TOPOLOGICAL AE(0)-GROUPS

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Abstract. We investigate topological AE(0)-groups class of which contains the class of Polish groups as well as the class of all locally compact groups. We establish the existence of an universal AE(0)-group of a given weight as well as the existence of an universal action of AE(0)-group of a given weight on a AE(0)-space of the same weight. A complete characterization of closed subgroups of powers of the symmetric group S_∞ is obtained. It is also shown that every AE(0)-group is Baire isomorphic to the product of Polish groups. These results are obtained by using the spectral descriptions of AE(0)-groups which are presented in Section 3.

1. Introduction

One of the main structure theorems for compact groups (see [10, Chapters 6 & 9]) can be formulated as follows.

Theorem A ([10], Theorem 9.24(ii))**.** *Let G be a connected compact group. Then there exists a continuous homomorphism*

$$p\colon Z_0(G) \times \prod\{L_t \colon t \in T\} \to G,$$

where $Z_0(G)$ stands for the identity component of the center of G, and L_t is a simple, connected and simply connected compact Lie group, $t \in T$, such that $\ker(p)$ *is a zero-dimensional central subgroup of $Z_0(G) \times \prod\{L_t \colon t \in T\}$.*

The above statement clearly shows that the classes of zero-dimensional groups, abelian groups and simple, simply connected Lie groups play a central role in the general theory of compact groups. Recently these classes of topological groups have been studied from the point of view of absolute extensors in dimension n (see [4] for a comprehensive introduction into the theory of absolute extensors in dimension n). The following three statements show that such an approach is quite effective.

Theorem B ([2])**.** *The following conditions are equivalent for a zero-dimensional topological group G:*

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(a) *G is topologically equivalent to the product $(\mathbb{Z}_2)^\tau \times \mathbb{Z}^\kappa$.*
(b) *G is an $AE(0)$-space.*

Theorem C ([6], Theorem E)**.** *The following conditions are equivalent for a compact abelian group G:*

(a) *G is a torus group (both in topological and algebraic senses).*
(b) *G is an $AE(1)$-compactum.*

Theorem D ([5], Corollary 1)**.** *The following conditions are equivalent for a non-trivial compact group G:*

(a) *G is a simple, connected and simply connected Lie group.*
(b) *G is an $AE(2)$-group with $\pi_3(G) = \mathbb{Z}$.*

The full classification problem for non-abelian (see Theorem C) and for non-simply connected compact AE(1)-groups remains open. On the other hand, the following two statements provide a complete classification of simply connected compact AE(1)-groups.

Theorem E ([5], Theorem C)**.** *The following conditions are equivalent for a compact group G:*

(a) *G is a simply connected $AE(1)$-compactum.*
(b) *G is an $AE(2)$-compactum.*
(c) *G is an $AE(3)$-compactum.*
(d) *G is a product of simple, connected and simply connected compact Lie groups.*

Theorem F ([5], Corollary 2)**.** *There is no non-trivial compact $AE(4)$-group.*

We complete this brief survey by pointing out that, as shows the following statement, for locally compact groups the restriction of being AE(0)-group is purely formal.

Theorem G (Pontryagin-Haydon, [14], [8]; [4])**.** *Every locally compact group is an $AE(0)$-space.*

Below we study AE(0)-groups. The class of AE(0)-groups contains the class of all (generally speaking, non-metrizable) locally compact groups (Theorem G) as well as the class of all Polish groups. Actually the class of Polish groups coincides with the class of metrizable AE(0)-groups and forms a foundation of the entire theory of AE(0)-groups. We hope that results presented in Section 3 do indicate a potential for a non-trivial theory of AE(0)-groups which unifies and generalizes theories of locally compact and Polish groups (thus offering a possible approach to the corresponding question posed in [13]).

In Section 3 we present a spectral characterization of AE(0)-groups in terms of well ordered continuous inverse spectra (Theorem 3.4). This characterization states that a non-metrizable topological group G of weight τ is a AE(0)-group if and only if it is the limit of a well ordered continuous inverse system

$\mathcal{S}_G = \{G_\alpha, p_\alpha^{\alpha+1}, \alpha < \tau\}$ of length τ, consisting of AE(0)-groups G_α and 0-soft limit homomorphisms $p_\alpha^{\alpha+1} \colon G_{\alpha+1} \to G_\alpha$, $\alpha < \tau$, so that G_0 is a Polish group and each homomorphism $p_\alpha^{\alpha+1}$, $\alpha < \tau$, has a Polish kernel.

Obviously this result can not be accepted as the one providing a satisfactory reduction of the non-metrizable case to the Polish one. Of course, everything is fine if the weight of G is ω_1 – in such a case all G_α's, $\alpha < \omega_1$, (and not only the very first one, i.e. G_0) are indeed Polish. But if the weight of G is greater than ω_1, then all G_α's, with $\alpha \geq \omega_1$, are non-metrizable.

In order to achieve our final goal and complete the reduction, we analyze 0-soft homomorphisms with Polish kernels between (generally speaking, non-metrizable) AE(0)-groups. A characterization of such homomorphisms, which is recorded in Proposition 3.6, states that a 0-soft homomorphism $f \colon G \to L$ of AE(0)-groups has a Polish kernel if and only if there exists a pullback diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & L \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ G_0 & \xrightarrow{f_0} & L_0, \end{array}$$

where G_0 and L_0 are Polish groups and the homomorphisms $p \colon G \to G_0$ and $q \colon L \to L_0$ are 0-soft. Theorem 3.4 and Proposition 3.6 together complete the required reduction.

Polish groups and their actions have been extensively studied in a variety of directions (ergodic theory, group representations, operator algebras; see [1] for further discussion and references). Some of the central themes of the theory of Polish groups counterparts of which (for arbitrary AE(0)-groups) are considered below are the existence of universal groups, the existence of universal actions and characterization of closed subgroups of the symmetric group S_∞.

In Section 4 we use above mentioned spectral characterizations and present extensions of some of these results for AE(0)-groups. We prove the existence of universal AE(0)-groups of a given weight (Proposition 4.1) and the existence of universal actions of AE(0)-groups of a given weight on compact AE(0)-spaces of the same weight (Theorem 4.2).

Theorem 4.4 characterizes AE(0)-groups which are isomorphic to closed subgroups of powers S_∞^τ of the symmetric group S_∞ - the group of all bijections of $\mathbb{N}$ under the relative topology inherited from $\mathbb{N}^\mathbb{N}$. This result extends the corresponding observation [1, Theorem 1.5.1] for the group S_∞ itself. As a corollary (Corollary 4.5) we note that if a Polish group G can be embedded (as a closed subgroup) into S_∞^τ for some τ, then G can be embedded into

S_∞ as well. In light of [7] this shows that there exist zero-dimensional Polish groups which can not be embedded into S^τ_∞ as closed subgroups for any cardinal number τ.

Finally we use Theorem 3.4 to prove (Theorem 4.8) that every AE(0)-group is Baire isomorphic to the product of Polish groups. In light of Theorem G this result appears to be new even for compact groups.

2. Preliminaries

All topological spaces below are assumed to be Tychonov (i.e. completely regular and Hausdorff) and all maps (except in Subsection 4.3) are continuous. We consider only Lebesgue dimension dim. Definitions of concepts related to inverse spectra can be found in [4]. $\mathbb{R}$ denotes the real line and $\mathbb{Q}$ stand for the Hilbert cube.

2.1. **Definitions of AE(n)-spaces and n-soft maps.** A comprehensive introduction into general theory of AE(n)-spaces and n-soft maps can be found in [4]. $C(X)$ denotes the set of all continous real-valued functions defined on X.

