ continues throughout the 1920's (generalising the logic from three truth values to an aribtrary n truth values) and later in 1930, Łukasiewicz and Tarski publish their work on this logic, notably demonstrating how Łukasiewicz logic can be extended to the countably-infinite valued case, denoted by $Ł_{\aleph_0}$ [24].

In 1931, Austrian logician Kurt Gödel revolutionized mathematics and logic with his *Incompleteness Theorems*[13] (see page 17-18), which fundamentally changed the understanding of formal systems and their limitations. These theorems transformed logic by demonstrating inherent limitations in formal mathematical systems. The first theorem reveals that *any consistent formal system capable of expressing basic arithmetic (Peano Arithmetic) is incomplete*, which means that for any consistent system capable of basic arithmetic, it contains true statements that cannot be proven within it, using a self-referential sentence constructed via Gödel numbering. The second theorem states that *no consistent formal system that includes arithmetic can prove its own consistency*. These results dismantled Hilbert's thought of a complete self-validating foundation for mathematics, instead showing that consistency and completeness are unattainable for systems which can express basic arithmetic. Gödel's work not only reshaped mathematical logic, but also influenced computer science and philosophy, illustrating the boundaries of formalization and the relationship between truth and provability.

It is around this time that Gödel begins his study into a truth-valued interpretation of Intuitionistic Logic, which culminates in 1932 when he publishes his theorem showing that Intuitionistic Logic cannot have finitely-many truth values (see pages 222-225 of *Feferman et al.* [13]).

Whilst Hilbert's Program came to an end with the publication of Gödel's Incompleteness Theorems, the ideas cultivated in the program were used by mathematicians such as Gentzen in the mid 1930's to begin the development of Proof Theory [25], continuing Hilbert's vision to prove the completeness and consistency of weaker formal systems.

Dutch mathematician L.E.J. Brouwer (1881–1966) founded *Intuitionism*[40] (see Chapter 8), a philosophy of mathematics that rejects non-constructive methods and emphasizes mental constructions as the basis of mathematical truth. Brouwer argued that mathematics is a creation of the human mind, where objects and proofs must be explicitly constructed. Intuitionism restructured mathematics by asserting that truth arises from mental constructions rather than abstract axioms, rejecting non-constructive principles like the Law of Excluded Middle. Although Brouwer's ideas were mostly purely philosophical, his student Arend Heyting (1898–1980) systematized intuitionism into a formal logic, providing a mathematical framework from which to study intuitionism. Intuitionistic Logic replaces classical axioms with rules that reflect constructive reasoning. Heyting redefining logical connectives[32] via constructive proofs and inspiring the *Brouwer-Heyting-Kolmogorov* (BHK) interpretation. For example, a proof of $P \Rightarrow Q$ is a method to convert proofs of P into proofs of Q. Heyting's work made intuitionism accessible to logicians, enabling its integration into Proof Theory and Type Theory.

It is also around this time in 1936 that Jaśkowski, follwoing his own studies into Intuitionistic Logic, publishes his theorem which states that Intuitionistic Logic can be given a truth-valued interpretation with countably infinitely many truth values [21].

In the 1960s, Saul Kripke (1940–2022) provided a model-theoretic interpretation of Intuitionistic Logic using possible worlds semantics. In Kripke models, truth is relativized to stages of knowledge (represented by nodes in a partially ordered set). A proposition is true at a "world" if it is verified at that stage or in any future stage. Together, Brouwer's philosophy, Heyting's formalism, and Kripke's models established Intuitionism as a foundational alternative to Classical Logic, emphasizing algorithmic provability over static truth and influencing fields from computer science to constructive mathematics.

Meanwhile in the late 1950's the concept of MV-algebras (MV is a abbreviation for Many Value) as models for MV-logics is developed. Notably the mathematician Chen Chang undertakes a careful study of MV-algebras [6] and in 1959 Chang proves the completeness of MV-logics by considering and working in their corresponding MV-algebras [5]. Work on MV-logics continues throughout the 20th century, as logicians continue the study of MV-algebras [27], and develop more exotic logics such as Fuzzy Logic [18].

Meanwhile, from the 1950's and 1960's the study of paraconsistent logics gains traction with many logicians around the globe, independently deriving important results. Florencio Asenjo in his 1954 thesis (published in 1966 [1]) has given a first formal account of the Logic of Paradox, which in due time would be popularised by Graham Priest. Around the same time Newton da Costa was developing his paraconsistent *C Systems*, he would later on collaborate with Ayda Arruda to develop the *P Systems*, and later still he would collaborate with Polish logicians to develop the theory of *Jaśkowski Systems*. In the 1970's the paraconsistent work has reached the international stage with different school's of thought developing. The work of Belnap and Dunn in paraconsistent logics culminates in the First Degree Entailment Logic [2] [12]. In 1997 the First World Congress on Paraconsistency is held, distinguishing the study of paraconsistent logics as its own sub-field within the field of mathematical logic.

In this paper we wish to give an account of Intuitionistic Logic and MV-Logics, giving definitions, examples and motivations for why we may prefer to work with these non-standard logics as opposed to Classical Logic, and link these two logics together through the use of Kripke models. More precisely our main goal is to show two theorems regarding this connection: the first by Kurt Gödel which shows that Intuitionistic Propositional Logic is not a finitely Many-Valued Logic and the second by Stanisław Jaśkowski which shows that Intuitionistic Propositional Logic is in fact an countably infinite Many-Valued Logic. Unless otherwise stated, we will be working with *propositional* logics throughout this paper, therefore whenever we simply talk about a logic L the reader should be aware that we mean the propositional logic L.

Chapter 1

Classical Propositional Logic

Since we do not want to presuppose that the reader will have studied Classical Propositional Logic before (from now on abbreviated as CPL), below we give a brief account of CPL, including definitions, important properties/inferences, a review of the logical calculus, and comment on the completeness and soundness of CPL, explaining what these terms and why we desire completeness and soundness in the logics we study. This is by no means a complete account of CPL, and we urge the curious reader to delve deeper into CPL which can be done by consulting *Swart's* Book [40].

1.1 Syntax and Validity

The following account is inspired by Chapter 1 of *Priest's* book [30]. We first need an object language with which to express our propositions and formulas. In CPL, the object language is made up of an infinite amount of *propositional variables* $P_1, P_2, \ldots,$ [where appropriate we may also use different capital letters of the latin alphabet, such as A, B, Q for example, to represent propositional varaibles]; the *propositional constants* $\top$ [read "truth"] and $\bot$ ["falsehood"]; connectives $\neg$ [read "Negation"], $\wedge$ [read "Conjunction"], $\vee$ [read "Disjunction"], $\Rightarrow$ [read "implies "], $\Leftrightarrow$ [read "if and only if"]; and punctuation marks [,.()]. We collect all the connectives into a set of connectives which we denote by $\mathcal{C} = \{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$, and we use capital letter of the Greek alphabet to denote sets of propositional variables [these will often be Σ and Γ]. We denote the set of propositional varaibles by At, to be thought of as the *atoms* of our language, and the set of formulas by Fm, where a formula is a combination of the propositional variables and connectives. For example $(P \wedge Q) \vee \neg P$ is a formula, where P and Q are propositional variables. The propositional constants $\top$ and $\bot$ should not be confused with propositional variables, importantly they do not represent arbitrary propositions, and they will be used later when we define the concepts of tautology and contradiction.

CPL is defined as the object language we discussed above together with the following list of axioms:

- A1: $A \Rightarrow (B \Rightarrow A)$
- A2: $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$
- A3: $A \Rightarrow (B \Rightarrow (A \wedge B))$
- A4a: $(A \wedge B) \Rightarrow A$
- A4b: $(A \wedge B) \Rightarrow B$
- A5a: $A \Rightarrow (A \vee B)$
- A5b: $B \Rightarrow (A \vee B)$
- A6: $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$
- A7: $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
- A8: $\neg\neg A \Rightarrow A$
- A9: $(A \Rightarrow B) \Rightarrow ((B \Rightarrow A) \Rightarrow (A \Leftrightarrow B))$
- A10a: $(A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$
- A10b: $(A \Leftrightarrow B) \Rightarrow (B \Rightarrow A)$

and the rule of inference called Modus Ponens (from now on abbreviated as MP), given by:

$$\frac{A, (A \Rightarrow B)}{B}. \tag{1.1}$$

The notation above should be interpreted as follows, given all the conditions above the line (here A and $(A \Rightarrow B)$) we can deduce what is below the line (here B). We see this kind of reasoning all the time in natural language, for example a hungry logician on a picnic may reason as follows using Modus Ponens,

> "(I am hungry), and (if I am hungry then I will a eat pineapple),
> therefore (I will eat a pineapple)."

The purpose of logic is to study *valid* arguments, where different logics will aim to preserve different notions of validity. However, common to all logics are the notions of *semantic validity* and *syntactic validity*. Semantic validity is what might be described as "preserving meaning", often this meaning is some notion of truth (as it the case in CPL), so a semantically valid argument is one in which all the premises have some meaning necessarily forces the conclusion to have the same meaning. In CPL the meaning that we are trying to preserve is truth, therefore in CPL a semantically valid argument is one in which all the premises being true necessarily forces the conclusion to be true. We denote semantic validity by $\models$, hence "$P \models Q$" should be interpreted as stating that P entails Q is a semantically valid argument. As we will look at more than one logic in this paper we need to distinguish symbolically between semantic validity in CPL and the semantic validity in other logics, therefore let us adopt the convention that $\vDash_L$ denotes semantic validity with respect to the logic L.

On the other hand, syntactic validity deals with propositions being derivable from the axioms and rules of inference via some formal procedure, for example tableaux, and we denote this type of validity by $\vdash$. Similarly we will write $\vdash_L$ to denote syntactic validity with respect to the logic L, and $P \vdash_L Q$ should be interpreted as stating that there exists a derivation from P to Q using the formal procedure of the logic L. Importantly, an argument does not have to preserve our notion of meaning in order to be syntactically valid, it merely has to be derivable from our axioms.

In any logic we work in, we want every meaning-preserving statement to be derivable from the axioms (that is $\Sigma \vDash \Gamma$ implies $\Sigma \vdash \Gamma$) and we want every statement derivable from the axioms to be meaning-preserving (that is $\Sigma \vdash \Gamma$ implies $\Sigma \vDash \Gamma$). The former condition is called *completeness*, the latter *soundness*, and an important result is that CPL is a complete and sound logic, for proofs of this result please see sections 2.6 and 2.9 in *Swart's* book [40].

1.2 Natural Deduction

The *formal procedure* we use to determine whether arguments are syntactically valid is Natural Deduction, in which every derivation is an ordered and numbered list in which each line contains a sentence derivable from the axioms, premises, or the sentences which came before it and the rules of inference or axioms from which the sentence was derived. The rules of inference for Natural Deduction in CPL are given below:

- Implication ($\Rightarrow$):
  $\Rightarrow I$: If assuming A leads to B, conclude $A \Rightarrow B$.
  $\Rightarrow E$ (a.k.a. Modus Ponens): From $A \Rightarrow B$ and A, derive B.
- Conjunction ($\wedge$):
  $\wedge I$: From A and B, derive $A \wedge B$.
  $\wedge E$: From $A \wedge B$, derive A and derive B.
- Disjunction ($\vee$):
  $\vee I$: From A, derive $A \vee B$; From B, derive $A \vee B$.
  $\vee E$: If $A \vee B$, and assuming A leads to C, and assuming B leads to C, conclude C.
- Negation ($\neg$):
  $\neg I$: Assuming A leads to $\perp$, conclude $\neg A$.
  $\neg E$: From A and $\neg A$, derive $\perp$.
- Truth ($\top$):
  Unit: at any point (depending on no premises) we can introduce $\top$

- Falsity ($\perp$):
  EFQ: From $\perp$, derive A
  RAA: Assuming $\neg A$ leads to $\perp$, derive A

Note that EFQ stands for *Ex Falso Quodlibet* and RAA stands for *Reductio Ad Absurdum*.

We now present an important theorem, the Deduction Theorem, which is a basis for conditional proofs and will aid us greatly in our derivations,

Theorem 1 (Deduction Theorem, Herbrand). *Let $A_1, \ldots, A_n, A, B \in Fm$ be formulas, if $A_1, \ldots, A_n, A \vdash B$ then $A_1, \ldots, A_n \vdash A \Rightarrow B$.*

Proof. See the proof of Theorem 2.24 in *Swart's* book [40]. $\square$

For example, let's say we wish to show $P \Rightarrow Q, \neg Q \vdash \neg P$, the Natural Deduction derivation would look as follows:

1.	~~Show~~ $\neg P$	2–6, $\neg$ I
2.	P	Assum. $\neg$ I
3.	$P \Rightarrow Q$	Pr 1
4.	Q	2,3, $\Rightarrow$ E
5.	$\neg Q$	Pr 2
6.	$\perp$	4,5, $\neg$ E

By the Deduction Theorem we see that $P \Rightarrow Q, \neg Q \vdash_{\text{CPL}} \neg P$ implies $P \Rightarrow Q \vdash_{\text{CPL}} \neg Q \Rightarrow \neg P$, and hence we can use this theorem to appropriately simplify our derivations.

