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Proof of a Symmetrized Trace Conjecture for the Abelian Born-Infeld Lagrangian

Paolo Aschieri[*†], Daniel Brace[‡], Bogdan Morariu[§] and Bruno Zumino[¶]

Department of Physics
University of California
and
Theoretical Physics Group
Lawrence Berkeley National Laboratory
University of California
Berkeley, California 94720

Abstract

In this paper we prove a conjecture regarding the form of the Born-Infeld Lagrangian with a $U(1)^{2n}$ gauge group after the elimination of the auxiliary fields. We show that the Lagrangian can be written as a symmetrized trace of Lorentz invariant bilinears in the field strength. More generally we prove a theorem regarding certain solutions of unilateral matrix equations of arbitrary order. For solutions which have perturbative expansions in the matrix coefficients, the solution and all its positive powers are sums of terms which are symmetrized in all the matrix coefficients and of terms which are commutators.

Keywords: Duality; Born-Infeld; Unilateral Matrix Equations

[*]present address: Sektion Physik LMU, Theresienstr. 37, D-80333 München
[†]email address: aschieri@theorie.physik.uni-muenchen.de
[‡]email address: DMBrace@lbl.gov
[§]email address: BMorariu@lbl.gov
[¶]email address: zumino@thsrv.lbl.gov

1 Duality Invariant Born-Infeld Lagrangians

In this note we prove the conjecture made in [1, 2] regarding the form of $Sp(2n,\mathbb{R})$ or $U(n,n)$ duality invariant Born-Infeld Lagrangians. See [2] for a more extensive list of references regarding the duality invariance of Born-Infeld theory. In [1, 2], inspired by [3], we exploited the fact that the square root in the $U(1)$ gauge group Born-Infeld Lagrangian can be eliminated using auxiliary fields. In the auxiliary field formulation one can generalize the theory to a higher rank abelian gauge group $U(1)^{2n}$ such that the duality group becomes $U(n,n)$. One complication discussed in [1, 2] is that one has to introduce complex gauge fields. However in [2] we also showed that after the elimination of the auxiliary fields one can impose a reality condition which preserves an $Sp(2n,\mathbb{R})$ subgroup of the duality group. For higher order matrices the elimination of the auxiliary fields is more complicated since the algebraic second order equation for the auxiliary field becomes a matrix second order equation.

The Born-Infeld Lagrangian introduced in [1, 2] with auxiliary fields is given by

$$L = \mathrm{Re}\,\mathrm{Tr}\,[\,\chi + i\lambda(\chi - \frac{1}{2}\chi\chi^\dagger + \alpha - i\beta)\,]\ ,$$

where α and β are given by the following Lorentz invariant hermitian matrices

$$\alpha^{ab} \equiv \frac{1}{2}F^a\bar{F}^b, \quad \beta^{ab} \equiv \frac{1}{2}\widetilde{F}^a\bar{F}^b.$$

Here $\widetilde{F}$ is the Hodge dual of F and a bar denotes complex conjugation. The auxiliary fields χ and λ are n dimensional complex matrices. For simplicity we have set the field S to the constant value i since as discussed in [2] it can be easily reintroduced. With this choice the duality group reduces to the maximal compact subgroup $U(n)\times U(n)$ of $U(n,n)$.

The equation of motion obtained by varying λ gives an equation for χ

$$\chi - \frac{1}{2}\chi\chi^\dagger + \alpha - i\beta = 0\ , \tag{1}$$

and after solving this equation the Lagrangian reduces to

$$L = \mathrm{Re}\,\mathrm{Tr}\,\chi\ .$$

Let $\chi = \chi_1 + i\chi_2$ where χ_1 and χ_2 are hermitian. The anti-hermitian part of (1) implies $\chi_2 = \beta$, thus $\chi^\dagger = \chi - 2i\beta$. This can be used to eliminate χ from (1) and obtain a quadratic equation for $\chi^\dagger$. Following [2], it is convenient to define $Q = \frac{1}{2}\chi^\dagger$ which then satisfies

$$Q = q + (p-q)Q + Q^2, \tag{2}$$

where

$$p \equiv -\frac{1}{2}(\alpha - i\beta) \ , \quad q \equiv -\frac{1}{2}(\alpha + i\beta) \ .$$

The Lagrangian is then

$$L = 2\,\mathrm{Re}\,\mathrm{Tr}\,Q \ . \tag{3}$$

If the degree of the matrices is one, we can solve for Q in the quadratic equation (2) and then (3) reduces to the Born-Infeld Lagrangian.