Definition 2.1. A space X is called an absolute extensor in dimension n (shortly, AE(n)-space), $n = 0, 1, \dots$, if for each at most n-dimensional space Z and each subspace Z_0 of Z, any map $f\colon Z_0 \to X$, such that $C(f)(C(X)) \subseteq \{\varphi|Z_0\colon \varphi \in C(Z)\}$, can be extended to Z.

For compact spaces this definition is equivalent to the standard one.

Proposition 2.2. *A compact space X is a AE(n)-space if and only if for each at most n-dimensional compactum Z and for each closed subspace Z_0 of Z, any map $f\colon Z_0 \to X$ has an extension to Z.*

It is known [4, Chapter 6] that the class of metrizable AE(0)-spaces coincides with the class of Polish spaces. Every AE(0)-spaces has a countable Suslin number.

Definition 2.3. A map $f\colon X \to Y$ between AE(n)-spaces is n-soft if and only if for each at most n-dimensional realcompact space Z, for its closed subspace Z_0, and for any two maps $g\colon Z_0 \to X$ and $h\colon Z \to Y$ such that $f \circ g = h|Z_0$ and $C(g)(C(X)) \subseteq \{\varphi|Z_0\colon \varphi \in C(Z)\}$, there exists a map $k\colon Z \to X$ such that $k|Z_0 = g$ and $f \circ k = h$.

It is easy to check that X is AE(n)-space if and only if the constant map $X \to \{\mathrm{pt}\}$ is n-soft. It is important to note that every 0-soft map between AE(0)-spaces is surjective and open ([4, Lemma 6.1.13 & Proposition 6.1.26]) and that for surjections between Polish spaces the converse of this fact is also true.

We say (see [4, Section 6.3]) that a map $f\colon X \to Y$ has a Polish kernel if there exists a Polish space P such that X is C-embedded in the product $Y \times P$ so that f coincides with the restriction $\pi_Y|X$ of the projection $\pi_Y\colon Y \times P \to Y$. Obviously any map between Polish spaces has a Polish kernel.

2.2. **Set-theoretical preliminaries.** Let A be a partially ordered directed set (i.e. for every two elements $\alpha, \beta \in A$ there exists an element $\gamma \in A$ such that $\gamma \geq \alpha$ and $\gamma \geq \beta$). We say that a subset $A_1 \subseteq A$ of A majorates another subset $A_2 \subseteq A$ of A if for each element $\alpha_2 \in A_2$ there exists an element $\alpha_1 \in A_1$ such that $\alpha_1 \geq \alpha_2$. A subset which majorates A is called cofinal in A. A subset of A is said to be a chain if every two elements of it are comparable. The symbol $\sup B$, where $B \subseteq A$, denotes the lower upper bound of B (if such an element exists in A). Let now τ be an infinite cardinal number. A subset B of A is said to be τ-closed in A if for each chain $C \subseteq B$, with $|C| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-complete if for each chain B of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A.

The standard example of a τ-complete set can be obtained as follows. For an arbitrary set A let $\exp A$ denote, as usual, the collection of all subsets of A. There is a natural partial order on $\exp A$: $A_1 \geq A_2$ if and only if $A_1 \supseteq A_2$. With this partial order $\exp A$ becomes a directed set. If we consider only those subsets of the set A which have cardinality $\leq \tau$, then the corresponding subcollection of $\exp A$, denoted by $\exp_\tau A$, serves as a basic example of a τ-complete set. Proofs of the following statements can be found in [4].

Proposition 2.4. *Let $\{A_t : t \in T\}$ be a collection of τ-closed and cofinal subsets of a τ-complete set A. If $|\, T \,| \leq \tau$, then the intersection $\cap\{A_t : t \in T\}$ is also cofinal (in particular, non-empty) and τ-closed in A .*

Corollary 2.5. *For each subset B, with $|\, B \,| \leq \tau$, of a τ-complete set A there exists an element $\gamma \in A$ such that $\gamma \geq \beta$ for each $\beta \in B$.*

Proposition 2.6. *Let A be a τ-complete set, $L \subseteq A^2$, and suppose the following three conditions are satisfied:*

Existence: *For each $\alpha \in A$ there exists $\beta \in A$ such that $(\alpha, \beta) \in L$.*
Majorantness: *If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$.*
τ-closeness: *Let $\{\alpha_t : t \in T\}$ be a chain in A with $|T| \leq \tau$. If $(\alpha_t, \beta) \in L$ for some $\beta \in A$ and each $t \in T$, then $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_t\colon t \in T\}$.*

*Then the set of all L-*reflexive *elements of A (an element $\alpha \in A$ is L-reflexive if $(\alpha, \alpha) \in L$) is cofinal and τ-closed in A.*

2.3. **Baire sets and Baire isomorphisms.** Recall that elements of the σ-algebra generated by functionally open subsets of a space X are called Baire sets of X. A map $f\colon X \to Y$ is a Baire map if inverse images of Baire sets are Baire sets. A bijection $f\colon X \to Y$ is Baire isomorphism if both f and f^{-1} are Baire maps.

The following statement ([3, Proposition 2.5]) is used in the proof of Theorem 4.8.

Proposition 2.7. *Let $f\colon X \to Y$ be a 0-soft map of $AE(0)$-spaces. Then there exists a Baire map $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$.*

If f is a continuous homomorphism of Polish groups, then the existence of such a g has been observed by Dixmier [1, Theorem 1.2.4], [11, Theorem 12.17]).

3. AE(0)-groups and actions of AE(0)-groups – spectral representations

In this section we present spectral characterizations of AE(0)-groups. We also present a spectral description of actions of AE(0)-groups on AE(0)-spaces.

Proposition 3.1. *Each topological $AE(0)$-group G is topologically and algebraically isomorphic to the limit of a factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consisting of Polish groups G_α, $\alpha \in A$, and 0-soft limit homomorphisms $p_\alpha\colon G \to G_\alpha$, $\alpha \in A$. In particular, all short projections $p_\alpha^\beta\colon G_\beta \to G_\alpha$, $\alpha \leq \beta$, $\alpha, \beta \in A$, are 0-soft homomorphisms.*

Proof. By [4, Theorem 6.3.2 or Proposition 6.3.5], the space G can be represented as the limit space of a factorizing ω-spectrum $\mathcal{S} = \{G_\alpha, p_\alpha^\beta, \widetilde{A}\}$ consisting of Polish spaces (i.e. AE(0)-spaces of countable weight) and 0-soft limit projections. Let us show that this spectrum contains ω-closed and cofinal subspectrum consisting of topological groups and limit projections that are (continuous) homomorphisms.

Let $\mu\colon G \times G \to G$ and $\nu\colon G \to G$ be continuous operations of multiplication and inversion given on G as on a topological group. We apply [4, Theorem 1.3.6] to both μ and ν. First consider the multiplication. Clearly, $G \times G$ is the limit space of the spectrum

$$\mathcal{S} \times \mathcal{S} = \{G_\alpha \times G_\alpha, p_\alpha^\beta \times p_\alpha^\beta, \widetilde{A}\}.$$

All projections of the spectrum $\mathcal{S} \times \mathcal{S}$ are 0-soft, and hence, by [4, Proposition 6.1.26], open. The Suslin number of the product $G \times G$ is obviously countable (see [4, proposition 6.1.8]). Consequently, by [4, Proposition 1.3.3], the spectrum $\mathcal{S} \times \mathcal{S}$ is factorizing. Next we apply [4, Theorem 1.3.6] to the spectra $\mathcal{S} \times \mathcal{S}$, $\mathcal{S}$ and to the map μ between their limit spaces. Then we get a ω-closed and cofinal subset A_μ of $\widetilde{A}$ such that for each $\alpha \in A_\mu$ there exists a continuous map $\mu_\alpha\colon G_\alpha \times G_\alpha \to G_\alpha$ such that the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle p_\alpha \times p_\alpha}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ G_\alpha \times G_\alpha & \xrightarrow{\mu_\alpha} & G_\alpha \end{array}$$

commutes. In other words, $p_\alpha \circ \mu = \mu_\alpha \circ (p_\alpha \times p_\alpha)$ for each $\alpha \in A_\mu$.