We now comment on features of Natural Deduction. We can at any point introduce a *show-line* by writing the words "Show" followed by a formula. Each new show-line is treated as a new derivation with the set of formulas above it (in the entire derivation) as premises, and hence we can include sub-derivations inside of our main derivation. Note that we can only use the formulas in show-lines once we have completed the derivation relevant to the show line, for instance once we have shown what the show line asks us to show. Once a derivation is complete, the relevant section used in the derivation is boxed-off and the relevant "Show" is crossed out. Importantly, we can not use any formulas inside of boxed-off sections in further derivations. Every line we derive contains in the margin the rule of inference we use. Note that when we write *Assum.* in the margins we are denoting an assumption required to apply one of the rules of inference. Moreover we can at any point introduce the n-th premise from the set of premises we assume in the proof, and we denote that in the margin by writing *Pr n*. If we are asked to show a conclusion B and in our derivation we manage to derive B then on the now scored-out show-line we write *DD*, for direct derivation, in the margin. If the conclusion we are asked to show is of the form $A \Rightarrow B$ then we will cite the rule $\Rightarrow I$, and if the conclusion we are asked to show is of the form $\neg B$ then we will cite the rule $\neg I$

An important idea in Natural Deduction is the idea of substitution. Let us suppose that we have a derivation of A from the set of formulas Γ, symbolically $\Gamma \vdash A$. Consider a derivation of B from the set of premises Σ, and consider that throughout the course of this derivation we manage to derive all formulas in the set Γ. Then using the feature of Natural Deduction that we can introduce show-lines anywhere in the derivation, we can introduce a show-line to derive A, and since we have all formulas in Γ above this show-line and since we already have a derivation of A from Γ, we can simply recreate that derivation of A inside of the larger derivation. Since we can always introduce these show-lines and any valid derivations, provided we have derived the set of premises in the larger derivation, by giving names to important derivations we can (and we will do so later) use them as rules of inference which will greatly simplify our proofs. Pictorially, this idea is captured below where we call the derivation $\Gamma \vdash A$, GA:

1.	~~Show~~ B	DD
2.	Σ	Pr
3.	$\cdots$	
4.	Γ	
5.	~~Show~~ A	DD
6.	Γ	R
7.	$\cdots$	
8.	A	
9.	$\cdots$	
10.	B	

1.	~~Show~~ B	DD
2.	Σ	Pr
3.	$\cdots$	
4.	Γ	
5.	A	GA
6.	$\cdots$	
7.	B	

Natural Deduction as a formal procedure was proposed by Gentzen in 1933, and over the 20th century it was refined by other logicians (Fitch, Jáskowski, Kalish, Montague, etc.) into the form seen above (the presentation above is known as the *Kalish-Montague method*). This type of formal procedure is meant to mimic the way we reason using logic, and it grew out of a dissatisfaction with the formal procedure used by Hilbert, Frege and Russell (to name a few) which was judged to be much more convoluted. To better understand some of this frustration experienced by the logicians of the 20th-century, we direct the curious reader to the Principia Mathematica Vol. I [36], in which the derivations follow the Hilbert-style.

1.3 Semantics

The above description of CPL is purely syntactic, however we can also give a semantic description of CPL. Given the object language, CPL is defined as the tuple $\langle \mathcal{V}, \mathcal{D}, \{f_c \mid c \in \mathcal{C}\}\rangle$ where here $\mathcal{V} = \{T, F\}$ is the set of truth values, $\{T\} = \mathcal{D} \subseteq \mathcal{V}$ is the set of designated values (the truth values which are preserved under valid inferences), we note that here $T \in \mathcal{V}$ is the only designated value, and f_c denotes the truth function for the connective $c \in \mathcal{C}$.

As remarked before, one of the distinguishing features of logics are the different semantics which they employ, that is the different meanings they attach to the symbols they use. A handy way of conveying the semantics of the connectives used in a logic is through truth tables, which unambiguously show how truth values are mapped under different connectives (which should be thought of as maps $c : \mathcal{V}^n \to \mathcal{V}$, where n denotes the arity of the connective c). For CPL we give below the truth-tables for its connectives:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	$\neg P$
T	F
F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 1. An interpretation is a map $\nu : At \to \mathcal{V}$, from the set of propositional variables At to the set of truth values $\mathcal{V}$. Intuitively it is a choice of assigning truth values for ν to each of the propositional variables P. Since formulas are recursively defined from propositional variables and connectives, we can extend any interpretation ν to a map from the set of formulas Fm to the set of truth values $\mathcal{V}$. Since any ν can be extended in this way, we will use ν to denote the extended map.

For example, let $P, Q \in At$, and let ν be the interpretation which fixes $\nu(P) = T$ and $\nu(Q) = F$. Then we can determine the truth value of the formula $P \wedge Q$ by looking at it's truth table, from which we get that $\nu(P \wedge Q) = f_\wedge(\nu(P), \nu(Q)) = f_\wedge(T, F) = F$. We can now define semantic validity as follows,

Definition 2. For sets of formulas Σ and Γ, $\Sigma \vDash \Gamma$ if and only if every interpretation ν that assigns a designated value to every formula $A \in \Sigma$ assigns a designated value to at least one $B \in \Gamma$.

One can check that given these truth tables we can derive all the axioms and rules of inference of CPL, and given the axioms and rules of inference of CPL we are forced to assign these interpretations to the connectives. This equivalence between the syntactic representation with axioms and rules of inference and the semantic representation with truth tables is due to the completeness and soundness of CPL. Moreover, this truth table presentation offers a powerful method to decide whether a given argument is valid or if a counter-model exists, for example, we can prove that $P, P \Rightarrow Q \nvDash \neg Q$ by considering the truth-table:

P	Q	$P \Rightarrow Q$	$\neg Q$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	T

Where we see that the top-line of this truth-table is a counter-model to the claim $P, P \Rightarrow Q \vDash \neg Q$ (note that for that interpretation $\nu(P) = \nu(P \Rightarrow Q) = T$, but $\nu(\neg Q) = F$), hence $P, P \Rightarrow Q \nvDash \neg Q$.

A particularly important set of arguments is the set of tautologies and the set of contradictions. More formally,

Definition 3. A tautology Q is a valid argument whose set of premises is empty, symbolically we write $\vDash Q$.
A contradiction Q is a valid argument such that its negation $\neg Q$ is a tautology, symbolically we write $Q \vDash$.

Semantically we can think of tautologies as formulas with no which are true for all valuations ν. An important tautology is the Law of Excluded Middle (from now on abbreviated LEM) which states that for any proposition P, $P \vee \neg P$ is true.To see why this is a tautology we consider the following truth-table,

P	$\neg P$	$P \vee \neg P$
T	F	T
T	F	T
F	T	T
F	T	T

and note that under any assignment of truth values to P, meaning under any valuation map ν, the conclusion $P \vee \neg P$ is always true, and hence by definition it is a tautology. Semantically we can think of contradictions in the dual sense as formulas which as false for all valuations ν. An important contradiction we wish to consider states that for any proposition P, $P \wedge \neg P$ is true. To see why this is a contradiction we consider the following truth table,

P	$\neg P$	$P \wedge \neg P$
T	F	F
T	F	F
F	T	F
F	T	F

and note that under any assignment of truth values to P, meaning under any valuation map ν, the conclusion $P \wedge \neg P$ is always false, hence the negation $\neg(P \wedge \neg P)$ is always true and so by definition it is a contradiction. The tautology $\neg(P \wedge \neg P)$ is known as the Law of Non-Contradiction.

Tautologies and contradictions are interesting to study since if A is a tautology then any argument of the form $P_1, P_2, \ldots P_n \vDash A$ is valid as there is no valuation which can make A false and hence no valuation which could serve as a counter-model. On the other hand if B is a contradiction then any argument of the form $B \vDash Q_1, Q_2, \ldots Q_m$ is valid as there is no valuation which can make B true and hence again no valuation which could serve as a counter-model. This second form of argument can be particularly troublesome, as if anywhere in our logic we discover a contradiction B, then for any proposition Q the argument $B \vDash Q$ is valid by the discussion above, and hence from the contradiction B we could entail any conclusion we like! This is known formally as the *Principle of Explosion*.

We note that the LEM and Axiom 8, known as the rule of double negation, are equivalent formulas, that is one can show that $P \vee \neg P \vdash \neg\neg P \Rightarrow P$ and $\neg\neg P \Rightarrow P \vdash P \vee \neg P$. Axiom 7 and 8 together form the basis for the technique of *Proof by Contradiction*, we see that if assuming $\neg A$ we manage to derive both formulas B and $\neg B$, the using Axiom 7 we can derive $\neg\neg A$, and using Axiom 8 we can thus derive A. Therefore logics which reject LEM, and equivalently double negation, also reject Proof by Contradiction as a valid technique of proof.

Chapter 2

Intuitionistic Propositional Logic

2.1 Introduction

We begin this chapter by considering the independence of Euclid's fifth postulate from the first four postulates, we will see how the lack of LEM in IPL makes it a more appropriate logic to base mathematics on compared to CPL. Then we introduce the axioms of Intuitionistic Propositional Logic (from now on abbreviated IPL), and compare it with CPL. After that, we will talk more on semantics and syntax, for instance, completeness and soundness. Finally, we will make some spoilers to motivate our discussion of Kripke models and Many-Valued Logics.

In this paragraph we will discuss *predicate* logic (logic with quantifiers) to showcase an important feature of Intuitionistic Logic, before switching back to propositional logic for the rest of the discussions in this paper. As we mentioned before, Intuitionistic logic is centred on constructive proofs, requiring explicit constructions for existential claims. In classical logic, "$\exists x, P(x)$" can be proven via contradiction. Intuitionism requires explicitly constructing an instance of x and verifying $P(x)$. For example, Euclid's proof of the existence of infinitely many primes is constructive: given a finite list of primes, we can constructs a new prime via prime factors of the product plus 1. That will leads to a question, whether the axiom we have in Classical Logic is valid in Intuitionistic Logic or not?

Now, let us talk more about LEM. By inspection, we can easily realize that all the axioms or theorems in Intuitionistic Logic are slightly different compare to Classical Logic due to lack of LEM. As we mentioned before, Intuitionistic Logic does not accept LEM, but Classical Logic does. In order to illustrate that, we will introduce Euclidean Postulates to make a comparison between those two Logic. There are five postulates of Euclidean:

- $P1$: A straight line segment can be drawn by joining any two points.
- $P2$: Any straight line segment can be extended indefinitely in a straight line.
- $P3$: Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

- $P4$: All right angles are equal to one another.
- $P5$: If two lines are not parallel, they eventually intersect.

For thousands of years mathematicians struggled to find a proof of the fifth postulate given the first four postulates. When no proof materialised, mathematicians turned their attention to finding a proof of the negation of the fifth postulate, and in their pursuits discovered spherical an hyperbolic geometries (so called non-euclidean geometries), which serve as models for the system consisting of the axioms $P1$, $P2$, $P3$, $P4$ and $\neg P5$. The existence of such models alongside the existence of Euclidean geometry (which itself is a model for the system consisting of the axioms $P1$, $P2$, $P3$, $P4$ and $P5$) shows that $P5$ is independent of the first four postulates, that is we can neither derive $P5$ nor derive $\neg P5$ from the first four postulates. Therefore, in the system axiomatised by the first four postulates, we have that $P5 \vee \neg P5$ is not a tautology. We have found an example in mathematics where LEM does not hold.

2.2 Syntax

We present the logical axioms of the Hilbert style proof system for Intuitionistic Logic where A, B, C are arbitrary formulas in IPL:

- A1: $A \Rightarrow (B \Rightarrow A)$
- A2: $(A \Rightarrow B) \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow C))$
- A3: $A \Rightarrow (B \Rightarrow (A \wedge B))$
- A4a: $(A \wedge B) \Rightarrow A$
- A4b: $(A \wedge B) \Rightarrow B$
- A5a: $A \Rightarrow (A \vee B)$
- A5b: $B \Rightarrow (A \vee B)$
- A6: $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \vee B) \Rightarrow C))$
- A7: $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A)$
- A8: $(A \wedge \neg A) \Rightarrow B$

We adopt the Modus Ponens rule, $\frac{A;(A\Rightarrow B)}{B}$ as the only inference rule. If we want to prove any axiom posted above, we need to assign constructive instance of any atom we are using. Comparing the above axioms with the axioms of CPL the reader might at first worry that we have missed out the axioms which contain the if-and-only-if connective $\Leftrightarrow$, however note that if we define $A \Leftrightarrow B$ as $(A \Rightarrow B) \wedge (B \Rightarrow A)$ for propositions A, B, then axioms A9, A10a and A10b of CPL follow by axioms A3, A4a and A4b of CPL. Here we can similarly use such a definition and the axioms of IPL to derive the additional three axioms of IPL involving $\Leftrightarrow$.