For matrices of higher degree, equation (2) can be solved perturbatively and by analyzing the first few terms in the expansion we conjectured in [1, 2] that the trace of Q can be obtained as follows. First, find the perturbative solution of equation (2) assuming p and q commute. Then the trace of Q is the trace of the symmetrized expansion

$$\mathrm{Tr}\,Q = \frac{1}{2}\,\mathrm{Tr}\left[1 + q - p - \mathcal{S}\sqrt{1 - 2(p+q) + (p-q)^2}\,\right] \ , \tag{4}$$

where the symmetrization operator $\mathcal{S}$ will be discussed in the next section. In the appendix of [2] we have also guessed an explicit formula for the coefficients of the expansion of the trace of Q

$$\mathrm{Tr}\,Q = \mathrm{Tr}\left[q + \sum_{r,s\geq 1} \begin{pmatrix} r+s-2 \\ r-1 \end{pmatrix} \begin{pmatrix} r+s \\ r \end{pmatrix} \mathcal{S}(\,p^r q^s\,)\right] \ . \tag{5}$$

In the next section we will prove that for a unilateral matrix equation of order N, the perturbative solution is a sum of terms which are symmetrized in all the matrix coefficients and of terms which are commutators. Since equation (2) is a unilateral matrix equation the trace of Q will be symmetrized in the matrix coefficients q and $p-q$. Since this is equivalent to symmetrization in q and p our conjecture (4) follows.

2 Unilateral Matrix Equations

In this section we prove a theorem regarding certain solutions of unilateral matrix equations. These are N^{th} order matrix equations for the variable ϕ with matrix coefficients A_i which are all on one side, e.g. on the left

$$\phi = A_0 + A_1\phi + A_2\phi^2 + \ldots + A_N\phi^N. \tag{6}$$

The matrices are all square and of arbitrary degree. We may equally consider the A_i's as generators of an associative algebra, and ϕ an element of this algebra which satisfies the above equation. We will prove that the formal perturbative solution of (6) around zero is a sum of symmetrized polynomials in the A_i and of terms which are commutators[a]. The same is true for all the positive powers of the solution.

By repeatedly inserting ϕ from the left hand side of (6) into the right hand side we obtain the perturbative expansion of ϕ as a sum

$$\phi = \sum_M D_M \ ,$$

where each D_M is a product of the A_i matrices. Any ordered product of these matrices will be referred to as a word. However not every word appears in the perturbative expansion of ϕ. We reserve the letter D for words that do appear[b].

Next we obtain the condition that a word must satisfy in order to be in the expansion. First note that because of (6) any word D_M can be written as the following product

$$D_M = A_s D_{M_1} \ldots D_{M_s} \tag{7}$$

[a] If the degree of the matrices is one the perturbative solution is convergent if A_0 and A_1 are sufficiently small.

[b] This notation originated from an earlier version of the proof where the perturbative expansion of ϕ was calculated diagrammatically and the diagrams were denoted by D. Although we will not use diagrams here, note that they are very useful in calculating the perturbative expansion of the solution.

for some value of s, where the D_{M_i}'s are also words in the expansion. Conversely, if all the D_{M_i}'s are words in the expansion, D_M defined in equation (7) is also a word in the expansion. By iterating (7) we obtain the following equivalent statement: for every splitting of D_M into two words $D_M = W_1 W_2$ the second word can be written as a product of terms in the expansion of ϕ

$$D_M = W_1 D_{N_1} \dots D_{N_k} \ .$$

It is convenient to assign to every matrix a dimension d such that $d(\phi) = -1$. Using (6), the dimension of the matrix A_i is given by $d(A_i) = i - 1$ and $d(D_M) = -1$. Then we obtain the following intrinsic characterization of a word in the expansion of ϕ. It is a word D such that for every splitting into two words $D = W_1 W_2$, where W_2 has at least one letter, we have

$$d(W_1) \geq 0 \quad \text{and} \quad d(D) = -1 \ . \tag{8}$$

Note that (8) is a necessary and sufficient condition for a word to be in the expansion of ϕ.

