THE POLYNOMIAL PROPERTY (V)

MANUEL GONZÁLEZ AND JOAQUÍN M. GUTIÉRREZ

Abstract. Given Banach spaces E and F, we denote by $\mathcal{P}(^kE,F)$ the space of all k-homogeneous (continuous) polynomials from E into F, and by $\mathcal{P}_{\rm wb}(^kE,F)$ the subspace of polynomials which are weak-to-norm continuous on bounded sets. It is shown that if E has an unconditional finite dimensional expansion of the identity, the following assertions are equivalent: (a) $\mathcal{P}(^kE,F)=\mathcal{P}_{\rm wb}(^kE,F)$; (b) $\mathcal{P}_{\rm wb}(^kE,F)$ contains no copy of c_0; (c) $\mathcal{P}(^kE,F)$ contains no copy of ℓ_∞; (d) $\mathcal{P}_{\rm wb}(^kE,F)$ is complemented in $\mathcal{P}(^kE,F)$. This result was obtained by Kalton for linear operators. As an application, we show that if E has Pełczyński's property (V) and satisfies $\mathcal{P}(^kE)=\mathcal{P}_{\rm wb}(^kE)$ then, for all F, every unconditionally converging $P\in\mathcal{P}(^kE,F)$ is weakly compact. If E has an unconditional finite dimensional expansion of the identity, then the converse is also true.

Given two Banach spaces E and F, we denote by $\mathcal{P}(^kE,F)$ the space of all k-homogeneous (continuous) polynomials from E into F, and by $\mathcal{P}_{\rm wb}(^kE,F)$ the subspace of polynomials which are weak-to-norm continuous on bounded sets. This subspace has been studied by many authors: see, for instance, [3, 4, 17, 19]. Clearly, every polynomial in $\mathcal{P}_{\rm wb}(^kE,F)$ takes bounded sets into relatively compact sets. Observe that $\mathcal{P}(^1E,F)=\mathcal{L}(E,F)$, the space of (linear bounded) operators from E into F, and that $\mathcal{P}_{\rm wb}(^1E,F)=\mathcal{K}(E,F)$, the space of compact operators. The spaces $\mathcal{P}(^0E,F)$ and $\mathcal{P}_{\rm wb}(^0E,F)$ may be identified with F.

Kalton studied in [23] the structure of the space $\mathcal{K}(E,F)$. In the present paper, we obtain versions of his results for the space $\mathcal{P}_{\rm wb}(^kE,F)$, showing that $\mathcal{P}_{\rm wb}(^kE,F)$ contains a copy of ℓ_∞ if and only if either E contains a complemented copy of ℓ_1 or F contains a copy of ℓ_∞. We also prove that, for E having an unconditional finite dimensional expansion of the identity, the following assertions are equivalent: (a) $\mathcal{P}(^kE,F)=\mathcal{P}_{\rm wb}(^kE,F)$; (b) $\mathcal{P}_{\rm wb}(^kE,F)$ contains no copy of c_0; (c) $\mathcal{P}(^kE,F)$ contains no copy of ℓ_∞; (d) $\mathcal{P}_{\rm wb}(^kE,F)$ is complemented in $\mathcal{P}(^kE,F)$.

As an application, we prove that, if E has property (V) (the definitions are given below) and $\mathcal{P}(^kE)=\mathcal{P}_{\rm wb}(^kE)$, then every k-homogeneous unconditionally converging polynomial on E is weakly compact. If E has an unconditional finite dimensional expansion of the identity, then the converse is also true.

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Throughout, E and F will denote Banach spaces, B_E is the closed unit ball of E and S_E is the unit sphere of E; E^* will be the dual of E. The set of natural numbers is denoted by $\mathbb{N}$. As usual, (e_n) stands for the unit vector basis of c_0.

A formal series $\sum x_n$ in E is *weakly unconditionally Cauchy* (*w.u.C.,* for short) if, for every $\phi \in E^*$, we have $\sum |\phi(x_n)| < +\infty$. Equivalent definitions may be seen in [7, Theorem V.6]. The series is *unconditionally convergent* if every subseries converges. Equivalent definitions may be seen in [8, Theorem 1.9].