Alternatively we can consider the Natural Deduction framework for IPL, it avoids LEM and uses introduction/elimination rules for logical connectives, presented below:

- Implication ($\Rightarrow$):
  $\Rightarrow I$: If assuming A leads to B, conclude $A \Rightarrow B$.
  $\Rightarrow E$ (Modus Ponens): From $A \Rightarrow B$ and A, derive B.
- Conjunction ($\wedge$):
  $\wedge I$: From A and B, derive $A \wedge B$.
  $\wedge E$: From $A \wedge B$, derive A and derive B.
- Disjunction ($\vee$):
  $\vee I$: From A, derive $A \vee B$; From B, derive $A \vee B$.
  $\vee E$: If $A \vee B$, and assuming A leads to C, and assuming B leads to C, conclude C.
- Negation ($\neg$):
  $\neg I$: Assuming A leads to $\perp$, conclude $\neg A$.
  $\neg E$: From A and $\neg A$, derive $\perp$.
- Truth ($\top$):
  Unit: at any point (depending on no premises) we can introduce $\top$
- Falsity ($\perp$):
  EFQ: From a contradiction, $\perp$, derive A

Similarly to the Natural Deduction system we have for Classical Logic, we also have a version of the Deduction Theorem for Intuitionistic Logic, which we present below.

Theorem 2. *If* $\Gamma, A \vdash B$ *(B is provable from assumptions* Γ *and* A*, then* $\Gamma \vdash A \Rightarrow B$ *($A \Rightarrow B$ is provable from assumption* Γ *alone).*
Conversely, if $\Gamma \vdash A \Rightarrow B$*, then* $\Gamma, A \vdash B$.

Proof. In order to prove this theorem, we need to prove it in both forward and backward directions. For the forward direction, there is a detailed proof by Herbrand in 1930 (see pages 73-75 of *Stewart* [40]). Although it is proving the Deduction Theorem in CPL, the proof is also valid for IPL as the proof does not use any inferences which are not valid in IPL. For the converse case: given that $\Gamma \vdash A \Rightarrow B$, we can assume A. Then using $\Rightarrow E$ on $A \Rightarrow B$ and since we have A, we can derive B. Therefore, $\Gamma, A \vdash B$. $\square$

To showcase the Natural Deduction system for IPL, we present a derivation of one of DeMorgan's laws, $\neg P \vee \neg Q \vdash_{\text{IPL}} \neg(P \wedge Q)$,

1.	~~*Show*~~ $\neg(P \wedge Q)$	2–12, $\neg I$
2.	$P \wedge Q$	Assum.
3.	$\neg P \vee \neg Q$	Pr 1
4.	~~*Show*~~ $\neg P \Rightarrow \bot$	5–7 $\Rightarrow I$
5.	$\neg P$	Assum.
6.	P	2, $\wedge E$
7.	$\bot$	5, 6, $\neg E$
8.	~~*Show*~~ $\neg Q \Rightarrow \bot$	9–11 $\Rightarrow I$
9.	$\neg Q$	Assum.
10.	Q	2, $\wedge E$
11.	$\bot$	9, 10 $\neg E$
12.	$\bot$	4, 8, $\vee E$

Importantly, the other direction of this entailment doesn't hold in IPL, $\neg(P \wedge Q) \nvdash_{\text{IPL}} \neg P \vee \neg Q$, we will give a counter-model to this in Chapter 4 (4.3.3), but for this we will need to discuss Kripke models first. It can be shown that this DeMorgan's Law is equivalent to the Weak Law of Excluded Middle (from now on abbreviated as WLEM), which for any proposition P states $\neg P \vee \neg\neg P$ is true. We can see that since this DeMorgan's Law isn't a tautology of IPL, the logical equivalence with WLEM implies that WLEM is also not a tautology in IPL. We note that all other DeMorgan's Laws are valid in IPL, that is,

Lemma 3. *The following are valid inferences in IPL,*

- $\vdash_{IPL} \neg P \vee \neg Q \Rightarrow \neg(P \wedge Q)$
- $\vdash_{IPL} \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

Proof. We have already presented the proof of the first inference. For the second set of inferences please see the appendix. □

2.3 Soundness, Completeness and Connections with CPL

Having covered the syntactic side of IPL, instead of now covering the semantics, we postpone this discussion until we cover the relevant material on Kripke frames in detail. Instead we will now comment positively on the completeness and soundness of IPL with respect to the syntax of Natural Deduction and the semantics given by Kripke frames. Our goal is to simply comment on the results, and for the proofs of the theorems we direct the curious reader to the references as they come up.

Theorem 4 (Soundness of IPL). *For any formula $A \in Fm$, $\vDash_{IPL} A$ implies $\vdash_{IPL} A$.*

Proof. Please see Theorem 2.4.7 of *Sørensen* and *Urzyczyn* [39]. □

Theorem 5 (Completeness of IPL). *For any formula $A \in Fm$, $\vdash_{IPL} A$ implies $\vDash_{IPL} A$.*

Proof. Please see Theorem 2.4.7 of *Sørensen* and *Urzyczyn* [39]. □

With soundness and completeness for IPL under our belts, we know move on to better understand the relationship between CPL and IPL.

Theorem 6. *Every formula that is derivable intuitionistically is classically derivable, symbolically if $\vdash_{IPL} A$ then $\vdash_{CPL} A$.*

Proof. Since the rules of inference of IPL Natural Deduction are a subset of the rules of inference of CPL Natural Deduction, it follows that any derivation in IPL can be reconstructed using the same rules but in the Natural Deduction system of CPL. □

Note that the converse to the above theorem is *not* true, as a counterexample we can consider the Weak Law of Excluded Middle, where it is derivable given no premises in CPL but it is not derivable without any premises in IPL, symbolically $\vdash_{\text{CPL}} WLEM$ and $\nvdash_{\text{IPL}} WLEM$.
As both IPL and CPL are complete and sound under respective semantics, we can state this as the following relationship between classical and intuitionistic tautologies.

Corollary 6.1. *For any formula $A \in F$, if $\vDash_{IPL} A$, then $\vDash_{CPL} A$.*

Proof. Using the completeness of IPL we have $\vDash_I A$ implies $\vdash_{\text{IPL}} A$, by Theorem 6 we have $\vdash_{\text{IPL}} A$ implies $\vdash_{\text{CPL}} A$, and lastly by the soundness of CPL we have $\vdash_{\text{CPL}} A$ implies $\vDash_{\text{CPL}} A$. Hence given $\vDash_{\text{IPL}} A$, by following the chain of implications we obtain $\vDash_{\text{CPL}} A$. □

The next relationships shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa. For this result we need the following Lemmas,

Lemma 7. *The rules of inference below are all equivalent,*

- *RAA,* $(\neg P \Rightarrow \bot) \Rightarrow P$
- *LEM,* $P \vee \neg P$
- *DN,* $\neg\neg P \Rightarrow P$

Proof. Please see Appendix A.3. $\square$

Lemma 8. *For any proposition* P*, we have* $\vdash_{IPL} \neg\neg(P \vee \neg P)$

Proof. Please see Appendix A.2. $\square$

and now we present the main theorems,

Theorem 9 (Glivenko). *For any formula* $A \in Fm$*,* A *is a classically provable if and only if* $\neg\neg A$ *is an intuitionistically provable, symbolically* $\vdash_{CPL} A$ *if and only if* $\vdash_{IPL} \neg\neg A$*.*

Proof. The following proof is inspired by Curtis Franks and can be found here [15].
For the forward implication, let $A \in Fm$ be a formula such that $\vdash_{\text{CPL}} A$. Importantly, this means we have a derivation of finite length, which means we can assume without loss of generality that there are n propositional variables, which we will denote here with P_i with $1 \leq i \leq n$. Knowing that double negation is equivalent to LEM, we can consider the derivation,

1.	~~Show~~ A	Derivaiton
2.	$P_1 \vee \neg P_1$	LEM
3.	$\cdots$	
4.	$P_n \vee \neg P_n$	LEM
5.	$\cdots$	
6.	A	

where after the line $P_n \vee \neg P_n$ we only apply rules of inference which are valid in both CPL and IPL. Using the fact that for any proposition P, $\vdash_{IPL} \neg\neg(P \vee \neg P)$ we can derive the result. The idea is to first introduce all formulas of the form $\neg\neg(P_i \vee \neg P_i)$ for each proposition P_i which is included in the derivation of A, then try to show $\neg\neg A$ which will let us assume $\neg A$ for a conditional derivation. We then try to show formulas $\neg(P_i \vee \neg P_i)$ for all $1 \leq i \leq n$, at each step letting us assume $P_i \vee \neg P_i$. Once we reach the inner most derivation box after assuming $P_n \vee \neg P_n$, we can now use the fact that we have a derivation for A in CPL to derive A, which coupled with the assumption we have made earlier that $\neg A$ gives us a contradiction using the rule $\neg E$. From this we derive $\neg(P_n \vee \neg P_n)$ which contradicts $\neg\neg(P_n \vee \neg P_n)$ and again we derive a contradiction using the rule $\neg E$.

This lets us derive $\neg(P_{n-1} \vee \neg P_{n-1})$, and so we continue in this fashion, at each step using the fact that we have derived $\neg(P_i \vee \neg P_i)$ to contradict $\neg\neg(P_i \vee \neg P_i)$ until we cross out all the show-lines and derive the result. In a Kalish-Montague styled derivation this looks as follows,

1.	~~Show~~ $\neg\neg A$	$\neg I$
2.	$\neg\neg(P_1 \vee \neg P_1)$	Unit
3.	$\ldots$	
4.	$\neg\neg(P_n \vee \neg P_n)$	Unit
5.	~~Show~~ $\neg\neg A$	$\neg I$
6.	$\neg A$	Assum.
7.	~~Show~~ $\neg(P_1 \vee \neg P_1)$	$\neg I$
8.	$P_1 \vee \neg P_1$	Assum.
9.	$\ldots$	
10.	~~Show~~ $\neg(P_{n-1} \vee \neg P_{n-1})$	$\neg I$
11.	$P_{n-1} \vee \neg P_{n-1}$	Assum.
12.	~~Show~~ $\neg(P_n \vee \neg P_n)$	$\neg I$
13.	$P_n \vee \neg P_n$	Assum.
14.	$\ldots$	
15.	A	
16.	$\perp$	$\neg$ E
17.	$\perp$	$\neg$ E
18.	$\ldots$	
19.	$\perp$	$\neg$ E
20.	$\perp$	$\neg$ E

Hence given that $\vdash_{CPL} A$ we can produce a proof of $\neg\neg A$ using only the rules of inference for the IPL Natural Deduction system, symbolically $\vdash_{IPL} \neg\neg A$.

