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Logarithmic Behaviours in the Feigin-Fuchs Construction of the $c = -2$ Conformal Field Theory

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Abstract

We obtain logarithmic behaviours of a four-point correlation function in the $c = -2$ conformal field theory by using the Feigin-Fuchs construction. It becomes an indeterminate form by a naive evaluation, but is obtained by introducing an appropriate regularization procedure.

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Conformal field theories whose correlation functions have logarithmic behaviour were first studied by Gurarie in the central charge $c = -2$ model [1]. This model is one of the simplest systems since the correlation function in the problem consists of only one kind of primary fields. Logarithmic conformal field theories have interesting properties: new operators are needed, which are called logarithmic operators and never appeared for ordinary conformal field theories. The origin of the logarithms of Ref. [1] is a hypergeometric function, which is a solution of the differential equation for the four-point correlation function and equivalent to the complete elliptic integral of the first kind. The origin should also be explained by the Feigin-Fuchs construction [2], but the approach only to the logarithmic operators is studied [3]. The construction also applies to other models [4].

In this paper the four-point correlation function of Ref. [1] is calculated by using the Feigin-Fuchs construction. In this construction the correlation function is given by an integral representation. We find that its integral value becomes an indeterminate form, $\frac{0}{0}$. In order to evaluate this form we introduce an appropriate regularization procedure: we perform an analytic continuation of a parameter in the hypergeometric function, namely, take the limit $c \to -2$ [5], and evaluate the form of the correlation function by using essentially the de l'Hospital theorem. In this way the logarithmic term appears, and our result is in agreement with that of Ref. [1]. Application of our method to generic four-point correlation functions with logarithms is under studying.

We now consider the Feigin-Fuchs construction [2] of conformal field theories. The action is given by

$$S = \frac{1}{8\pi} \int d^2\xi \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 2i\alpha_0 R\phi \right), \tag{1}$$

where ϕ is a real scalar field and R is the scalar curvature on a sphere with fixed reference metric $g_{\mu\nu}$. The parameter $2\alpha_0$ can be interpreted as the background charge. On the complex plane, the energy momentum tensor is of the form

$$T(z) = -\frac{1}{2} \partial_z \phi \partial_z \phi + i\alpha_0 \partial_z^2 \phi, \tag{2}$$

and two-point function of the field ϕ is $\langle \phi(z, \overline{z}) \phi(w, \overline{w}) \rangle = -\ln |z - w|^2$. Thus the central charge of the system is written in terms of α_0 as

$$c = 1 - 12\alpha_0^2. \tag{3}$$

When considering correlation functions of primary fields Φ, we treat the correlation function in terms of the vertex operators $V_\alpha \equiv e^{i\alpha\phi}$ instead of Φ. The conformal weight h of the operator V_α is

$$h(V_\alpha) = h(V_{2\alpha_0-\alpha}) = -\frac{1}{2}\alpha(2\alpha_0 - \alpha), \tag{4}$$

so two admissible operators exist for one field Φ. For the (p, q) primary fields $\Phi_{p,q}$ [6], the conformal weight takes the discrete value

$$h_{p,q} = -\frac{1}{2}\alpha_0^2 + \frac{1}{8}(p\alpha_+ + q\alpha_-)^2, \tag{5}$$

where $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 2}$. The corresponding operator V_α has the parameter α given by

$$\alpha_{p,q} = \alpha_0 - \frac{1}{2}\left(p\alpha_+ + q\alpha_-\right). \tag{6}$$

We also need the screening charges

$$Q_\pm = \int d^2u\, e^{i\alpha_\pm\phi(u,\overline{u})}. \tag{7}$$

A certain number of screening charges should be inserted in the correlation function so that the charge neutrality condition, required by the zero mode integration of ϕ, is satisfied. Since the screening charges are the integrals of the operators with conformal weight one, the insertion have no effects on the conformal properties of correlation functions.