A polynomial $P \in \mathcal{P}(^kE,F)$ is *unconditionally converging* [15, 16] if, for each w.u.C. series $\sum x_n$ in E, the sequence $\left(P\left(\sum_{i=1}^n x_i\right)\right)_n$ is convergent in F. The space of all unconditionally converging polynomials is denoted by $\mathcal{P}_{\mathrm{uc}}(^kE,F)$. This class has been very useful for obtaining polynomial characterizations of Banach space properties (see [21]). We say that $P \in \mathcal{P}(^kE,F)$ is *(weakly) compact* if $P(B_E)$ is relatively (weakly) compact in F. Every weakly compact polynomial is unconditionally converging. For the general theory of polynomials on Banach spaces, we refer to [9, 26].

To each polynomial $P \in \mathcal{P}(^kE,F)$ we can associate a unique symmetric k-linear mapping $\hat{P} : E \times \overset{(k)}{\cdots} \times E \to F$ so that $P(x) = \hat{P}(x,\dots,x)$ and an operator $T_P : E \to \mathcal{P}(^{k-1}E,F)$ given by $T_P(x)(y) := \hat{P}(x, y, \overset{(k-1)}{\dots}, y)$. It is well known that $P \in \mathcal{P}_{\mathrm{wb}}(^kE,F)$ if and only if T_P is compact [3, Theorem 2.9].

Denote by $\mathcal{L}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$ the space of all operators $C : E \to \mathcal{P}(^{k-1}E,F)$ such that

$$(C(x))^\wedge(y_1,\dots,y_{k-1}) = (C(y_1))^\wedge(x,y_2,\dots,y_{k-1}) \qquad (x,y_1,\dots,y_{k-1} \in E),$$

where $(C(x))^\wedge$ stands for the symmetric $(k-1)$-linear mapping associated to $C(x)$.

Proposition 1. *The mapping $T : \mathcal{P}(^kE,F) \to \mathcal{L}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$ given by $T(P) = T_P$ is a surjective linear isomorphism.*

Proof. Clearly, T is well defined, linear and injective. Since

$$\|T_P\| = \sup_{x \in B_E} \|T_P(x)\| = \sup_{x,y \in B_E} \|\hat{P}(x,y,\overset{(k-1)}{\dots},y)\| \le \|\hat{P}\| \le \frac{k^k}{k!}\,\|P\|$$

[26, Theorem 2.2], we have that T is continuous. To see that it is surjective, take $C \in \mathcal{L}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$, and define $A : E \times \overset{(k)}{\cdots} \times E \to F$ by $A(y_1,\dots,y_k) := (C(y_1))^\wedge(y_2,\dots,y_k)$, and $P(x) = A(x,\overset{(k)}{\dots},x)$. Then

$$T_P(x)(y) = A(x,y,\overset{(k-1)}{\dots},y) = C(x)(y) \qquad (x,y \in E).$$

Hence, $T_P = C$. □

The subspace of all operators in $\mathcal{L}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$ which are compact (resp. weakly compact) will be denoted by $\mathcal{K}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$ (resp. $\mathcal{W}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$). Given $C \in \mathcal{K}_{\mathrm{s}}(E,\mathcal{P}(^{k-1}E,F))$, the symmetry of C easily implies that $C(E) \subseteq \mathcal{P}_{\mathrm{wb}}(^{k-1}E,F)$.

The proof of [5, Proposition 5.3] gives:

Proposition 2. *For $n \ge m$, the space $\mathcal{P}(^mE,F)$ (resp. $\mathcal{P}_{\mathrm{wb}}(^mE,F)$) is isomorphic to a complemented subspace of $\mathcal{P}(^nE,F)$ (resp. $\mathcal{P}_{\mathrm{wb}}(^nE,F)$).*

Theorem 3. *The space $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ contains a copy of ℓ_∞ if and only if either F contains a copy of ℓ_∞ or E contains a complemented copy of ℓ_1.*

Proof. If $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ contains a copy of ℓ_∞, a fortiori the space $\mathcal{K}\left(E,\mathcal{P}_{\mathrm{wb}}({}^{k-1}E,F)\right)$ contains it. Therefore [23, Theorem 4], either E contains a complemented copy of ℓ_1 or $\mathcal{P}_{\mathrm{wb}}({}^{k-1}E,F)$ contains a copy of ℓ_∞. Repeating the process, we conclude that either E contains a complemented copy of ℓ_1 or $\mathcal{P}_{\mathrm{wb}}({}^0E,F)\equiv F$ contains a copy of ℓ_∞.