For the reverse direction, we note that by Theorem 6 if $\vdash_{IPL} \neg\neg A$ then $\vdash_{CPL} \neg\neg A$, and since the double negation rule holds in CPL, we have that if $\vdash_{CPL} \neg\neg A$ then $\vdash_{CPL} A$ as we can take the derivation of $\neg\neg A$ and apply the double negation rule at the end to obtain A. Following this chain of implicaiton, we see that if $\vdash_{IPL} \neg\neg A$ then $\vdash_{CPL} A$. □

Theorem 10 (Tarski). *For any formula $A \in F$, A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, symbolically $\vDash_{CPL} A \Leftrightarrow \vDash_{IPL} \neg\neg A$.*

Proof. This effectively becomes a corollary of Glivenko's Theorem.
For the forward direction, assuming $\vDash_{\text{CPL}} A$, we first use the completeness of CPL

to deduce that $\vdash_{\text{CPL}} A$, then use Glivenko's Theorem to deduce $\vdash_{\text{IPL}} \neg\neg A$, and lastly we use the soundness of IPL to deduce $\vDash_{\text{IPL}} \neg\neg A$.
For the reverse direction, assuming $\vDash_{\text{IPL}} \neg\neg A$, we first use the completeness of IPL to deduce that $\vdash_{\text{IPL}} \neg\neg A$, then use Glivenko's Theorem to deduce $\vdash_{\text{CPL}} A$, and lastly we use the soundness of CPL to deduce $\vDash_{\text{CPL}} A$. □

We have seen by Theorem 6 and by Corollary 6.1 that any valid inference (either syntactically or semantically) in IPL is a valid inference in CPL. In this way we can view CPL as an extension of IPL; Glivenko's and Tarski's Theorems indicate just how far this extension takes us. The semantics for CPL, as we have seen in the introduction, are given by truth tables, thus one could conjencture that, given how similar IPl and CPL are, we can give similar truth table semantics for IPL. Any such truth table will require a set of truth values $\mathcal{V}$ for IPL, however looking back at the example of Euclid's postulates and the independence of the 5th postulate, we see that the classical two truth values interpretation will not suffice. If we represent the 5th postulate by P then we simultaneously don't have a proof of P, hence P can not be true, and we don't have a proof of $\neg P$, hence $\neg P$ can not be true which is classically equivalent to P can not be false. In this way we may be tempted to extend the set of truth values $\mathcal{V}$ and introduce more truth values, for example we may wish to introduce the truth value i (independent), fix $\nu(P) = i$ and try to write down sensible truth tables for logical connectives and hence give them semantic meaning. We will see in Chapter 5 the complications of this approach, and in Chapter 4 we will give another formulation of the semantics of IPL using Kripke models, but for now we will explore this idea of extending $\mathcal{V}$ and see what kind of logic arises from many truth values.

Chapter 3

Many Valued Logics

3.1 Motivation

Consider the following sentence: “This sentence is true”, and imagine that I ask you to determine its truth value. Working in Classical Logic, we know that this sentence can take either T (for true) or F (for false) as its truth value, nothing else. But we quickly see that we can assign both T *and* F as truth values for this sentence, so we have gotten nowhere closer to answering our question, as to assign a truth value of T or a truth value of F doesn't convey the full logical picture.

This example might motivate us to expand the set of truth values from $\{T, F\}$ to $\{T, F, i\}$ where i can be interpreted as meaning; *“both true and false”*. With this modification we can now assign a single truth value to our sentence, namely i. Logics which have more than two truth values are known as *Many-Valued Logics* (from now on abbreviated MV Logics) and in this section we give examples of famous MV Logics and discuss their semantics and syntax.

For further motivation (and a glimpse into some modern research on the subject!) we note that MV Logics can be used in the study of abstract algebras [26] as well as in rethinking how time is thought of in physics [17]. Moreover if we were to further generalise from a discrete set of truth values to a continuous set of truth values (e.g. the interval $[0, 1]$, where any real number $x \in [0, 1]$ is a truth value), then we would arrive at Fuzzy Logic which currently enjoys great popularity due to its applicability in areas such as artificial intelligence research, see [38] and [11]. The point we're trying to make here is that MV Logics aren't just a fun abstraction that logicians with too much time on their hands entertain themselves with, but an important object in mathematics worth attention and study.

3.2 Definitions and Syntax

A Many-Valued Logic, given some propositional language and a set of connectives for this language (denoted $\mathcal{C}$), is the ordered three tuple $\langle \mathcal{V}, \mathcal{D}, \{f_c \mid c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is the set of truth values, $\mathcal{D} \subseteq \mathcal{V}$ is the set of designated values (the truth values which are preserved under valid inferences), and f_c denotes the truth function for the connective $c \in \mathcal{C}$.. Note that if $\mathcal{V} = \{T, F\}$, $\mathcal{D} = \{T\}$ and $\mathcal{C}$ is the set of connectives of CPL, $\mathcal{C} = \{\wedge, \vee, \neg, \Rightarrow\}$, then this definition matches the one we have given for CPL.

A map $\nu : At \to \mathcal{V}$ is called an interpretation. Intuitively, a map ν is a choice of which truth values to assign to the atoms in At, which then will recursively determine the truth values of formulas in our language. For the set of connectives $\mathcal{C} = \{\wedge, \vee, \neg, \Rightarrow\}$ and a valuation ν the following equations hold,

$$\begin{aligned}
\nu(P \wedge Q) &= f_{\wedge}(\nu(P), \nu(Q)),\\
\nu(P \vee Q) &= f_{\vee}(\nu(P), \nu(Q)),\\
\nu(\neg P) &= f_{\neg}(\nu(P)),\\
\nu(P \Rightarrow Q) &= f_{\Rightarrow}(\nu(P), \nu(Q)).
\end{aligned}$$

This recursive determination of a formula's truth value is captured in the general equation,

$$\nu(c(A_1, \ldots, A_n)) = f_c(\nu(A_1), \ldots, \nu(A_n)),$$

where $c \in \mathcal{C}$ is an n-place connective and $A_1, \ldots, A_n \in At$ are propositional variables. In our example above $\wedge \in \mathcal{C}$ is a 2-place connective and $P, Q \in At$ are the atoms.

The notions above were already defined in the introduction and the only difference in the definitions for MV-Logics is that in general $|\mathcal{V}| \geq 3$ and $|\mathcal{D}| \geq 1$.

Just like CPL, MV-Logics are decidable, as given any formula with m propositional parameters in a MV-Logic with $|\mathcal{V}| = n$ truth-values, one could write down a truth table with n^m possible rows (this is essentially due to the fact that since in this instance $\nu : \{A_1, \ldots, A_m\} \to \mathcal{V}$ and there are exactly n^m maps whose domain has cardinality m and codomain has cardinality n), and in finite time recursively evaluate the truth value of the formula for each of the n^m cases.

With this mini-exposition on MV-Logics and their basic definition, we proceed with a review of some famous and historical examples.

3.3 Łukasiewicz Logic Ł$_3$

One of the earliest examples of an MV-Logic is the 3-valued Łukasiewicz Logic, it uses the same language as CPL, with the difference, of course, being in the set of truth values $\mathcal{V} = \{T, i, F\}$ whilst the set of designated values remains the same as for CPL, $\mathcal{D} = \{T\}$. The truth functions f_c for the connectives $c \in \mathcal{C} = \{\neg, \wedge, \vee, \Rightarrow\}$ in Ł$_3$ can be defined by truth tables as follows:

$f_\neg$	
T	F
i	i
F	T

$f_\wedge$	T	i	F
T	T	i	F
i	i	i	F
F	i	F	F

$f_\vee$	T	i	F
T	T	T	T
i	T	i	i
F	T	i	F

$f_\Rightarrow$	T	i	F
T	T	i	F
i	T	T	i
F	T	T	T

Łukasiewicz was motivated to develop this MV-logic by Aristotle's claim that the 2-valued Logics couldn't accurately capture the truthfulness of sentences describing future events, meaning CPL does not accurately capture the truthfulness of the sentence "Tomorrow I will prove the Riemann Hypothesis", as it's inaccurate to argue that we have complete certainty over tomorrow's events which we require to evaluate the truth value of any sentence. Łukasiewicz gave the following semantics for the truth values: T denotes "True", F denotes "False" and i denotes "neither True nor False".

At this point one may be tempted to take Ł$_3$ and declare it equivalent to Intuitionistic Propositional Logic (IPL), after all, for a proposition P, interpreting $\nu(P) = T$ as provable, $\nu(P) = F$ as $\nu(\neq P) = T$, the negation of P is provable, and $\nu(P) = i$ as neither P or its negation is provable, seems like an accurate way to capture the "provability" of propositions in constructive mathematics, including those which are independent of the system we work in (e.g Euclid's 5th axiom being independent of the system defined by the first 4 of Euclid's axioms). However this is false! We will see later in Chapter 4 an elegant proof by Gödel which states that IPL is not a finitely MV-Logic.

3.4 (Strong) Kleene Logc K_3

For our next example we present the (Strong)[1] Kleene logic, denoted K_3, which is another 3-valued logic. It is in fact almost identical to Ł$_3$, however the implication truth function is defined differently:

$f_\Rightarrow$	T	i	F
T	T	i	F
i	T	i	i
F	T	T	T

Comparing the truth tables of $f_\supset$ in Ł$_3$ and K_3 we note that they're the same apart from the middle entry, in K_3 we have $f_\Rightarrow(i, i) = i$. This seemingly small

[1]This is in contrast to *Weak Kleene Logic* which differs to *Strong Kleene Logic.* only by defining the truth functions slightly differently

change has quite powerful consequences, one of which we provide as a theorem below:

Theorem 11. *There are no tautologies K_3. More precisely, for any sentence of the form $\models \Sigma$, where Σ is a collection of formulas, the valuation ν which assigns i to each atom which occurs in Σ is a counter-model.*

Proof. The key observation, is that $f_{\neg}(i) = f_{\wedge}(i,i) = f_{\vee}(i,i) = f_{\Rightarrow}(i,i) = i$. Therefore, since any formula is simply a finite recursive composition of connectives in the language, the valuation which assigns i to each atom will assign the truth value i to these connectives which recursively will will assign each formula in Σ the truth value $i \notin \mathcal{D}$. Thus by definition $\not\models \Sigma$, where ν is the appropriate counter-model. □

3.5 Logic of Paradox LP

In this example we present the Logic of Paradox, denoted LP which is again a 3-valued logic defined in the same way as K_3 but with the change that in LP, $\mathcal{D} = \{1, i\}$. In LP, the truth value i is given new semantics, it is interpreted as "both True and False", and the name for this logic is explained by the fact that it seems to capture the "truthfulness" of paradoxical statements such as *"this statement is false"* which we have already seen in the introduction to this chapter.

An important difference between LP and MV-Logics such as K_3 and Ł$_3$ is that LEM holds in LP whilst not holding in K_3 and Ł$_3$. On the other hand, Modus Ponens doesn't hold in LP but it does hold in K_3 and Ł$_3$. First to show the claims about LEM we note that since for LP, Ł$_3$ and K_3 the disjunction $\vee$ has the same definition, we can consider the single truth table below,

P	$\neg P$	$P \vee \neg P$
T	F	T
i	i	i
F	T	T

and deduce that since $i \notin \mathcal{D}_{Ł_3}$ and $i \notin \mathcal{D}_{K_3}$ then the valuation $\nu(P) = i$ is a counter model to LEM in Ł$_3$ and K_3 respectively. Note that here $\mathcal{D}_L$ denotes the designated set of the logic L. Meanwhile $i, T \in \mathcal{D}_{LP}$, and so by examining the truth table and remarking that for every valuation ν we have $\nu(P \vee \neg P) \in \mathcal{D}_{LP}$ we see that LEM is a tautology in LP. Similarly we consider the truth table for Modus Ponens, where we take $\Rightarrow$ to be implication defined for LP and K_3 whilst $\Rightarrow_{Ł_3}$ will denote the implication as defined for Ł$_3$,

P	$P \Rightarrow Q$	$P \Rightarrow_{Ł_3} Q$	Q
T	T	T	T
T	i	i	i
T	F	F	F
i	T	T	T
i	i	T	i
i	i	i	F
F	T	T	T
F	T	T	i
F	T	T	F

From the truth table above and the definition of semantic validity we see that modus ponens is semantically valid in both K_3 and $Ł_3$, and we see that the valuation ν which fixes $\nu(P) = i$ and $\nu(Q) = F$ is a counter-model to the claim that Modus Ponens is valid in LP.

3.6 First-Degree Entailment FDE

For the last example in this section we present the First Degree Entailment logic (abbreviated FDE from now on), a creation of Dunn [12] and Belnap [2], which in contrast to the previous three is a 4-valued logic. In his paper Belnap discusses how such a 4-valued logic could better protect a computer reasoning agent when it is supplied contradictory information. Imagine a computer is given contradictory information from two different credible sources, i.e source A gives the computer information of the form P, whilst source B gives the computer information of the form $\neg P$. When humans are the sources of information for computers, this situation of feeding the computer contradictory information is not that far fetched! The computer treats both source A and source B as supplying truthful information so now the logical system in the computer contains a statement which is both **True** and **False**. Now, in the two-valued CPL this leads to the well-known *Principle of Explosion*, which in short says that for any propositions P and Q, $P \wedge \neg P \vDash Q$. In a slogan, the Principle of Explosion says that "From a contradiction, anything follows". Now, this is problematic for our computer system if it cannot discern which of the two sources (A or B) is wrong here. Belnap in conceiving of this logic has sought to contain such contradictions, such that the Principle of Explosion is contained to a small portion of the system, instead of entailing any possible proposition, as would be the case if CPL.

The truth values of this logic are $\mathcal{V} = \{T, F, B, N\}$, which are given the semantics T: "True", F: "False", B: "Both true and false" and N: "Neither true nor false". If a computer has been told a statement P is true, it will mark $\nu(P) = T$, and similarly if it has been told P is false it will mark $\nu(P) = F$. If the computer has been told P is true and subsequently been told that P is false, it will mark $\nu(P) = B$, to reflect the contradictory information it was given. If the computer has been given no information about P, it will mark $\nu(P) = N$.