Next consider a continuous inversion $\nu\colon G \to G$. By [4, Theorem 1.3.6], applied to ν and the spectrum $\mathcal{S}$, there exists a ω-closed and cofinal subset A_ν of $\widetilde{A}$ such that for each $\alpha \in A_\nu$ there exists a continuous map $\nu_\alpha\colon G_\alpha \to G_\alpha$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\nu} & G \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ G_\alpha & \xrightarrow{\nu_\alpha} & G_\alpha \end{array}$$

commutes. In other words, $p_\alpha \circ \nu = \nu_\alpha \circ p_\alpha$ for each $\alpha \in A_\nu$.

By Proposition 2.4, the intersection $A = A_\mu \cap A_\nu$ is still ω-closed and cofinal in $\widetilde{A}$. This guarantees that G is topologically and algebraically isomorphic to the limit of the factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$. Also for each $\alpha \in A$ we have two maps

$$\mu_\alpha\colon G_\alpha \times G_\alpha \to G_\alpha \ \text{ and } \ \nu_\alpha\colon G_\alpha \to G_\alpha$$

which allow us to define

(a) a continuous multiplication operation on G_α by letting

$$x_\alpha \cdot y_\alpha = \mu_\alpha(x_\alpha, y_\alpha) \ \text{ for each } \ (x_\alpha, y_\alpha) \in G_\alpha \times G_\alpha;$$

(b) a continuous inversion on G_α by letting

$$x_\alpha^{-1} = \nu_\alpha(x_\alpha) \ \text{ for each } \ x_\alpha \in G_\alpha.$$

It is easy to see that $G_\alpha,\ \alpha \in A$, becomes a topological group with respect to these operations. Moreover, for each $\alpha \in A$ the limit projection $p_\alpha\colon G \to G_\alpha$ becomes a homomorphism with respect to the above defined operations. $\square$

Corollary 3.2. *Let G be a AE(0)-group such that* $\dim G \leq n$. *Then G is topologically and algebraically isomorphic to the limit of a factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consisting of at most n-dimensional Polish groups G_α, $\alpha \in A$, and 0-soft limit homomorphisms $p_\alpha \colon G \to G_\alpha$, $\alpha \in A$.*

Proof. By Proposition 3.1, $G = \lim \mathcal{S}_1$, where $\mathcal{S}_1 = \{G_\alpha, p_\alpha^\beta, A_1\}$ is a factorizing ω-spectrum consisting of Polish groups and 0-soft limit homomorphisms. By [4, Theorem 1.3.10], $G = \lim \mathcal{S}_2$, where $\mathcal{S}_2 = \{G_\alpha, p_\alpha^\beta, A_2\}$ is a factorizing ω-spectrum consisting of at most n-dimensional Polish spaces. It follows directly from the proofs of Proposition 3.1 and [4, Theorem 1.3.10], that both indexing sets A_1 and A_2 can be assumed to be closed and ω-complete subsets of a ω-complete set B. By Proposition 2.4, $A = A_1 \cap A_2$ is still cofinal and ω-closed subset of B. Consequesntly, the factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ consists of at most n-dimensional Polish groups and 0-soft limit homomorphisms. It only remains to note that $G = \lim \mathcal{S}_G$. □

Corollary 3.3. *Let $\tau \geq \omega$. Every AE(0)-group of weight $\tau \geq \omega$ is topologically and algebraically isomorphic to a closed and C-embedded subgroup of the product $\prod\{G_t \colon t \in T\}$, where G_t, $t \in T$, is a Polish group and $|T| = \tau$.*

Proof. Let G be an AE(0)-group of weight τ. If $\tau = \omega$, then G is Polish (see [4, Chapter 6] and consequently there is nothing to prove.

If $\tau > \omega$, then by Proposition 3.1, G is topologically and algebraically isomorphic to the limit of a factorizing ω-spectrum $\mathcal{S}_G = \{G_t, p_t^{t'}, T\}$ with $|T| = \tau$. Clearly $\lim \mathcal{S}_G$ is isomorphic to a closed subgroup of the product $\prod\{G_t, t \in A\}$. Since the spectrum $\mathcal{S}_G$ is factorizing, it follows that $\lim \mathcal{S}_G$ is C-embedded in $\prod\{G_t, t \in T\}$. □

Theorem 3.4. *Let G be a topological group of weight $\tau > \omega$. Then the following conditions are equivalent:*

(a) *G is a AE(0)-group.*
(b) *There exists a well-ordered inverse spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ satisfying the following properties:*
   (1) *G_α is a AE(0)-group and $p_\alpha^{\alpha+1} \colon G_{\alpha+1} \to G_\alpha$ is a 0-soft homomorphism with Polish kernel, $\alpha < \tau$.*
   (2) *If $\beta < \tau$ is a limit ordinal, then the diagonal product*
   $$\triangle\{p_\alpha^\beta \colon \alpha < \beta\} \colon G_\beta \to \lim\{G_\alpha, p_\alpha^{\alpha+1}, \alpha < \beta\}$$
   is a topological and algebraic isomorphism.
   (3) *G is topologically and algebraically isomorphic to* $\lim \mathcal{S}_G$.
   (4) *G_0 is a Polish group.*

Proof. (a) $\Longrightarrow$ (b). By Corollary 3.3, we may assume that G is a closed and C-embedded subgroup of the product $\prod\{X_a \colon a \in A\}$, $|A| = \tau$, of Polish groups X_a, $a \in A$. There exists a proper, functionally closed and 0-invertible map $f\colon Y \to \prod\{X_a\colon a \in A\}$, where Y is a spectrally complete (see [4, p.247]) realcompact space of weight τ and dimension $\dim Y = 0$ (see [4, Proposition 6.2.13] for details). Consider the inverse image $f^{-1}(G) \subseteq Y$ of G and the map $f|f^{-1}\colon f^{-1}(G) \to G$. Since G is C-embedded in the product $\prod\{X_a\colon a \in A\}$, since $\dim Y = 0$ and since G, according to (a), is an $AE(0)$-space, there exists a map $g\colon Y \to G$ such that $g|f^{-1}(G) = f|f^{-1}(G)$.

Next let us denote by

$$\pi_B\colon \prod\{X_a\colon a \in A\} \to \prod\{X_a\colon a \in B\}$$

and

$$\pi_C^B\colon \prod\{X_a\colon a \in B\} \to \prod\{X_a\colon a \in C\}$$

the natural projections onto the corresponding subproducts ($C \subseteq B \subseteq A$). We call a subset $B \subseteq A$ admissible (compare with the proof of [4, Theorem 6.3.1]) if the following equality

$$\pi_B(g(f^{-1}(x))) = \pi_B(x)$$

is true for each point $x \in \pi_B^{-1}(\pi_B(G))$. We need the following properties of admissible sets.

Claim 1. The union of arbitrary collection of admissible sets is admissible.

