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Two-Time Physics with Gravitational and Gauge Field Backgrounds[1]

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Abstract

It is shown that all possible gravitational, gauge and other interactions experienced by particles in ordinary d-dimensions (one-time) can be described in the language of two-time physics in a spacetime with $d+2$ dimensions. This is obtained by generalizing the worldline formulation of two-time physics by including background fields. A given two-time model, with a fixed set of background fields, can be gauged fixed from $d+2$ dimensions to $(d-1)+1$ dimensions to produce diverse one-time dynamical models, all of which are dually related to each other under the underlying gauge symmetry of the unified two-time theory. To satisfy the gauge symmetry of the two-time theory the background fields must obey certain coupled differential equations that are generally covariant and gauge invariant in the target $d+2$ dimensional spacetime. The gravitational background obeys a closed homothety condition while the gauge field obeys a differential equation that generalizes a similar equation derived by Dirac in 1936. Explicit solutions to these coupled equations show that the usual gravitational, gauge, and other interactions in d dimensions may be viewed as embedded in the higher $d+2$ dimensional space, thus displaying higher spacetime symmetries that otherwise remain hidden.

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1 Introduction

Two-Time Physics [1]-[6] is an approach that provides a new perspective for understanding ordinary one-time dynamics from a higher dimensional, more unified point of view including two timelike dimensions. This is achieved by introducing new gauge symmetries that insure unitarity, causality and absence of ghosts. The new phenomenon in two-time physics is that the gauge symmetry can be used to obtain various one-time dynamical systems from the same simple action of two-time physics, through gauge fixing, thus uncovering a new layer of unification through higher dimensions.

The principle behind two-time physics is the gauge symmetry [1]. The basic observation in its simplest form is that for any theory the Lagrangian has the form $L = \frac{1}{2}\left(\dot{x}p - \dot{p}x\right) - H\left(x,p\right)$ up to an inessential total time derivative. The first term has a global $\mathrm{Sp}(2,R)$ symmetry that transforms (x,p) as a doublet. The basic question we pose is: what modification of the Lagrangian can turn this global symmetry into a local symmetry? The reason to be interested in such a local symmetry is that duality symmetries in M-theory and N=2 super Yang-Mills theory have similarities to gauge symplectic transformations, and their origin in the fundamental theories in physics remains a mystery. Understanding them may well be the key to constructing M-theory. Independent of M-theory, the question is a fundamental one in its own right, and its investigation has already led to a reformulation of ordinary one-time dynamical systems in a new language of two-time physics. This has uncovered previously unnoticed higher symmetries in well known one-time dynamical systems, and provided a new level of unification through higher dimensions for systems that previously would have been considered unrelated to each other [2]. The simplest $\mathrm{Sp}(2,R)$ gauge symmetry has generalizations when spin [3], supersymmetry [4][5], and extended objects (branes) [6] are part of the theory. Recent works have given an indication that the domain of unification of two-time physics can be enlarged in additional directions in field theory [7] including interactions, and in the world of branes [8].

In the two-time physics approach the familiar one-time is a gauge dependent concept. From the point of view of a two-time observer the true gauge invariants are identical in a variety of one-time dynamical systems that are unified by the same two-time action. Such gauge invariant quantities can be used to test the validity of the underlying unification. An important gauge invariant concept is the global symmetry of the two-time action, which must be shared by all the gauge fixed one-time dynamical systems. In the simplest case the global symmetry is $\mathrm{SO}(d,2)$, but this can be different in the presence of background fields as we will see in the current paper. In the simple case, the $\mathrm{SO}(d,2)$ symmetry has been shown to be present *in the same irreducible representation* in all the one-time dynamical systems derived from the same two-time action. The presence of such symmetries, which remained unknown even in elementary one-time systems until the advent of two-time physics, can be considered as a test of the underlying unification within a two-time theory [2].

Two-Time Physics has been generalized to include global space-time supersymmetry and local kappa supersymmetry with two-times [4]. This led to a framework which suggests that M-theory could be embedded in a two-time theory in 13 dimensions, with a global OSp(1|64) symmetry. In this scenario the different corners of M-theory correspond to gauge fixed sectors of the 13D theory, and the dualities in M-theory are regarded as gauge transformations from one fixed gauge to another fixed gauge. Then the well known supersymmetries of various corners of M-theory appear as subsupergroups of OSp(1|64). This mechanism has been illustrated through explicit examples of dynamical particle models [5][9] which may be regarded as a toy-M-theory. In the 11D-covariant gauge fixed corner, the supergroup OSp(1|64) is interpreted as the conformal supergroup in 11-dimensions, with 32 supersymmetries and 32 superconformal symmetries. But in other gauge fixed sectors, the same OSp(1|64) symmetry of two-time physics is realized and interpreted differently, thus revealing various corners of toy-M-theory on which a subsupergroup is linearly realized while the rest is non-linearly realized. Indeed OSp(1|64) contains various embeddings that reveal 13,12,11 dimensional supersymmetries, as well as the usual 10-dimensional type-IIA, type-IIB, heterotic, type-I, and $\mathrm{AdS}_D \otimes \mathrm{S}^k$ type supersymmetries in $D+k=11,10$ and lower dimensions. The explicit models provided by [5] [9] illustrate these ideas while beginning to realize dynamically some of the observations that suggested two-time physics in the framework of branes, dualities and extended supersymmetries in M-theory, F-theory, and S-theory [10]-[20].

In this paper we generalize the worldline formulation of two-time physics by including background gravitational and gauge fields and other potentials. To keep the discussion simple we concentrate mainly on particles without supersymmetry. For spinless particles, as in the case of the free theory, local $\mathrm{Sp}(2,R)$ gauge symmetry is imposed as the underlying principle. For the gauge symmetry to be valid, the gravitational and gauge fields and other potentials must obey certain differential equations. We show that the gauge field obeys an equation that generalizes a similar one discovered by Dirac in 1936 [21] in the flat background, while the gravitational field satisfies a closed homothety condition. When all fields are simultaneously present they obey coupled equations. Examples of background fields that solve these equations are provided.

A similar treatment for spinning particles in background fields is given. As in the free theory, local $\mathrm{OSp}(n|2)$ gauge symmetry is imposed as the underlying principle. The set of background fields is now richer. The generalizations of Dirac's equation and the closed homothety conditions in the presence of spin are derived. Instead of $\mathrm{OSp}(n|2)$ gauge symmetry it may also be possible to consider other supergroups that contain $\mathrm{Sp}(2,R) \equiv SL(2,R)$.

In the presence of the background fields one learns that much larger classes of one-time dynamical systems can now be reformulated as gauge fixed versions of the same two-time theory. This extends the domain of unification of one-time systems through higher dimensions and a sort of duality symmetry (the $\mathrm{Sp}(2,R)$ gauge symmetry and its generalizations in systems with spin and/or spacetime supersymmetry, and branes). Furthermore, with the results of this paper it becomes evident that all one-time particle dynamics can be reformulated as particle dynamics in two-time physics. This provides a much broader realm of possible applications of the two-time physics formalism.

One possible practical application of the formulation is to provide a tool for solving problems by transforming a complicated one-time dynamical system (one fixed gauge) to a simpler one-time dynamical system (another fixed gauge), as in duality transformations in M-theory. Although this may turn out to be the computationally useful aspect of this formulation, it is not explored in the present paper since our main aim here is the formulation of the concepts.