Conversely, if F contains a copy of ℓ_∞, since $F\equiv\mathcal{P}_{\mathrm{wb}}({}^0E,F)$ is isomorphic to a subspace of $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ (Proposition 2), we obtain that $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ contains a copy of ℓ_∞. If E contains a complemented copy of ℓ_1, then E^* contains a copy of ℓ_∞; since $E^*=\mathcal{P}_{\mathrm{wb}}({}^1E)$ is isomorphic to a subspace of $\mathcal{P}_{\mathrm{wb}}({}^1E,F)$, which is in turn isomorphic to a subspace of $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$, we obtain that the latter contains a copy of ℓ_∞. $\square$

The proof of [23, Lemma 2] yields:

Lemma 4. *Assume E is separable, $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ is complemented in $\mathcal{P}({}^kE,F)$, and an operator $\Phi:\ell_\infty\to\mathcal{P}({}^kE,F)$ is given with the following properties:*

(a) *$\Phi(e_n)\in\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ for all n;*

(b) *the subset $\{\Phi(\xi)(x):\xi\in\ell_\infty, x\in E\}\subset F$ is separable.*

Then, for every infinite subset $M\subseteq\mathbb{N}$, there exists an infinite subset $M_0\subseteq M$ with $\Phi(\xi)\in\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ for all $\xi\in\ell_\infty(M_0)$.

Lemma 5. *Suppose E contains a complemented copy of ℓ_1. Then $\mathcal{P}_{\mathrm{wb}}({}^kE,F)$ is uncomplemented in $\mathcal{P}({}^kE,F)$ for every F and $k\in\mathbb{N}$ $(k>1)$.*

Proof. As in [23, Lemma 3], we can reduce the problem to the case $E=\ell_1$.

Fix $v\in S_F$ and define the operator

$$\Phi:\ell_\infty\longrightarrow\mathcal{P}({}^k\ell_1,F)$$

by

$$\Phi(\xi)(x)=\sum_{i=1}^\infty \xi_i x_i^k v \quad \text{for } \xi=(\xi_i)_{i=1}^\infty\in\ell_\infty \text{ and } x=(x_i)_{i=1}^\infty\in\ell_1.$$

Since

$$\|\Phi(\xi)\|=\sup_{x\in B_{\ell_1}}\left\|\sum_{i=1}^\infty \xi_i x_i^k v\right\|\le\|\xi\|\cdot\sup_{x\in B_{\ell_1}}\sum_{i=1}^\infty\left|x_i^k\right|=\|\xi\|,$$

Φ is continuous (easily, it is even an isometric embedding).

We claim that $\Phi(\xi)\in\mathcal{P}_{\mathrm{wb}}({}^k\ell_1,F)$ if and only if $\xi\in c_0$. Indeed, let

$$T_{\Phi(\xi)}:\ell_1\longrightarrow\mathcal{P}({}^{k-1}\ell_1,F)$$

be the associated operator given by

$$T_{\Phi(\xi)}(x)(y)=\sum_{i=1}^\infty \xi_i x_i y_i^{k-1} v \qquad \text{for } x=(x_i), y=(y_i)\in\ell_1.$$

Since, for $n \neq m$,

$$\left\| \left(T_{\Phi(\xi)}(e_n) - T_{\Phi(\xi)}(e_m)\right)(y) \right\| =$$

$$\left\| T_{\Phi(\xi)}(e_n - e_m)(y) \right\| = \left\| \sum_{i=1}^{\infty} \xi_i(\delta_{in} - \delta_{im}) y_i^{k-1} v \right\| = \left| \xi_n y_n^{k-1} - \xi_m y_m^{k-1} \right|,$$

we have

$$\max\{|\xi_n|, |\xi_m|\} \leq \left\| T_{\Phi(\xi)}(e_n) - T_{\Phi(\xi)}(e_m) \right\| \leq |\xi_n| + |\xi_m|,$$

it follows that $T_{\Phi(\xi)}$ is compact if and only if $\xi \in c_0$ (see [7, Exercise VII.5]).