Indeed let $\{B_t\colon t \in T\}$ be a collection of admissible sets and $B = \cup\{B_t\colon t \in T\}$. Let $x \in \pi_B^{-1}(\pi_B(G))$. Clearly $x \in \pi_{B_t}^{-1}(\pi_{B_t}(G))$ for each $t \in T$ and consequently

$$\pi_{B_t}(g(f^{-1}(x))) = \pi_{B_t}(x) \ \text{ for each } \ t \in T.$$

Obviously, $\pi_B(x) \in \pi_B(g(f^{-1}(x)))$ and it therefore suffices to show that the set $\pi_B(g(f^{-1}(x)))$ contains only one point. Assuming that there is a point $y \in \pi_B(g(f^{-1}(x)))$ such that $y \neq \pi_B(x)$ we conclude (having in mind that $B = \cup\{B_t\colon t \in T\}$) that there must be an index $t \in T$ such that $\pi_{B_t}^B(y) \neq \pi_{B_t}^B(\pi_B(x))$. But this is impossible

$$\pi_{B_t}^B(y) \in \pi_{B_t}^B\left(\pi_B(g(f^{-1}(x)))\right) = \pi_{B_t}(g(f^{-1}(x))) = \pi_{B_t}(x) = \pi_{B_t}^B\left(\pi_B(x)\right).$$

Claim 2. If $B \subseteq A$ is admissible, then the restriction $\pi_B|G\colon G \to \pi_B(G)$ is 0-soft.

Let $\varphi\colon Z \to \pi_B(G)$ and $\varphi_0\colon Z_0 \to G$ be two maps defined on a realcompact space Z, with $\dim Z = 0$, and its closed subset Z_0 respectively. Assume that $\pi_B\varphi_0 = \varphi|Z_0$ and $C(\varphi_0)(C(G)) \subseteq C(Z)|Z_0$. We wish to construct a map $\phi\colon Z \to G$ such that $\phi|Z_0 = \varphi_0$ and $\pi_B\phi = \varphi$, i.e. ϕ makes the diagram

$$\begin{tikzcd}
Z \arrow[r, "\varphi_0"] \arrow[d, hook, "\mathrm{incl}"'] & G \arrow[d, "\pi_B|G"] \\
Z_0 \arrow[r, "\varphi"] \arrow[ur, "\phi"] & \pi_B(G)
\end{tikzcd}$$

commutative. Since, according to our choice, all X_a's are AE(0)-spaces (recall that each X_a is a Polish space), so is the product $\prod\{X_a \colon a \in A - B\}$. This implies the 0-softness of the projection π_B and hence of its restriction

$$\pi_B|\pi_B^{-1}(\pi_B(G)) : \pi_B^{-1}(\pi_B(G)) \to \pi_B(G).$$

Then there exists a map $\phi'' \colon Z \to \pi_B^{-1}(\pi_B(G))$ such that $\phi''|Z_0 = \varphi_0$ and $\pi_B\phi'' = \varphi$. Since f is 0-invertible (and $\dim Z = 0$), there exists a map $\phi' \colon Z \to Y$ such that $f\phi' = \phi''$. Now let $\phi = g\phi'$. Since $g|f^{-1}(G) = f|f^{-1}(G)$, we have $\varphi_0 = \phi|Z_0$. Finally observe that the admissibility of B implies $\varphi = \pi_B\phi$ as required.

Claim 3. For each countable subset $C \subseteq A$ there exists a countable admissible subset $B \subseteq A$ such that $C \subseteq B$.

Since $w(Y) = \tau$ and $\dim Y = 0$, it follows (consult [4, Theorem 1.3.10]) that Y can be represented as the limit space of a factorizing ω-spectrum $\mathcal{S}_Y = \{Y_B, q_C^B, \exp_\omega A\}$ consisting of zero-dimensional Polish spaces Y_B, $B \in \exp_\omega A$, and continuous surjections $q_C^B \colon Y_B \to Y_C$, $C \subseteq B$, $C, B \in \exp_\omega A$. Consider also the standard factorizing ω-spectrum $\mathcal{S}_X = \{\prod\{X_a \colon a \in B\}, \pi_C^B, \exp_\omega A\}$ consisting of countable subproducts of the product $\prod\{X_a \colon a \in A\}$ and corresponding natural projections. Obviously the full product coincides with the limit of $\mathcal{S}_X$. One more factorizing ω-spectrum arises naturally. This is the spectrum $\mathcal{S}_G = \{\pi_B(G), \pi_C^B|\pi_B(G), \exp_\omega A\}$ the limit of which coincides with G.

Consider the map $f \colon \lim \mathcal{S}_Y \to \lim \mathcal{S}_X$. By [4, Theorem 1.3.4], there is a cofinal and ω-closed subset $\mathcal{T}_f$ of $\exp_\omega A$ such that for each $B \in \mathcal{T}_f$ there is a map $f_B \colon Y_B \to \prod\{X_a \colon a \in B\}$ such that $f_B \circ q_B = \pi_B \circ f$. Moreover, these maps form a morphism

$$\{f_B; B \in \mathcal{T}_f\} \colon \mathcal{S}_Y \to \mathcal{S}_X$$

limit of which coincides with f. Since f is proper and functionally closed, we may assume (see [4, Proposition 6.2.9]) without loss of generality (considering a smaller cofinal and ω-subset of $\mathcal{T}_f$ if necessary) that the above indicated morphism is bicommutative. This simply means that $q_B f^{-1}(K) = f_B^{-1}(\pi_B(K))$ for any $B \in \mathcal{T}_f$ and any closed subset K of the product $\prod\{X_a \colon a \in A\}$.

Similarly, applying [4, Theorem 1.3.4] to the map $g \colon \lim \mathcal{S}_Y \to \lim \mathcal{S}_G$, we obtain a cofinal and ω-closed subset $\mathcal{T}_g$ of $\exp_\omega A$ and the associated to it morphism

$$\{g_B \colon Y_B \to \pi_B(G); B \in \mathcal{T}_g\} \colon \mathcal{S}_Y \to \mathcal{S}_G$$

limit of which coincides with the map g.

By Proposition 2.4, the intersection $\mathcal{T} = \mathcal{T}_f \cap \mathcal{T}_g$ is still a cofinal and ω-closed subset of $\exp_\omega A$. It therefore suffices to show that each $B \in \mathcal{T}$ is an admissible subset of A. Consider a point $x \in \pi_B^{-1}(\pi_B(G))$. First observe that the bicommutativity of the morphism associated with $\mathcal{T}_f$ implies that $q_B(f^{-1}(x)) = f_B^{-1}(\pi_B(x))$. Since the maps f_B and g_B coincide on $f_B^{-1}(\pi_B(G))$ we have

$$\pi_B(g(f^{-1}(x))) = g_B(q_B(f^{-1}(x))) = g_B(f_B^{-1}(\pi_B(x))) = \\ f_B(f_B^{-1}(\pi_B(x))) = \pi_B(x)$$

as required.

Claim 4. If C and B are admissible subsets of A and $C \subseteq B$, then the map $\pi^B_C|\pi_B(G) \colon \pi_B(G) \to \pi_C(G)$ is 0-soft.

This property follows from Claim 2 and [4, Lemma 6.1.15].

After having all the needed properties of admissible subsets established we proceed as follows. Since $|A| = \tau$ we can write $A = \{a_\alpha \colon \alpha < \tau\}$. By Claim 3, each $a_\alpha \in A$ is contained in a countable admissible subset $B_\alpha \subseteq A$. Let $A_\alpha = \cup\{B_\beta \colon \beta \leq \alpha\}$. We use these sets to define a transfinite inverse spectrum $\mathcal{S} = \{G_\alpha, p_\alpha^{\alpha+1}, \tau\}$ as follows. Let $G_\alpha = \pi_{A_\alpha}(G)$ and $p_\alpha^{\alpha+1} = \pi_{A_\alpha}^{A_{\alpha+1}}|G_{\alpha+1}$ for each $\alpha < \tau$. All the required properties of the spectrum $\mathcal{S}_G$ are satisfied by construction.