The two-time formulation also has deeper ramifications. By providing the perspective of two-time physics for ordinary physical phenomena, the familiar "time" dimension appears to play a less fundamental role in the formulation of physics. Since the usual "time" is a gauge dependent concept in the new formulation, naturally one is led to a re-examination of the concept of "time" in this new setting.

2 Local and global symmetry

We start with a brief summary of the worldline formulation of two-time physics for the simplest case of spinless particle dynamics without background fields and without a Hamiltonian [1]-[5] (i.e. the "free" case). Just demanding local symmetry for the first term in the Lagrangian gives a surprisingly rich model based on $Sp(2,R)$ gauge symmetry described by the action

$$\begin{aligned} S_0 &= \frac{1}{2}\int d\tau\, D_\tau X_i^M X_j^N \varepsilon^{ij}\eta_{MN} & (1)\\ &= \int d\tau (\partial_\tau X_1^M X_2^N - \frac{1}{2}A^{ij}X_i^M X_j^N)\eta_{MN}. & (2)\end{aligned}$$

Here $X_i^M(\tau)$ is an $Sp(2,R)$ doublet, consisting of a coordinate and its conjugate momentum ($X_1^M \equiv X^M$ and $X_2^M \equiv P^M$). The indices $i,j=1,2$ denote the doublet of $Sp(2,R)$; they are raised and lowered by the antisymmetric Levi-Civita symbol ε_{ij}. The covariant derivative $D_\tau X_i^M$ that appears in (1) is defined as

$$D_\tau X_i^M = \partial_\tau X_i^M - \varepsilon_{ik}A^{kl}X_l^M, \tag{3}$$

where $A^{ij}(\tau)$ are the three $\mathrm{Sp}(2,R)$ gauge potentials in the adjoint representation written as a $2{\times}2$ symmetric matrix. The local $Sp(2,R)$ acts as $\delta X_i^M = \varepsilon_{ik}\omega^{kl}X_l^m$ and $\delta A^{ij} = \omega^{ik}\varepsilon_{kl}A^{kj} + \omega^{jk}\varepsilon_{kl}A^{ik} + \partial_\tau\omega^{ij}$, where $\omega^{ij}(\tau)$ are the $\mathrm{Sp}(2,R)$ gauge parameters. The second form of the action (2) is obtained after an integration by parts so that only X_1^M appears with derivatives. This allows the identification of X,P by the canonical procedure ($X_1^M \equiv X^M$ and $X_2^M \equiv P^M = \partial S_0/\partial \dot{X}_{1M}$). A third form of the action can be obtained by integrating out X_2^M and writing it in terms of X^M and $\dot{X}^M$ [1][22]. Then the local $\mathrm{Sp}(2,R)$ $=\mathrm{SO}(1,2)$ can also be regarded as the local conformal group on the worldline (including τ reparametrization, local scale transformations, and local special conformal transformations) and the theory can be interpreted as conformal gravity on the worldline [1][6].

The gauge fields A^{11}, A^{12}, and A^{22} act as Lagrange multipliers for the following three first class constraints

$$Q_{ij}^0 = X_i \cdot X_j = 0 \ \rightarrow \ X^2 = P^2 = X \cdot P = 0, \tag{4}$$

as implied by the local $Sp(2,R)$ invariance. From the basic quantum rules for $\left(X^M, P^M\right)$ one can verify that the Q_{ij}^0 form the $\mathrm{Sp}(2,R)$ algebra

$$\begin{aligned} [Q_{ij}, Q_{kl}] &= i\varepsilon_{jk}Q_{il} + i\varepsilon_{ik}Q_{jl} + i\varepsilon_{jl}Q_{ik} + i\varepsilon_{il}Q_{jk}\,, \quad \mathit{or} & (5) \\ [Q_{11}, Q_{22}] &= 4iQ_{12}, \quad [Q_{11}, Q_{12}] = 2iQ_{11}, \quad [Q_{22}, Q_{12}] = -2iQ_{22}. & (6) \end{aligned}$$

The two timelike dimensions are not put in by hand, they are implied by the local $Sp(2,R)$ symmetry. It is precisely the solution of the constraints $Q_{ij}^0 = 0$ that require the global metric η_{MN} in (1) to have a signature with two-time like dimensions: if η_{MN} were purely Euclidean the only solution would be vanishing vectors X_i^M, if it had Minkowski signature (one time) the only solution would be two lightlike parallel vectors X_i^M without any angular momentum, if it had more than two timelike dimensions there would be ghosts that would render the theory non-unitary. The local $\mathrm{Sp}(2,R)$ is just enough gauge symmetry to remove the ghosts due to two timelike dimensions. Thus, η_{MN} stands for the flat metric on a $(d,2)$ dimensional space-time. It is the only signature consistent with absence of ghosts, unitrarity or causality problems.

We now turn to the global symmetries that are gauge invariant under $\mathrm{Sp}(2,R)$. The metric η_{MN} is invariant under $\mathrm{SO}(d,2)$. Hence the action (1-2) has an explicit global $SO(d,2)$ invariance. Like the two times, the $SO(d,2)$ symmetry of the action (1) is also implied by the local $Sp(2,R)$ symmetry when background fields are absent. The $\mathrm{SO}(d,2)$ Lorentz generators

$$L^{MN} = X^M P^N - X^N P^M = \varepsilon^{ij} X_i^M X_j^N \tag{7}$$

commute with the $\mathrm{Sp}(2,R)$ generators, therefore they are gauge invariant. As we mentioned above, different gauge choices lead to different one-time particle dynamics (examples: free massless and massive particles, H-atom, harmonic oscillator, particle in $\mathrm{AdS}_D \times \mathrm{S}^k$ etc.) all of which have $SO(d,2)$ *invariant actions* that are directly obtained from (1-2) by gauge fixing. Since the action (1-2) and the generators L^{MN} are gauge invariant, the global symmetry $\mathrm{SO}(d,2)$ is not lost by gauge fixing. This explains why one should expect a hidden (previously unnoticed, non-linearly realized) global symmetry $\mathrm{SO}(d,2)$ for each of the one-time systems that result by gauge fixing[2]. Furthermore all of the resulting one-time dynamical systems are quantum mechanically realized in the *same unitary representation* of $\mathrm{SO}(d,2)$ [1]-[2]. This fact can be

[2] A well known case is the $\mathrm{SO}(4,2)$ conformal symmetry of the massless particle. Less well known is the $\mathrm{SO}(4,2)$ symmety of the H-atom action, which acts as the dynamical symmetry for the quantum H-atom. Previously unknown is the $\mathrm{SO}(4,2)$ symmetry of the massive non-relativistic particle action $S = \int d\tau\ \dot{\mathbf{x}}^2\ /2m$. Others are the $\mathrm{SO}(10,2)$ symmetry of a particle in the $\mathrm{AdS}_5 \times S^5$ background, or the $\mathrm{SO}(11,2)$ symmetry in the $\mathrm{AdS}_7 \times S^4$ and the $\mathrm{AdS}_4 \times S^7$ backgrounds, etc. These and more examples of such non-linearly realized $\mathrm{SO}(d,2)$ hidden symmetries for familiar systems in any space-time dimension d are explicitly given in [2].

understood again as a simple consequence of representing the same quantum mechanical two-time system in various fixed gauges. The gauge choices merely distinguish one basis versus another basis within the same unitary representation of SO$(d,2)$ without changing the Casimir eigenvalues of the irreducible representation. Such relations among diverse one-time systems provide evidence that there is an underlying unifying principle behind them. The principle is the *local* Sp$(2,R)$ symmetry, and its unavoidable consequence of demanding a spacetime with two timelike dimensions which provides a basis for the *global* symmetry.

