Optimal Advertising Expenditure

Justin Abeles[1], Worku T. Bitew[1⋆⋆⋆], Radika Latchman[1]
Nicholas Seaton[1], Michael Tartamella[2], and Richard Vogel[1†]

[1] State University of New York, Farmingdale, New York, USA,
`abeljm@farmingdale.edu, latcrd@farmingdale.edu,`
`seatnk@farmingdale.edu`
[2] Stony Brook University, New York, USA,
`tartmr@farmingdale.edu`

Abstract. The main purpose of advertising is to impact buying behavior of consumers which can differ significantly between individuals based on their memories and experiences. Multiple factors impact a firm's decision to advertise such as the expense involved, the best medium to reach the target consumer, and the return rate of the advertising effort. In this paper, we developed a socioeconomic model to determine the firm's optimal advertising and marketing effort levels in order to maximize overall profit. Using numerical simulations, we examined the optimal solutions based on the performance of advertising activities. We also found a potential increase in sales relative to the optimized advertising expenditures. This helps the firm in deciding when to discontinue the old and introduce a new product or brand.

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⋆⋆⋆ corresponding author: biteww@farmingdale.edu
† corresponding author: vogelrm@farmingdale.edu

1 Introduction

There are multiple factors that affect a firm's decision to advertise such as the expense involved, and the medium to be used to reach the target consumer. Advertising provides several important functions for a firm, from communicating important product information that introduces the product and enables consumers to make informed choices, to brand differentiation and awareness. Halder and Shakib in [3] asserts that the main purpose of advertising is to impact buying behavior and impact that differs between individual consumers based upon their memories. In other words, the main purpose of advertising is to impact buying behavior which can differ significantly between individuals based on their memories. Niazi et al in [5] have shown that consumers' memories are formed through associations with a particular brand name. These brands continuously influence consideration, evaluation, and finally purchases (Romaniuk and Sharp, 2004).

Based on past studies we know that advertising can have a significant impact on a consumer's memories creating a link or association between an individual to a particular brand or product (Katke, 2007). The impact, however, is changed or strengthened frequently through the emotional attachment individuals form to these products (Niazi et al, 2012). A good example of the link between memory and emotional attachment is the case of the Samsung Galaxy Note 7 cellphone introduced in 2016. During its initial introduction, a number of devices randomly burst into flame with significant media coverage of this problem. As a result, Samsung phones fell into temporary disfavor in the market.

In launching an advertising campaign for a product or multiple products, the firm must determine the target audience for the product, and the medium (online, print, billboards, television and radio, etc.) to use to reach their target audience. As Arvindakshan, Peters, and Naik in [1] pointed out in their analysis, the traditional method of establishing an advertising budget using a Brand Development Index may not be the most optimal. The focus of their study is the development of a dynamic optimization model to determine the optimal advertising expenditures in a spatial-temporal market. Similarly, Dayanik and Parlar in [2] developed a dynamic optimization model to evaluate online advertising with a focus on maximizing sales subject to a strict budget constraint. This is an especially acute issue when firms are engaged in online search-based advertising where they face a varying charge based upon their search position and how often individuals follow through and click on the web link. Daynik and Parlar solved the problem using a dynamic programming approach.

In this paper, we develop a dynamic optimization model that describes the firm's advertising function and returns. We determine the optimal expenditure on advertising a product that maximizes the overall profit. This is vital in evaluating a firm's advertising decision and deciding when to discontinue a product and introduce a new brand. Applications of Dynamic Programming and Optimal Control Theory applied to this issue first began to appear in the literature about 30 years ago and in the last decade have begun to be applied more frequently. One early example of this analysis is a study conducted by Pritchett, Liu, and Kaiser in [6] in analyzing generic milk advertising and the question of the optimal mix of advertising expenditure across media formats. This paper adds to the growing literature on optimal advertising expenditures. We develop a generalized model focusing on the interaction between overall profitability and advertising expenditures and then evaluate and illustrate the solution numerically. We also performed a sensitivity analysis of the model with respect to the parameters involved in the model. The next section of this paper presents an overview of the optimization model. In Section 3, we discuss the necessary conditions for optimality and present a numerical simulation of the model. The conclusion of the study is presented in Section 4.

