Gravity and the Newtonian limit in the Randall–Sundrum model

R. Dick, D. Mikulovicz
Sektion Physik der Ludwig–Maximilians–Universität
Theresienstr. 37, 80333 München, Germany

Abstract: We point out that the gravitational evolution equations in the Randall–Sundrum model appear in a different form than hitherto assumed. As a consequence, the model yields a correct Newtonian limit in a novel manner.

1 Introduction

Randall and Sundrum recently explained the large hierarchy between the Planck scale and the weak scale in terms of a model where our observable universe corresponds to a (3+1)-dimensional boundary of a (4+1)-dimensional manifold. The extra spacelike dimension in this model is non-periodic and finite [1] or eventually even of infinite length [2], and ordinary matter is restricted to the boundary while gravitational modes may propagate in the bulk. This model attracted a lot of attention, see [3] and references there.

Usually the Randall–Sundrum model is investigated under the assumption that the coupling of gravitational modes to the matter on the boundary is governed by the restriction of the Einstein tensor to the boundary. This led in particular to the claim that the Randall–Sundrum model with a small non-periodic dimension would yield antigravity [4].

In the present paper we reconsider the gravitational evolution equations and the Newtonian limit of the Randall–Sundrum model. Contrary to the common assumption, the coupling of matter to gravity in this model does not appear through an Einstein equation on the boundary but through Neumann type boundary conditions for the Einstein equation in the bulk. This deviation from Einstein gravity on the boundary cures the antigravity problem: Gravity on the boundary is attractive and has a correct Newtonian limit.

The observation that the Randall–Sundrum model implies first order equations for the metric on the boundary instead of second order equations does not depend on whether one uses the Einstein–Hilbert term or an Einstein term for the gravitational action in the bulk. However, with the Einstein–Hilbert term the system of gravitational evolution equations is overdetermined and therefore we use an Einstein term. To explain this, we point out several differences between the Randall–Sundrum model and Kaluza–Klein theory in the next section before we address the equations of motion for the metric and the Newtonian limit in the Randall–Sundrum model.

To avoid confusion, we count the dimension of spaces with Minkowski signature explicitly in the form $d+1$, with $d = 3, 4$.

2 Differences to Kaluza–Klein theory

One might presume that gravity should contribute an Einstein–Hilbert term to the action of the Randall–Sundrum model, and that gravity in $3+1$ dimensions should arise in a similar way as in a Kaluza–Klein theory with periodic dimensions.

In Kaluza–Klein theories with small periodic internal dimensions the low-energy degrees of freedom are restricted to zero modes which are separated by a large mass gap from the massive modes. The zero modes are indepedent from the internal coordinates and the resulting low-energy theory is genuine (3+1)-dimensional. However, the periodicity constraints are instrumental for the emergence of the mass gap. Contrary to Kaluza–Klein theory, boundary conditions in a bounded *non-periodic* dimension have to be fixed dynamically by the equations of motion, and *a priori* this does not imply a restriction to zero modes separated by a mass gap. By the same token, we have to subtract a complete divergence from the Einstein–Hilbert term in the action of the Randall–Sundrum model.

For an explanation of this point, consider the Einstein–Hilbert action with a cosmological term in the $(4+1)$-dimensional universe of the Randall–Sundrum model ($d^5x = d^4x dx^5$, $0 \leq x^5 \leq L$):

$$\begin{aligned} S_{EH} &= \int d^5x \sqrt{-g}\Big(\frac{\mu^3}{2} g^{MN} R_{MN} - \Lambda\Big), \\ \delta S_{EH} &= \int d^5x \sqrt{-g}\delta g^{MN}\left(\frac{\mu^3}{2}\Big(R_{MN} - \frac{1}{2} g_{MN} g^{KL} R_{KL}\Big) + \frac{1}{2} g_{MN}\Lambda\right) \\ &+ \frac{\mu^3}{2}\oint d^4x \sqrt{-g}\Big(g^{MN}\delta\Gamma^5{}_{MN} - g^{5N}\delta\Gamma^M{}_{MN}\Big)\Big|_{x^5=0}^{L}. \end{aligned} \tag{1}$$

If matter degrees of freedom could propagate on the whole manifold and if the fields would be periodic, then

• the boundary terms would cancel

and

• we could perform a Fourier decomposition of the degrees of freedom and throw away the massive Kaluza–Klein modes.