To describe spinning systems, worldline fermions $\psi_\alpha^M(\tau)$, with $\alpha = 1,2,\cdots,n$ are introduced. Together with X^M, P^M, they form the fundamental representation $\left(\psi_\alpha^M, X^M, P^M\right)$ of the supergroup OSp$(n/2)$. Gauging this supergroup [3] instead of Sp$(2,R)$ produces a Lagrangian that has n local supersymmetries plus n local conformal supersymmetries on the worldline, in addition to local Sp$(2,R)$ and local SO(n). The full set of first class constraints that correspond to the generators of these gauge (super)symmetries are, at the classical level,

$$X \cdot X = P \cdot P = X \cdot P = X \cdot \psi_\alpha = P \cdot \psi_\alpha = \psi_{[\alpha} \cdot \psi_{\beta]} = 0. \tag{8}$$

The classical solution of these constraints, with a flat spacetime metric η^{MN}, require a signature with two timelike dimensions. Therefore, as in the spinless case the global symmetry of the theory is SO$(d,2)$. It is applied to the label M on $\left(\psi_\alpha^M, X^M, P^M\right)$. The global SO$(d,2)$ generators J^{MN} that commute with all the OSp$(n/2)$ gauge generators (8) now include the spin

$$J^{MN} = L^{MN} + S^{MN}\,, \quad S^{MN} = \frac{1}{2i}\left(\psi_\alpha^M \psi_\alpha^N - \psi_\alpha^N \psi_\alpha^M\right). \tag{9}$$

As in the spinless case, by gauge fixing the bosons as well as the fermions, one finds a multitude of spinning one-time dynamical systems that are unified by the same two-time system both at the classical and quantum levels. All of these have SO$(d,2)$ hidden symmetry realized in the same representation, where the representation is different for each n (number of local supersymmetries on the worldline, which is related also to the spin of the particle).

3 Interactions with background fields

The simple action in (2) is written in a flat two-time spacetime with metric η_{MN} which could be characterized as a "free" theory. Interactions in the one-time systems emerged because of the first class constraints $X^2 = P^2 = X \cdot P = 0$, not because of explicit interactions in the two time theory. The constraints generate the Sp$(2,R)$ gauge symmetry. This symmetry was realized linearly on the doublet $X_i^M = \left(X^M, P^M\right)$ and its generators were $Q_{ij}^0 = X_i \cdot X_j$.

We now generalize the "free" theory to an "interacting" theory by including background gravitational and gauge fields and other potentials. This will be done by generalizing the worldline Hamiltonian (canonical conjugate to τ) $Q_{22}^0 = P_M P_N \eta^{MN}$ to a more general form that includes a metric $G^{MN}(X)$, a gauge potential[3] to gauge-covariantize the momentum $P_M + A_M(X)$, and an additional potential $U(X)$ that is added to the kinetic term. Generalizing Q_{22} in this way requires also generalizing all Q^0_{ij} to $Q_{ij}(X,P)$ whose functional form will be determined. The Lagrangian is formally similar to the "free" case (2)

$$S = \int d\tau (\partial_\tau X^M P_M - \frac{1}{2} A^{ij} Q_{ij}(X,P)). \tag{10}$$

Whatever the expressions for $Q_{ij}(X,P)$ are, by the equations of motion of the gauge potentials A^{ij}, they are required to form first class constraints that close under the $\mathrm{Sp}(2,R)$ commutation rules (5), which should follow from the basic commutation rules of $\left(X^M, P^M\right)$. Furthermore, the local $\mathrm{Sp}(2,R)$ transformation properties of the dynamical variables should be given by these generators under commutation rules

$$\begin{aligned}
\delta X^M &= \frac{i}{2}\omega^{ij}(\tau)\left[Q_{ij}(X,P), X^M\right] = \frac{1}{2}\omega^{ij}(\tau)\frac{\partial Q_{ij}(X,P)}{\partial P_M} && (11)\\
\delta P^M &= i\omega^{ij}(\tau)\left[Q_{ij}(X,P), P^M\right] = -\frac{1}{2}\omega^{ij}(\tau)\frac{\partial Q_{ij}(X,P)}{\partial X^M} && (12)\\
\delta A^{ij} &= \partial_\tau \omega^{ij} + \omega^{ik}\varepsilon_{kl}A^{lj} + \omega^{jk}\varepsilon_{kl}A^{li}. && (13)
\end{aligned}$$

These certainly hold for the free case with $Q^0_{ij} = X_i \cdot X_j$, but now we discuss the general case. Substituting these transformation laws into the Lagrangian we have (ignoring orders of operators at the classical level)

$$\delta L = \partial_\tau \left(\delta X^M\right) P_M + \partial_\tau X^M \delta P_M - \frac{1}{2}\delta A^{ij} Q_{ij}(X,P) - \frac{1}{2}A^{ij}\delta Q_{ij}(X,P) \tag{14}$$

where $\delta Q_{ij}(X,P) = \frac{\partial Q_{ij}}{\partial X^M}\delta X^M + \frac{\partial Q_{ij}}{\partial P_M}\delta P_M$. After an integration by parts of the first term, using (11-13) this becomes

$$\delta L = -\frac{1}{2}\partial_\tau\left(\omega^{ij}Q_{ij}\right) - \frac{1}{2}\left(\omega^{ik}\varepsilon_{kl}A^{lj} + \omega^{jk}\varepsilon_{kl}A^{li}\right)Q_{ij} - \frac{1}{4}A^{ij}\omega^{kl}\left\{Q_{ij}, Q_{kl}\right\}, \tag{15}$$

where $\{Q_{ij}, Q_{kl}\}$ is the Poisson bracket

$$\{Q_{ij}, Q_{kl}\} = \frac{\partial Q_{ij}}{\partial X^M}\frac{\partial Q_{kl}}{\partial P_M} - \frac{\partial Q_{ij}}{\partial P_M}\frac{\partial Q_{kl}}{\partial X^M}. \tag{16}$$

Thus, if the Q_{ij} satisfy the $\mathrm{Sp}(2,R)$ algebra (5), then the Poisson bracket term cancels the second term, and δL is a total derivative. Hence to insure the gauge invariance of the action S we must require the differential constraints

$$\frac{\partial Q_{ij}}{\partial X^M}\frac{\partial Q_{kl}}{\partial P_M} - \frac{\partial Q_{ij}}{\partial P_M}\frac{\partial Q_{kl}}{\partial X^M} = \varepsilon_{jk}Q_{il} + \varepsilon_{ik}Q_{jl} + \varepsilon_{jl}Q_{ik} + \varepsilon_{il}Q_{jk}. \tag{17}$$

[3] It is possible to generalize this discussion by promoting A to a non-Abelian Yang-Mills potential coupled to a non-Abelian charge, which is an additional dynamical degree of freedom. To keep the discussion simple we take an Abelian A in the present paper.