2 Mathematical Model

Let $M(t)$ be the maximum market volume or the number of customers who has the potential of owning or buying a new product under consideration and $S(t)$ be the number of consumers who have already owned the product at a time, t. If $a(t)$ is the advertising effort level at a time, t, following the Vidale-Wolfe advertising model the dynamics of the market share becomes

$$S'(t) = p(1 - \frac{S(t)}{M(t)})[a(t)]^k - \beta S(t). \tag{1}$$

Dividing both sides of equation(1) by $M(t)$, we get

$$x'(t) = \frac{p}{M(t)}(1 - x(t))[a(t)]^k - \beta x(t), \tag{2}$$

where $x(t) = \frac{S(t)}{M(t)}$ is the portion of the target group using the product, $k \in (0, 1)$ is a measure of the impact of the advert effort, β the decay factor with $T = \frac{1}{\beta}$ be the expected lifetime of the item in the market, and p is the reaction factor, number of new buyers per unit advertising effort. We assume that the market saturation level M is constant so that $r = \frac{p}{M}$ is constant. Hence, the dynamic equation becomes

$$x'(t) = r(1 - x(t))[a(t)]^k - \beta x(t). \tag{3}$$

Let Π be the total net profit expected from the sale of the product at the end of the cycle and $C[a(t)]$ be the cost of advertisement at time t, where C is per unit cost of advertisement. Our objective is to maximize the overall discounted net benefit functional integrate our function from 0 to T, with respect to t.

$$J = \int_0^T \left(\Pi r(1-x)a^k - Ca\right) e^{-\rho t} dt \tag{4}$$

subject to

$$x' = r(1-x)a^k - \beta x,\ x(0) = x_0 \text{ and } x(T) = x_T \text{ is free final state.} \tag{5}$$

To simplify the model further, let $u = a^k$. Implies $a = u^{\frac{1}{k}}$. Substituting this in the integral, we can reformulate performance functional as

$$J = \int_0^T \left(\Pi r(1-x)u - Cu^{\frac{1}{k}} \right) e^{-\rho t} dt \tag{6}$$

Note that $x' = r(1-x)u - \beta x$. Implies $x' + \beta x = r(1-x)u$. Hence, we can replace $r(1-x)u$ in the integral with $x' + \beta x$ to get:

$$J = \int_0^T (\Pi\,(x' + \beta x) - Cu^{\frac{1}{k}}) e^{-\rho t} dt \tag{7}$$

Using integration by parts we can set up the model in a simplified and manageable form

$$\int_0^T (\Pi\,(x' + \beta x) - Cu^{\frac{1}{k}}) e^{-\rho t} dt = \int_0^T \left(\Pi(\rho + \beta)x - Cu^{\frac{1}{k}} \right) e^{-\rho t} dt + x_T e^{(-\rho T)} - x_0$$

$$= \Pi \int_0^T \left((\rho + \beta)x - \frac{C}{\Pi} u^{\frac{1}{k}} \right) e^{-\rho t} dt + x_T e^{(-\rho T)} - x_0$$

Therefore, since x_0 is a constant, our maximization problem becomes

$$\text{Max}_u J = \int_0^T \left((\rho + \beta)x - cu^{\frac{1}{k}} \right) e^{-\rho t} dt + x_T e^{(-\rho T)} \tag{8}$$

subject to

$x' = r(1-x)u - \beta x, x(0) = x_0$ and $x(T) = x_T$ is free final state,

where $c = \frac{C}{\Pi}$.

3 The Necessary Conditions and Optimal Solutions

3.1 The Necessary Conditions

The current value Hamiltonian corresponding to our problem is

$$H(x, u, \lambda) = (\rho + \beta)x - cu^{\frac{1}{k}} + \lambda\left(r(1-x)u - \beta x\right), \tag{9}$$

where λ is the current shadow value. By the maximum principle, the necessary condition for optimality with respect to the control variable, u, is

$$\frac{\partial H}{\partial u} = 0. \tag{10}$$

Implies that

$$-\frac{cu^{\frac{1}{k}-1}}{k} + \lambda r(1-x) = 0. \tag{11}$$

Solving Equation (11) for λ :

$$\lambda = \frac{cu^{(\frac{1}{k}-1)}}{kr(1-x)}. \tag{12}$$

From Equation (12), the current shadow value depends on the ratio of cost per unit effort, c, to the return rate parameter, r. This shows that if the return from a per-unit advertisement effort is better than expected, the weekly advertisement effort can be lowered.
From the boundary condition for optimality, the current shadow value

$$\lambda(T) = 1 = m(T)e^{\rho T}, \tag{13}$$

where $m(T)$ is the present shadow value. Equations (12) and (13) implies

$$x(T) = 1 - \frac{c}{kr}(u(T))^{(\frac{1}{k}-1)} \text{ , provided } x_0 \leq x(T) \leq 1. \tag{14}$$

