

Relaxation properties of small-world networks

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Abstract

Recently, Watts and Strogatz introduced the so-called *small-world networks* in order to describe systems which combine simultaneously properties of regular and of random lattices. In this work we study diffusion processes defined on such structures by considering explicitly the probability for a random walker to be present at the origin. The results are intermediate between the corresponding ones for fractals and for Cayley trees.

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1 Introduction

Networks of the real world often seem to combine aspects from regular and from completely random lattices. Social networks, neural networks, electrical powergrids, and traffic networks [1–3] are all examples of patterns not described satisfactorily by conventional regular lattices, nor by completely random lattices. Social structures, for instance, do not behave as regular lattices, since (as is well known) randomly chosen people are connected in general by a small number of intermediary bilateral ties. Here, as in random graphs, the minimal (chemical) distance between any two points in the system scales logarithmically with the system size [4].

To combine these two properties, Watts and Strogatz recently introduced the idea of *small-world networks* [1]. This construction is a superposition of a regular lattice with a random lattice, and includes simultaneously well defined local clusters and short global connections. As we will demonstrate, these systems also display properties intermediate between those of regular and tree-like (loop-less) lattices, already under a small number of global connections, provided the system size is large enough.

Much work has already been done on the properties of small-world networks [1,2,5–14]

but most of it has focused on static (geometric) properties. We shall not address these issues, but rather concentrate on a *dynamical* model defined on the structure. Treatments of the dynamics of small-world networks include for instance the study of an Ising model defined on the lattice [5], spectral properties of the small-world Laplacian, [6], percolation [7], spreading of diseases [8] and neural networks [9]. In the following we will examine the properties of random walks on small-world networks, in particular the relaxation, exemplified by the probability for a random walker of being at the original site at a later time. This is a simple quantity to extract numerically, and very relevant for various physical properties: It is sensitive to the topology of the network, and is related to its vibrational modes.

2 Definition of the model and presentation of the results.

The small-world networks we consider are built as follows: We start from a regular lattice with L vertices in 1 dimension under periodic boundary conditions, each site being connected symmetrically to its $2k$ nearest neighbours, i.e. having as coordination number $z = 2k$. Then we add to each of the sites with probability p a new bond. The other end gets attached with equal probability to any of the lattice sites; this allows also the possi-

bility of vertices to become connected to themselves. In this way we add, independent of k , on the average pL new bonds to the underlying regular lattice.

This construction follows [7] for $k = 1$ and is simpler than the original procedure [1], by which one rewires with probability p each of the original kL bonds randomly.

A step-wise diffusion process is now defined by specifying all the transition probabilities $W_{i,j}$ entering the master-equation:

$$P(i, n + 1) - P(i, n) = \sum_j W_{i,j} P(j, n) - P(i, n) \sum_j W_{j,i} \quad (1)$$

The $W_{f,i}$ is the probability to go from site i to site f during one time step, and the probability $P(i, n)$, $i = 1 \dots L$ is just the probability of being at site i after the n th step. The process defined in Eq. (1) is the discrete variant of diffusion on an arbitrary lattice, a topic interesting in its own right. Diffusion on regular lattices is ubiquitous, and diffusion on random graphs has (among other things) also been studied in the context of glassy relaxation [15]. We are therefore inspired to investigate what happens on the small-world model, which interpolates between these two extremes. Previously a lot of interest has also been seen in the related problem of diffusion on fractals (see for example [16–19] and references therein). As we proceed to show, diffusion on Cayley trees [20–22] shows also features closely related to the present problem. Furthermore, the motion of charge carriers or of excitons over polymer chains, where steps between spatially close sites can connect regions far apart along the chemical backbone, also involves global shortcuts [23, 24].

The transition probabilities $W_{i,j}$ in Eq. (1) are as follows: First $W_{i,j} = 0$ if there are no bonds between i and j . For i connected to j by one or more direct bonds, $W_{i,j}$ is proportional to the number of such bonds. The same holds for the probability of remaining at the same site after one time unit, i.e. we allow “sticking”. Formally

$$W_{i,j} = \frac{z_{i,j} + \delta_{i,j}}{z_j + 1} \quad (2)$$

In this equation, $z_{i,j}$ is the number of bonds between the two sites i and j , and z_i is the total number of bonds emanating from vertex i , i.e. the coordination number of the site. Hence $z_i = \sum_j z_{i,j}$. Note that the $z_{i,j}$ -values are determined both by the additional wiring as well as by the underlying lattice. The $\delta_{i,j}$ and the 1 in the denominator appear because we allow for the possibility of the walker to remain at site i during a time step. This procedure renders the numerically determined $P(i,n)$ smoother in n . We remark that the rates defined according to Eq. (2) are not symmetrical in i and j , i.e. in general $W_{i,j} \neq W_{j,i}$.