By Lemma 4, there is an infinite $M \subseteq \mathbb{N}$ such that $\Phi(\xi) \in \mathcal{P}_{\mathrm{wb}}({}^k\ell_1, F)$ for all $\xi \in \ell_\infty(M)$, which contradicts the above claim. □

In the linear case ($k = 1$), Kalton needs to assume that F is infinite dimensional for the validity of Lemma 5 [23, Lemma 3]. Taking $k > 1$, we can drop this condition. Observe that, if $k = 1$ and $\dim(F) < \infty$, we have

$$\mathcal{P}_{\mathrm{wb}}({}^1E, F) = \mathcal{K}(E, F) = \mathcal{L}(E, F) = \mathcal{P}({}^1E, F),$$

so the conclusion of the Lemma is not true.

An *unconditional finite dimensional expansion of the identity* for a Banach space E is a sequence of finite dimensional operators $A_n : E \to E$ such that for each $x \in E$,

$$x = \sum_{n=1}^{\infty} A_n(x)$$

unconditionally. This condition is a bit more general than having an unconditional basis [22].

Lemma 6. *Suppose E has an unconditional finite dimensional expansion of the identity and let $P \in \mathcal{P}({}^kE, F)$. Then there is a w.u.C. series $\sum P_i$ in $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ such that, for all $x \in E$, $P(x) = \sum_{m=1}^{\infty} P_m(x)$ unconditionally.*

Proof. There is a sequence $(A_i) \subset \mathcal{K}(E, E)$ such that, for every $x \in E$, we have $x = \sum_{i=1}^{\infty} A_i(x)$ unconditionally. Then,

$$\begin{aligned} P\left(\sum_{i=1}^{n} A_i(x)\right) &= \sum_{i_1,\ldots,i_k=1}^{n} \hat{P}\left(A_{i_1}(x), \ldots, A_{i_k}(x)\right) \\ &= \sum_{m=1}^{n} \left(\sum_{\max\{i_1,\ldots,i_k\}=m} \hat{P}\left(A_{i_1}(x), \ldots, A_{i_k}(x)\right) \right) \\ &= \sum_{m=1}^{n} P_m(x) \end{aligned}$$

where $P_m \in \mathcal{P}_{\mathrm{wb}}({}^kE, F)$.

Choosing finite subsets $I_1, \ldots, I_k$ of integers, we have

$$\begin{aligned}\left\| \sum_{i_1 \in I_1, \ldots, i_k \in I_k} \hat{P}\left(A_{i_1}(x), \ldots, A_{i_k}(x)\right) \right\| &= \left\| \hat{P}\left(\sum_{i_1 \in I_1} A_{i_1}(x), \ldots, \sum_{i_k \in I_k} A_{i_k}(x) \right) \right\| \\ &\leq \|\hat{P}\| \cdot \left\| \sum_{i_1 \in I_1} A_{i_1}(x) \right\| \cdot \ldots \cdot \left\| \sum_{i_k \in I_k} A_{i_k}(x) \right\|.\end{aligned}$$

Hence, the series

$$\sum_{i_1, \ldots, i_k = 1}^{\infty} \hat{P}\left(A_{i_1}(x), \ldots, A_{i_k}(x)\right)$$

is unconditionally convergent for all $x \in E$ [8, Theorem 1.9]. Therefore, $P(x) = \sum_{m=1}^{\infty} P_m(x)$ unconditionally.