The implication (b) $\Longrightarrow$ (a) immediately follows from [4, Proposition 6.3.4] $\square$

Remark 3.5. Actually 0-soft homomorphism $p_\alpha^{\alpha+1} \colon G_{\alpha+1} \to G_\alpha$, $\alpha < \tau$, in Theorem 3.4(b)1 has Polish kernel in a somewhat stonger sense than the original definition presented in Subsection 2.1. Namely a Polish space P (from the definition), such that $G_{\alpha+1}$ admist a C-embedding into the product $G_\alpha \times P$ in such a way that $p_\alpha^{\alpha+1}$ coincides with the restriction of the projection $\pi_{G_\alpha} \colon G_\alpha \times P \to G_\alpha$, can be chosen to be a Polish group and the embedding of $G_{\alpha+1} \to G_\alpha \times P$ can be assumed to be a homomorphism. Of course this implies that $\ker p_\alpha^{\alpha+1}$, as a closed subgroup of P, is itself a Polish group. It would be interesting to see whether the converse of this observation is also true, i.e. is it true that if the kernel $\ker p$ of a 0-soft homomorphism $p \colon G \to L$

of AE(0)-groups is Polish, then p has a Polish kernel in the sense of Subsection 2.1.

Next we characterize 0-soft homomorphisms of AE(0)-groups with Polish kernels.

Proposition 3.6. *A 0-soft homomorphism $f\colon G \to L$ between AE(0)-groups has a Polish kernel if and only if there exist factorizing ω-spectra $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$, $\mathcal{S}_L = \{L_\alpha, q_\alpha^\beta, A\}$, consisting of Polish groups and 0-soft limit homomorphisms, and a morphism $\{f_\alpha\}\colon \mathcal{S}_G \to \mathcal{S}_L$, consisting of 0-soft homomorphisms, such that the following conditions are satisfied:*

(1) *$G = \lim \mathcal{S}_G$, $L = \lim \mathcal{S}_L$ and $f = \lim\{f_\alpha\}$.*
(2) *All limit projections of the spectra $\mathcal{S}_G$ and $\mathcal{S}_L$ are 0-soft.*
(3) *All limit square diagrams, generated by limit projections of spectra $\mathcal{S}_G$ and $\mathcal{S}_L$, by elements of the morphism $\{f_\alpha\}$ and by the map f, are the Cartesian squares.*

Proof. By Proposition 3.1, we may assume, without loss of generality, that both G and L are topologically and algebraically isomorphic to the limits of factorizing ω-spectra $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ and $\mathcal{S}_L = \{L_\alpha, q_\alpha^\beta, A\}$, consisting of Polish groups and 0-soft limit homomorphisms. By [4, Theorem 1.3.6], we may also assume that the map f is the limit of a morphism $\{f_\alpha\colon G_\alpha \to L_\alpha; A\}$, consisting of continuous maps $f_\alpha\colon G_\alpha \to L_\alpha$. Since f itself and all the limit projections $p_\alpha\colon G \to G_\alpha$ and $q_\alpha\colon L \to L_\alpha$ are homomorphisms between respective groups, it follows easily that $f_\alpha\colon G_\alpha \to L_\alpha$ is also a homomorphism, $\alpha \in A$. Finally since f is 0-soft and has a Polish kernel, it follows, by [4, Theorem 6.3.1(vi)], that all limit square diagrams

$$\begin{array}{ccc} G & \xrightarrow{f} & L \\ {\scriptstyle p_\alpha}\downarrow & & \downarrow{\scriptstyle q_\alpha} \\ G_\alpha & \xrightarrow{f_\alpha} & L_\alpha, \end{array}$$

generated by limit projections of spectra $\mathcal{S}_G$ and $\mathcal{S}_L$, by elements of the morphism $\{f_\alpha\}$ and by the map f, are the Cartesian squares. □

Below, in Subsection 4.1, we consider actions of AE(0)-groups on AE(0)-spaces. Main tool here is the following statement (see [4, Theorem 8.7.1]).

Proposition 3.7. *Let $\lambda\colon G \times X \to X$ be a continuous action of an AE(0)-group G on an AE(0)-space X. Suppose that X is homeomorphic to the limit space of a factorizing ω-spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of Polish spaces and 0-soft limit projections. Suppose also that G is topologically and algebraically isomorphic to the limit of the factorizing ω-spectrum $\mathcal{S}_G =$*

$\{G_\alpha, s_\alpha^\beta, A\}$ consisting of Polish groups and 0-soft limit homomorphisms. Then λ is the limit of "level actions", i.e. $\lambda = \lim \lambda_\alpha$, where

$$\{\lambda_\alpha \colon G_\alpha \times X_\alpha \to X_\alpha, B\} \colon \mathcal{S}_G|B \times \mathcal{S}_X|B \to \mathcal{S}_X|B$$

is a morphism between the spectra $\mathcal{S}_G|B \times \mathcal{S}_X|B$ and $\mathcal{S}_X|B$ and B is a cofinal and ω-closed subset of the indexing set A.

4. Applications

4.1. **Universal AE(0)-groups and universal actions of AE(0)-groups.**

In this Subsection we prove the existence of universal AE(0)-groups of a given weight as well as the existence of a universal action of a AE(0)-group of a given weight on a compact AE(0)-space of the same weight.

Proposition 4.1. *Let $\tau \geq \omega$. The class of AE(0)-groups of weight $\leq \tau$ contains a universal element. More formally, every AE(0)-group is topologically and algebraically isomorphic to a closed and C-embedded subgroup of the power $(\mathrm{Aut}(\mathbb{Q}))^\tau$, where $\mathrm{Aut}(\mathbb{Q})$ denotes the group of autohomeomorphisms of the Hilbert cube $\mathbb{Q}$.*

Proof. Let G be a AE(0)-group of weight τ. By Corollary 3.3, G is topologically and algebraically isomorphic to a closed and C-embedded subgroup of the product $\prod\{G_t \colon t \in T\}$, where G_t is a Polish group for each $t \in T$ and $|T| = \tau$. By Uspenskii's theorem [15], G_t can be identified with a closed subgroup of $\mathrm{Aut}(\mathbb{Q})$. Obviously $\prod\{G_t \colon t \in T\}$, and consequently G, is a closed and C-embedded subgroup of $(\mathrm{Aut}(\mathbb{Q}))^\tau$. □

Let $\tau > \omega$. Clearly the AE(0)-group $(\mathrm{Aut}(\mathbb{Q}))^\tau$ (i.e. the τ-th power of the group $\mathrm{Aut}(\mathbb{Q})$) continuously acts on the Tychonov cube $\mathbb{Q}^\tau$ via the natural action ("coordinatewise evaluation")

$$\mathrm{ev}_\tau \colon (\mathrm{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \to \mathbb{Q}^\tau,$$

which is defined by letting

$$\mathrm{ev}_\tau\left(\{g_\alpha \colon \alpha < \tau\}, \{q_\alpha \colon \alpha < \tau\}\right) = \{g_\alpha(q_\alpha) \colon \alpha < \tau\} \quad \text{for each} \quad (\{g_\alpha \colon \alpha < \tau\}, \{q_\alpha \colon \alpha < \tau\}) \in (\mathrm{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau.$$

Theorem 4.2. *Let $\tau > \omega$. The action $\mathrm{ev}_\tau \colon (\mathrm{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \to \mathbb{Q}^\tau$ is universal in the category of actions of AE(0)-groups of weight τ on compact AE(0)-spaces of weight τ. More formally, let $\lambda \colon G \times X \to X$ be a continuous action of a AE(0)-group G of weight τ on a compact AE(0)-space X of weight τ. Then there exists a topological and algebraic embedding $i_G \colon G \to (\mathrm{Aut}(\mathbb{Q}))^\tau$ with a* *closed image and an embedding* $i_X\colon X \to \mathbb{Q}^\tau$ *such that the following diagram*

$$\begin{array}{ccc} (\mathrm{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau & \xrightarrow{\mathrm{ev}_\tau} & \mathbb{Q}^\tau \\ {\scriptstyle i_G \times i_X}\Big\uparrow & & \Big\uparrow{\scriptstyle i_X} \\ G \times X & \xrightarrow{\lambda} & X \end{array}$$

commutes.

