Naked Singularities in Higher Dimensional Inhomogeneous Dust Collapse

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Abstract

We investigate the occurrence and nature of a naked singularity in the gravitational collapse of an inhomogeneous dust cloud described by a non self-similar higher dimensional Tolman spacetime. The necessary condition for the formation of a naked singularity or a black hole is obtained. The naked singularities are found to be gravitationally strong in the sense of Tipler and provide another example that violates the cosmic censorship conjecture.

Key Words: Naked singularity, cosmic censorship, higher dimensions
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1 Introduction

The cosmic censorship hypothesis (CCH) [1] is an important source of inspiration for research in general relativity. It states that the singularities produced by gravitational collapse must be hidden behind an event horizon. Moreover, according to the strong version of the CCH, such singularities are not even locally naked, i.e., no non-spacelike curve can emerge from such singularities. Since the conjecture has as yet no precise mathematical formulation, a significant amount of attention has been given to studying examples of gravitational collapse which lead to naked singularities with matter content that satisfies the energy conditions (see [2] for recent

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reviews). In particular, the collapse of spherical matter in the form of dust forms naked shell focusing singularities violating the CCH [3]-[11].

Recently, significant efforts have been expended to study gravitational collapse models in higher dimensional spacetime. llha *et al* [12] have constructed Oppenheimer-Snyder models in higher dimensional spacetime. The self-similar solution of spherically symmetric gravitational collapse of a scalar field in higher dimensions is obtained in [13]. Gravitational collapse of a perfect fluid in higher dimensional spacetime is studied by Rocha and Wang [14]. In particular, one would like to understand the role played by extra dimensions in the formation and the nature of singularities. The solution for the higher dimensional spherically symmetric dust collapse is obtained in [15], which reduces to the well known Tolman-Bondi solution when the dimension of the spacetime becomes four. We shall call it the higher dimensional Tolman spacetime. They have shown the occurrence of naked singularities for the self-similar case. However, self similarity is a strong geometric condition on the spacetime and thus gives rise to a possibility that the naked singularity could be the artifact of a geometric condition rather than the gravitational dynamics of matter therein.

The objective of this paper is to analyze in some detail the collapse of an inhomogeneous dust cloud in higher dimensions to cover both non self-similar and self-similar models. We also access the curvature strength of central shell focusing singularities. We find that gravitational collapse of a non self-similar higher dimensional spacetime gives rise to a naked strong-curvature shell-focusing singularity, providing an explicit counter-example to the CCH.

2 Higher Dimensional Tolman Solution

The idea that spacetime should be extended from four to higher dimensions was introduced by Kaluza and Klein [16] to unify gravity and electromagnetism. Five-dimensional ($5D$) spacetime is particularly more relevant because both $10D$ and $11D$ supergravity theories yield solutions where a $5D$ spacetime results after dimensional reduction [17]. The metric for the $5D$ case, in comoving coordinates coordinates, assumes the form:

$$ds^2 = -dt^2 + \frac{R'^2}{1+f(r)}dr^2 + R^2 d\Omega^2 \tag{1}$$

where $d\Omega^2 = d\theta_1^2 + sin^2\theta_1(d\theta_2^2 + sin^2\theta_2 d\theta_3^2)$ is the metric of a 3 sphere, r is the comoving radial coordinate, t is the proper time of freely falling shells, R is a function of t and r with $R > 0$ and a prime denotes a partial derivative with respect to r. The energy momentum tensor is of the form:

$$T_{ab} = \epsilon(t,r)u_a u_b \tag{2}$$

where u_a is the five velocity. The function $R(t,r)$ is the solution of

$$\dot{R}^2 = \frac{F(r)}{R^2} + f(r) \tag{3}$$

where an overdot denotes the partial derivative with respect to t. The functions $F(r)$ and $f(r)$ are arbitrary, and result from the integration of the field equations. They are referred to as the mass and energy functions, respectively. Since in the present discussion we are concerned with gravitational collapse, we require that $\dot{R}(t,r) < 0$.