Suppose that W is an arbitrary word such that $d(W) = -1$. Then, as we will show, there is a unique cyclic permutation D of W such that D is a term in the expansion of ϕ. Let us write $W = D_{N_1} D_{N_2} \dots D_{N_k} W_1$, where D_{N_1} is the shortest word starting from the first letter such that $d(D_{N_1}) = -1$. D_{N_i} is defined in the same way, except we start from the first letter after the word $D_{N_{i-1}}$. Finally W_1 is whatever is left over. We use the notation D_{N_i} since they correspond to terms in the ϕ expansion. To see this, note that the total dimension of a word can increase or decrease when a letter is added on the right, but if it decreases it can only do so by one unit. This is when the letter added is A_0. Combining this with the fact that D_{N_i} is the shortest word which satisfies $d(D_{N_i}) = -1$ then implies that if D_{N_i} is a product of two words the dimension of the first word is greater than or equal to zero. This is just the condition (8). Then using the fact that $d(W_1) = k - 1$ one can check that the cyclic permutation of W defined as $D = W_1 D_{N_1} \dots D_{N_k}$ satisfies (8), thus it belongs to the expansion of ϕ. Note that all the other cyclic permutations lead to words that are not in the expansion. Assuming the converse implies that two distinct terms in the expansion can be related by a cyclic permutation. But this is impossible: if we write $D = W_1 W_2$, then $d(W_1) \geq 0$ and thus $d(W_2) \leq -1$, so that its cyclic permutation $W_2 W_1$ does not satisfy (8). A similar argument can be used to show that all different cyclic permutations of a term in the expansion of ϕ lead to distinct words.

Consider the trace of the sum of all distinct words of dimension $d = -1$ and of order a_i in A_i. We can group together all words that are cyclic permutations of each other, and replace each group by a single word with coefficient $\sum_{i=0}^{N} a_i$. Using the result of the previous paragraph, we can choose this word to satisfy (8). Thus we have

$$\mathrm{Tr}\left(\sum_{\text{order }\{a_i\}} D_M\right) = \left(\sum_{i=0}^{N} a_i\right)^{-1} \mathrm{Tr}\left(\sum_{\text{order }\{a_i\}} W\right), \tag{9}$$

where the sum in the right hand side is over all distinct words of some fixed order $\{a_i\}$ and of dimension $d(W) = -1$.

We define the symmetrization operator $\mathcal{S}$ as a linear operator acting on monomials as

$$\mathcal{S}(A_0^{a_0} A_1^{a_1} \ldots A_N^{a_N}) = \frac{a_0! a_1! \ldots a_N!}{\left(\sum_{i=0}^{N} a_i\right)!} \left(\sum_{\text{order }\{a_i\}} W\right), \tag{10}$$

where the sum is over distinct words of fixed order $\{a_i\}$. Equivalently, a word can be symmetrized by averaging over all permutations of its letters. Not all permutations give distinct words and this accounts for the numerator on the right side of equation (10). The normalization of $\mathcal{S}$ is such that on commutative A_i's $\mathcal{S}$ acts as the identity.

Combining (9) and (10), we can obtain the solution for the trace of ϕ to all orders

$$\mathrm{Tr}\,\phi = \sum_{\substack{\{a_i\} \\ \sum (i-1)a_i = -1}} \frac{\left(\sum_{i=0}^{N} a_i - 1\right)!}{a_0! a_1! \ldots a_N!} \mathrm{Tr}\,\mathcal{S}(A_0^{a_0} A_1^{a_1} \ldots A_N^{a_N}), \tag{11}$$

where the sum is over all sets $\{a_i\}$ restricted to words of dimension $d = -1$. More generally, if the A_i's are considered to be the generators of an associative algebra, we can replace the trace in (11) with the cyclic average operator which was defined in [2]. This is true since in the proof we only used the cyclic property of the trace which also holds for the cyclic average operator. Therefore, the solution ϕ can be written as a sum of symmetric polynomials and terms which are commutators. This is the statement we set out to prove. Notice that our derivation implies that the coefficients in (11) are all integers.

