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CONTINUATION OF DIRECT PRODUCTS OF DISTRIBUTIONS

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Preamble

If, in some problems, one has to deal with the "product" of distributions $\mathrm{f_i}$ (also called generalized functions) $\overline{\mathrm{T}} = \Pi_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{f_i}$, this product has a priori no definite meaning as a functional $(\overline{\mathrm{T}}, \varphi)$ for $\varphi \in \mathrm{S}$. But if $\mathrm{x}^{\kappa+1}\Pi_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{f_i}$ exists, whatever the associativity is between some powers $\mathrm{r_i}$ of x $(\mathrm{r_i} \in \mathbb{N}, \sum_{\mathrm{i}} \mathrm{r_i} \leq \kappa + 1, \mathrm{r_i} \geq 0)$ and the various $\mathrm{f_i}$, then a continuation of the linear functional $\overline{\mathrm{T}}$ from M onto $\mathrm{S}^{(\mathrm{N})}$ for some N is shown to exist[1] in such a way that $\mathrm{x}^{\kappa+1}\overline{\mathrm{T}}$ is defined unambiguously, and $(\overline{\mathrm{T}}, \varphi), \varphi \in \mathrm{S}$, significant, though not unique.

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[1] M is a closed subspace of S^{N} for some N. It is a Banach space with norm $\|\|_{\mathrm{N}}$.

1 Existence

In the sense of convergence in the space S^* (distributions),

$$f_\kappa = \lim_{y \to 0,\ y \in C^+} F_\kappa^y(x)\ ;\quad \kappa = 1, 2, \cdots, m\ ,$$

with $F_\kappa^y(x) = f_\kappa^+(x + iy) - f_\kappa^-(x - iy)$, $\Big($$f_\kappa^\pm(x)$ are holomorphic in tabular domains $T^{C_R^\pm}$ and satisfy

$$|f(x + iy)| \leq C(R', C')|y|^{-\alpha}(1 + |x|)^\beta \tag{1}$$

and

$$z \in R^n + i(C' \cap U(0, R'))$$

$\alpha, \beta \geq 0$, independent of R' and C'. From this, it follows that there exists in S^* a unique boundary value

$$f(x) = \lim_{y \to 0,\ y \to c} f(x + iy) \in S^{(m)*};\ m = \alpha + \beta + n + 3\ .\Big)$$

Let us suppose that for arbitrary $\varphi \in S$ there exists a finite limit

$$\lim_{y \to 0,\ y \in C^+} \int F_1^y(x) \cdots F_m^y(x) \cdot \varphi(x) dx \tag{2}$$

independent of the sequence $y \to 0,\ y \in C^+$. Then, since the space S^* is dense, this limit defines a distribution in S^* which we call the product $f_1.f_2.\cdots.f_m$ of the distributions $f_1, f_2, \cdots, f_m$. Thus

$$f_1.f_2.\cdots.f_m = \lim_{y \to 0,\ y \in C^+} F_1^y \cdots F_m^y\ \ (\text{in } S^*) \tag{3}$$

if the limit of the RHS exists and is independent of the sequence $y \to 0, y \in C^+$. This product is obviously commutative and associative. So the set of boundary values that are holomorphic in $T^{C_R^+}$ and satisfy (1) constitute a commutative ring with unity, without zero divisors with respect to the multiplication defined above.

We note that the existence of the lim in (2) for $\varphi \in S$ implies the existence of the limit in (3) with respect to the norm of the functional in $S^{(N)*}$ for some N, which depends on $f_1 \ldots f_m$ (notice that weak convergence in S^* implies strong convergence).

2 General case

Suppose now that (2) does not exist for all $\varphi \in S$, but that it exists for all φ in a closed subspace M of $S^{(N)}$ for some N. (Since M is closed in $S^{(N)}$ it is a Banach space with norm $\|\ \|_N$). From the Banach-Steinhaus theorem, (2) defines a continuous linear functional $\overline{T}$ on M. We use now the term 'product' $f_1.\cdots.f_m$ of the distributions $f_1, f_2, \cdots, f_m$ for <u>any</u> continuous linear functional in the space $S^{(N)*} \subset S^*$ that is a continuation of $\overline{T}$ from M to $S^{(N)}$. According to the Hahn-Banach theorem, such an extension always exists but is not unique in general.

We shall concentrate now on the case of those φ in $S^{(N)}$ that vanish together with all derivatives of order $p \leq N$ inclusively, at $x = 0$. In this case, all continuations $f_1.f_2.\cdots.f_m$ of $\overline{T}$ from M onto $S^{(N)}$ are given by

$$(f_1.f_2.\cdots.f_m, \varphi) = (\overline{T}, \overline{\varphi}) + \sum_{\kappa \leq p} c_\kappa (\delta^{(\kappa)}, \varphi) \tag{4}$$

where

$$\overline{\varphi}(x) = \varphi(x) - \sum_{\kappa \leq p} \varphi^{(\kappa)}(o)\omega(x)\frac{x^\kappa}{\kappa!}$$

and $\omega(x)$ is an arbitrary function, $\omega \in S$, identically equal to 1 in a neighbourhood of the point $x = 0$; the c_κ are arbitrary constants. (Notice that the extension (4) is actually independent of $\omega(x)$).

In conclusion, the formula (4) represents the desired result, given at the end of the preamble with $\sum_{\kappa \leq p} c_\kappa \delta^{(\kappa)}$ the general solution of $(f_1.\cdots.f_m, \varphi) = 0$ and $(T, \overline{\varphi}) = (\overline{T}, x^{\kappa+1}\psi) = (x^{\kappa+1}\overline{T}, \psi), \psi \in S$, a particular solution of $(f_1.\cdots.f_m, \varphi)$.

It is therefore shown that the solution (4) is not unique, the c_κ being arbitrary constants.