Moreover, by the uniform boundedness principle [26, Theorem 2.6], we have

$$\sup_{I \subset \mathbb{N} \text{ finite}} \left\| \sum_{i \in I} P_i \right\| < \infty.$$

So $\sum P_i$ is w.u.C. in $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ [7, Theorem V.6]. □

Theorem 7. *Suppose E has an unconditional finite dimensional expansion of the identity and let $k \in \mathbb{N}$ $(k > 1)$. Then the following assertions are equivalent:*

(a) $\mathcal{P}({}^kE, F) = \mathcal{P}_{\mathrm{wb}}({}^kE, F)$;

(b) $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ *contains no copy of* c_0;

(c) $\mathcal{P}({}^kE, F)$ *contains no copy of* ℓ_∞;

(d) $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ *is complemented in* $\mathcal{P}({}^kE, F)$.

Proof. (a) ⇒ (c): Assume $\mathcal{P}({}^kE, F) = \mathcal{P}_{\mathrm{wb}}({}^kE, F)$ contains a copy of ℓ_∞. By Theorem 3, either E contains a complemented copy of ℓ_1 or F contains a copy of ℓ_∞. Lemma 5 implies that E contains no complemented copy of ℓ_1, so F contains a copy of ℓ_∞. Take a normalized basic sequence $(x_i) \subset E$ and a bounded sequence of coefficient functionals $(\phi_n) \subset E^*$ $(\phi_i(x_j) = \delta_{ij})$. Define $P \in \mathcal{P}({}^kE, \ell_\infty)$ by $P(x) := \left(\phi_n(x)^k\right)_n$. Then, for $i \neq j$, we have $\|P(x_i) - P(x_j)\| \geq |\phi_i(x_i)^k - \phi_i(x_j)^k| = 1$. Hence, denoting by $J : \ell_\infty \to F$ an isomorphism, we get $J \circ P \in \mathcal{P}({}^kE, F) \backslash \mathcal{P}_{\mathrm{wb}}({}^kE, F)$, a contradiction.

(c) ⇒ (b): The proof is the same of the linear case [23, Theorem 6] with slight modifications.

(b) ⇒ (a): Take $P \in \mathcal{P}({}^kE, F)$. Consider the w.u.C. series $\sum P_i$ given by Lemma 6 with $P_i \in \mathcal{P}_{\mathrm{wb}}({}^kE, F)$. Since this space contains no copy of c_0, the series is unconditionally convergent [7, Theorem V.8]. Clearly, its sum must be P. Since the space $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ is closed, we conclude that $P \in \mathcal{P}_{\mathrm{wb}}({}^kE, F)$.

(a) ⇒ (d) is trivial.

(d) ⇒ (a): If $\mathcal{P}_{\mathrm{wb}}({}^kE, F)$ is complemented in $\mathcal{P}({}^kE, F)$, Lemma 5 implies that E contains no complemented copy of ℓ_1. Suppose there is $P \in \mathcal{P}({}^kE, F) \backslash \mathcal{P}_{\mathrm{wb}}({}^kE, F)$. By Lemma 6, we can find a sequence $(P_i) \subset \mathcal{P}_{\mathrm{wb}}({}^kE, F)$ so that $P(x) = \sum_{i=1}^{\infty} P_i(x)$ unconditionally for each $x \in E$, and $\sum P_i$ is w.u.C. but is not unconditionally convergent since $P \notin \mathcal{P}_{\mathrm{wb}}({}^kE, F)$. Hence, we can find $\epsilon > 0$, and an increasing sequence (m_j) of integers such that, for each j, the polynomial

$$C_j := \sum_{i=m_j+1}^{m_{j+1}} P_i$$

satisfies $\|C_j\| > \epsilon$.

Define now $\Phi : \ell_\infty \to \mathcal{P}({}^kE, F)$ by $\Phi(\xi)(x) = \sum \xi_j C_j(x)$ for $\xi = (\xi_j)_{j=1}^\infty \in \ell_\infty$ and $x \in E$. Since the series $\sum P_i(x)$ is unconditionally convergent, so is the series $\sum \xi_j C_j(x)$. The set $\{\Phi(\xi)(x) : \xi \in \ell_\infty, x \in E\}$ is contained in the closed linear span of $\{C_j(E) : j \in \mathbb{N}\}$ which is separable by the compactness of C_j. On the other hand, $\Phi(e_j) = C_j \in \mathcal{P}_{\mathrm{wb}}({}^kE, F)$ for all j. By Lemma 4, there is an infinite subset $M \subset \mathbb{N}$ such that $\Phi(\xi) \in \mathcal{P}_{\mathrm{wb}}({}^kE, F)$ for each $\xi \in \ell_\infty(M)$.