Proof. By Proposition 3.1, G can be represented as the limit of a factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, s_\alpha^\beta, A\}$ consisting of Polish groups and 0-soft limit homomorphisms. Similarly, by [4, Proposition 6.3.5], X can be represented as the limit space of a factorizing ω-spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^\beta, A\}$ consisting of metrizable compacta and 0-soft limit projections. Without loss of generality we may assume that these spectra $\mathcal{S}_G$ and $\mathcal{S}_X$ have the same indexing set A and $|A| = \tau$. By Proposition 3.7, the given action $\lambda\colon G \times X \to X$ is the limit of level actions, i.e. $\lambda = \lim \lambda_\alpha$, where

$$\{\lambda_\alpha\colon G_\alpha \times X_\alpha \to X_\alpha, B\}\colon \mathcal{S}_G|B \times \mathcal{S}_X|B \to \mathcal{S}_X|B$$

is a morphism between the spectra $\mathcal{S}_G|B \times \mathcal{S}_X|B$ and $\mathcal{S}_X|B$ and B is a cofinal and ω-closed subset of the indexing set A. We may also assume that $|B| = \tau$.

Since $G = \lim \mathcal{S}_G|B$ it follows that the diagonal product

$$s = \triangle\{s_\alpha\colon G \to G_\alpha, \alpha \in B\}\colon G \to \prod\{G_\alpha\colon \alpha \in B\}$$

is a topological and algebraic isomorphism with a closed image. Similarly the diagonal product

$$p = \triangle\{p_\alpha\colon X \to X_\alpha, \alpha \in B\}\colon X \to \prod\{X_\alpha\colon \alpha \in B\}$$

is an embedding. Consider also the product action

$$\widetilde{\lambda}\colon \prod\{G_\alpha\colon \alpha \in B\} \times \prod\{X_\alpha\colon \alpha \in B\} \to \prod\{X_\alpha\colon \alpha \in B\}$$

defined by letting

$$\widetilde{\lambda}(\{g_\alpha\colon \alpha \in B\}, \{x_\alpha\colon \alpha \in B\}) = \{\lambda_\alpha(g_\alpha, x_\alpha)\colon \alpha \in B\} \text{ for each}$$
$$(\{g_\alpha\colon \alpha \in B\}, \{x_\alpha\colon \alpha \in B\}) \in \prod\{G_\alpha\colon \alpha \in B\} \times \prod\{X_\alpha\colon \alpha \in B\}.$$

Note that $\widetilde{\lambda} \circ (s \times p) = p \circ \lambda$, i.e. the following diagram

$$\begin{array}{ccc} \prod\{G_\alpha\colon \alpha \in B\} \times \prod\{X_\alpha\colon \alpha \in B\} & \xrightarrow{\widetilde{\lambda}} & \prod\{X_\alpha\colon \alpha \in B\} \\ {\scriptstyle s \times p}\Big\uparrow & & \Big\uparrow{\scriptstyle p} \\ G \times X & \xrightarrow{\lambda} & X \end{array}$$

is commutative.

Since for each $\alpha \in B$ the group G_α is Polish, it follows by [12] (see also [1, Theorem 2.6.7]) that there exist a topological and algebraic embedding $j_\alpha \colon G_\alpha \to \mathrm{Aut}(\mathbb{Q}_\alpha)$ with a closed image and an embedding $i_\alpha \colon X_\alpha \to \mathbb{Q}_\alpha$ (here $\mathbb{Q}_\alpha$ denotes a copy of the Hilbert cube $\mathbb{Q}$) such that $\mathrm{ev}_\alpha \circ (j_\alpha \times i_\alpha) = i_\alpha \circ \lambda_\alpha$. This simply means that the diagram

$$\begin{tikzcd}
\mathrm{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha \arrow[r, "\mathrm{ev}_\alpha"] & \mathbb{Q}_\alpha \\
G_\alpha \times X_\alpha \arrow[u, "j_\alpha \times i_\alpha"] \arrow[r, "\lambda_\alpha"] & X_\alpha \arrow[u, "i_\alpha"']
\end{tikzcd}$$

commutes for each $\alpha \in B$. Here $\mathrm{ev}_\alpha \colon \mathrm{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha \to \mathbb{Q}_\alpha$ is the evaluation action, i.e. $\mathrm{ev}_\alpha(g_\alpha, x_\alpha) = g_\alpha(x_\alpha)$ for each $(g_\alpha, x_\alpha) \in \mathrm{Aut}(\mathbb{Q}_\alpha) \times \mathbb{Q}_\alpha$. Let

$$j = \times\{j_\alpha \colon \alpha \in B\} \colon \prod\{G_\alpha \colon \alpha \in B\} \to \prod\{\mathrm{Aut}(\mathbb{Q}_\alpha) \colon \alpha \in B\}$$

and

$$i = \times\{i_\alpha \colon \alpha \in B\} \colon \prod\{X_\alpha \colon \alpha \in B\} \to \prod\{\mathbb{Q}_\alpha \colon \alpha \in B\}.$$

Finally consider the commutative diagram

$$\begin{tikzcd}
\prod\{\mathrm{Aut}(\mathbb{Q}_\alpha) \colon \alpha \in B\} \times \prod\{\mathbb{Q}_\alpha \colon \alpha \in B\} \arrow[r, "\times\{\mathrm{ev}_\alpha \colon \alpha \in B\}"] & \prod\{\mathbb{Q}_\alpha \colon \alpha \in B\} \\
\prod\{G_\alpha \colon \alpha \in B\} \times \prod\{X_\alpha \colon \alpha \in B\} \arrow[u, "j \times i"] \arrow[r, "\widetilde{\lambda}"] & \prod\{X_\alpha \colon \alpha \in B\} \arrow[u, "i"'] \\
G \times X \arrow[u, "s \times p"] \arrow[r, "\lambda"] & X \arrow[u, "p"']
\end{tikzcd}$$

and note that since $|B| = \tau$ the upper horizontal arrow is actually the action $\mathrm{ev}_\tau \colon (\mathrm{Aut}(\mathbb{Q}))^\tau \times \mathbb{Q}^\tau \to \mathbb{Q}^\tau$. Clearly it suffices to let $i_G = (j \times i) \circ (s \times p)$ and $i_X = i \circ p$. This completes the proof. $\square$

It would be very interesting to prove that for a AE(0)-group G of weight $\tau > \omega$ the category of $AE(0)$-spaces (of weight τ) admitting actions of the group G and their G-maps contains a universal object. For $\tau = \omega$ this fact has recently been proved in [9].

4.2. **Closed subgroups of powers of the symmetric group S_∞.** The following result gives an embeddability criterion into the symmetric group S_∞ - the group of all bijections of $\mathbb{N}$ under the relative topology inherited from $\mathbb{N}^{\mathbb{N}}$. It is important to note [7] that there exist zero-dimensional Polish groups which can not be embedded into S_∞ as closed subgroups.

Theorem 4.3 ([1])**.** *Let G be a Polish group. Then the following conditions are equivalent:*
- (i) *G is isomorphic to a closed subgroup of S_∞;*
- (ii) *G admits a (countable) neighborhood basis at the identity consisting of open subgroups;*
- (iii) *G admits a (countable) basis closed under left multiplication (or a countable basis closed under right multiplication);*
- (iv) *G admits a compatible left-invariant ultrametric.*

Next we characterize those topological $AE(0)$-groups which are isomorphic to closed subgroups of infinite powers S_∞^τ, $\tau \geq \omega$, of S_∞.