The energy density $\epsilon(t,r)$ is given by

$$\epsilon(t,r) = \frac{3F'}{2R^3R'} \tag{4}$$

We have used units which fix the speed of light and the gravitational constant via $8\pi G = c^4 = 1$. For physical reasons, one assumes that the energy density ϵ is everywhere nonnegative (≥ 0). Eq. (3) can easily be integrated to

$$t - t_c(r) = -\frac{R^2}{\sqrt{F}} G(fR^2/F) \tag{5}$$

where $G(x)$ is the function given by

$$G(x) = \begin{cases} \frac{\sqrt{1+x}}{x}, & x \neq 0, \\ \frac{1}{2} & x = 0. \end{cases} \tag{6}$$

and where $t_c(r)$ is a function of integration which represents the time taken by the shell with coordinate r to collapse to the centre. This is unlike the $4D$ case, where the functional form of G is rather complicated [6, 9]. As it is possible to make an arbitrary relabeling of spherical dust shells by $r \to g(r)$, without loss of generality, we fix the labeling by requiring that, on the hypersurface $t = 0$, r coincide with the radius

$$R(0,r) = r \tag{7}$$

This corresponds to the following choice of $t_c(r)$

$$t_c(r) = \frac{r^2}{\sqrt{F}} G(fr^2/F) \tag{8}$$

We denote by $\rho(r)$ the initial density:

$$\rho(r) \equiv \epsilon(0,r) = \frac{3F'}{2r^3} \Rightarrow F(r) = \frac{2}{3} \int \rho(r) r^3 dr \tag{9}$$

Given a regular initial surface, the time for the occurrence of the central shell-focussing singularity for the collapse developing from that surface is reduced as compared to the $4D$ case for the marginally bound collapse. The reason for this stems from the form of the mass function in Eq. (9). In a ball of radius 0 to r, for any given initial density profile $\rho(r)$, the total mass contained in the ball is greater than in the corresponding $4D$ case. In the $4D$ case, the mass function $F(r)$ involves the integral $\int \rho(r)r^2dr$ [2], as compared to the factor r^3 in the $5D$ case. Hence, there is relatively more mass-energy collapsing in the spacetime as compared to the $4D$ case, because of the assumed overall positivity of mass-energy (energy condition). This explains why the collapse is faster in the $5D$ case.

The easiest way to detect a singularity in a spacetime is to observe the divergence of some invariant of the Riemann tensor. Next we calculate one such quantity, the Kretschmann scalar ($K = R_{abcd}R^{abcd}$, R_{abcd} the Riemann tensor). For the metric (1), it reduces to

$$K = 7\frac{F'^2}{R^6 R'^2} - 36\frac{FF'}{R^7 R'} + 78\frac{F^2}{R^8} \tag{10}$$

The Kretschmann scalar and energy density both diverge at $t = t_c(r)$ confirming the presence of a scalar polynomial curvature singularity [18]. Thus the time coordinate and radial coordinate are respectively in the ranges $-\infty < t < t_c(r)$ and $0 \leq r < \infty$. It has been shown [6] that shell crossing singularities (characterized by $R' = 0$ and $R > 0$) are gravitationally weak and hence such singularities cannot be considered seriously. Christodoulou [4] pointed out in the $4D$ case that the non-central singularities are not naked. Hence, we shall confine our discussion to the central shell focusing singularity.

3 Existence and Nature of Naked Singularity

It is known that, depending upon the inhomogeneity factor, the $4D$ Tolman-Bondi metric admits a central shell focusing naked singularity in the sense that outgoing geodesics emanate from the singularity. Here we wish to investigate the similar situation in our higher dimensional spacetime. In what follows, we shall confine ourselves to the marginally bound case ($f = 0$). Eq. (5), by virtue of eq. (7), leads to

$$R^2 = r^2 - 2\sqrt{F}t \tag{11}$$

and the energy density becomes

$$\epsilon(t,r) = \frac{3/2}{\left[t - \frac{2r\sqrt{F}}{F'}\right]\left[t - \frac{r^2}{2\sqrt{F}}\right]} \tag{12}$$

We are free to specify $F(r)$ and we consider a class of models which are non self-similar in general, and as a special case, the self-similar models can be constructed from them. In particular, we suppose that $F(r) = r^2\lambda(r)$ and $\lambda(0) = \lambda_0 > 0$ (finite). With this choice of $F(r)$, the density behaves as inversely proportional to the square of time at the centre, and $F(r) \propto r^2$ in the neighborhood of $r = 0$. For spacetime to be self-similar, we require that $\lambda(r) = const.$ This class of models for $4D$ spacetime is discussed in refs. [9]. From eq. (4) it is seen that the density at the centre (r=0) behaves with time as $\epsilon = 3/2t^2$. This means that the density becomes singular at $t = 0$ and finite at any time $t = t_0 < 0$. Thus the singularity arises from dust collapse which had a finite density distribution in the past on an initial epoch. At this initial nonsingular epoch, all the physical parameters, including the density, are finite and well-behaved. Thus our collapse starts from regular initial data.