The algorithm we have used is the exact (cellular automaton) enumeration of random walks [18], corresponding to the implementation of Eq. (1). All the results plotted are averaged over 200 disorder configurations. We have worked mostly with the value $k = 1$. This is also the value implied if we do not state otherwise.

We focus on the probability $P(i,n|i,0)$ that a particle initially at site i is found at

Figure 1: The relaxation or probability of presence at the origin $P_n(0)$ as a function of number of steps for $p = 0.05$ and several system sizes L , which from upper to lower right are $L = 1000$, $L = 2000$, $L = 5000$ and $L = 10000$.

the same site just after the n th step. In the figures below we plot $\langle P(i, n|i, 0) \rangle$, i.e. the average of $P(i, n|i, 0)$ over the different realisations of the small-world lattice. Since all sites are equivalent in an ensemble of small-world networks, this quantity does not depend the particular site i chosen, and we hereafter denote it by $P_n(0)$. In Fig. 1 we have chosen $p = 0.05$ and plotted $P_n(0)$ in double-logarithmic scales for system sizes ranging from $L = 1000$ to $L = 10000$. This allows us to examine the dependence of $P_n(0)$ on the size of the system.

From Fig. 1 we infer that initially all the curves fall on one curve and that for large n they saturate at their respective equilibrium values, $1/L$. However, Eq. (2) implies an

Figure 2: Plot of $P_n(0) - P_\infty(0)$ as a function of n , the number of time steps for $p = 0.05$ and L as in Fig. 1. The curves fall nicely on a master curve.

inhomogenous equilibrium distribution

$$P^{eq}(i) \propto (z_i + 1). \quad (3)$$

Therefore $P(i, \infty|i, 0)$ depends on the specific small-world realisation, and will fluctuate from realisation to realisation around its average value $1/L$.

To find out how much of the behavior is due to finite size effects, we subtract from each average curve in Fig. 1 its corresponding average equilibrium value $P_\infty(0) \equiv 1/L$, and replot $P_n(0) - P_\infty(0)$ in Fig. 2. From Fig. 2 we see that all curves collapse nicely onto what we view as representing $P_n(0)$ on small-world networks in the limit $L \rightarrow \infty$. Both Figs. 1 and 2 display initially a quasi-linear decay in the chosen double logarithmic

Figure 3: $P_n(0)$ for $p = 0.01$ as well as a power law approximation. The fit in the region $10 < t < 100$ gives as least-squares-fit exponent -0.52 .

scales, and this may be viewed as being an approximate power law decay. Depending on p , the exponents range from around -0.5 for the smallest p to around -0.6 for the largest. This regime is followed by a steeper decay at larger n . To highlight the power law character we have plotted in Fig.3 $P_n(0)$ for $p = 0.01$. As is evident from the figure, the power-law domain extends well over two orders of magnitude in n .

The results can be understood qualitatively in the following way: For a fractal one has [25]

$$P_n(0) \sim n^{-d_s/2}, \quad (4)$$

where d_s is the spectral dimension. Thus the initial decay in Figs. 1 to 3 follows that of a fractal with a d_s close to 1, i.e. that of a quasi 1-d system. This is reasonable given

our construction: for sufficiently small p and small n , only relatively few random walkers encounter any long-range connections (shortcuts). Therefore in the beginning the behavior of $P_n(0)$ closely reflects the character of the underlying 1-d lattice. However for larger n , the random walkers probe larger and larger portions of the graph, and thus follow more and more shortcuts. This speeds up progressively the decay of $P_n(0)$ as more regions at larger and larger length scales are visited, and the fractal picture is lost. One would thus expect that the concept of a d_s begins to be invalid when the random walkers visit enough short cuts, i.e. when the 1-d diffusion extends longer than the typical distance between shortcuts. This is the fundamental length scale ξ of small-world networks, besides the lattice constant, which is less important here. In our case we have

$$\xi = p^{-1}, \quad (5)$$

measuring ξ in units of the lattice constant. For diffusion on scales smaller than ξ one furthermore has in terms of the diffusion constant D of the regular lattice $\xi^2 \sim 2Dn$, so that

$$n \sim \frac{1}{2Dp^2}. \quad (6)$$

Given that we allow random walkers to stay at a site during a time step, $D = 1/3$ and thus $n = 2/3p^{-2}$. However some walkers do encounter shortcuts at length-scales below $\sim p^{-1}$,