Let us now concretely consider the four-point correlation function of Ref. [1]

$$G^{(4)} \equiv \langle \mu(z_1, \overline{z}_1)\mu(z_2, \overline{z}_2)\mu(z_3, \overline{z}_3)\mu(z_4, \overline{z}_4)\rangle \tag{8}$$

in the $c = -2$ model. Here $\mu(z, \overline{z}) \equiv \Phi_{1,2}(z, \overline{z})$ is the primary field with the conformal weight $h_{1,2} = -\frac{1}{8}$. In the Feigin-Fuchs construction, four-point correlation functions of primary fields Φ_i's in general take the form [2]

$$\langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\Phi_3(z_3, \overline{z}_3)\Phi_4(z_4, \overline{z}_4)\rangle = \left\langle \prod_{i=1}^{4} e^{i\alpha_i\phi(z_i,\overline{z}_i)}(Q_+)^m(Q_-)^n \right\rangle, \tag{9}$$

where the charge neutrality condition is

$$\sum_{i=1}^{4} \alpha_i + m\alpha_+ + n\alpha_- = 2\alpha_0. \tag{10}$$

In our model with $\alpha_0 = \frac{1}{2},\ \alpha_+ = 2,\ \alpha_- = -1$ and $\alpha_i = \alpha_{1,2} = \frac{1}{2}$ we choose $m = 0$ and $n = 1$. The correlation function (8) is, therefore, evaluated as

$$G^{(4)} = \left\langle \prod_{i=1}^{4} e^{i\frac{1}{2}\phi(z_i,\overline{z}_i)} Q_- \right\rangle = \prod_{i<j} |z_{ij}|^{\frac{1}{2}} \int d^2u \prod_{i=1}^{4} |z_i - u|^{-1}, \tag{11}$$

where $z_{ij} = z_i - z_j$. On the other hand, from the SL(2,**C**) Ward identity [6], the correlation function (8) can be written as

$$G^{(4)} = F(x, \overline{x}) \prod_{i<j} |z_{ij}|^{\frac{1}{6}}, \tag{12}$$

where F is an arbitrary function of the SL(2,**C**) invariant cross ratios

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \overline{x} = \frac{\overline{z}_{12}\overline{z}_{34}}{\overline{z}_{13}\overline{z}_{24}}. \tag{13}$$

From Eqs. (11) and (12), by fixing $z_1 = 0,\ z_2 = x,\ z_3 = 1,\ z_4 = \infty$, $F(x, \overline{x})$ can be evaluated and we obtain

$$G^{(4)} = |z_{13}z_{24}|^{\frac{1}{2}} |x(1-x)|^{\frac{1}{2}} I\left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}; x\right), \tag{14}$$

where

$$I(a, b, c; x) = \int d^2u\, |u|^{2a} |1-u|^{2b} |u-x|^{2c}. \tag{15}$$

The integral $I(a, b, c; x)$ can be transformed into a sum of squares of line integrals [2]

$$I(a, b, c; x) = \frac{\sin[\pi(a+b+c)]\sin(\pi b)}{\sin[\pi(a+c)]} |I_1(x)|^2 + \frac{\sin(\pi a)\sin(\pi c)}{\sin[\pi(a+c)]} |I_2(x)|^2, \tag{16}$$

where

$$\begin{aligned}
I_1(x) &= \int_1^\infty du\, u^a (u-1)^b (u-x)^c \\
&= \frac{\Gamma(-a-b-c-1)\,\Gamma(b+1)}{\Gamma(-a-c)}\, F(-c,-a-b-c-1,-a-c;x), \qquad (17)\\
I_2(x) &= \int_0^x du\, u^a (1-u)^b (x-u)^c \\
&= \frac{\Gamma(a+1)\,\Gamma(c+1)}{\Gamma(a+c+2)}\, x^{a+c+1}\, F(-b,a+1,a+c+2;x). \qquad (18)
\end{aligned}$$

The coefficients of $|I_1(x)|^2$ and $|I_2(x)|^2$ in Eq. (16) are determined by the monodoromy invariance of $I(a,b,c;x)$. We see that the result (14) is naively an indeterminate form:

$$I\left(-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2};x\right) = \frac{1}{\sin(-\pi)}\left[-|I_1(x)|^2 + |I_2(x)|^2\right] \sim \frac{0}{0}, \qquad (19)$$

since $I_1(x) = I_2(x) = \pi F\left(\tfrac{1}{2},\tfrac{1}{2},1;x\right)$.