We wish to investigate if the singularity, when the central shell with comoving coordinate $(r = 0)$ collapses to the centre at time $t = 0$, is naked. The singularity is naked iff there exists a null geodesic which emanates from the singularity. Let $K^a = dx^a/dk$ be the tangent vector to the radial null geodesic, where k is an affine parameter. Then we derive the following equations

$$\frac{dK^t}{dk} + \dot{R}' K^r K^t = 0 \tag{13}$$

$$\frac{dt}{dr} = \frac{K^t}{K^r} = R' \tag{14}$$

The last equation, upon using eq. (3), turns out to be

$$\frac{dt}{dr} = \frac{2r - \frac{tF'}{\sqrt{F}}}{2\sqrt{r^2 - 2t\sqrt{F}}} \tag{15}$$

Clearly this differential equation becomes singular at $(t, r) = (0, 0)$. We now wish to put eq. (15) in a form that will be more useful for subsequent calculations. To this end, we define two new functions $\eta = rF'/F$ and $P = R/r$. From eq. (11), for $f = 0$, we have $\dot{R} = -\sqrt{F}/R$, and we can express F in terms of r by $F(r) = r^2\lambda(r)$. Eq. (15) can thus be re-written as

$$\frac{dt}{dr} = \left[\frac{t}{r}\eta - 2\sqrt{\lambda}\right]\dot{R} = -\left[\frac{t}{r}\eta - 2\sqrt{\lambda}\right]\frac{\sqrt{\lambda}}{P} \tag{16}$$

It can be seen that the functions $\eta(r)$ and $P(r, t)$ are well defined when the singularity is approached.

The nature (a naked singularity or a black hole) of the singularity can be characterised by the existence of radial null geodesics emerging from the singularity. The singularity is at least locally naked if there exist such geodesics, and if no such geodesics exist, it is a black hole. Let us define $X = t/r$. If the singularity is naked, then there exists a real and positive value of X_0 as a solution to the algebraic equation [2]

$$X_0 = \lim_{t\to 0\, r\to 0} X = \lim_{t\to 0\, r\to 0} \frac{t}{r} = \lim_{t\to 0\, r\to 0} \frac{dt}{dr} = R' \tag{17}$$

We insert eq. (16) into (17) and use the result $\lim_{r\to 0}\eta = 2$ to get

$$X_0 = \frac{2}{Q_0}\left[\lambda_0 - X_0\sqrt{\lambda_0}\right] \tag{18}$$

where $Q(X) = P(X,0)$. From the definitions of P and X, and eq. (11), we can derive the following equation

$$X - \frac{1}{2\sqrt{\lambda}} = \frac{-P^2}{2\sqrt{\lambda}} \tag{19}$$

from which it is clear that $X\sqrt{\lambda} < 1/2$, as P is a positive function. Since $Q(X) = P(X,0)$, from eq. (19) we get $Q_0 = \sqrt{1 - 2X_0\sqrt{\lambda_0}}$. Substituting this into eq. (18) leads to the cubic equation

$$2z^3 + (4\lambda_0 - 1)z^2 - 8\lambda_0^2 z + 4\lambda_0^3 = 0 \tag{20}$$

where $z = X_0\sqrt{\lambda_0}$.

We are interested in positive roots of eq. (20) subject to the constraint that $z < 1/2$, in which case outgoing null geodesics terminate at the singularity in the past. It is verified numerically that that for $\lambda_0 \leq 0.5480$ (correct to four decimal places) eq. (20) has two positive real roots which satisfy the constraint that $z = X_0\sqrt{\lambda_0} < 1/2$. For example, if $\lambda_0 = 0.25$, then eq. (20) has the two roots $z = 0.1348$ and 0.4188 (which correspond to two values $X_0 = 0.2696$ and 0.8376). Thus it follows that the singularity will be at least locally naked when $\lambda_0 \leq 0.5480$. On the other hand, if the inequality is reversed, i.e., $\lambda_0 > 0.5480$, no naked singularity occurs and gravitational collapse of the dust cloud must result in a black hole. In the analogous $4D$ case, one gets a quartic equation and the shell focusing singularity is naked iff $\lambda_0 < 0.1809$ [9]. The global nakedness of the singularity can then be seen by making a junction onto the higher dimensional Schwarzschild spacetime, analogously to the $4D$ case (see [15]). Jhingan and Magli [2] have pointed out that if locally naked singularities occur in dust spacetimes, then these spacetimes can be matched to spacetimes containing globally visible spacetimes.