The co-state equation corresponding to the state is

$$\lambda'(t) - \rho\lambda(t) = -\frac{\partial H}{\partial x} \tag{15}$$

Substituting λ from Equation (12) in the left side of Equation (15)

$$\lambda'(t) - \rho\lambda(t) = \frac{d}{dt}\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right) - \rho\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right), \tag{16}$$

and using

$$-\frac{\partial H}{\partial x} = -\left((\beta+\rho) + \lambda(-ru-\beta)\right), \tag{17}$$

we get one of the necessary conditions

$$\frac{d}{dt}\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right) - \rho\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right) = -(\beta+\rho) + (\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)})(ru+\beta). \tag{18}$$

The other necessary condition is the dynamic equation which governs the rate of change in the market share

$$x' = r(1-x)u - \beta x. \tag{19}$$

Therefore, the optimal state and control variables satisfy the following necessary conditions:

$$\frac{d}{dt}\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right) - \rho\left(\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)}\right) = -(\beta+\rho) + (\frac{cu^{(\frac{-k+1}{k})}}{kr(1-x)})(ru+\beta), \tag{20}$$

$$x' = r(1-x)u - \beta x, \tag{21}$$

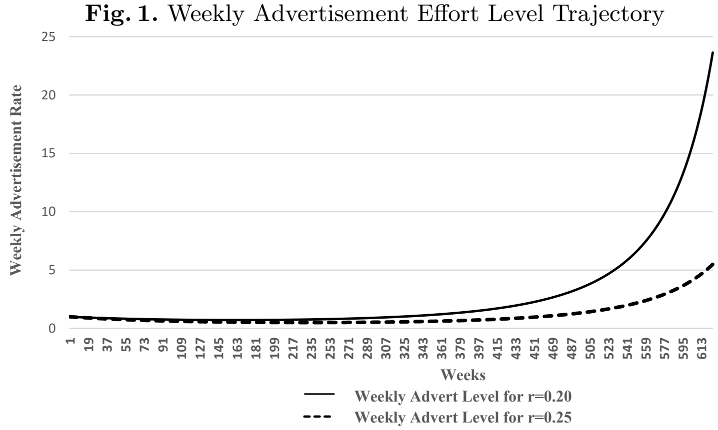
and $x(0) = x_0,\ u(0) = u_0$.
The above system of equations with the given initial values can be potentially solved for the optimal trajectory of advertisement effort level and market share. The analytic solutions are complicated, so we use numerical methods to generate results.

3.2 Numerical Simulations and Sensitivity Analysis

We numerically solve the above nonlinear system of equations, Equations (20) and (21) with given initial values, using MAPLE™ by assigning the following values for the parameters: the normalized per unit advertisement cost value, $c = 0.05$, the reaction factor, $r = 0.25$, and $k = \frac{1}{2}$. We also use the discount rate, $\rho = 0.023$. Assuming that the expected life in the market of the product, $T = 12$ years, we use $\beta = \frac{1}{12}$.

The numerical simulation results displayed in Figure 1 are the advertisement plan on a weekly basis for $r = 0.20$ and $r = 0.25$. In both cases, the results showed that we kick off the advertising activity at an initial value based on the competitiveness of the market for our product. Then continue advertising at almost the same level is an effective method to increase the number of sales. We

Fig. 1. Weekly Advertisement Effort Level Trajectory



also observed that as the market share is close to its saturation level, $x(T)$, or as $t \to T$, it takes a lot of effort to convince a customer to buy the product. These customers have either already decided not to buy the product or moved on to another similar product. Therefore, it does not pay to spend another dollar on advertising. In our numerical simulation results, there is a threshold where the difference between market profit share and advertisement cost becomes negative and the shadow value increases drastically. This behavior mainly depends on the return rate parameter, r. For Example, when r increases from $r = 0.2$ to $r = 0.25$, the weekly advertisement rate decreases. This resulted in a lower overall cost.

Figure 2 is the predicted weekly sales return from the advertisement for $r = 0.25$. The number of weekly sales starts to increase around the end of the lifetime of the product, T, because we advertise at a higher rate to clear the inventory and start introducing the new brand to the market. Most companies prefer to give discounts instead of spending more money on advertising since the marginal increase in the market share is too low relative to advertising expenditure causing net profit to become negative.