With these restrictions we look for $Q_{ij}(X,P)$ that can be interpreted as dynamics with background fields, as opposed to dynamics in flat spacetime. To be able to integrate out the momenta P^M we restrict these expressions to contain at the most two powers of P^M (this restriction could be lifted to construct even more general systems [4]). Also, keeping the analogy to the flat case, we will take Q_{11} to have no powers of P^M, Q_{12} to have at most one power of P^M, and Q_{22} to have at the most two powers of P^M, as follows

$$Q_{11} = W(X), \quad Q_{12} = \frac{1}{2}V^M(P_M + A_M) + \frac{1}{2\sqrt{G}}(P_M + A_M)\sqrt{G}V^M, \tag{18}$$

$$Q_{22} = \frac{1}{\sqrt{G}}(P_M + A_M)\sqrt{G}G^{MN}(P_N + A_N) + U(X). \tag{19}$$

The functions $W(X), V^M(X), G^{MN}(X), A_N(X), U(X)$ will satisfy certain constraints. The expression for Q_{22} is a generalization of the free worldline "Hamiltonian" in flat space $\eta^{MN}P_M P_N$. The factors of $\sqrt{G}$ are inserted to insure hermiticity of the operators in a quantum theory as applied on wavefunctions with a norm $\int \sqrt{G}\psi^*\psi$. In the classical theory the factors of $\sqrt{G}$ in Q_{12}, Q_{22} cancel since orders of operators are neglected, but in any case a reordering amounts to a redefinition of $A_M(X)$ and $U(X)$.

The combination $P_M + A_M(X)$ is gauge invariant under $\delta_\Lambda A_M(X) = \partial_M\Lambda(X)$ and $\delta_\Lambda P_M = -\partial_M\Lambda(X)$, where $\Lambda(X(\tau))$ is a gauge function of spacetime. The Lagrangian has this gauge symmetry since it transforms into a total derivative under the gauge transformation $\delta_\Lambda L = -\partial_\tau X^M\partial_M\Lambda(X) = -\partial_\tau\Lambda$. Furthermore, the Lagrangian is a scalar under spacetime general coordinate transformations, since the Q_{ij} are scalars when all the background fields are transformed as tensors, while the term $\partial_\tau X^M P_M$ is invariant under $\delta_\varepsilon X^M = -\varepsilon^M(X)$ and $\delta_\varepsilon P_M = \partial_M\varepsilon^N P_N$. Of course, if the background fields are fixed, the general covariance and gauge symmetries are not generally valid, and only a subgroup that corresponds to Killing symmetries of the combined gauge and reparametrization transformations survive.

By integrating out P_M we can rewrite the Lagrangian purely in terms of $X^M(\tau)$ and its derivatives $\dot{X}^M(\tau)$

$$L = \frac{1}{2A^{22}}\left(\dot{X}^M - A^{12}V^M\right)G_{MN}\left(\dot{X}^N - A^{12}V^N\right) - \frac{A^{22}}{2}U - \frac{A^{11}}{2}W - \dot{X}^M A_M. \tag{20}$$

By inspection of (19) or (20) we interpret $A_M(X)$ as a gauge field, $G_{MN}(X)$ as a spacetime metric and $U(X)$ as an additional potential. The function $W(X) \sim 0$ is the constraint that replaces $X \cdot X \sim 0$ and the vector $V^M(X)$ can be thought of as a general coordinate transformation since the action of Q_{12} on phase space is $\delta_{12}X^M = V^M(X)$ and $\delta_{12}P_M = \partial_M V^K P_K + \partial_M(V \cdot A)$ which looks like a general coordinate transformation up to a gauge transformation.

The classical local Sp$(2,R)$ transformation laws for $\left(X^M, P_M\right)$ in phase space follow from (11, 13)

$$\delta X^M = \omega^{12}(\tau)V^M + \omega^{22}(\tau)G^{MN}(P_N + A_N) \tag{21}$$

[4] The coefficients of higher powers of P^M have the interpretation of higher spin fields

$$\delta P_M = -\frac{1}{2}\omega^{11}(\tau)\,\partial_M W - \omega^{12}(\tau)\left[\left(\partial_M V^N\right)P_N + \partial_M (V\cdot A)\right] \tag{22}$$
$$-\frac{1}{2}\omega^{22}(\tau)\left[\left(\partial_M G^{KL}\right)(P_K+A_K)(P_L+A_L) + \partial_M U + 2G^{KL}\partial_M A_K (P_L+A_L)\right]$$

This, together with (13), is a local symmetry of the action provided (17) is satisfied. These conditions give the following differential constraints on the functions $W(X)$, $V^M(X)$, $G^{MN}(X)$, $A_N(X)$, $U(X)$. From $\{Q_{11}, Q_{22}\} = 4Q_{12}$ we learn

$$V^M = \frac{1}{2}G^{MN}\partial_N W. \tag{23}$$

From $\{Q_{11}, Q_{12}\} = 2Q_{11}$ we learn

$$V^M \partial_M W = 2W, \quad \mathit{or} \quad G^{MN}(\partial_M W)(\partial_N W) = 4W. \tag{24}$$

Finally from $\{Q_{22}, Q_{12}\} = -2Q_{22}$ we learn (from the coefficients of each power of P_M) that

$$\pounds_V G^{MN} = -2G^{MN}, \quad V^M \partial_M U = -2U, \quad V^M F_{MN} = 0, \tag{25}$$

where $\pounds_V G^{MN}$ is the Lie derivative of G^{MN} (an infinitesimal general coordinate transformation)

$$\pounds_V G^{MN} \equiv V^K \partial_K G^{MN} - \partial_K V^M G^{KN} - \partial_K V^N G^{MK}, \tag{26}$$

and $F_{MN} = \partial_M A_N - \partial_N A_M$ is the gauge field strength. The differential equation $\pounds_V G^{MN} = -2G^{MN}$ together with (23) was called a "closed homothety" condition on the geometry [5]. We have the added generalization of the gauge field A_M in our case. When all fields are present they are coupled to each other.

The differential equation for the gauge field may also be rewritten in terms of the Lie derivative on the vector $\pounds_V A_M = \partial_M (V\cdot A)$, where the Lie derivative on the vector is $\pounds_V A_M = V^K \partial_K A_M + \partial_M V^K A_K$ (an infinitesimal general coordinate transformation). Using the gauge invariance of the physics, without loss of generality one may choose an axial gauge $V \cdot A = 0$. There still is a remaining gauge symmetry $\delta_\Lambda A_M = \partial_M \Lambda$, for all Λ that satisfy $V^K \partial_K \Lambda = 0$. Thus, the gauge field equation may be rewritten in the form

$$\pounds_V A_M = 0, \quad V\cdot A = 0, \tag{27}$$

[5] I learned this term when I came across ref.[26], after having derived these equations independently sometime ago. The physical problem in the present paper is quite different than [26] where our spacetime index M (with (d,2) signature) is replaced by a particle label for multiparticles in [26] (with Euclidean signature); nevertheless the mathematics formally coincide with ref.[26]. After the current paper was submitted for publication, I was informed that similar equations were obtained in [27] in the context of conformally invariant sigma models on a p+1 dimensional worldvolume, using a very different approach than ours. Although the case of p=0 (worldline) relevant for our case was missed by these authors, when their expressions are continued to p=0 they agree with our results. While there are formal similarities, an important difference between our work and those of [26] and [27] is that we have local SO(1,2)=Sp(2) symmetry as opposed to their global symmetry. This requires the constraints $Q_{ij}(X,P) = 0$ which demand a spacetime with two timelike dimensions, thus leading to conceptually very different physics.

with a remaining gauge symmetry of these equations $\left\{\Lambda;\ V^K\partial_K\Lambda = 0\right\}$ which we will make use of later.