Theorem 4.4. *Let $\tau \geq 1$ be a cardinal number. The following conditions are equivalent for any topological $AE(0)$-group G of weight $\tau \geq \omega$:*
- (i) *G is isomorphic to a closed subgroup of S_∞^τ;*
- (ii) *G admits a neighborhood basis at the identity consisting of open subgroups.*

Proof. If $\tau = 1$ our statement coincides with Theorem 4.3. Next consider the case $1 < \tau \leq \omega$. It is easy to see that the group S_ω^τ admits a countable neighborhood basis at the identity consisting of open subgroups. Obviously every closed subgroup of S_τ^ω has the same property (and, consequently, by Theorem 4.3, can be embedded into S_∞). Conversely if a Polish group G admits a countable neighborhood basis at the identity consisting of open subgroups, then, by Theorem 4.3, G is isomorphic to a closed subgroup of S_∞. It only remains to note that G_∞ is isomorphic to a closed subgroup of S_∞^τ for any τ.

Next we assume that $\tau > \omega$. By [4, Lemma 8.2.1], G is isomorphic to the limit space of a factorizing ω-spectrum $\mathcal{S}_G = \{G_\alpha, p_\alpha^\beta, A\}$ all spaces G_α, $\alpha \in A$, of which are Polish groups and all limit projections $p_\alpha \colon G \to G_\alpha$, $\alpha \in A$, of which are 0-soft homomorphisms.

Now consider the following relation $L \in A^2$:

$$L = \Big\{(\alpha,\beta) \in A^2 \colon \alpha \leq \beta \text{ and there exists a countable neighborhood basis } \mathcal{V}_\alpha \text{ at the identity } e_\alpha \text{ of } G_\alpha \text{ containing intersections of its finite subcollections and such that for each } V \in \mathcal{V}_\alpha \text{ there is an open subgroup } U_V^{\beta,\alpha} \text{ of } G_\beta \text{ with } U_V^{\beta,\alpha} \subseteq \left(p_\alpha^\beta\right)^{-1}(V)\Big\}$$

Let us verify conditions of Proposition 2.6.

Existence. For each $\alpha \in A$ we need to find $\beta \in A$ such that $(\alpha, \beta) \in L$. Let $\mathcal{V}_\alpha$ be a countable neighborhood basis at $e_\alpha \in G_\alpha$ which contains intersections of its finite subcollections. For each $V \in \mathcal{V}_\alpha$ the set $p_\alpha^{-1}(V)$ is a neighborhood of the identity $e \in G$. By (ii), there exists an open subgroup U_V of G such that $U_V \subseteq p_\alpha^{-1}(V)$. Since every open subgroup in G is closed, it follows that U_V, as an open and closed subset of G, is a functionally open in G. Recall that the spectrum $\mathcal{S}_G$ is factorizing and consequently there exist an index $\beta_V \in A$ and an open subset $U_V^{\beta_V} \subseteq G_{\beta_V}$ such that $\beta_V \geq \alpha$ and $U_V = p_{\beta_V}^{-1}\left(U_V^{\beta_V}\right)$. By Corollary 2.5, there exists an index $\beta \in A$ such that $\beta \geq \beta_V$ for each $V \in \mathcal{V}_\alpha$. Let $U_V^{\beta,\alpha} = \left(p^\beta_{\beta_V}\right)^{-1}\left(U_V^{\beta_V}\right)$, $V \in \mathcal{V}_\alpha$. Note that

$$U_V^{\beta,\alpha} = \left(p^\beta_{\beta_V}\right)^{-1}\left(U_V^{\beta_V}\right) = p_\beta\left(p_{\beta_V}\left(U_V^{\beta_V}\right)\right) = p_\beta(U_V) \subseteq p_\beta\left(p_\alpha^{-1}(V)\right) = \left(p^\beta_\alpha\right)^{-1}(V) \text{ for each } V \in \mathcal{V}_\alpha.$$

Since the limit projection $p_\beta\colon G \to G_\beta$ is a homomorphism it follows that $U_V^{\beta,\alpha} = p_\beta(U_V)$ is a subgroup of G_β. Finally since $U_V^{\beta_V}$ is open in G_{β_V}, we conclude that $U_V^{\beta,\alpha} = \left(p^\beta_{\beta_V}\right)^{-1}\left(U_V^{\beta_n}\right)$ is open in G_β. This shows that $(\alpha, \beta) \in L$.

Majorantness. If $(\alpha, \beta) \in L$ and $\gamma \geq \beta$, then $(\alpha, \gamma) \in L$. Since $(\alpha, \beta) \in L$ it follows that for each $V \in \mathcal{V}_\alpha$ there exists an open subgroup $U_V^{\beta,\alpha}$ of G_β such that $U_V^{\beta,\alpha} \subseteq \left(p^\beta_\alpha\right)^{-1}(V)$ where $\mathcal{V}_\alpha$ is a countable neighborhood basis at the identity $e_\alpha \in G_\alpha$ containing intersections of its finite subcollections. Let $U_V^{\gamma,\alpha} = \left(p^\gamma_\beta\right)^{-1}(U_V^{\beta,\alpha})$ for each $V \in \mathcal{V}_\alpha$. Since the projection $p^\gamma_\beta\colon G_\gamma \to G_\beta$ is a continuous homomorphism, it follows that $U_V^{\gamma,\alpha}$ is an open subgroup of G_γ. Obviously

$$U_V^{\gamma,\alpha} = \left(p^\gamma_\beta\right)^{-1}(U_V^{\beta,\alpha}) \subseteq \left(p^\gamma_\beta\right)^{-1}\left(\left(p^\beta_\alpha\right)^{-1}(V)\right) = \left(p^\gamma_\alpha\right)^{-1}(V), \quad V \in \mathcal{V}_\alpha,$$

which shows that $(\alpha, \gamma) \in L$ as required.

ω-closeness. Let $\{\alpha_i : i \in \omega\}$ be a countable chain in A and $(\alpha_i, \beta) \in L$ for some $\beta \in A$ and each $i \in \omega$. We need to show that $(\alpha, \beta) \in L$ where $\alpha = \sup\{\alpha_i\colon i \in \omega\}$.

Let $\mathcal{V}_{\alpha_i}$ be a countable neighborhood basis at $e_{\alpha_i} \in G_{\alpha_i}$ and U_V^{β,α_i}, $V \in \mathcal{V}_{\alpha_i}$, be an open subgroup of G_β witnessing the fact that $(\alpha_i, \beta) \in L$.

Consider the collection

$$\mathcal{V}_\alpha = \bigcup\{\left(p^\alpha_{\alpha_i}\right)^{-1}(\mathcal{V}_{\alpha_i}) : i \in \omega\}.$$

Since the spectrum $\mathcal{S}_G$ is an ω-spectrum and since $\alpha = \sup\{\alpha_i \colon i \in \omega\}$ it follows that $\mathcal{V}_\alpha$ forms a neighborhood basis at $e_\alpha \in G_\alpha$. For each $\widetilde{V} \in \mathcal{V}_\alpha$ choose $V \in \mathcal{V}_{\alpha_i}$ such that $\widetilde{V} = \left(p^\alpha_{\alpha_i}\right)^{-1}(V)$ and let $U^{\beta,\alpha}_{\widetilde{V}} = U^{\beta,\alpha_i}_V$. Since $(\alpha_i, \beta) \in L$, we have

$$U^{\beta,\alpha}_{\widetilde{V}} = U^{\beta,\alpha_i}_V \subseteq \left(p^\beta_{\alpha_i}\right)^{-1}(V) = \left(p^\beta_\alpha\right)^{-1}\left(\left(p^\alpha_{\alpha_i}\right)^{-1}(V)\right) = \left(p^\beta_\alpha\right)^{-1}(\widetilde{V}).$$

This proves that $(\alpha, \beta) \in L$.