In FDE the negation operation has the following truth table:

P	T	F	B	N
$\neg P$	F	T	B	N

where we crucially note that $\neg B = B$ and $\neg N = N$. To understand how conjunctions and disjunctions behave in this logic we consider the following lattice of truth values:

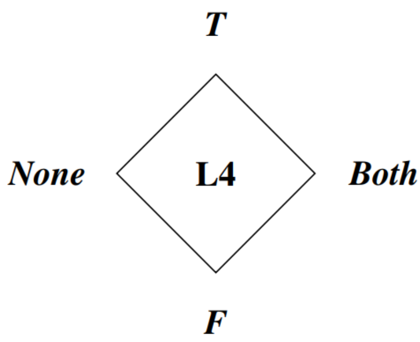


Figure 3.1: Image taken from Belnap's Paper [2]

and we define the conjunction of truth values to be the meet (the greatest lower bound) of the above lattice, and the disjunction of truth values to be the join (the least upper bound) of the above lattice. Notationally we may express this as follows for propositional variables P and Q,

$$\nu(P \wedge Q) = \nu(P) \sqcap \nu(Q) \qquad\qquad \nu(P \vee Q) = \nu(P) \sqcup \nu(Q),$$

where $\sqcap$ denotes the meet operation and $\sqcup$ denotes the join operation. The implication is defined through entailment by Belnap, that is $P \Rightarrow Q$ if and only if $P \vDash Q$. This has a nice interpretation for the L4 lattice above, where $P \vDash Q$ is equivalent to $P \leq_{L4} Q$, where $\leq_{L4}$ is the partial ordering on the set of truth values $\mathcal{V} = \{T, B, N, F\}$ implied by the L4 lattice image. Explicitly $F \leq_{L4} B \leq_{L4} T$ and $F \leq_{L4} N \leq_{L4} T$ is the partial oredr. From this equivalence we can immediately see a problem with this definition of the implication by Belnap, how do we evaluate $P \vDash Q$ when $\nu(P) = B$ and $\nu(Q) = N$? By the above discussion this question is equivalent to determining how does $\leq_{L_4}$ order the truth values B and N? But by the definition of $\leq_{L_4}$ we see that this partial order in fact does not order the truth values B and N, and so it seems that Belnap's definition of the implication here is not well-defined. To fix this, FDE has three different well-defined types of implication, and whilst we will not discuss them here we direct the curious reader to *Ultlog* [41] for further study.

With the logical operations defined we now provide a counterexample to the Principle of Explosion $(P \wedge \neg P) \vDash Q$. Consider the valuation ν which assigns $\nu(P) = B$ and $\nu(Q) = F$, by the definitions above we see that $\nu(\neg P) = B$

and thus $\nu(P \wedge \neg P) = B \sqcap B = B \in \mathcal{D}$, meanwhile $\nu(Q) = F \notin \mathcal{D}$. This counterexample shows that $(P \wedge \neg P) \nvDash Q$, and in this way FDE is able to "contain" contradictions in such a way as to not entail everything.

3.7 Connection Between Logics

Now we wish to point out the connections between the three logics K_3, LP and FDE. By interpreting the i truth value in K_3 Note that by examining the truth tables below for FDE, K_3 and LP,

(FDE)

P	$\neg P$
T	F
B	B
N	N
F	T

$\wedge$	T	B	N	F
T	T	B	N	F
B	B	B	F	F
N	N	F	N	F
F	F	F	F	F

$\vee$	T	B	N	F
T	T	T	T	T
B	T	B	T	B
N	T	T	N	N
F	T	B	N	F

($\mathbf{K_3}$)

P	$\neg P$
T	F
N	N
F	T

$\wedge$	T	N	F
T	T	N	F
N	N	N	F
F	F	F	F

$\vee$	T	N	F
T	T	T	T
N	T	N	N
F	T	N	F

(LP)

P	$\neg P$
T	F
B	B
F	T

$\wedge$	T	B	F
T	T	B	F
B	B	B	F
F	F	F	F

$\vee$	T	B	F
T	T	T	T
B	T	B	B
F	T	B	F

Note how restricting the truth-tables of FDE to values $\{T, N, F\}$ gives the truth tables of K_3, and restricting the truth-tables of FDE to values $\{T, B, F\}$ gives the truth tables of LP. Therefore a semantic inference $\Sigma \vDash_{\text{FDE}} \Gamma$ implies $\Sigma \vDash_{K_3} \Gamma$ and $\Sigma \vDash_{\text{LP}} \Gamma$, as a valid semantic inference in FDE means that every valuation ν satisfies the property,

$$\text{if } \nu(\Sigma) \in \mathcal{D} \text{ then } \nu(\Gamma) \in \mathcal{D},$$

which implies that the restrictions $\nu|_{\mathcal{V}_{K_3}}$ and $\nu|_{\mathcal{V}_{LP}}$ also satisfy this property, more precisely

$$\text{if } \nu|_{\mathcal{V}_{K_3}}(\Sigma) \in \mathcal{D} \text{ then } \nu|_{\mathcal{V}_{K_3}}(\Gamma) \in \mathcal{D}, \quad \text{and,} \quad \text{if } \nu|_{\mathcal{V}_{LP}}(\Sigma) \in \mathcal{D} \text{ then } \nu|_{\mathcal{V}_{LP}}(\Gamma) \in \mathcal{D}.$$

We will see towards the end of this section that FDE, K_3 and LP are all complete and sound logic with respect to the semantics described here and appropriate syntactic systems. Hence using the soundness of FDE, the completeness of K_3 and LP, and the above semantic relationship we can show that if $\Sigma \vdash_{\text{FDE}} \Gamma$ then $\Sigma \vdash_{K_3} \Gamma$ and $\Sigma \vdash_{\text{LP}} \Gamma$. In this way, we can see that both K_3 and LP are extensions of FDE, and we urge the reader to compare this with the way in which CPL is an extension of IPL.

3.8 Truth-Value Gaps and Gluts

Having given plenty of examples of MV logics, we wish to give a bank of examples in everyday language which can't be assigned a simply T or F truth value, and hence motivate the study of the logics we have seen.

Logics such as K_3 and Ł$_3$ (that is logics which interpret i as *neither true nor false*) are motivated by *truth-value gaps*, that is propositions which are neither true nor false. As examples consider statements about the future such as,

P : It will be sunny tomorrow in Edinburgh.

To assign $\nu(P) = T$ or $\nu(P) = F$ would entail us having certainty over what the whether in Edinburgh will be like tomorrow, a certainty which we can't have, hence for such statements we would be urged to use a logic such as K_3 or Ł$_3$ and assign $\nu(P) = i$. On the other hand, logics such as LP (that is logics which interpret i as *both true and false*) are motivated by *truth-value gluts*, that is propositions which are both true and false. We already have seen an example of such a sentence in the motivation in the beginning of this chapter, and in general self-referential statements tend to result in truth-value gluts, for example consider Russell's Paradox. In general this view towards the nature of truth, that there exist proposition that are both true and false, meaning that some contradictions are true, is called dialetheism. For more on dialetheism we direct the curious reader to the work of Graham Priest, a proponent of dialetheism who has wrote extensively on this subject [31].

3.9 Soundness and Completeness Results

Theorem 12. *The logics Ł$_3, K_3, LP, FDE$ are sound and complete with respect to the semantics presented here and the appropriate syntactical system.*

Proof. For soundness and completeness of Ł$_3$ please see Chapter 3 of *Hájek* [18]. Note that the syntactic system here is given by algebras, called Łukasiewicz algebras.
For soundness and completeness of K_3, LP, FDE please see Chapter 8 of *Priest* [30]. Note that the syntactic system here is given by tableaux which one can read more about in the referred Priest's Book. □

Chapter 4

Kripke Models

Kripke models exhibit different structures across various logics. In this section, we will explore the Kripke semantics of Intuitionistic Logic and Many-Valued Logic, particularly focusing on Many-Valued Modal Logic within Many-Valued Logic; therefore, we begin with the Kripke semantics for Modal Logic.

4.1 Kripke Semantics for Modal Logic

Modal Logic studies necessity ($\Box$) and possibility ($\Diamond$). To formalize its semantics, Saul Kripke introduced the Possible World Semantics, embedding modal operators into set-theoretic structures via the Accessibility Relation, which revolutionized the research paradigm of Modal Logic.

4.1.1 Kripke Model

Kripke model is a triple $M = (W, R, \nu)$[22], where $W \neq \emptyset$ is a set of worlds, each world $w \in W$ represent a logical state or situation, what we called possible world.

One can consider possible worlds as 'different situations', for example 'today is a sunny day' and 'today is a rainy day', these are two possible worlds, and they are identical except for weather. And under these two possible worlds or different situations, statements will have different truth values. Consider 'the ground is dry', this is true in the possible world 'today is a sunny day', but is not true when 'today is a rainy day'.

$R \subseteq W \times W$ is the Accessibility Relation,which defines the accessibility between worlds, it acts as a bridge between worlds, defining how knowledge or truth "flows" from one possible world to another.

Take the same example, consider possible worlds w: 'today is a sunny day' and v: 'today is a rainy day'. Suppose the weather forecast shows: If today is sunny, then tomorrow might be rainy; but if today is rainy, then tomorrow will definitely be rainy. By this we can define an accessibility relation $R = \{(w, w)(w, v), (v, v)\}$, that is, from world w (today is sunny day), we could access to w(today), or we can access to v(tomorrow could be rainy day); but from world v, we could only access to v(tomorrow will definitely be rainy day).

Accessibility Relation has the properties of reflexivity, transitivity, symmetry and euclideanity[16], which means $\forall w, v, u \in W$, we have:

- Reflexive: $(w, w) \in R$
- Transitive: if $(w, v) \in R$ and $(v, u) \in R$, then $(w, u) \in R$
- Symmetric: if $(w, v) \in R$, then $(v, w) \in R$
- Euclidean: if $(w, v) \in R$ and $(w, u) \in R$, then $(v, u) \in R$

The euclideanity is derived by transitivity and symmetry, consider if $(w, v) \in R$ and $(w, u) \in R$, then $(v, w) \in R$, and thus $(v, u) \in R$.

And $\nu : W \times At \to \{0, 1\}$ is a valuation function, At is the set of symbols of the logic language. ν assigns a truth value to each atomic proposition $P \in At$ at every world $w \in W$. For example, $\nu\,(w, P) = 1\,(or\ 0)$, it means, in the possible world w, P is true (or false).

In our example, let proposition P be 'the ground is dry', so we can get $\nu(w, P) = 1$ and $\nu(v, P) = 0$: if today is sunny day, the ground is dry, and if today is rainy day, the ground is not dry.

4.1.2 Satisfiability Relation

The semantic recursive satisfiability relation ($\Vdash$) is the key mechanism for defining the truth value of formulas in each possible world, for example $w \Vdash A$ means 'in the world w, formula A is true'. Its core idea is to recursively decompose the structure of formulas, combined with the accessibility relation between possible worlds, to gradually determine the semantic truth value of formulas.

Recursive Definition of the Semantics[3]:

- Atomic Proposition: $w \Vdash P$ if and only if $\nu\,(w, P) = 1$
- Negation: $w \Vdash \neg A$ if and only if $w \nVdash A$
- Conjunction: $w \Vdash A \wedge B$ if and only if $w \Vdash A$ and $w \Vdash B$
- Disjunction: $w \Vdash A \vee B$ if and only if $w \Vdash A$ or $w \Vdash B$
- Necessity: $w \Vdash \Box A$ if and only if $\forall v, (w, v) \in R, v \Vdash A$.
- Possibility: $w \Vdash \Diamond A$ if and only if $\exists v, (w, v) \in R, v \Vdash A$

In the Weather Example, consider proposition P 'ground is dry', we have:

For the sunny day w, we can access to sunny day w and rainy day v, and P is true in w, P is false in v. Thus we have $w \Vdash \Diamond P$ since there exist an accessible possible world w such that P is true; $w \nVdash \Box P$ since P is not true in all accessible possible worlds; and $w \nVdash \Box\neg P$ since P is true in sunny day v.

For the rainy day v, we can only access to rainy day. Thus $v \nVdash \Diamond P$ since there does not exist accessible world such that P is true; and $v \Vdash \Box\neg P$, since P is absolutely not true in the only accessible world v.