Any solution to the coupled equations (23, 24, 25, 27) gives an action with local $\mathrm{Sp}(2,R)$ symmetry. Such an action provides a two-time physics theory including interactions with background fields. The global symmetries correspond to Killing symmetries in the presence of backgrounds, which is a subgroup embedded in general coordinate transformations combined with gauge transformations. This is the global symmetry, which in the flat and free case becomes $\mathrm{SO}(d,2)$.

The $\mathrm{Sp}(2,R)$ gauge symmetry may be gauge fixed to define a "time" and analyze the system from the point of view of one-time physics. The global symmetry described in the previous paragraph survives after gauge fixing the $\mathrm{Sp}(2,R)$ local symmetry, since it commutes with it (recall the Q_{ij} are invariant under general coordinate and gauge transformations). This global symmetry would then become the non-linearly realized hidden global symmetries in each of the one-time dynamical systems that emerge after gauge fixing (in the "free" case it is $\mathrm{SO}(d,2)$). The symmetry must be realized in the same representation for each one-time dynamical system that belongs to the same class, where the class is fixed by a given set of background fields.

4 Pure gauge field background

When the background metric is flat $G^{MN} = \eta^{MN}$ the only solution of the homothethy condition $\pounds_V G^{MN} = -2G^{MN}$ is $V^M = X^M$. This immediately gives $W = X\cdot X$, and U is any homogeneous function of X^M of degree -2. The global symmetry of the metric is $\mathrm{SO}(d,2)$. If we want to keep the $\mathrm{SO}(d,2)$ symmetry, U could only be $U = g/X\cdot X$ (however, without the $\mathrm{SO}(d,2)$ symmetry one can allow some other U of degree -2).

The equations for the gauge field (27) simplify in flat space. The remaining gauge symmetry parameter is homogeneous of degree zero $X\cdot\partial\Lambda = 0$ in $d+2$ dimensions. This is sufficient to fix further the gauge $\partial_M A^M = 0$ since according to the equations A_M also is homogeneous of degree -1 in this gauge. The three equations satisfied by the gauge field are now

$$X\cdot A\left(X\right) = 0,\quad \left(X\cdot\partial + 1\right)A_M\left(X\right) = 0,\quad \partial_M A^M = 0. \tag{28}$$

There still remains gauge symmetry in these equations for Λ that satisfy $X\cdot\partial\Lambda = \partial\cdot\partial\Lambda = 0$. The content of these equations for Λ is still non-trivial.

These equations were proposed by Dirac in 1936 [21] as subsidiary conditions to describe the usual 4-dimensional Maxwell theory of electromagnetism (in the Lorentz gauge), as a theory in 6 dimensions which automatically displays $\mathrm{SO}(4,2)$ symmetry. Dirac's aim was to linearize the conformal symmetry of the 4 dimensional Maxwell theory. The subsidiary conditions can be regarded as "kinematics" while dynamics is given by a Klein-Gordon type equation in 6-dimensions that may include interactions with other fields. As Dirac showed, the linear $\mathrm{SO}(4,2)$

Lorentz symmetry of the 6 dimensional theory is indeed the non-linear conformal symmetry of the Maxwell theory.

Actually, in the framework of two-time physics, conformal symmetry is only one of the possible interpretations of the SO$(4,2)$ global symmetry of these equations. In two-time physics this interpretation relies on a particular choice of "time" among the two available timelike dimensions, while with other gauge choices the interpretation of the SO$(4,2)$ symmetry is completely different than conformal symmetry. To illustrate this, denote the components of the 6 dimensions as $X^M = \left(X^{+'}, X^{-'}, X^\mu\right)$ with metric $X \cdot X = -2X^{+'}X^{-'} + X_\mu X^\mu$. The Sp$(2,R)$ gauge choices $P^{+'}(\tau) = 0,\ X^{+'}(\tau) = 1$ eliminates one timelike and one spacelike dimensions and brings down the two-time formulation in $d+2$ dimension to a one time formulation in d dimensions. It is convenient to use the electromagnetic gauge choice $A^{+'}(X) = 0$ (instead of Dirac's $\partial_M A^M = 0$). Then the solution of the gauge choices and constraints (including $Q_{11} = Q_{12} = 0$), $X \cdot X = X \cdot P = X \cdot A = 0$, is given in the following form

$$X^M(\tau) = \left(1,\ x^2/2,\ x^\mu(\tau)\right), \quad P^M = (0,\ x \cdot p,\ p^\mu(\tau)), \tag{29}$$

$$A^M(X) = (\,0,\ x \cdot A,\ A^\mu(x(\tau))\,). \tag{30}$$

The dynamics of the remaining degrees of freedom $(x^\mu(\tau), p^\mu(\tau))$ are obtained by substituting these solutions into the gauge invariant 6-dimensional action (20). The result is the standard 4-dimensional action for the massless relativistic particle coupled to the electromagnetic gauge potential $A_\mu(x)$

$$L = \frac{1}{2A^{22}}(\dot{x}^\mu)^2 - \dot{x}^\mu A_\mu(x). \tag{31}$$

Thus the original two-time action displays explicitly the hidden SO$(4,2)$ symmetry of the one-time action. The general coordinate transformation of the previous section, specialized to $\varepsilon^M = \varepsilon^{MN} X_N$ with constant antisymmetric ε^{MN}, is the SO$(4,2)$ global Lorentz symmetry of the 6-dimensional action, including the gauge field. This 6-dimensional Lorentz symmetry is also the non-linearly realized conformal symmetry of the gauge fixed action above, since the global symmetry commutes with the gauge symmetry, and gauge fixing of the gauge invariant action could not destroy the global symmetry. Indeed the generators of conformal transformations are the gauge invariant $L^{MN} = X^M P^N - X^N P^N$ now expressed in terms of the gauge fixed coordinates and momenta as shown in [1][2]. This agrees with Dirac's interpretation of the conformal SO$(4,2)$ symmetry as being the Lorentz symmetry of 6 dimensions.

However, if one chooses another gauge for time instead of $X^{+'}(\tau) = 1$, as was done with many illustrations in [1] [2], other d-dimensional dynamical systems arise, which now are coupled to a gauge potential. Then the SO$(d,2)$ symmetry generated by the same L^{MN} has a different interpretation than conformal symmetry, as explained in [1][2]. The presence of the gauge field background now produces a large class of dynamical systems with hidden SO$(d,2)$ symmetries, and Sp$(2,R)$ duality relations among them.