This would then correspond to original Kaluza–Klein theory and yield low-dimensional Einstein gravity in the usual way. However, the space-time points $x^0, \ldots x^3, x^5 = 0$ and $x^0, \ldots x^3, x^5 = L$ are different physical points in the Randall–Sundrum model and periodicity is not required (and cannot be required by causality). Furthermore, matter degrees of freedom are supposed to be fixed to the $(3+1)$-dimensional boundaries, and therefore variation of corresponding action principles yields homogeneous equations for the gravitational degrees of freedom in the bulk, while the coupling to the matter degrees of freedom arises from the variation on the boundaries. Below we will point out that the gravitational potential in this theory does not correspond to a three-dimensional Greens function for Dirichlet boundary conditions at infinity, but to a four-dimensional Greens function for Neumann boundary conditions on three-dimensional boundaries. *A priori* this implies deviations from the ordinary Newton potential in three dimensions. However, for a small non-periodic extra dimension of length L the leading terms in the Newtonian limits of the Randall–Sundrum model and Einstein gravity agree if the naive estimate on the relation between the $(3+1)$- and $(4+1)$-dimensional Planck masses (inferred from the corresponding relation in Kaluza–Klein theory) is augmented by a factor 3. Another difference to Kaluza–Klein theory concerns the fact, that for a small non-periodic extra dimension the deviations from the Newtonian limit of Einstein gravity are not suppressed by a term $\exp(-r/L)$ but correspond to an expansion in $r/V_3^{1/3} \ll 1$, where V_3 is the 3-volume of a time slice of the $(3+1)$-dimensional boundary. As a consequence, in the large V_3 limit the gravitational potential has the usual form, with the correction term corresponding to a renormalization of the Planck mass.

3 The gravitational potential in the Randall–Sundrum model

We have seen that the excitation of space-time curvature in the Randall–Sundrum model does not arise due to matter sources in the bulk equations, but through boundary conditions on the gravitational field arising from boundary equations of motion. This raises the issue of the Newtonian limit for the gravitational field on the boundary, which we examine through the $(4+1)$-dimensional action

$$S = \int_{\mathcal{V}} d^5x\sqrt{-g}\left(\frac{\mu^3}{2}g^{KL}(\Gamma^M{}_{NK}\Gamma^N{}_{ML} - \Gamma^M{}_{NM}\Gamma^N{}_{KL}) - \Lambda\right) + \sum_{i=1}^{2}\int_{\partial\mathcal{V}_i} d^4x\mathcal{L}_i. \quad (2)$$

Here $\partial\mathcal{V}_i$ are the two connected components of the $(3+1)$-dimensional boundary and $\mathcal{L}_i$ denotes the Lagrangians for the matter degrees of freedom on the boundary components. The Lagrangians $\mathcal{L}_i$ may also contain cosmological terms on the boundary. Coordinates x^0, x^1, x^2, x^3, x^5 are chosen such that the two boundary components correspond to $x^5 = 0$ and $x^5 = L$, respectively.

The gravitational part in (2) is fixed from two requirements:

• The Einstein tensor is the leading derivative term in any evolution equation for the metric on a Riemannian manifold, and this should also hold true in the present model, since there is no symmetry prohibiting this leading curvature term.

• At the same time, the full Einstein–Hilbert Lagrangian (as well as a leading higher curvature term in the action) would not give consistent boundary equations of motion, due to the second derivatives on the metric tensor:

The divergence included in the Einstein–Hilbert Lagrangian yields boundary terms $\sim \partial_5\delta g_{\mu\nu}$ which have no counterpart in the $\delta\mathcal{L}_i$ terms and overdetermine the boundary value problem for the metric. Therefore, we used Einstein's well-known Lagrangian

(adapted to $4+1$ dimensions)

$$\mathcal{L}_E = \frac{\mu^3}{2}\sqrt{-g}g^{KL}(\Gamma^M{}_{NK}\Gamma^N{}_{ML} - \Gamma^M{}_{NM}\Gamma^N{}_{KL}).$$

This subtracts the divergence term from $\sqrt{-g}R$ and yields the full Einstein tensor in the bulk.