Using the same kind of arguments we used to derive equation (11), we can also prove that the trace of positive powers of ϕ is given by

$$\mathrm{Tr}\,\phi^r \;=\; r \sum_{\substack{\{a_i\} \\ \sum (i-1)a_i = -r}} \frac{\left(\sum_{i=0}^N a_i - 1\right)!}{a_0! a_1! \dots a_N!} \;\mathrm{Tr}\;\mathcal{S}(A_0^{a_0} A_1^{a_1} \dots A_N^{a_N}) \;. \tag{12}$$

Furthermore we can write a generating function for (12)

$$\mathrm{Tr}\;\log(1-\phi) = \mathrm{Tr}\;\log(1 - \sum_{i=0}^N A_i)\Big|_{d<0} \;. \tag{13}$$

On the right hand side of (13) one must expand the logarithm and restrict the sum to words of negative dimension. Since $d(\phi^r) = -r$ we can obtain (12) by extracting the dimension $d = -r$ terms from the right hand side of (13). Note that all the terms in the expansion of Tr $\log(1 - \sum_{i=0}^N A_i)$ are automatically symmetrized.

It is possible to give a simple proof of (13) without going through the combinatoric arguments above, which however give a construction of the solution and its powers themselves, not only their trace. First note that we can rewrite equation (6) as

$$1 - \sum_{i=0}^N A_i \;=\; 1 - \phi - \sum_{k=1}^N A_k(1-\phi^k)$$

The right hand side factorizes

$$1 - \sum_{i=0}^N A_i \;=\; (1 - \sum_{k=1}^N \sum_{m=0}^{k-1} A_k \phi^m)(1-\phi) \;.$$

Under the trace we can use the fundamental property of the logarithm, even for noncommutative objects, and obtain

$$\mathrm{Tr}\,\log(1-\sum_{i=0}^{N}A_i) \;=\; \mathrm{Tr}\,\log(1-\sum_{k=1}^{N}\sum_{m=0}^{k-1}A_k\phi^m)+\mathrm{Tr}\,\log(1-\phi)\;.$$

Using $d(A_k)=k-1$ and $d(\phi)=-1$ we have $d(A_k\phi^m)=k-m-1$ and we see that all the words in the argument of the first logarithm on the right hand side have semi-positive dimension. Since all the words in the expansion of the second term have negative dimension we obtain (13).

If the coefficient A_N is unity, we have the following identity for the symmetrization operator

$$\mathcal{S}(A_0^{a_0}A_1^{a_1}\dots A_N^{a_N})|_{A_N=1}=\mathcal{S}(A_0^{a_0}A_1^{a_1}\dots A_{N-1}^{a_{N-1}})\;.$$

This is obviously true up to normalization; the normalization can be checked in the commutative case.

The trace of the solution of (2) can now be obtained from (11) by taking $N=2$ and setting A_2 to unity. The restriction on the sum of (11) in this case reads $a_0-a_2=1$. The sum can then be rewritten

$$\mathrm{Tr}\,\phi \;=\; \sum_{a_0=1}^{\infty}\sum_{a_1=0}^{\infty}\frac{(2a_0+a_1-2)!}{a_0!\,a_1!\,(a_0-1)!}\;\mathrm{Tr}\;\mathcal{S}(A_0^{a_0}A_1^{a_1})\;. \tag{14}$$

Using $\phi=Q$, $A_0=q$, $A_1=p-q$, the combinatoric identity

$$\begin{pmatrix} a+b \\ c \end{pmatrix} = \sum_{m=\max(0,c-b)}^{\min(a,c)} \begin{pmatrix} a \\ m \end{pmatrix}\begin{pmatrix} b \\ c-m \end{pmatrix}$$

and the resummation identities

$$\begin{aligned} \sum_{r\geq 1}\quad \sum_{a=0}^{r} &= \sum_{a=0}^{\infty}\quad \sum_{r=\max(a,1)}^{\infty}\;, \\ \sum_{r=\max(a,1)}^{\infty}\quad \sum_{b=r-a+1}^{\infty} &= \sum_{b=\max(1,2-a)}^{\infty}\quad \sum_{r=\max(a,1)}^{a+b-1} \end{aligned}$$

one can show that (14) reduces to (5).

3 Discussion

After completing the first version of this paper [4], where we only proved the symmetrization theorem for the trace of ϕ, we learned through private communications that A. Schwarz was developing another method [5] of proving the theorem (for a slightly different, but related equation). Using his method he was able to show that the theorem is true for arbitrary powers of the solution. Inspired by this, we also extended the theorem, using our method, to positive powers of ϕ, see (12). In the process we discovered the simpler proof using the generating function (13).

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