4.2 Kripke semantics for Many-Valued Modal Logic

For many-valued modal logic, a linear order is defined on the set of truth values, $\mathcal{V} = \{q_1, q_2, ..., q_n\}$, $n \geq 2$ is a set of truth valued ordered by $q_1 < q_2 < \cdots < q_n$ (for example $\mathcal{V} = \{0, 0.5, 1\}$). Formulas of many-valued modal logic are built of atomic propositions combined by propositional connectives $(\neg, \wedge, \vee, \cdots)$ and modal connectives $\Box$ and $\Diamond$, the set of all formulas is denoted by Fm.
The semantics of propositional connectives is given by truth tables, where the truth table of an l-ary propositional connective c is a function $c : \mathcal{V}^l \to \mathcal{V}$.
A labelled formula is a pair (A, k), where A is a formula and $k = 1, ..., n$, and it means that v_k is the truth value associated with A.
Sequents are expressions of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite sets of labelled formulas.

4.2.1 Many-Valued Kripke Model

A many-valued Kripke model is a triple $M = (W, R, \nu)$[22] where W is nonempty set of possible worlds, $R \subseteq W \times W$ is binary accessibility relation, and $\nu : W \times At \to \mathcal{V}$ is valuation function.

For a world $w \in W$, the set of successors of w, $S(w)$ is defined by $S(w) = \{v \in W : (v, w) \in R\}$.

Extend V to $W \times Fm$ recursively, as follows.

- $\nu(w, c(A_1, ..., A_l)) = c(\nu(w, A_1), ..., \nu(w, A_l))$
- $\nu(w, \Box A) = \inf(\{\nu(v, A) : v \in S(w)\})$
- $\nu(w, \Diamond A) = \sup(\{\nu(v, A) : v \in S(w)\})$

We write $w \vDash (A, k)$ if $\nu(w, A) = q_k$.

Example 4.2.1. *Let $\mathcal{V} = \{0, 0.5, 1\}$ and set a model with $W = \{w, v\}$, $R = \{(w, v)\}$ and truth values $\nu(w, P) = 0.5$, $\nu(v, P) = 1$. Then we have,*

- $\nu(w, \Box P) = \inf\{0.5, 1\} = 0.5$
- $\nu(w, \Diamond P) = \sup\{0.5, 1\} = 1$

4.2.2 Satisfiability Relation

The satisfiability relation $\Vdash$ between worlds of W and sequents of labelled formulas is defined: A world w satisfies a sequent $\Gamma \Rightarrow \Delta$, denoted $M, w \Vdash \Gamma \Rightarrow \Delta$, if the following holds.

if for each $(A, k) \in \Gamma$, $\nu(w, A) = q_k$, then for some $(A, k) \in \Delta$, $\nu(w, A) = q_k$.

A Kripke model M satisfies a sequent $\Gamma \Rightarrow \Delta$, if each world in W satisfies $\Gamma \Rightarrow \Delta$; and M satisfies a set of sequents Σ, if it satisfies each sequent in Σ. Finally, a set of sequents Σ semantically entails a sequent $\Gamma \rightarrow \Delta$, $\Sigma \vDash \Gamma \Rightarrow \Delta$, if each many-valued Kripke model satisfying Σ also satisfies $\Gamma \Rightarrow \Delta$.

4.3 Kripke Semantics for Intuitionistic Logic

Intuitionistic Logic rejects the classical Law of Excluded Middle and claims that the truth value of a proposition must be constructively verified. In Intuitionistic Logic, the Kripke frame is endowed with a unique structure and properties to reflect its philosophical ideals of Constructive Truth and Cumulative Information Growth.

4.3.1 Intuitionistic Kripke Model

The intuitionistic Kripke model is a triple $M = (W, \preceq, \nu)$[8] where $\preceq \subseteq W \times W$ is a partial order representing the accumulation of information. That is, if $w \preceq v$ (v extends w), then v contains all the information of w and may include additional new information.

Partial Order $\preceq$ has the properties of reflexivity and transitivity same as Accessibility Relation R, but have different properties of antisymmetry. $\forall w, v, u \subseteq W$, we have properties:

- Reflexive: $w \preceq w$
- Transitive: if $w \preceq v$ and $v \preceq u$, then $w \preceq u$
- Antisymmetric: if $w \preceq v$ and $v \preceq w$, then $w = v$

The valuation function $\nu : W \times At \rightarrow \{0, 1\}$ must satisfy the Heredity Condition:

$$\forall w \in W, \forall P \in At,\ if\ \nu(w, P) = 1\ and\ w \preceq v,\ then\ \nu(v, P) = 1. \qquad (4.1)$$

The Heredity Condition is a core constraint of the intuitionistic Kripke frame: if a proposition is true in a given world, then it must be true in every extension of that world.This property ensures the irreversibility of knowledge accumulation, thereby ruling out scenarios in which truth could be overturned by later extensions.

4.3.2 Semantic satisfiability Relation

In an intuitionistic Kripke model, $w \Vdash A$ means that at the possible world w, the formula A is constructively forced to be true. Once knowledge is acquired, it is irreversible; the truth persists in all extensions of the world.

It has the property of heredity: if $w \preceq v$ and $w \Vdash A$, then $v \Vdash A$; yet the converse is not true, $w \preceq v$ and $v \Vdash A$ does not imply $w \Vdash A$.

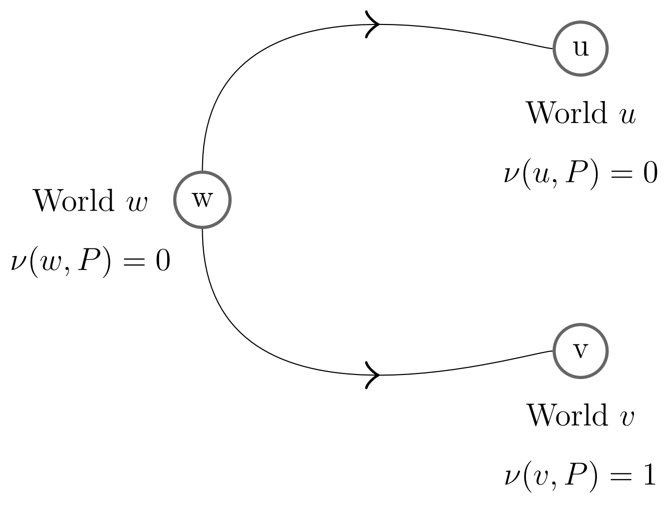
Recursive Definition of the Semantics:

- Atomic Proposition: $w \Vdash P \Leftrightarrow \nu\,(w, P) = 1$
- Conjunction:$w \Vdash A \wedge B \Leftrightarrow w \Vdash A \textit{ and } w \Vdash B$
- Disjunction:$w \Vdash A \vee B \Leftrightarrow w \Vdash A \textit{ or } w \Vdash B$
- Negation:$w \Vdash \neg A \Leftrightarrow \forall v \succeq w, v \nVdash A$
- Implication: $w \Vdash A \Rightarrow B \Leftrightarrow \forall v \succeq w$, if $v \Vdash A$ then $v \Vdash B$

4.3.3 Examples

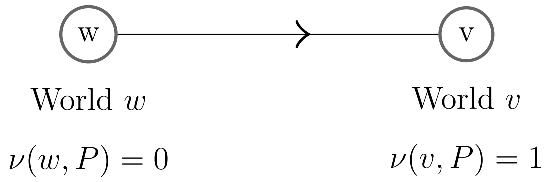
We will discuss some examples to explore how LEM, double negation elimination, and other classically valid inference fail in Intuitionistic Logic. In the diagrams below we denote possible worlds as nodes, and accessibility relations as directed edges. Moreover, since the accessibility relation is reflexive, for ease of notation, we will neither explicitly specify them when defining our counter-models nor illustrate them in the diagrams.

Example 4.3.1. *Consider a model with worlds $W = \{w, v, u\}$, and order $w \preceq v$, $w \preceq u$. Set $\nu(w, P) = 0, \nu(v, P) = 1, \nu(u, P) = 0$. Then the law of excluded middle is rejected: $w \nVdash P \vee \neg P$.*



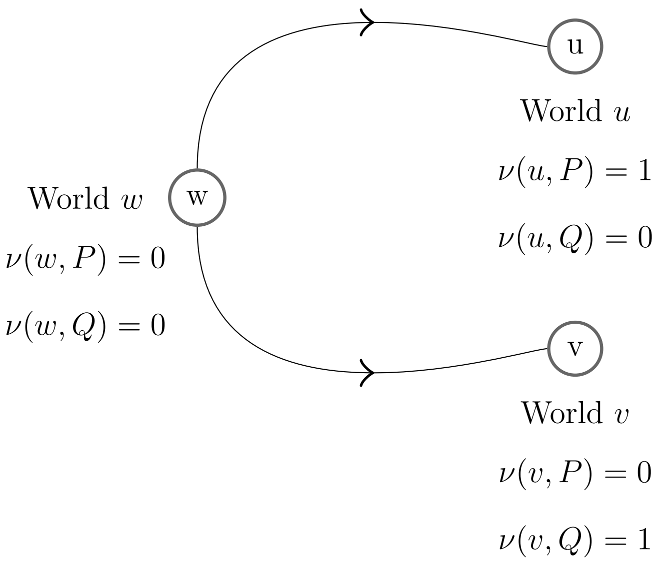
Proof. According to the recursive definition we know that $w \Vdash P \vee \neg P$ if and only if $w \Vdash P$ or $w \Vdash \neg P$. Since $\nu(w, P) = 0$, we have $w \nVdash P$. And $w \Vdash \neg P$ if and only if for all $w' \geq w$, we have $w' \nVdash P$. Since there exists v such that $\nu(v, P) = 1$, $v \Vdash P$, so we have $w \nVdash \neg P$. Thus the law of excluded middle is rejected. $\square$

Example 4.3.2. *Consider a model with worlds $W = \{w, v\}$, and order $w \preceq v$. Set $\nu(w, P) = 0, \nu(v, P) = 1$. Then the double negation elimination is rejected: $w \nVdash \neg\neg P \Rightarrow P$.*



Proof. By the recursive definition, we have $w \Vdash \neg\neg P \Rightarrow P$ if and only if $\forall w' \succeq w$, if $w' \Vdash \neg\neg P$ then $w' \Vdash P$. Since $w \Vdash \neg P$ if and only if $\forall v \succeq w, v \nVdash P$, and since $\nu(v, P) = 1$, we have $w \nVdash \neg P$, and thus $w \Vdash \neg\neg P$. Yet since $\nu(w, P) = 0$, $w \nVdash P$. Then $w \Vdash \neg\neg P$ but $w \nVdash P$, and so the double negation elimination is rejected. $\square$

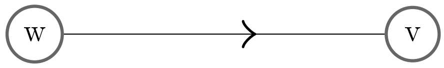
Example 4.3.3. *Here we fulfil our promise in chapter 2 (2.2) and give a counter-model to the claim that $\neg(P \wedge Q) \vdash_{IPL} \neg P \vee \neg Q$. Consider the set of world $W = \{w, v, u\}$, and order $w \preceq v$, $w \preceq u$. Set $\nu(w, P) = 0, \nu(w, Q) = 0$; $\nu(v, P) = 1, \nu(v, Q) = 0$ and $\nu(u, P) = 0, \nu(u, Q) = 1$. We claim that this is an appropriate counter-model.*



Proof. Working from the definitions, trying to make the inference false at world w, we see that our counter-model needs to satisfy the condition $\forall x \succeq w$, $\nu(x, P) = 0$ or $\nu(x, Q) = 0$ and the condition $\exists y \succeq w$, $\nu(y, P) = 1$ and $\exists z \succeq w$, $\nu(z, Q) = 1$.

Now, in every world accessible from w, call it x, we see that $\nu(x,P) = 0$ or $\nu(x,Q) = 0$, therefore we satisfy the first condition. We also note that world v is accessible from world w and $\nu(w,P) = 1$ and world u is accessible from world w and $\nu(w,Q) = 1$, therefore we also satisfy the second condition. Hence we have shown that this is a valid counter-model, and so we conclude that $\neg(P \wedge Q) \nvdash_{IPL} \neg P \vee \neg Q$. □

Example 4.3.4. *We give a counter-model to the claim that* $\neg(P \Rightarrow Q) \vdash_{IPL} P \wedge \neg Q$. *Consider the set of world* $W = \{w, v\}$, *and order* $w \preceq v$. *Set* $\nu(w,P) = 0, \nu(w,Q) = 1$ *and* $\nu(v,P) = 1, \nu(v,Q) = 1$. *We claim that this is an appropriate counter-model.*



World w	World v
$\nu(w,P) = 0$	$\nu(v,P) = 1$
$\nu(w,Q) = 1$	$\nu(v,Q) = 1$

Proof. Working from the definitions, trying to make the inference false at world w, we see that our counter-model needs to satisfy the condition $\forall x \succeq w$, $\exists y \succeq x$, $\nu(y,P) = 1$ or $\nu(y,Q) = 0$ and the condition $\nu(w,P) = 0$ or $\exists z \succeq w$, $\nu(z,Q) = 1$. Now, the only worlds accessible from w are itself and world v, both w and v access v, and v satisfies $\nu(v,P) = 1$ or $\nu(v,Q) = 0$, therefore we satisfy the first condition. We also note that world v, which is accessible from world w, satisfies $\nu(v,Q) = 1$, therefore we also satisfy the second condition. Hence we have shown that this is a valid counter-model, and so we conclude that $\neg(P \Rightarrow Q) \nvdash_{IPL} P \wedge \neg Q$. □

Chapter 5

Links Between Both Logics

5.1 Introduction

From previous chapters, we know that Many-Valued Logics generalize Classical Logic by allowing more than two truth values; Intuitionistic Logic are constructive logic that rejects the law of excluded middle (not every proposition is necessarily true or false).
In this chapter, we will be focusing on the links between those two logics - we will first introduce the Gödel Logic, and then investigate a key concept which explained the relation of two logics even more by proving that Intuitionistic Propositional Logic (IPL) is countably infinite Many-Valued Logic in the following sections.