3.1 Self-Similar Case

To support our analysis, we now specialise to the case of self-similar spacetime, which has been analyzed earlier [15] by a different approach. As already mentioned, for the spacetime to be self-similar, we require that $F(r) = \lambda r^2$ ($\lambda = const.$) so that

$$R = r\sqrt{1 - 2\sqrt{\lambda}\frac{t}{r}} \tag{21}$$

and R' can be expressed in terms of the quantity $X = t/r$ as

$$R' = \frac{1 - \sqrt{\lambda}X}{\sqrt{1 - 2\sqrt{\lambda}X}} \tag{22}$$

Eq. (17) with this results in:

$$2y^3 + (\lambda - 1)y^2 - 2\lambda y + \lambda = 0 \tag{23}$$

where $y = X\sqrt{\lambda}$. Eq. (23) has positive roots, subject to constraint that $y < 1/2$, if $\lambda \leq 0.0901$. (The two roots $y = 0.2349$ and 0.4679 of Eq. (23) correspond to $\lambda = 0.05$). Thus referring to our above discussion, self-similar collapse lead to a naked singularity for $\lambda \leq 0.0901$ and to formation of a black hole otherwise. It is well known that the formation of a naked singularity can be understand in terms of the inhomogeneity of the collapse. If collapse is homogeneous no naked singularity occurs. On the other hand a naked singularity occurs if the collapse is sufficiently inhomogeneous, i.e., if the outer shells collapse much later than the central shells [5]. The parameter B which gives a measure of the inhomogeneity of the collapse is usually defined as $t_c(r) = Br$. In our case, on comparing this with eq. (8) (with $f = 0$), $B = 1/2\sqrt{\lambda}$. So for the singularity to be naked we must have $B > 1.6657$. This is in agreement with earlier work [15].

3.2 Strength of Naked Singularity

An important aspect of a singularity is its gravitational strength [19]. A singularity is gravitationally strong or simply strong if it destroys by crushing or stretching any object which fall into it. It is widely believed that a spacetime does not admit an extension through a singularity if it is a strong curvature singularity in the sense of Tipler [20]. Clarke and Królak [21] have shown that a sufficient condition for a strong curvature singularity as defined by Tipler [20] is that for at least one non-space like geodesic with affine parameter k, in the limiting approach to the singularity, we must have

$$\lim_{k \to 0} k^2 \psi = \lim_{k \to 0} k^2 R_{ab} K^a K^b > 0 \tag{24}$$

where R_{ab} is the Ricci tensor. Our purpose here is to investigate the above condition along future directed radial null geodesics which emanate from the naked singularity. Now $k^2\psi$, with the help of eqs. (2) can be expressed as

$$k^2\psi = k^2\frac{3F'(K^t)^2}{2R^3R'} = \frac{3F'}{2rPR'}\left[\frac{kK^t}{R}\right]^2 \tag{25}$$

Using our previous results, we find that

$$\lim_{k\to 0} k^2\psi = \frac{3\lambda_0 X_0}{Q_0(X_0-\sqrt{\lambda_0})^2} > 0 \tag{26}$$

as the singularity is naked for $X_0\sqrt{\lambda_0} < 1/2$. Thus along radial null geodesics coming out from singularity, the strong curvature condition is satisfied.

4 Conclusion

The occurrence and curvature strength of a shell focusing naked singularity in a non self-similar higher dimensional spherically symmetric collapse of a dust cloud has been investigated. We found that naked singularities in our case occur for a slightly higher value of the inhomogeneity parameter in comparison to the analogous situation in the $4D$ case. Along the null ray emanating from the naked singularity, the strong curvature condition (24) is satisfied. The models constructed here are non self-similar in general and for the special case $\lambda = const.$, reduce to the self-similar case. Whereas we have applied this formalism to the $5D$ case, there is no reason to believe that it cannot be extended to a spacetime of any dimension ($n \geq 4$). The formation of these naked singularities violates the strong CCH. We do not claim any particular physical significance to the $5D$ metric considered. Nevertheless we think that the results obtained here have some interest in the sense that they do offer the opportunity to explore properties associated with naked singularities.

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