According to Proposition 2.6 the set $\widetilde{A}$ of L-reflexive elements is cofinal and ω-closed in A. The L-reflexivity of an element $\alpha \in A$ means precisely that there exists a countable neighborhood basis $\mathcal{V}_\alpha$ at $e_\alpha \in G_\alpha$ containing intersections of its finite subcollections and such that for each $V \in \mathcal{V}_\alpha$ there exists an open subgroup $U^\alpha_V \subseteq G_\alpha$ with $U^\alpha_V \subseteq V$. This obviously means that for each $\alpha \in \widetilde{A}$ the Polish group G_α satisfies condition (ii) of Theorem 4.3. Consequently, by Theorem 4.3, G_α is topologically isomorphic to a closed subgroup of S_∞. Next note that since $\widetilde{A}$ is cofinal in A the limit space of the spectrum $\mathcal{S} = \{G_\alpha, p^\beta_\alpha, \widetilde{A}\}$ is topologically isomorphic to G. This obviously implies that G is isomorphic to a closed subgroup of the product $\prod\{G_\alpha \colon \alpha \in \widetilde{A}\}$, which in turn is topologically isomorphic to a closed subgroup of S^τ_∞ (note that $|\widetilde{A}| = w(G) = \tau$). This completes the proof of implication (ii) $\Longrightarrow$ (i).

Verification of the implication (i) $\Longrightarrow$ (ii) is trivial. Proof is completed. □

Corollary 4.5. *Let $\tau \geq 2$. The following conditions are equivalent for any Polish group G:*

(i) *G is topologically isomorphic to a closed subgroup of S_∞;*
(ii) *G is topologically isomorphic to a closed subgroup of S^τ_∞.*

Corollary 4.6. *There exists a zero-dimensional Polish group which is not topologically isomorphic to a closed subgroup of S^τ_∞ for any cardinal number τ.*

Proof. It is known [7] that there exists a zero-dimensional Polish group G which is not topologically isomorphic to a closed subgroup of S_∞. By Corollary 4.6, G can not be topologically isomorphic to a closed subgroup of S^τ_∞ for any cardinal $\tau \geq 2$. □

4.3. **Baire isomorphisms.** The main result of this Subsection (Theorem 4.8) allows us to reduce in many instances (descriptive) set theoretical considerations of general AE(0)-groups to those for Polish groups.

Lemma 4.7. *Let $f \colon X \to Y$ be a 0-soft homomorphism between $AE(0)$-groups. Then there exists a Baire isomorphism $h \colon Y \times \ker f \to X$ such that $f \circ h = \pi_Y$, where $\pi_Y \colon Y \times \ker f \to Y$ stands for the projection onto the first coordinate.*

Proof. By Proposition 2.7, there exists a Baire map $g\colon Y \to X$ such that $f \circ g = \mathrm{id}_Y$. The required Baire isomorphism $h\colon Y \times \ker f \to X$ (not a homomorphism unless g is a homomorphism itself) can now be defined by letting

$$h(y,a) = g(y) \cdot a, \quad \text{for each } \ (y,a) \in Y \times \ker f,$$

where $\cdot$ denotes the multiplication operation in X. □

Theorem 4.8. *Every AE*(0)*-group is Baire isomorphic to the product of Polish groups.*

Proof. Let X be a AE(0)-group. If $w(X) = \omega$, then X itself is Polish and there is nothing to prove.

Let now $w(X) = \tau > \omega$. According to Theorem 3.4, X is topologically and algebraically isomorphic to the limit of a well-ordered continuous spectrum $\mathcal{S}_X = \{X_\alpha, p_\alpha^{\alpha+1}, \tau\}$ such that X_0 is a Polish group and the 0-soft homomorphism $p_\alpha^{\alpha+1}\colon X_{\alpha+1} \to X_\alpha$ has a Polish kernel for each $\alpha < \tau$.

Our goal is to prove that X is Baire isomorphic to the product $X_0 \times \prod\{\ker p_\alpha^{\alpha+1}\colon \alpha < \tau\}$. We proceed by induction. By Lemma 4.7, there exists a Baire isomorphism $h_1\colon X_0 \times \ker p_0^1 \to X_1$ such that $p_0^1 \circ h_1 = \pi_{X_0}$. Suppose that for each α, where $1 \le \alpha < \beta < \tau$, we have already constructed Baire isomorphism $h_\alpha\colon X_0 \times \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha\} \to X_\alpha$ in such a way that

(1) If $\alpha + 1 < \beta$, then

$$\begin{array}{ccc}
X_0 \times \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha+1\} & \xrightarrow{h_{\alpha+1}} & X_{\alpha+1} \\
\Big\downarrow{\scriptstyle \mathrm{id}_{X_0} \times \pi_\alpha^{\alpha+1}} & & \Big\downarrow{\scriptstyle p_\alpha^{\alpha+1}} \\
X_0 \times \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha\} & \xrightarrow{h_\alpha} & X_\alpha,
\end{array}$$

where

$$\pi_\alpha^{\alpha+1}\colon \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha+1\} = \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha\} \times X_\alpha \to \prod\{\ker p_\delta^{\delta+1}\colon \delta < \alpha\}$$

is the natural projection.

(2) $h_\alpha = \lim\{h_\gamma\colon \gamma < \alpha\}$, whenever $\alpha < \beta$ is a limit ordinal number.

We now construct Baire isomorphism $h_\beta\colon X_0 \times \prod\{\ker p_\delta^{\delta+1}\colon \delta < \beta\} \to X_\beta$
If β is a limit ordinal number, then we let $h_\beta = \lim\{h_\alpha\colon \alpha < \beta\}$.
If $\beta = \alpha + 1$, then consider the following commutative diagram

$$\begin{array}{ccccc}
\left(X_0\times\prod\{\ker p_\delta^{\delta+1}\colon \delta<\alpha\}\right)\times\ker p_\alpha^{\alpha+1} & \xrightarrow{h_\alpha\times\mathrm{id}} & X_\alpha\times\ker p_\alpha^{\alpha+1} & \xrightarrow{h} & X_{\alpha+1}\\
{\scriptstyle\pi_1}\big\downarrow & & {\scriptstyle\pi_{X_\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle p_\alpha^{\alpha+1}}\\
X_0\times\prod\{\ker p_\delta^{\delta+1}\colon \delta<\alpha\} & \xrightarrow{h_\alpha} & X_\alpha & \xrightarrow{\mathrm{id}_{X_\alpha}} & X_\alpha,
\end{array}$$

where

(a) $\mathrm{id}\colon \ker p_\alpha^{\alpha+1}\to\ker p_\alpha^{\alpha+1}$ stands for the identity map;
(b)

$$\pi_1\colon \left(X_0\times\prod\{\ker p_\delta^{\delta+1}\colon \delta<\alpha\}\right)\times\ker p_\alpha^{\alpha+1}\to X_0\times\prod\{\ker p_\delta^{\delta+1}\colon \delta<\alpha\}$$

denotes the projection onto the first coordinate and
(c) $h\colon X_\alpha\times\ker p_\alpha^{\alpha+1}\to X_{\alpha+1}$ is a Baire isomorphism existence of which is guaranteed by Lemma 4.7.

The required Baire isomorphism $h_{\alpha+1}\colon X_0\times\prod\{\ker p_\delta^{\delta+1}\colon \delta<\alpha+1\}\to X_{\alpha+1}$ can now be defined as the composition $h_{\alpha+1}=h\circ(h_\alpha\times\mathrm{id})$. This completes induction and finishes the construction of Baire isomorphisms h_α, $\alpha<\tau$. It is now easy to see that

$$h=\lim\{h_\alpha\colon \alpha<\tau\}\colon X_0\times\prod\{\ker p_\alpha^{\alpha+1}\colon \alpha<\tau\}\to X$$

is the required Baire isomorphism. Note here that X_0 as well as $\ker p_\alpha^{\alpha+1}$, $\alpha<\tau$, are Polish groups. Proof is completed. □

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