Figure 4: $P_n(0) - P_\infty(0)$ for $p = 0.05$ and $L = 10000$ on a semi-logarithmic scale. At longer times the decay appears to be slower than exponential. Also shown is a fit to a stretched exponential, indistinguishable from the data.

and numerically the cross-over to a region that does not have approximate power-law character is seen to take place earlier than $n \sim p^{-2}$.

We turn now to the analysis of this region. To be able to follow the it more closely, we replot the results of Fig. 2 for $L = 10000$ on semi-logarithmic scales in Fig. 4. Evidently, the decay for larger n is slower than exponential.

The decay of $P_n(0)$ is hence quicker than a power-law, but slower than the one for Cayley-trees, for which one has (for coordination numbers greater than 2) [21, 22]

$$P_n(0) \sim n^{-3/2} \exp[-Cn], \quad (7)$$

Figure 5: The relaxation or probability of presence at the origin as a function of the number of steps for $L = 2000$ and from upper right to lower left $p = 0.01, 0.05, 0.1, 0.2, 0.4$ and 0.8 .

where C is a constant. Comparing this behavior to the one displayed in Figs. 1 and 4 we remark that in our case, for a relatively small number of steps, the decay goes approximately as $n^{-\alpha}$ with $\alpha \gtrsim 1/2$, whereas at larger n , a more adequate description would be a stretched exponential, $\exp[-Cn^\beta]$. One may even suspect that the decay in Fig. 4 obeys $P_n(0) \sim n^{-\alpha} \exp[-Cn^\beta]$. A fit of the data in Fig. 4 to this functional form (keeping $\alpha \equiv 0.5$ fixed) is also shown; the fit turns out to be indistinguishable from the numerical data, when we choose $\beta = 0.56$ and $C = 0.04$, as plotted in Fig. 4.

We now consider the dependence of the decay on the value of p . For this we plot in Fig. 5 the decay law $P_n(0)$ for $L = 2000$ and p ranging from $p = 0.01$ to $p = 0.8$. We note that

the initial power-law like region diminishes with increasing p . Furthermore, the plateau-region $P_n(0) \simeq 1/L$ is reached earlier for larger p . This is in accordance with our argument above, that the long-range connections (short-cuts) interrupt the simple diffusion on the underlying lattice, such that the crossover length decreases with increasing p (c.f. also Eq. (6)). As p becomes large enough, the power law regime practically disappears. This is so because the random walker rapidly meets a short-cut. As before, the influence of finite size effects can be reduced by plotting, as in Fig. 2, $P_n(0) - P_\infty(0)$.

We have also performed simulations of the random walk on small-world networks where the underlying lattice has a k value larger than 1. In Fig. 6 we plot the results for $p = 0.1$ and $L = 2000$ in the cases of $k = 1$, $k = 2$, $k = 3$ and $k = 4$. The findings reproduce the general picture: $P_n(0)$ behaves like a power law for small n , while decaying more rapidly as n gets larger. The curves for different k are mainly shifted with respect to each other, and the network with the largest coordination number (largest k) also displays the quickest relaxation. To be noted, however, is that the case $k = 1$ has the largest dynamical range and thus shows best the decay forms, while also being the one simplest to implement; hence $k = 1$ may be the ideal small-world model.

Figure 6: $P_n(0)$ plotted for $L = 2000$, $p = 0.1$ and $k = 1, 2, 3$ and 4 , from right to left.

3 Conclusions

In this work we have studied numerically the behavior of random walks on small-world lattices. Our work has focused on the probability of being at the initial site $P_n(0)$ as a function of the number of steps n . This quantity is found to show a complex, very interesting pattern: initially $P_n(0)$ displays a power-law, “quasi-fractal” regime. At larger n a quicker decay takes over, reminiscent of stretched exponentials. In this respect the $P_n(0)$ decay is intermediate between the decays found for fractal structures and the ones found for tree-like (loop-less) structures, exemplified here by Cayley trees.

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