5.2 Gödel Logic

First of all, Gödel Logic was introduced by Gödel in 1932, and was designed to explore the relationship between intuitionistic and classical logic. In *All Collected Kurt Gödel Works* [13] pages 222-225, he figure out, for any finite $n \geq 2$, there exists an $(n+1)$-valued logic whose theorems are strictly *weaker* than IPL. In other words, for the theorems within $(n+1)$-valued logic system, they cannot explain all theorems in IPL with logical operators. Hence, on Intuitionistic Propositional Logic (IPL), Gödel claims that:

Theorem 13. *IPL cannot be viewed as a system of Many-Valued Logics.*

In other words, there does not exist a finite set M of truth values containing a designated subset $D \subset M$, along with interpretations of the connectives $\Rightarrow$, A, V as binary operations on M, as well as an additional interpretation of $\Rightarrow$ as a unary operation on M, such that $H \vdash A$ holds precisely when, for every valuation ν mapping formulas into M, the value $\nu(A)$ belongs to D.

This theorem from Gödel makes Gödel Logic to be an intermediate propositional logic (between Intuitionistic and Many-Valued Logics), which extends Intuitionistic Logic and connects to Many-Valued Logics.

By the soundness[32] and completeness[39] of IPL with respect to finite Kripke models, the tautologies of IPL are the formulas that are satisfied in all the elements of any finite Kripke model. However, there is still a problem. Although IPL is decidable via finite Kripke models, infinite-valued logics are generally undecidable.

In order to solve that, Gödel claims that:

Theorem 14. *There exists an infinite descending hierarchy of logics with strengths intermediate between A (classical propositional logic) and IPL.*

Hence, we can see that:

$$A = L_2 \supset L_3 \supset L_4 \supset ...,$$

where L_n denotes the set of propositional formulas that are universally valid when interpreted over an n-element linearly ordered Heyting algebra (also termed a pseudo-Boolean algebra).

Theorem 14 signifies that infinitely many logical systems can be constructed between IPL and classical propositional calculus (A), forming a strictly decreasing sequence. Each system in this sequence contains IPL as a subset and is itself contained within A, thereby establishing a nested loop of progressively weaker logics. With Theorem 13 and Theorem 14, we have that IPL can be viewed as an infinitely Many-Valued Logics, but actually we can do more than that.

In order to find a more explicit semantic modeling, Stanisław Jaśkowski (1936) further argued that Intuitionistic Logic is not just infinitely many-valued but *countably* infinitely many-valued.

5.3 Proof for IPL

In this section, we will prove that: IPL is countably many-valued by following the similar structure of the proof made by Salehi[37]. Throughout this section, we will be dealing with propositional logic only, and the disjunction operation is assumed to be commutative and associative.
The definition of Many-Valued Logics (Section 3.2) from previous chapter provides a solid foundation for us to express the following concepts from mathematical point of view, and for the notion of tautology, we need to expand it to the n-valued cases in order to generalize our proof.

Lemma 15. *(A Tautology in n-Valued Logics):*
For every integer $n > 1$, the formula $\bigvee_{0 \leq i < j \leq n}(P_i \Rightarrow P_j)$, where the disjunction $\bigvee$ represents the logical ($\vee$) applied to all implication pairs with indices $0 \leq i < j \leq n$, is a tautology in any n-valued logic if the formula $(P \Rightarrow P) \vee Q$ is a tautology in the same system.

Proof. In n-valued logic, the $n+1$ atoms $\{P_0, P_1, ..., P_n\}$ can assumed to be one of n truth values. By *the Pigeonhole Principle*, there must exist some $0 \leq i < j \leq n$, such that P_i and P_j receive the same truth value under any valuation. If the formula $(P \Rightarrow P) \vee Q$ is a tautology which always evaluates to a designated T value under all valuations. Then, the formula $\bigvee_{0 \le i < j \le n}(P_i \Rightarrow P_j)$ must also evaluate to T for every valuation. This is because the equality $P_i = P_j$ ensures $(P_i \Rightarrow P_j)$ is true as $P \Rightarrow P$ is tautological, and the disjunction $\bigvee$ guarantees at least one such implication holds. Therefore, the formula is a tautology. □

This lemma highlights that formula like $(A \Rightarrow B) \vee (A \Rightarrow C) \vee (B \Rightarrow C)$ is tautology in CPL, where excluded middle holds. However, such disjunctive tautology fails in IPL, as intuitionism rejects non-constructive verification of disjunctions.

Theorem 16 (**Gödel**). *Intuitionistic Propositional Logic is not finitely many valued.*

Proof. Using the Lemma 15 we just proven, and since Kripke models are models of IPL, it is sufficient to show that $\bigvee_{0 \le i < j \le n}(P_i \Rightarrow P_j)$ for $0 \le i < j \le n$ is not a tautology for a finite Kripke model in IPL.
Consider the Kripke model $F = \langle W, \succeq, \Vdash \rangle$ with $W = \{w, w_0, w_1, ..., w_{n-1}\}$,
$\succeq = \{(w_i, w) | i < n\} \cup \{(w_i, w_i) | i < n\} \cup \{(w, w)\}$, and
$\Vdash = \{(w_0, P_0), (w_1, P_1), ..., (w_{n-1}, P_{n-1})\}$.

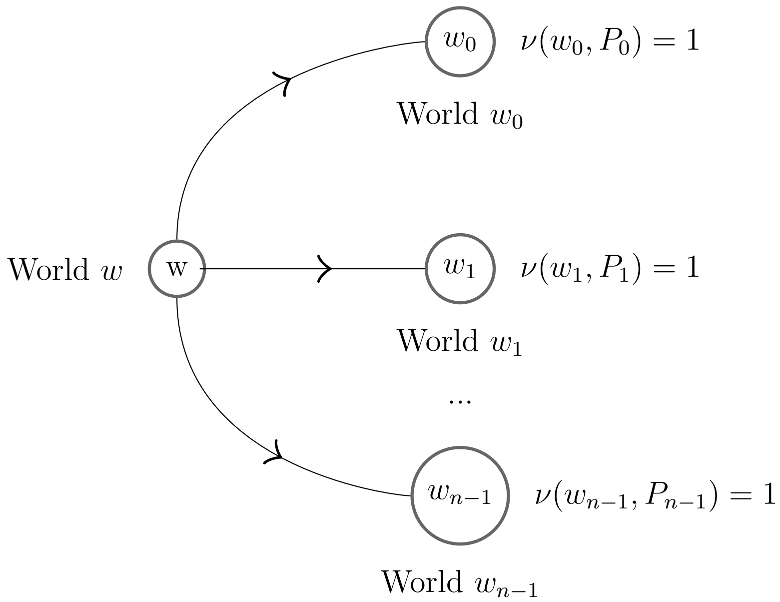


Figure 5.1: Illustration of proof through the Kripke model

As the Figure 5.1 shown, there is only one edge from all possible world w to every distinct possible world w_i, which leads to a one-to-one correspondence between the sets $\{w_0, w_1, \ldots, w_{n-1}\}$ and $\{(w_i, w) | i < n\}$, and it means that for any $0 \le i < n$ we have $w_i \Vdash P_i$. Also there is no edge between any two w_i, which indicates that $w_i \nVdash P_j$ for any $j \ne i$. So, $w_i \nVdash P_i \Rightarrow P_j$ for any $0 \le i < j \le n$, which implies that $w \nVdash \bigvee_{0 \le i < j \le n}(P_i \Rightarrow P_j)$. Thus, the formula $\bigvee_{0 \le i < j \le n}(P_i \Rightarrow P_j)$ for $0 \le i < j \le n$ is not a tautology in IPL. □

The failure of finite truth-value semantics for IPL underscores the necessity of a different algebraic framework. To formalize the relationship between Kripke semantics and many-valued logics, we now introduce some foundational definitions for monotone functions and their logical operations in Kripke point of view. These constructions will serve as the building blocks for encoding intuitionistic truth conditions within a countable valuation structure.

Definition 4. (Monotone Functions):
A function $f : W \to \{0, 1\}$ on a Kripke frame $(W, \succeq)$ is defined as *monotone*, if for all $w, w' \in W$, $w' \succeq w$ implies $f(w') \geq f(w)$.

Definition 5. $(\neg, \wedge, \vee, \Rightarrow)$:
For a Kripke frame $(W, \succeq)$, and monotone functions $f, g : W \to \{0, 1\}$, then we have:
$\neg f : W \to \{0, 1\}$ is defined as

$$(\neg f)(w) = \begin{cases} 1, \text{if } \forall w' \succeq w, f(w') = 0 \\ 0, \text{if } \exists w' \succeq w, f(w') = 1 \end{cases}$$

$f \wedge g : W \to \{0, 1\}$ is defined as $(f \wedge g)(w) = min\{f(w), g(w)\}$;
$f \vee g : W \to \{0, 1\}$ is defined as $(f \vee g)(w) = max\{f(w), g(w)\}$;
$f \Rightarrow g : W \to \{0, 1\}$ is defined as

$$(f \Rightarrow g)(w) = \begin{cases} 1, \text{if } \forall w' \succeq w, f(w') = 1 \Rightarrow g(w') = 1 \\ 0, \text{if } \exists w' \succeq w, f(w') = 1 \wedge g(w') = 0 \end{cases}$$

for all $w \in W$.

Definition 6. (Constant functions):
Let $1_W : W \to \{0, 1\}$ be a constant function of 1, which means that $1_W(w) = 1$ for all $w \in W$.
Similarly, Let $0_W : W \to \{0, 1\}$ be a constant function of 0, which means that $0_W(w) = 0$ for all $w \in W$.

With some definitions chasing and manipulating, it is easy to see that both 1_W and 0_W obey the law of $(\neg, \wedge, \vee, \Rightarrow)$. For instance, $\neg 1_W = 0_W$, $(0_W \wedge 0_W) = 0_W$, $(0_W \vee 1_W) = 1_W$, $(1_W \Rightarrow 0_W) = 0_W$ and so on.
Equipped with monotonicity and logical operations, we can comment on their closure properties. The upcoming Lemma 17 will ensure that these operations preserve the monotonic structure of functions on Kripke frames, a critical step toward assembling a coherent many-valued system. This stability under logical operations guarantees that the resulting valuation space remains well-defined and algebraically consistent.

Lemma 17. *For any Kripke model, 1_W and 0_W are monotonic constant functions. If $f, g : W \to \{0, 1\}$ are monotonic functions, then $\neg f$, $f \wedge g$, $f \vee g$ and $f \Rightarrow g$ are also monotonic functions.*

With all the frameworks we had constructed, we are able to discuss the countability for modeling IPL.

Definition 7. (Countably Many Values for IPL):
Consider the family of finite Kripke frames $(W_n, \succeq)$ where $W_n \subset \mathbb{N}$ and $n \in \mathbb{N}$. Define $\mathcal{F}$ as the set of all sequences $\{(f_0, f_1, ...)\}$ where:

- Each f_n is a monotonic function on $(W_n, \succeq)$.
- The sequence converge to constant which implies that for some $N > n \in \mathbb{N}$, f_n are either all 1_W or 0_W eventually.

Let $T = (1_{W_0}, 1_{W_1}, 1_{W_2}, ...)$ represent the "truth" valuation.
For $F = (f_0, f_1, f_2, ...) \in \mathcal{F}$ and $G = (g_0, g_1, g_2, ...) \in \mathcal{F}$, define the logical operations component-wise:
$\neg F = (\neg f_0, \neg f_1, \neg f_2, ...)$;
$F \wedge G = (f_0 \wedge g_0, f_1 \wedge g_1, f_2 \wedge g_2, ...)$;
$F \vee G = (f_0 \vee g_0, f_1 \vee g_1, f_2 \vee g_2, ...)$, and
$F \Rightarrow G = (f_0 \Rightarrow g_0, f_1 \Rightarrow g_1, f_2 \Rightarrow g_2, ...)$.