In analyzing (2) it is convenient to choose the bulk coordinate orthogonal to the boundaries: $g_{\mu 5}|_{x^5=0,L} = 0$, $0 \leq \mu \leq 3$. Variation of (2) yields again a sum of a $(4+1)$-dimensional integral and an integral over the boundary, implying gravitational equations of motion in the bulk

$$R_{MN} = \frac{2\Lambda}{3\mu^3}g_{MN} \tag{3}$$

and on the boundary:

$$g^{\lambda\nu}\partial_\mu g_{\lambda\nu}|_{x^5=0,L} = g^{55}\partial_\mu g_{55}|_{x^5=0,L}, \tag{4}$$

$$\partial_5 g_{\mu\nu}|_{x^5=0} = -\frac{2}{\mu^3}g_{55}\Big(T^{(1)}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}g^{\kappa\lambda}T^{(1)}_{\kappa\lambda}\Big), \tag{5}$$

$$\partial_5 g_{\mu\nu}|_{x^5=L} = \frac{2}{\mu^3}g_{55}\Big(T^{(2)}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}g^{\kappa\lambda}T^{(2)}_{\kappa\lambda}\Big). \tag{6}$$

Here $g_{\mu\nu}$ denotes the tangent components of the metric tensor on the boundary, and (4) arises from boundary terms $\sim \delta g^{5\mu}$, while (5) and (6) arise from boundary terms $\sim \delta g^{\mu\nu}$. No boundary terms $\sim \delta g^{55}$ appear.

The energy momentum tensors on the boundary components are

$$T^{(i)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_i}{\delta g^{\mu\nu}},$$

and as usual in this kind of variational problems, the boundary equations amount to boundary conditions for the bulk equations of motion.

Eq. (4) has two implications: On the one hand it tells us that the determinant $-g_{(4)}$ of the metric induced on the boundary determines the boundary value of g_{55} up to a constant factor, and on the other hand it ensures invariance of (2) under diffeomorphisms $x^M \to x^M - \epsilon^M(x)$ which leave the boundary invariant: $\epsilon^5(x)|_{x^5=0,L} = 0$.

To examine the gravitational potential emerging in the Randall–Sundrum model, it is useful to reformulate the evolution equations for spatially closed $(3+1)$-dimensional boundary universes, i.e. we consider x^5 as a radial coordinate between two spherical shells at radii $a \leq x^5 = r \leq b$. Eqs. (5,6) then read

$$\frac{\partial}{\partial r}g_{\mu\nu}\Big|_{r=a} = -\frac{2}{\mu^3}g_{55}\Big(T^{(1)}_{\mu\nu} - \frac{1}{3}g_{\mu\nu}g^{\kappa\lambda}T^{(1)}_{\kappa\lambda}\Big). \tag{7}$$

$$\frac{\partial}{\partial r}g_{\mu\nu}\Big|_{r=b}=\frac{2}{\mu^3}g_{55}\Big(T^{(2)}_{\mu\nu}-\frac{1}{3}g_{\mu\nu}g^{\kappa\lambda}T^{(2)}_{\kappa\lambda}\Big). \tag{8}$$

In the Newtonian approximation we consider weakly coupled gravitational systems on time scales much shorter than the age of the universe and length scales far below the Hubble radius. In this approximation cosmological background metrics can very well be approximated by a local Minkowski background.
The weak field approximation $g_{MN}=\eta_{MN}+h_{MN}$ for static sources $T^{(i)}_{00}=\varrho_i$ on the boundary yields a Neumann type boundary problem for the gravitational potential $U=-h_{00}/2$:

$$\Delta U=0, \tag{9}$$

$$\frac{\partial}{\partial r}U\Big|_{r=a}=\frac{2}{3\mu^3}\varrho_1, \tag{10}$$

$$\frac{\partial}{\partial r}U\Big|_{r=b}=-\frac{2}{3\mu^3}\varrho_2. \tag{11}$$

As a consequence, the gravitational interaction between matter components on the boundary arises through a four-dimensional Greens function adapted to Neumann boundary conditions:

$$\begin{aligned} U(\mathbf{r}) &= \oint_{\partial V} d^3\mathbf{r}'\Big(G(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r'}U(\mathbf{r}')-U(\mathbf{r}')\frac{\partial}{\partial r'}G(\mathbf{r},\mathbf{r}')\Big)\Big|_{r'=a}^{r'=b} \\ &= \langle U\rangle-\frac{2}{3\mu^3}\int_{r'=a}d^3\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')\varrho_1(\mathbf{r}')-\frac{2}{3\mu^3}\int_{r'=b}d^3\mathbf{r}'\,G(\mathbf{r},\mathbf{r}')\varrho_2(\mathbf{r}'). \end{aligned} \tag{12}$$