Therefore, for each $\xi \in \ell_\infty(M)$, the series $\sum \xi_j C_j$ is weak subseries convergent. The Orlicz-Pettis theorem then implies that it is unconditionally convergent. In particular,

$$\lim_{\substack{j \in M \\ j \to \infty}} \|C_j\| = 0,$$

a contradiction. $\square$

Observe that the unconditional finite dimensional expansion of the identity is used only in (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (a).

In the linear case ($k = 1$), the restriction $\dim(F) = \infty$ is required for the validity of Theorem 7 [23, Theorem 6].

Remark 8. In order to highlight the difference between Theorem 3 and Theorem 7, let us consider the spaces $\mathcal{P}({}^k\ell_p, \ell_q)$ and $\mathcal{P}_{\mathrm{wb}}({}^k\ell_p, \ell_q)$ for $1 < p, q < \infty$ and $k > 1$.

If $kq > p$, then the space $\mathcal{P}({}^k\ell_p, \ell_q)$ contains a copy of ℓ_∞. Indeed, define $J : \ell_\infty \to \mathcal{P}({}^k\ell_p, \ell_q)$ by

$$J(\xi)(x) = \left(\xi_i x_i^k\right)_{i=1}^\infty .$$

Since, for all n,

$$|\xi_n| = \|J(\xi)(e_n)\| \le \|J(\xi)\| = \sup_{x \in B_{\ell_p}} \|J(\xi)(x)\| \le \|\xi\|,$$

we get that J is an isometric embedding.

However, the space $\mathcal{P}_{\mathrm{wb}}({}^k\ell_p, \ell_q)$ contains no copy of ℓ_∞ since it is separable. In fact, it is the norm closure of the space of all finite type polynomials generated by the mappings of the form $\phi^n \otimes y$, for $\phi \in (\ell_p)^*$, $y \in \ell_q$ [4, Proposition 2.7].

On the other hand, if $\xi \in c_0$, then $J(\xi)$ is in the closure of the finite type polynomials, so $J(\xi) \in \mathcal{P}_{\mathrm{wb}}({}^k\ell_p, \ell_q)$. Hence, this space contains $J(c_0)$, an isometric copy of c_0.

Recall moreover that, if $kq < p$, then the space $\mathcal{P}({}^k\ell_p, \ell_q)$ is reflexive [1, 4.3].

Every polynomial $P \in \mathcal{P}({}^kE, F)$ has an extension, called *the Aron-Berner extension*, to a polynomial $\overline{P} \in \mathcal{P}({}^kE^{**}, F^{**})$ (see [2, 18, 21]). If $P \in \mathcal{P}_{\mathrm{wb}}({}^kE)$, then $\overline{P} \in \mathcal{P}({}^kE^{**})$ is weak-star continuous on bounded sets [25].

Recall that a Banach space has *property (V)*, introduced in [27], if every unconditionally converging operator on E is weakly compact. Every $C(K)$ space has property (V) [27].

We shall need the following result:

Theorem 9. [21] *The following assertions are equivalent:*

(a) *The space E has property (V);*

(b) *for all F and $k \in \mathbb{N}$, the Aron-Berner extension of every $P \in \mathcal{P}_{\mathrm{uc}}({}^kE, F)$ is F-valued;*

(c) *There is $k \in \mathbb{N}$ such that, for all F, the Aron-Berner extension of every $P \in \mathcal{P}_{\mathrm{uc}}({}^kE, F)$ is F-valued.*

Easily, if a polynomial is weakly compact, then its Aron-Berner extension is F-valued [6].

Theorem 10. *For $k \in \mathbb{N}$, consider the assertions:*

(a) *E has property (V) and $\mathcal{P}({}^kE) = \mathcal{P}_{\mathrm{wb}}({}^kE)$;*

(b) *for each F, every $P \in \mathcal{P}_{\mathrm{uc}}({}^kE, F)$ is weakly compact.*

Then (a) $\Rightarrow$ (b). *If, moreover, E has an unconditional finite dimensional expansion of the identity, then* (b) $\Rightarrow$ (a).