The set $\mathcal{F}$ is a countable set due to the finite structure of each W_n and the eventually convergence of sequences. Moreover, the Lemma 17 implies that $\mathcal{F}$ is closed under the operations $\neg, \wedge, \vee,$ and $\Rightarrow$, which means that applying these operations to elements of $\mathcal{F}$ yields another element in $\mathcal{F}$.
Besides, the countable set $\mathcal{F}$, comprising sequences of monotone functions, provides the semantic machinery to interpret IPL formulas. To establish a precise correspondence between Kripke satisfaction and truth values in $\mathcal{F}$, we define valuation mappings that encode local forcing relations into global sequences. The Theorem 18 will bridge these concepts, formalizing how truth in a Kripke model translates component-wise into the many-valued structure.

Theorem 18. *For a sequence, let $(f_i)_n$ denotes the n-th element of that sequence for any $n \in \mathbb{N}$.*
For a given valuation function $\nu : At \to \mathcal{V}$, the satisfaction $\Vdash_\nu$ is defined as $w \Vdash_\nu P \Leftrightarrow (\nu(P))_n(w) = 1$ for any atom $P \in At$ and w be any possible world.
For a Kripke model $F = (W_m, \succeq_m, \Vdash)$, define the valuation
$\nu_n^{\Vdash}(P) = (1_{W_0}, ..., 1_{W_{m-1}}, f^*, 1_{W_{m+1}})$ for any $P \in At$, and

$$f^* : W \to \{0, 1\} := \begin{cases} 1, \text{if } w \Vdash \phi \\ 0, \text{if } w \nVdash \phi \end{cases}$$

where ϕ is an arbitrary formula.

Lemma 19. *(On two relations $\Vdash_\nu$, $\nu_n^{\Vdash}$):*
For the satisfaction defined as 18, we also have: $w \Vdash_\nu \phi \Leftrightarrow (\nu(\phi))_n(w) = 1$ for any formula $\phi \in Fm$.
Similarly, for the valuation defined as 18, we also have: $w \nVdash \phi \Leftrightarrow (\nu_m^{\Vdash}(\phi))_m(w) = 0$ for any formula $\phi \in Fm$.

The Lemma 19 is rigorously linked between the valuations and the satisfaction relations, which also crystallizes their interaction. By demonstrating equivalence between forcing conditions and truth-value assignments, this lemma ensures that the validity in IPL aligned with the preservation of the designated value T across all components of $\mathcal{F}$. With all the recipes introduced above, we can now prove what we want.

Theorem 20 (Jaśkowski). *Intuitionistic Propositional Logic is countably infinite many valued.*

Proof. In order to prove this theorem, we want to show that for any formula $\phi \in Fm$ is valid in Intuitionistic Logic if and only if it is universally satisfied in all finite Kripke models. This equivalence can be demonstrated in two cases:

1. If ϕ is satisfied in every world of every finite Kripke model, then for any valuation ν by Lemma 19, we have $(\nu(\phi)_n) = 1_W$ for all $n \in \mathbb{N}$. Hence, $\nu(\phi) = T$.
2. If ϕ fails in some world w of a finite Kripke model $F = (W_m, \succeq_m, \Vdash)$, Lemma 19 implies that $(\nu_m^{\Vdash}(\phi))_m(w) = 0$. Therefore, $\nu_m^{\Vdash}(\phi) \neq T$.

These cases confirm that validity in Intuitionistic Logic corresponds precisely to preservation of the designated truth value T across the countable valuation structure $\mathcal{F}$. □

Finally, this concludes the proof that IPL can be characterized by many-valued semantics using Kripke model. By the reasoning above, we have proven that the IPL is countably infinite Many-Valued Logics as we expected.

Conclusion

Throughout the course of this paper we have discussed the motivations for studying both IPL and MV-logics, highlighting the particular philosophical concerns that these logics aim to address, assigning truth values to statements independent of an axiomatic system, self-referential paradoxes, assigning truth values to future events, and so forth. Afterwards we have discussed the syntax and semantics of IPL, presenting the reader with Intuitionistic Natural Deduction and Kripke models, focusing on how IPL relates to CPL. Likewise we have discussed the semantics of MV-Logics, presenting and discussing the truth tables for these logics, and again we have focus on how these logics relate to each other (in this case viewing K_3 and LP as extensions of FDE). Towards the end we have shown that a truth-table like semantics of IPL would require a countably infinite amount of truth values, by proving the theorems of Gödel and Jaśkowski, and hence this justifies Kripke models as the paradigm for giving semantics to IPL. We note that since our motivation for studying this connection between Intuitionistic logic and Many-Valued logic was to investigate the semantics for Intuitionistic Logic, which as discussed previously is a more reliable logic for mathematics, it is pertinent to address that the work in this paper examines *propositional* logics whilst the day-to-day mathematician is likely to work with *predicate* logics, that is logics in which we introduce predicates P, variables x and quantifiers $\exists, \forall$. The language of predicate logics is richer than that of propositional logics, for a quick example consider the sentence,

There exists a smallest natural number.

In propositional logic we are forced to denote this simply as a proposition P, but in predicate logic we can define S to be the predicate "[ ] is the smallest", N to be the predicate "[ ] is a natural number" and express the above sentence using the existential quantifies $\exists$ as, $\exists x(Sx \wedge Nx)$, which directly translating reads as,

(There exists an x) such that ((x is the smallest) and (x is a natural number)).

Therefore we can see that predicate logic better captures the *strcture* of our example sentence than propositional logic. The next step onwards from this paper would be to give a similar treatment of syntax and semantics to Intuitionistic Predicate Logic and Many-Valued Predicate Logic.

Appendix A

Derivations

Below we present numerous useful derivations we refer to throughout our paper. Please note that unless otherwise stated the derivations below are valid in both IPL and CPL.

A.1 Proofs of Rules of Inference

Identity: $\vdash P \Rightarrow P$.

1.	~~*Show*~~ $P \Rightarrow P$	2, $\Rightarrow I$
2.	$\boxed{P}$	Assum.

True Consequent (abbreviated as TC): $Q \vdash P \Rightarrow Q$.

1.	~~*Show*~~ $P \Rightarrow Q$	2–3, $\Rightarrow I$
2.	P	Assum.
3.	Q	Pr 1

False Antecedent (Abbreviated as FA): $\neg P \vdash P \Rightarrow Q$

1.	~~*Show*~~ $P \Rightarrow Q$	4, EFQ
2.	P	Assum.
3.	$\neg P$	Pr 1
4.	$\bot$	2,3, $\neg E$

Negation of the Conditional (abbreviated as NC, also not Intuitionistically Valid), symbolically $\neg(P \Rightarrow Q) \vdash_{\text{CPL}} P \wedge \neg Q$

1.	~~*Show*~~ $P \wedge \neg Q$	2–13, DD
2.	$\neg(P \Rightarrow Q)$	Pr 1
3.	~~*Show*~~ $\neg Q$	4–7, $\neg I$
4.	Q	Assum.
5.	$P \Rightarrow Q$	4, TC
6.	$\neg(P \Rightarrow Q)$	2, R
7.	$\bot$	5,6, $\neg E$
8.	~~*Show*~~ P	9–12, RAA
9.	$\neg P$	Assum.
10.	$P \Rightarrow Q$	9, FA
11.	$\neg(P \Rightarrow Q)$	2, R
12.	$\bot$	10,11, $\neg E$
13.	$P \wedge \neg Q$	3,8, $\wedge I$

Modus Tollens Ponens (Abbreviated as MTP), symbolically $\neg P, P \vee Q \vdash_{\text{IPL}} Q$, which by the Deduction Theorem is equivalent to $P \vee Q \vdash \neg P \Rightarrow Q$

1.	~~*Show*~~ $\neg P \Rightarrow Q$	2–9, $\Rightarrow I$
2.	$\neg P$	Assum.
3.	$P \vee Q$	Pr 1
4.	~~*Show*~~ $P \Rightarrow Q$	5–7, EFQ
5.	P	Assum.
6.	$\neg P$	2, R
7.	$\bot$	5,6, $\neg E$
8.	$Q \Rightarrow Q$	Identity
9.	Q	3,4,8, $\vee E$

DeMorgan's Law (Not valid in IPL): $\neg(P \wedge Q) \vdash_{\mathrm{CPL}} \neg P \vee \neg Q$. By the Deduction Theorem this is equivalent to $\vdash_{\mathrm{CPL}} \neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$

1.	~~*Show*~~ $\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q$	2–7, RAA
2.	$\neg(\neg(P \wedge Q) \Rightarrow \neg P \vee \neg Q)$	Assum.
3.	$\neg(P \wedge Q) \wedge \neg(\neg P \vee \neg Q)$	2, NC
4.	$\neg(\neg P \vee \neg Q)$	3, $\wedge E$
5.	$\neg\neg(P \wedge Q)$	4, DM
6.	$\neg(P \wedge Q)$	3, $\wedge E$
7.	$\perp$	5,6 $\neg E$

DeMorgan's Law: $\neg P \wedge \neg Q \vdash \neg(P \vee Q)$

1.	~~*Show*~~ $\neg(P \vee Q)$	2–7, EFQ
2.	$P \vee Q$	Assum.
3.	$\neg P \wedge \neg Q$	Pr 1
4.	$\neg P$	3, $\wedge E$
5.	Q	2,4, MTP
6.	$\neg Q$	3, $\wedge E$
7.	$\perp$	5,6, $\neg E$

DeMorgan's Law: $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$

1.	~~*Show*~~ $\neg P \wedge \neg Q$	2–13, DD
2.	$\neg(P \vee Q)$	Pr 1
3.	~~*Show*~~ $\neg P$	4–7, $\neg I$
4.	P	Assum.
5.	$P \vee Q$	4, $\vee I$
6.	$\neg(P \vee Q)$	2, R
7.	$\bot$	5,6, $\neg E$
8.	~~*Show*~~ $\neg Q$	9–12, $\neg I$
9.	Q	Assum.
10.	$P \vee Q$	9, $\vee I$
11.	$\neg(P \vee Q)$	2, R
12.	$\bot$	10,11, $\neg E$
13.	$\neg P \wedge \neg Q$	3,8, $\wedge I$

A.2 Double Negation of LEM Tautology

Double Negation of LEM, symbolically $\vdash \neg\neg(P \vee \neg P)$.

1.	~~*Show*~~ $\neg\neg(P \vee \neg P)$	2–8, $\neg I$
2.	$\neg(P \vee \neg P)$	Assum.
3.	~~*Show*~~ $\neg P$	4–6, $\neg I$
4.	P	Assum.
5.	$P \vee \neg P$	4, $\vee I$
6.	$\bot$	2,5, $\neg E$
7.	$P \vee \neg P$	3, $\vee I$
8.	$\bot$	2,7, $\neg E$

A.3 LEM Equivalence Proofs

From RAA we can derive LEM, symbolically $(\neg P \Rightarrow \bot) \Rightarrow P \vdash P \vee \neg P$.

1.	~~Show~~ $P \vee \neg P$	2–6, RAA
2.	$\neg(P \vee \neg P)$	Assum.
3.	$\neg P \wedge \neg\neg P$	2, DM
4.	$\neg P$	3, $\wedge E$
5.	$\neg\neg P$	3, $\wedge E$
6.	$\bot$	4,5, $\neg E$

From LEM we can derive DN, symbolically $P \vee \neg P \vdash \neg\neg P \Rightarrow P$.

1.	~~Show~~ $\neg\neg P \Rightarrow P$	2–10, $\Rightarrow I$
2.	$\neg\neg P$	Assum.
3.	$P \vee \neg P$	LEM
4.	$P \Rightarrow P$	Identity
5.	~~Show~~ $\neg P \Rightarrow P$	6–9, EFQ
6.	$\neg P$	Assum.
7.	$\neg\neg P$	2, R
8.	$\bot$	7,8, $\neg E$
9.	P	3,4,5, $\vee E$

From DN we can derive RAA, symbolically $\neg\neg P \Rightarrow P \vdash (\neg P \Rightarrow \bot) \Rightarrow P$.

1.	~~Show~~ $(\neg P \Rightarrow \bot) \Rightarrow P$	2–4, $\Rightarrow I$
2.	$\neg P \Rightarrow \bot$	Assum.
3.	$\neg\neg P$	2, $\neg I$
4.	P	3, DN

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