Here $\langle U\rangle$ is the average value of U on the boundary, and $d^3\mathbf{r}'$ is the spatial volume element on the boundary ∂V of a time slice V of $\mathcal{V}$.
The Greens function for the Neumann boundary problem is defined by the requirements

$$\Delta' G(\mathbf{r},\mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'),$$

$$\frac{\partial}{\partial r'}G(\mathbf{r},\mathbf{r}')\Big|_{r'=a}=-\frac{\partial}{\partial r'}G(\mathbf{r},\mathbf{r}')\Big|_{r'=b}=\frac{1}{2\pi^2(a^3+b^3)}$$

and we have calculated it for a spatial four-manifold bounded by two concentric three-spheres:

$$4\pi^2G(\mathbf{r},\mathbf{r}')=\frac{1}{a^3+b^3}\left(\frac{b^3}{r_>^2}-\frac{a^3}{r_<^2}\right)+\sum_{l=1}^{\infty}\left(\frac{r_<^l}{r_>^{l+2}}+\frac{l+2}{l}\frac{r^l r'^l}{b^{2l+2}-a^{2l+2}}\right. \tag{13}$$

$$+\frac{a^{2l+2}}{b^{2l+2}-a^{2l+2}}\left(\frac{r^l}{r'^{l+2}}+\frac{r'^l}{r^{l+2}}+\frac{l}{l+2}\frac{b^{2l+2}}{r^{l+2}r'^{l+2}}\right)\bigg)\frac{\sin((l+1)\theta)}{\sin\theta}.$$

Here θ denotes the angle between the four-dimensional vectors $\mathbf{r}$ and $\mathbf{r}'$, and like in three-dimensional multipole expansions $r_<$ is the smaller of the two radii r and r', while $r_>$ is the larger radius.

If we choose the three-sphere at $r=a$ as the time slice of our $(3+1)$-dimensional universe in this scenario, the gravitational potential between ordinary matter sources and probes arises in the limit $r, r' \to a$, and the distance between source and probe within this three-sphere is $d = a\sin\theta$. Up to an irrelevant constant term the gravitational potential of a mass m on the 3-sphere S_a^3 of radius a follows from (13,12)

$$U(\theta) = -\frac{m}{6\pi^2\mu^3a^2}\sum_{l=1}^{\infty}\left(1+\frac{a^{2l+2}}{b^{2l+2}-a^{2l+2}}\left(3+\frac{2}{l}+\frac{l}{l+2}\frac{b^{2l+2}}{a^{2l+2}}\right)\right)\frac{\sin((l+1)\theta)}{\sin\theta}. \quad (14)$$

Here θ is the angle between the source m of the gravitational field and the point where it is probed.

A large internal dimension corresponds to $b \gg a$ and yields

$$U(\theta)|_{b\gg a} = -\frac{m}{3\pi^2\mu^3a^2}\sum_{l=1}^{\infty}\frac{l+1}{l+2}\frac{\sin((l+1)\theta)}{\sin\theta}. \quad (15)$$

In the other case of small internal length $L \ll a, b = a+L$ we find

$$U(\theta)|_{a,b=a+L\gg L} = -\frac{m}{3\pi^2\mu^3aL}\sum_{l=1}^{\infty}\frac{l+1}{l(l+2)}\frac{\sin((l+1)\theta)}{\sin\theta}. \quad (16)$$

For comparison, the genuine three-dimensional gravitational potential on a 3-sphere of radius a is:

$$\mathcal{U}(\theta) = -\frac{m}{4\pi m_{Pl}^2a}\cot\theta = -\frac{2m}{\pi^2m_{Pl}^2a}\sum_{l=1}^{\infty}\frac{l}{(2l-1)(2l+1)}\frac{\sin(2l\theta)}{\sin\theta}, \quad (17)$$

where $m_{Pl} = (8\pi G_N)^{-1/2}$ is the reduced Planck mass on S_a^3.