The two-time physics approach [1]-[6] was developed without being aware of the field equations invented by Dirac. While Dirac was interested in linearizing conformal symmetry[6], the motivation for the work in [1]-[6] came independently from duality, and signals for two-timelike dimensions in M-theory and its extended superalgebra including D-branes [11][12][13]. Driven by different motivations, and unaware of Dirac's approach to conformal symmetry, two-time physics produced new insights that include conformal symmetry but go well beyond it. Besides providing a deeper Sp$(2,R)$ gauge symmetry as the fundamental basis for Dirac's approach (see further [7]), two-time physics unifies classes of one-time physical systems in d dimensions that previously would have been thought of as being unrelated to each other. The SO$(d,2)$ symmetry is interpreted as conformal symmetry in a certain one-time system, but in other dually related dynamical systems it is a hidden symmetry with a different interpretation, but realized in exactly the same irreducible representation. . The unifying aspect in all the interpretations is that the symmetry is the underlying spacetime symmetry in a spacetime that includes two timelike dimensions.

5 Gravitational background

We now seek a solution of (23-27) that includes gravity in d dimensions. It is convenient to make a change of variables $X^M = X^M(\kappa, w, x^\mu)$ such that the function $W(X)$ is identified with the product of new coordinates $-2w\kappa$, while the coordinate x^μ is in d dimensions. The inverse of this change of variables is, $\kappa = K(X)$, $w = -W(X)/2K(X)$ and $x^\mu = x^\mu(X)$. Before we look for a solution to (23-27) it is instructive to consider the example of the flat case that has components $X^M = \left(X^{+'}, X^{-'}, X^\mu\right)$ with the constraint $W(X) = X \cdot X = -2X^{+'}X^{-'} + X_\mu X^\mu$. The change of variables and the inverse relations for this case are

$$X^{+'} = \kappa, \quad X^{-'} = \frac{\kappa x^2}{2} + w, \quad X^\mu = \kappa x^\mu \tag{32}$$

$$\kappa = X^{+'}, \quad w = \frac{X \cdot X}{-2X^{+'}}, \quad x^\mu = \frac{X^\mu}{X^{+'}} \tag{33}$$

This change of variables is a special case of a general coordinate transformation. The flat metric in the new variables takes the form

$$ds^2 = dX^M dX^N \eta_{MN} = -2dX^{+'}dX^{-'} + dX^\mu dX^\nu \eta_{\mu\nu} \tag{34}$$

$$= -2d\kappa dw + \kappa^2 dx^\mu dx^\nu \eta_{\mu\nu}. \tag{35}$$

[6] I thank Vasilev for informing me of Dirac's work and the line of research that followed the same trend of thought in relation to conformal symmetry [21][23][24][25]. A field theoretic formulation of two-time physics has been derived recently [7] and its relation to Dirac's work has been established. It is shown in [7] that two-time physics in a field theoretic setting, as in the particle dynamics setting, unifies different looking one-time field theories as being the same two-time field theory, while simultaneously revealing previously unnoticed hidden symmetries in field theory, including interactions. Such duality and global symmetry properties of two-time physics go well beyond Dirac's goal of linearizing conformal symmetry.

For this choice of basis we have $V^M = (\kappa, w, 0)$ and $W = -2\kappa w$ and the homothety conditions are easily verified. Taking this form as a model we seek a similar solution. With a choice of coordinates we can always take $V^M = (\kappa, w, 0)$. In the new coordinate system $W(\kappa, w, x^\mu)$ needs to be determined consistently with the closed homothety conditions. We will make an ansatz which may not be the most general, but is adequate to provide a sufficiently large set of solutions. Thus, we will take $W(\kappa, w, x) = -2w\kappa$ to have the same form as the free case, and insert these forms of V, W in the closed homothety conditions with a general G^{MN}. The homothety condition reads

$$(\kappa\partial_\kappa + w\partial_w) G^{MN} - \delta^M_\kappa G^{\kappa N} - \delta^M_w G^{wN} - \delta^N_\kappa G^{\kappa M} - \delta^N_w G^{wM} = -2G^{MN}. \tag{36}$$

From $V^M = \frac{1}{2} G^{MN}\partial_N W(X)$ we learn further

$$\begin{aligned} V^\mu &= 0 = -G^{\mu\kappa} w - G^{\nu w}\kappa \quad \to G^{\mu\kappa} = \frac{1}{\kappa} W^\mu, \quad G^{\mu w} = -\frac{w}{\kappa^2} W^\mu, & (37)\\ V^\kappa &= \kappa = -G^{\kappa\kappa} w - G^{\kappa w}\kappa, \quad \to \; G^{\kappa\kappa} = -\frac{\kappa}{w}\left(1 + G^{\kappa w}\right) & (38)\\ V^w &= w = -G^{w\kappa} w - G^{ww}\kappa, \quad \to G^{ww} = -\frac{w}{\kappa}\left(1 + G^{\kappa w}\right) & (39) \end{aligned}$$

Specializing the indices in the homothety condition gives the solutions for all components of G^{MN} in the form

$$G^{MN} = \begin{pmatrix} \frac{\kappa}{w}(\gamma - 1) & -\gamma & \frac{1}{\kappa} W^\nu \\ -\gamma & \frac{w}{\kappa}(\gamma - 1) & -\frac{w}{\kappa^2} W^\nu \\ \frac{1}{\kappa} W^\mu & -\frac{w}{\kappa^2} W^\mu & \frac{g^{\mu\nu}}{\kappa^2} \end{pmatrix} \tag{40}$$

where the functions $\gamma\left(x, \frac{w}{\kappa}\right)$, $W^\mu\left(x, \frac{w}{\kappa}\right)$, $g^{\mu\nu}\left(x, \frac{w}{\kappa}\right)$ are arbitrary functions of only x^μ and the ratio $\frac{w}{\kappa}$.

In this coordinate system we can also solve the kinematic conditions for the gauge field (27), which become

$$(w\partial_w + \kappa\partial_\kappa) A_M + \delta^w_M A_w + \delta^\kappa_M A_\kappa = 0, \quad wA_w + \kappa A_\kappa = 0. \tag{41}$$

The general solution is

$$A_w = \frac{1}{\kappa} B\left(\frac{w}{\kappa}, x\right), \quad A_\kappa = -\frac{w}{\kappa^2} B\left(\frac{w}{\kappa}, x\right), \quad A_\mu = A_\mu\left(\frac{w}{\kappa}, x\right).$$

The remaining gauge symmetry $V^M\partial_M\Lambda = 0$ is just sufficient to set $B = 0$ in this solution, if so desired. Finally the solution for $U(w, \kappa, x)$ that satisfies $V^M\partial_M U = -2U$ is

$$U = \frac{1}{\kappa^2} u\left(\frac{w}{\kappa}, x\right). \tag{42}$$

For this solution, the generators of $\mathrm{Sp}(2, R)$ in $(18, 19)$ become, in the gauge $B = 0$,

$$\begin{aligned} Q_{11} &= -2\kappa w, \quad Q_{12} = \kappa p_\kappa + w p_w, & (43)\\ Q_{22} &= -2\gamma p_w p_\kappa + \left(p_\kappa^2 \frac{\kappa}{w} + p_w^2 \frac{w}{\kappa}\right)(\gamma - 1) & (44)\\ &\quad + \frac{2}{\kappa^2}\left(\kappa p_\kappa - w p_w\right) W^\mu p_\mu + \frac{H}{\kappa^2}, \end{aligned}$$

where

$$H = \frac{1}{\sqrt{-g}} \left(p_\mu + A_\mu \right) \sqrt{-g} g^{\mu\nu} \left(p_\nu + A_\nu \right) + u. \tag{45}$$

It is easy to verify directly that they close correctly for any background fields $\gamma, g_{\mu\nu}, W^\mu, A_\mu, u$ that are arbitrary functions of $\left(\frac{w}{\kappa}, x^\mu \right)$.