To evaluate the above indeterminate form we now introduce a regularization procedure as follows:

$$\begin{aligned}
I\left(-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2};x\right) &\equiv \lim_{a\to -\frac{1}{2}} I\left(a,-\tfrac{1}{2},-\tfrac{1}{2};x\right) \\
&= \left.\frac{\frac{d}{da}\left\{-\sin[\pi(a-1)]|I_1(x)|^2 - \sin(\pi a)|I_2(x)|^2\right\}}{\frac{d}{da}\sin[\pi(a-\frac{1}{2})]}\right|_{a=-\frac{1}{2}}. \qquad (20)
\end{aligned}$$

Note that $I_1(x)$ and $I_2(x)$ are the functions of the regularization parameter a, which are given by Eqs. (17) and (18) with $b=c=-\frac{1}{2}$. Since $I_1(x) = I_2(x)$ when $a=-\frac{1}{2}$, Eq. (20) becomes

$$I\left(-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2};x\right) = \frac{1}{\pi}\left[I_1(\overline{x})\left\{\left.\frac{d}{da}\left(I_1(x)-I_2(x)\right)\right|_{a=-\frac{1}{2}}\right\} + (x \leftrightarrow \overline{x})\right]. \qquad (21)$$

The factor of differential in the braces is evaluated as

$$\begin{aligned}
\left.\frac{d}{da}\left(I_1(x)-I_2(x)\right)\right|_{a=-\frac{1}{2}} &= -\pi \ln\left(\frac{x}{16}\right) F\left(\tfrac{1}{2},\tfrac{1}{2},1;x\right) - 2\pi M(x) \\
&\equiv 2\pi \tilde{F}(x), \qquad (22)
\end{aligned}$$

where

$$M(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\frac{\Gamma\left(n+\frac{1}{2}\right)}{n!} \right]^2 \left[\psi(1) - \psi(n+1) - \psi(\tfrac{1}{2}) + \psi(n+\tfrac{1}{2}) \right] x^n \tag{23}$$

and $\psi(x)$ is the digamma function. Anti-holomorphic part can be evaluated in the same way. Therefore we finally obtain

$$I\left(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}; x\right) = 2\pi \left[F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; x\right) \tilde{F}(\overline{x}) + (x \leftrightarrow \overline{x}) \right]. \tag{24}$$

Notice that from Appendix C of Ref. [7] $\tilde{F}(x)$ satisfies the following relation

$$\tilde{F}(x) = \frac{\pi}{2} F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-x\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-(1-x)\sin^2\theta}}. \tag{25}$$

This is just the function $G(1-x)$ in Eq. (10) of Ref. [1], which is the origin of logarithm. Thus our result (14) has logarithmic behaviour as

$$\begin{aligned} G^{(4)} &= \pi^2 \left| z_{13} z_{24} \right|^{\frac{1}{2}} \left| x(1-x) \right|^{\frac{1}{2}} \\ &\quad \times \left[F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; x\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-\overline{x}\right) + F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; \overline{x}\right) F\left(\tfrac{1}{2}, \tfrac{1}{2}, 1; 1-x\right) \right]. \end{aligned} \tag{26}$$

This agrees with that of Ref. [1] up to overall constant, which was obtained by directly solving the hypergeometric differential equation.

In the above procedure we performed an analytic continuation of the first parameter a of the function $I(a,b,c;x)$. We can reproduce the same result by using the third parameter c but not by the second one b. The fact depends on the choice of the two independent contours of integrals (17) and (18) since the coefficients of $|I_1(x)|^2$ and $|I_2(x)|^2$ in Eq. (16) are determined by monodoromy invariance of Eq. (16).

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