Fig. 2. Weekly Sales Trajectory

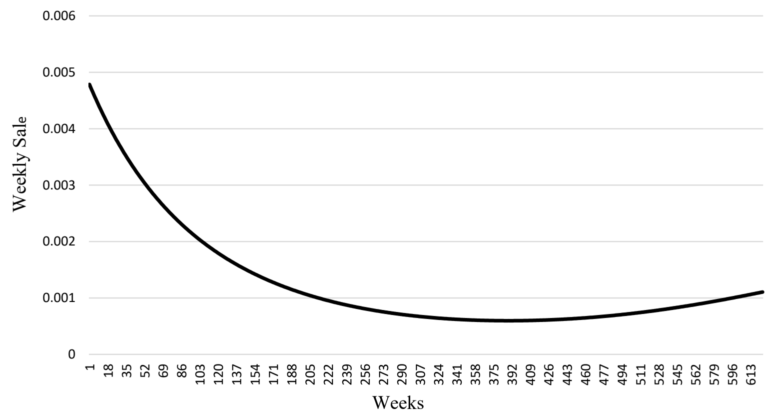
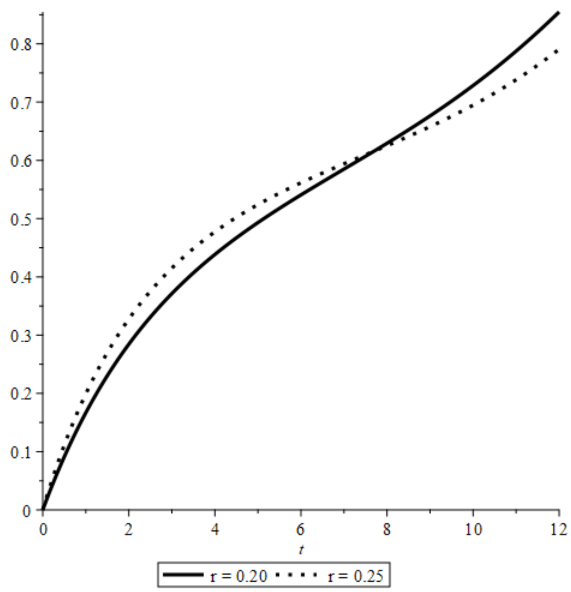


Figure 3 shows the trajectory of the market share for $r = 0.20$ and $r = 0.25$. As we mentioned earlier, Equation (14), the boundary value at T depends on the per unit advertisement return rate, r, which controls the advertising behavior around the end of a cycle. In both cases, in the last quarter of the T year, the market share changes rapidly at the expense of aggressive advertisement. There is also an inflection point in the trajectory. Therefore, around the end of the cycle, after the inflection time, the company needs to make a cost-benefit analysis to decide the appropriate time to end the advertisement activity or give some discount to customers before discontinuing the product.

Fig. 3. Market Share Trajectory



4 Conclusion

From the analysis presented above, we were able to find an optimal solution for the firm aligning profitability and the gains from advertising that can be used to measure the impact of adverting across any time frame, $[0, T]$. Using the Maximum Principle, we derived the necessary conditions for optimality, solved the system for an optimal solution using MAPLE™, and performed a sensitivity analysis of the model with respect to the advertisement return rate parameter.

The results of the numerical simulation showed that we kick off the advertising activity at the initial value based on the competitiveness of the market for our product. Then continuing advertising at almost the same level and then gradually increase in advertising expenditures was an effective method to increase sales. Advertising costs rose at an increasing rate, becoming less cost-effective over the course of the product's lifespan, especially as new competing products became available. At the end of the plan, the increased expense to attract a new customer will become too high causing net profit to become negative if we were to continue advertising. It is at this point that advertising expenditures for the particular product should be reduced consistent with overall profitability.

Overall, our research provides a deeper understanding of product life cycles and advertising. to be successful, firms must continue to innovate, creating new products and updating existing products to remain competitive in the market. This type of behavior is readily observed in markets such as the electronics industry when a new cellphone is released every year and advertising for the previous and older models is substantially reduced if not eliminated completely. Our research described this notion clearly. Thus, a firm will need to ramp up its marketing and advertising expenditures and effort over some time to effectively begin to curtail them as it approaches market saturation or reaches the end of the product life cycle.

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