As expected from a higher-dimensional potential, $U(\theta)|_{b\gg a}$ has a stronger singularity for $\theta \to 0$ than the ordinary Newton potential on S_a^3.

The case of small non-periodic extra dimension is more subtle: The odd-l modes of $U(\theta)|_{a,b=a+L\gg L}$ are absent in the classical inherently three-dimensional potential $\mathcal{U}(\theta)$, but the even modes agree if the naive Kaluza–Klein type relation between μ and m_{Pl} is augmented by a factor 3:

$$3\mu^3L = m_{Pl}^2. \quad (18)$$

Contrary to Kaluza–Klein theory, the relation (18) in the present theory eliminates the parameter L completely from the correction term to the ordinary Newton potential:

$$U(\theta)|_{a,b=a+L\gg L} - \mathcal{U}(\theta) = -\frac{m}{4\pi^2 m_{Pl}^2 a}\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}\frac{\sin((2l+1)\theta)}{\sin\theta}, \tag{19}$$

and therefore the corrections to the Newton potential are not suppressed by a factor $\exp(-d/L)$ but correspond to an expansion in d/a. In the limit $a\to\infty$, the correction term corresponds to a renormalization of the three-dimensional Planck mass, but the functional dependence on the distance between source and probe is just that of the three-dimensional Newton potential.

We finally would like to point out that the 4D Poincaré invariant metric of Randall and Sundrum [1] complies with the boundary equations (4–6):

In the present conventions the metric arises from the *Ansatz*

$$g_{\mu\nu} = \exp(-2\sigma(x^5))\eta_{\mu\nu}, \quad g_{55} = 1$$

under the assumption of boundary cosmological terms:

$$\mathcal{L}_1 = -\lambda_1 \exp(-4\sigma(0)),$$

$$\mathcal{L}_2 = -\lambda_2 \exp(-4\sigma(L))$$

corresponding to boundary energy momentum tensors

$$T_{\mu\nu}|_{x^5=0} = -\lambda_1 g_{\mu\nu}|_{x^5=0},$$

$$T_{\mu\nu}|_{x^5=L} = -\lambda_2 g_{\mu\nu}|_{x^5=L}.$$

The Einstein equation in the volume yields again (cf. eq. (7) in Ref. [1])

$$\sigma'^2 = -\frac{\Lambda}{6\mu^3},$$

and the boundary equations imply

$$\sigma' = \frac{1}{3\mu^3}\lambda_1 = -\frac{1}{3\mu^3}\lambda_2,$$

i.e. eq. (11) from Ref. [1] is only rescaled by a factor 4

$$\Lambda = -\frac{2}{3\mu^3}\lambda_i^2.$$

We conclude that the criticism of the small-L Randall–Sundrum model was based on an incorrect set of gravitational evolution equations and not justified. Gravity in the Randall–Sundrum model is not repulsive, and it has a correct Newtonian limit.

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References

[1] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370.

[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690.

[3] T. Nihei, Phys. Lett. B 465 (1999) 81;
N. Arkani-Hamed, S. Dimopoulos, G. Dvali and N. Kaloper, hep-th/9907209;
N. Kaloper, Phys. Rev. D 60 (1999) 123506, hep-th/9912125v2;
H. Hatanaka, M. Sakamoto, M. Tachibana and K. Takenaga, hep-th/9909076;
E. Halyo, hep-th/9909127;
A. Chamblin and G.W. Gibbons, hep-th/9909130v3;
A. Chamblin, S.W. Hawking and H.S. Reall, hep-th/9909205;
E.E. Flanagan, S.-H.H. Tye and I. Wasserman, hep-th/9910498v2;
J. Garriga and T. Tanaka, hep-th/9911055;
D. Youm, hep-th/9911218v4;
T.-j. Li, hep-th/9911234;
C. Csáki, M. Graesser, L. Randall and J. Terning, hep-ph/9911406;
D. Ida, gr-qc/9912002;
T. Ozeki and N. Shimoyama, hep-th/9912276;
M. Cvetič, H. Lü and C.N. Pope, hep-th/0001002;
Y.S. Myung and Gungwon Kang, hep-th/0001003.

[4] T. Shiromizu, K.-i. Maeda and M. Sasaki, gr-qc/9910076v2.