Imposing the $\mathrm{Sp}(2,R)$ constraints $Q_{ij} = 0$ is now easy. It is convenient to choose a $\mathrm{Sp}(2,R)$ gauge, which we know will produce a one-time theory. A gauge choice that is closely related to the massless relativistic particle is taken by analogy to the flat theory. At the classical level we choose the $\mathrm{Sp}(2,R)$ gauges $\kappa(\tau) = 1$ and $p_w(\tau) = 0$, and solve $Q_{11} = Q_{12} = 0$ in the form $w(\tau) = p_\kappa(\tau) = 0$. There remains unfixed one gauge subgroup of $\mathrm{Sp}(2,R)$ which corresponds to τ reparametrization, and the corresponding Hamiltonian constraint $H \sim 0$, which involves the background fields $g_{\mu\nu}(x)$, $A_\mu(x)$, $u(x)$ that now are functions of only the d dimensional coordinates x^μ, since $w/\kappa = 0$. In this gauge, the background fields γ, W^μ decouple from the dynamics that govern the time development of $x^\mu(\tau)$. The two-time theory described by the original Lagrangian (20) reduces to a one-time theory

$$L = \frac{1}{2A^{22}} \dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) - \frac{A^{22}}{2} u(x) - \dot{x}^\mu A_\mu(x).$$

which controls the dynamics of the remaining degrees of freedom $x^\mu(\tau)$. Evidently this Lagrangian describes a particle moving in arbitrary gravitational, electromagnetic gauge fields and other potential $g_{\mu\nu}(x)$, $A_\mu(x)$, $u(x)$ in the remaining d dimensional spacetime.

We have therefore demonstrated that *all usual interactions* experienced by a particle, as described in the one-time formulation of dynamics, can be embedded in two time physics as a natural solution of the two-time equations (23-27), taken in a fixed $\mathrm{Sp}(2,R)$ gauge.

6 Spinning particles in background fields

To describe spinning particles in two time physics we need local superconformal symmetry instead of local conformal symmetry, as demonstrated in flat space in [3]. There the $\mathrm{Sp}(2,R)$ gauge group was replaced by the supergroup $\mathrm{OSp}(n|2)$ as described at the end of section 2 of this paper. To generalize this approach to curved space we need a soldering form E_M^a and its inverse E_a^M (analog of vierbein) that transforms curved base space indices to flat tangent space indices and vice versa. The metric in tangent space is η_{ab} while the general metric is given by $G_{MN} = E_M^a E_N^b \eta_{ab}$. Next consider phase space including spin degrees of freedom $\left(X^M, P_M, \psi_\alpha^a \right)$ where a is a tangent space index and $\alpha = 1, 2, \cdots, n$ denote the n supersymmetries. The canonical commutation rules are

$$\left[X^M, P_N \right] = i\delta_N^M, \quad \left\{ \psi_\alpha^a, \psi_\beta^b \right\} = \eta^{ab} \delta_{\alpha\beta}. \tag{46}$$

The ψ_α^a form a Clifford algebra and may be represented by gamma matrices if so desired.

A Lagrangian that has the desired OSp$(n|2)$ local symmetry has the same form as the flat case given in [3] with some modifications

$$L = \dot{X}^M P_M + \frac{i}{2}\psi_\alpha^a \dot{\psi}_\alpha^b \eta_{ab} - \frac{1}{2}A^{ij}Q_{ij} + iF^{i\alpha}Q_{i\alpha} - \frac{1}{2}B^{\alpha\beta}Q_{\alpha\beta}, \tag{47}$$

The OSp$(n|2)$ gauge fields may be arranged into the form of a $(n+2)\times(n+2)$ supermatrix

$$\begin{pmatrix} B^{[\alpha\beta]} & F^{\alpha i} \\ \varepsilon_{ij}F^{j\beta} & A^{ij} \end{pmatrix}, \quad A, B = bose, \quad F = fermi \tag{48}$$

They obey the standard transformation rules for gauge fields, as given in [3]. The OSp$(n|2)$ generators $Q_{ij}, Q_{i\alpha}, Q_{\alpha\beta}$ are to be taken as non-linear functions in phase space, including background fields. As in the purely bosonic case, our task is to find the forms of the background fields that have an interpretation as gravitational, gauge or other interactions experienced by spinning particles in two-time physics. The gauge field equations of motion require the first class constraints $Q_{ij} \sim Q_{i\alpha} \sim Q_{\alpha\beta} \sim 0$, whose solution will require two timelike dimensions, as in the flat theory or as in the curved purely bosonic theory. These are then the generators of infinitesimal transformations that tell us how to transform $\delta X^M, \delta P_M, \delta\psi_\alpha^M$ under the local OSp$(n|2)$. As in the purely bosonic theory treated earlier in this paper, it is easy to show that the Lagrangian has the local symmetry provided these first class constraints close into the algebra of OSp$(n|2)$. This requirement gives the differential equations for the background fields.

In the flat case the OSp$(n|2)$ generators are given by $Q_{ij}^0 = X_i \cdot X_j$, $Q_{i\alpha}^0 = X_i \cdot \psi_\alpha$, and $Q_{\alpha\beta}^0 = \frac{i}{2}\psi_{[\alpha}\cdot\psi_{\beta]}$. To include background fields we first generalize the fermionic generators $P\cdot\psi_\alpha$ (n local supersymmetries) and $X\cdot\psi_\alpha$ (n local superconformal symmetries) by introducing a tangent space vector $V_a(X)$, a soldering from $E_M^a(X)$, a spin connection $\omega_M^{ab}(X)$, a gauge field $A_M(X)$, and replacing the momentum by the covariant momentum

$$\Pi_a(X,P,\psi) = E_a^M\left(P_M + A_M + \frac{1}{2}\omega_M^{ab}S_{ab}\right) \tag{49}$$

The spin connection, which generally includes torsion, is coupled to the spin operator $S^{ab} = \frac{1}{2i}\left(\psi_\alpha^a\psi_\alpha^b - \psi_\alpha^b\psi_\alpha^a\right)$ to form the covariant momentum. The generalized fermionic generators are as follows

$$Q_{1\alpha} = \psi_\alpha^a V_a(X), \quad Q_{2\alpha} = \frac{1}{2}\left(\psi_\alpha^a\Pi_a + \tilde{\Pi}_a\psi_\alpha^a\right). \tag{50}$$

The bosonic generators are computed from the closure of the OSp$(n|2)$ commutation relations

$$\{Q_{1\alpha}, Q_{1\beta}\} = \delta_{\alpha\beta}Q_{11}, \quad \{Q_{2\alpha}, Q_{2\beta}\} = \delta_{\alpha\beta}Q_{22}, \quad \{Q_{1\alpha}, Q_{2\beta}\} = \delta_{\alpha\beta}Q_{12} + Q_{\alpha\beta}. \tag{51}$$

where $Q_{\alpha\beta}$ is the antisymmetric SO(n) generator, and Q_{ij} are the symmetric Sp(2) generators. Note that $Q_{2\alpha}$ contains up to cubic terms in the fermions. $\tilde{\Pi}_a$ is given by $\tilde{\Pi}_a = \left(\sqrt{G}\right)^{-1}\Pi_a\sqrt{G}$, where the factors of $\sqrt{G}$ insure hermiticity in a quantum theory with correct factor ordering, but for the invariance of the classical action, where we only need Poisson brackets instead of the commutators as explained in the spinless case, these factors may be neglected.