Proof. (a) $\Rightarrow$ (b): Given $P \in \mathcal{P}_{\mathrm{uc}}({}^kE, F)$, by Theorem 9, the range of its Aron-Berner extension $\overline{P}$ is contained in F. Take a net $(x_\alpha) \subset B_E$. We can assume that (x_α) is weak Cauchy and so it converges in the weak-star topology to some $z \in E^{**}$.

Let $\psi \in F^*$. Then $\psi \circ P \in \mathcal{P}_{\mathrm{wb}}({}^kE)$ and so,

$$\psi \circ P(x_\alpha) = \psi \circ \overline{P}(x_\alpha) = \overline{\psi \circ P}(x_\alpha) \longrightarrow \overline{\psi \circ P}(z) = \psi \circ \overline{P}(z).$$

Therefore, the net $(P(x_\alpha))$ is weakly convergent to $\overline{P}(z) \in F$. So, $P(B_E)$ is relatively weakly compact.

(b) $\Rightarrow$ (a): By the comment preceding this Theorem, and by Theorem 9, (b) implies that E has property (V). Also, every polynomial in $\mathcal{P}({}^kE, \ell_1)$ is compact. From this, we obtain that $\mathcal{P}({}^kE)$ contains no copy of c_0 [10, Corollary 8]. A fortiori, $\mathcal{P}_{\mathrm{wb}}({}^kE)$ contains no copy of c_0. Since E has an unconditional finite dimensional expansion of the identity, we conclude from Theorem 7 that $\mathcal{P}_{\mathrm{wb}}({}^kE) = \mathcal{P}({}^kE)$. □

We do not know if the condition on the existence of an unconditional finite dimensional expansion of the identity may be removed from Theorem 10. In fact, if (b) is satisfied, since $\mathcal{P}({}^{k-1}E)$ contains no copy of c_0 and E has property (V), we have

$$\begin{aligned} \mathcal{P}({}^kE) &= \mathcal{L}_{\mathrm{s}}\left(E, \mathcal{P}({}^{k-1}E)\right) = \mathcal{W}_{\mathrm{s}}\left(E, \mathcal{P}({}^{k-1}E)\right), \\ \mathcal{P}_{\mathrm{wb}}({}^kE) &= \mathcal{K}_{\mathrm{s}}\left(E, \mathcal{P}({}^{k-1}E)\right). \end{aligned}$$

So, we only have to show that $\mathcal{K}_{\mathrm{s}}\left(E, \mathcal{P}({}^{k-1}E)\right) = \mathcal{W}_{\mathrm{s}}\left(E, \mathcal{P}({}^{k-1}E)\right)$. There are many conditions on E (apart from the existence of an unconditional finite dimensional expansion of the identity) that imply this equality [13, 14], and it is not known if there are Banach spaces X, Y such that $\mathcal{K}(X, Y) \neq \mathcal{W}(X, Y)$ while $\mathcal{K}(X, Y)$ contains no copy of c_0 [11].

Recall that the condition $\mathcal{P}({}^kE) = \mathcal{P}_{\mathrm{wb}}({}^kE)$ implies that E contains no copy of ℓ_1 [20].

It is proved in [15] that if $\mathcal{P}({}^kE) = \mathcal{P}_{\mathrm{wb}}({}^kE)$ and E has property (u) then, for all F, every $P \in \mathcal{P}_{\mathrm{uc}}({}^kE, F)$ is weakly compact. Part (a) $\Rightarrow$ (b) of Theorem 10 is stronger than the result of [15]. The latter cannot be applied, for instance, to the space $c_0\hat{\otimes}_\pi c_0$ which fails property (u) [24] while it does have property (V) [12]. For the definition of property (u) and its relationship to property (V), see [27]. Recall in particular that, if E has property (u) and contains no copy of ℓ_1, then E has property (V) [27, Proposition 2].

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander (Spain)
E-mail address: gonzalem@ccaix3.unican.es

Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid (Spain)
E-mail address: jgutierrez@etsii.upm.es