For simplicity we will impose the flat $Q_{\alpha\beta} = Q^0_{\alpha\beta}$

$$Q_{\alpha\beta} = \frac{i}{2}\psi_{[\alpha} \cdot \psi_{\beta]} \tag{52}$$

but will compute Q_{ij} as a function of the background fields[7]. This condition requires that E^a_M be determined in terms of V^a, ω^{ab}_M

$$E^a_M = D_M V^a = \partial_M V^a + \omega^{ab}_M V_b, \tag{53}$$

while

$$Q_{11} = V^a V^b \eta_{ab}, \quad Q_{12} = \frac{1}{2}\left(V^a \Pi_a + \tilde{\Pi}_a V^a\right), \quad Q_{22} = \frac{1}{n}\left[\frac{1}{2}\left(\psi^a_\alpha \Pi_a + \tilde{\Pi}_a \psi^a_\alpha\right)\right]^2. \tag{54}$$

Note that Q_{22} contains several powers of the fermions. The closure (51) is possible provided the gauge field strength and the curvature are transverse to V

$$V^M F_{MN} = 0, \quad V^M R^{ab}_{MN} = 0, \tag{55}$$

where

$$V^M = E^M_a V^a, \tag{56}$$

and

$$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N], \quad R^{ab}_{MN} = \partial_M \omega^{ab}_N - \partial_N \omega^{ab}_M + [\omega_M, \omega_N]^{ab}. \tag{57}$$

Furthermore, since $E^a_M = D_M V^a$ the torsion is determined in terms of the curvature and V as

$$T^a_{MN} = D_M E^a_N - D_N E^a_M = R^{ab}_{MN} V_b, \tag{58}$$

and is automatically transverse to V provided the curvature is.

There remains to check the $\mathrm{Sp}(2) \times SO(n)$ closure of the bosonic generators. The $\mathrm{SO}(n)$ part is trivial. The Sp(2) part is similar to the purely bosonic case of the previous section and is subject to the same conditions (23)-(25) discussed there. However now W, G^{MN} are given by $W = V^a V_a$ and $G_{MN} = E^a_M E^b_N \eta_{ab}$ and $U = 0$. These forms automatically satisfy (23)-(25) provided E^a_M is of the form (53). In particular, (23) is satisfied as follows

$$V^M = \frac{1}{2} G^{MN} \partial_N W = G^{MN} (D_N V^a) V_a = G^{MN} E^a_N V_a = E^M_b V^b \tag{59}$$

[7]We could have included also $E^M_I W^{ab}_M Q^0_{ab}$ as part of Π_I, with W^{ab}_M a gauge field that acts in the $\mathrm{SO}(n)$ space within $\mathrm{OSp}(n|2)$. In that case we could also introduce a vielbein E^A_a for an internal space. For simplicity we will omit these complications and seek a solution with a "flat" $\mathrm{SO}(n)$ space, implying that the metric in $\mathrm{SO}(n)$ space is δ_{ab} instead of a curved space metric $G_{AB} = E^a_A E^a_B$. Recall that in the final analysis we are interested in imposing $Q_{ab} = 0$ as part of the singlet condition. I the presence of non-singlet background fields such as E^a_A, W^{ab}_M this condition is harder to satisfy.

which agrees with the definition (56). Meanwhile, the homothety condition (25) is equivalent to

$$\pounds_V E_M^a = E_M^a \tag{60}$$

where $\pounds_V E_M^a = V^N D_N E_M^a + \partial_M V^N E_N^a$. This is also satisfied automatically for the geometry constructed above in terms of V^a and ω_M^{ab} as follows

$$\begin{aligned} \pounds_V E_M^a &= V^N D_N E_M^a + \partial_M V^N E_N^a = V^N T_{NM}^a + V^N D_M E_N^a + \partial_M V^N E_N^a &(61)\\ &= V^N T_{NM}^a + D_M \left(V^N E_N^a \right) = V^N T_{NM}^a + D_M V^a &(62)\\ &= E_M^a &(63) \end{aligned}$$

where we have used the orthogonality of V to the curvature or torsion. Related equations appear in [25], but our approach provides a OSp$(n|2)$ gauge symmetry basis for introducing Eq.(53) and the rest of the geometrical equations. Also, a similar problem was discussed in [26] in a less geometrical formalism and in the absence of the gauge field A_M. In our case we are interested in solutions of the equations that permit the imposition of the constraints $Q_{ij} \sim Q_{i\alpha} \sim Q_{\alpha\beta} \sim 0$.

The geometry described by E_M^a is fully determined by the functions $\omega_M^{ab}(X)$ and $V^a(X)$ which are constrained only by the transversality condition $V^M R_{MN}^{ab} = 0$, but are otherwise arbitrary. The solution space includes the most general gravitational metric in d dimensions as already seen in the previous section. The formalism in this section provides a more covariant solution and permits the construction of the general interacting two-time physics for spinning particles.

7 Conclusion and discussion

The choice of coordinates κ, w, x^μ and solution of background fields used above emphasizes a basis that is convenient for deriving the free massless relativistic particle from two time physics in the case of zero background fields. In this basis it was easy to eliminate one timelike and one spacelike coordinates through the gauge choices $\kappa(\tau) = 1$, $p_w = 0$, leaving the usual timelike coordinate as a component of the d-dimensional vector $x^\mu(\tau)$. With this choice of time we interpreted the theory and the background fields, as discussed above. However, as we have already seen in the flat case, other choices of the time coordinate produce very different physical interpretations from the point of view of the one-time observer, even though the two time physics theory is the same. In the general theory it is also possible to work in other coordinates that are convenient to solve the Sp$(2,R)$ constraints in other Sp$(2,R)$ gauges. Then the choice of "time" embedded in the two-time theory is different.

It follows that the *same background fields* given above would give rise to very different interpretation of the dynamics in one-time physics in different Sp$(2,R)$ gauges. For example, in the flat spinless case, with $\gamma = g_{\mu\nu} = W^\mu = A_\mu = u = 0$, different Sp$(2,R)$ gauges produced a class of related one-time dynamics that included the free massless relativistic particle, the free massive relativistic particle, the free massive non-relativistic particle, the H-atom, the harmonic oscillator in one less dimension, the particle in AdS$_{d-k}\times$S^k backgrounds for any $k = 0, 1, \cdots, d-2$, etc. In a similar way, in the general theory all possible choices of time define a class of one-time dynamical theories related to the same two-time dynamics with a *fixed set of background fields*. Changing the background fields changes the class of related one-time dynamical models.

In the flat case the global symmetry was SO$(d,2)$. In the general case the Killing symmetries of the background fields (which is embedded in the general coordinate and gauge transformations) replaces the global SO$(d,2)$ symmetry. The global symmetries should be realized in the same representation for all of the different one-time dynamical models in the same class derived from the same two-time physics theory.

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