Frobenius-Schur Indicators, the Klein-bottle Amplitude, and the Principle of Orbifold Covariance

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Abstract

The "orbifold covariance principle", or OCP for short, is presented to support a conjecture of Pradisi, Sagnotti and Stanev on the expression of the Klein-bottle amplitude.

Frobenius-Schur indicators had been introduced in [1] to distinguish between real and pseudo-real primary fields of a CFT, i.e. those primaries whose two-point function is symmetric (resp. antisymmetric) with respect to braiding. They have a simple expression in terms of the usual data of a CFT, i.e. the fusion rule coefficients N_{pq}^r, the exponentiated conformal weights (or statistic phases) $\omega_p = \exp\left(2\pi i(\Delta_p - c/24)\right)$ and the matrix elements of the modular transformation $S : \tau \mapsto \frac{-1}{\tau}$, which reads

$$\nu_p = \sum_{q,r} N_{qr}^p S_{0q} S_{0r} \left(\frac{\omega_q}{\omega_r}\right)^2 \tag{1}$$

where the sum runs over the primary fields, and the label 0 refers to the vacuum. The basic result about the Frobenius-Schur indicator ν_p is that it is three-valued : its value is $+1$ for real primaries, -1 for pseudo-real ones, and 0 if $p \neq \bar{p}$.

Besides the original motivation to characterize simply the symmetry properties of two-point functions, Frobenius-Schur indicators had been applied previously in the study of simple current extensions [2], of boundary conditions [3], and of WZNW orbifolds [4]. They have also appeared in the work of Pradisi, Sagnotti and Stanev on open string theory [5], although in a disguised form, as the coefficients of the Klein-bottle amplitude for a CFT whose torus partition function is the charge conjugation modular invariant. This connection has been noticed in several papers since then, and arguments were presented to support this Ansatz [6][7]. Recently, the Klein-bottle amplitude had been computed using 3D techniques in [8], and the result agrees with the Ansatz of Pradisi, Sagnotti and Stanev. Unfortunately, there is still an important piece of

evidence missing, namely the validity of the Ansatz depends on the positivity conjecture

$$N_{pqr}\nu_p\nu_q\nu_r \geq 0 \tag{2}$$

for any three primaries p, q, r. Although in some special cases the positivity conjecture can be shown to hold [9], no general proof is available at the moment. Therefore, it seems relevant to present another argument strongly supporting the Ansatz of Pradisi, Sagnotti and Stanev, which is completely independent of the previous ones. This argument is based on what we call the "orbifold covariance principle", which we'll explain in a moment.

First of all, let's summarize some basics of permutation orbifolds which will be needed in the sequel. For any Conformal Field Theory $\mathcal{C}$ and any permutation group Ω, one can construct a new CFT $\mathcal{C} \wr \Omega$ by orbifoldizing the n-fold tensor power of $\mathcal{C}$ by the twist group Ω, where n is the degree of Ω, and the resulting CFT is called a permutation orbifold [10][11]. One can compute most interesting quantities of the permutation orbifold $\mathcal{C} \wr \Omega$ from the knowledge of $\mathcal{C}$, e.g. one has explicit expressions for the torus partition function, the characters, the matrix elements of modular transformations, etc. [12]. Not only may one compute the relevant quantities, but the resulting expressions have a simple geometrical meaning : besides the obvious symmetrizations involved, one has to include instanton corrections arising from the twisted sectors, related to the non-trivial coverings of the world-sheet. This recipe works for arbitrary oriented surfaces [13], and may be generalized to the unoriented case, in particular the Klein-bottle. But to obtain the explicit expression of the Klein-bottle amplitude, one has first to make a short detour into uniformization theory.

In case of orientable surfaces, uniformization theory tells us that a closed surface is obtained by quotienting its universal covering surface - which is either the sphere for genus 0, the plane for genus 1, or the upper half-plane for genus bigger than 1 - by a suitable discrete group of holomorphic transformations isomorphic to its fundamental group : in case the genus is greater than one, this is a hyperbolic Fuchsian group, for genus 1 this is a group of translations, and the genus 0 case is trivial [14]. For non-orientable surfaces one has to include orientation reversing, i.e. antiholomorphic transformations as well. A suitable presentation of the fundamental group of the Klein-bottle looks as follows :

$$\left\langle a, b \,|\, b^{-1}ab = a^{-1} \right\rangle \tag{3}$$

i.e. the fundamental group is generated by two elements a and b satisfying the single defining relation $aba = b$ [15]. So we have to look for one holomorphic and one antiholomorphic affine transformation that satisfy the above relation. As the uniformizing group is only determined up to conjugacy, we may use this freedom to transform the generators into the following canonical form :

$$a: \quad z \mapsto \quad z + it \tag{4}$$

$$b: \quad z \mapsto \quad \overline{z} + \frac{1}{2} \tag{5}$$

whith $t < 0$. The meaning of the parameter t may be recovered by considering the oriented (two-sheeted) cover of the Klein-bottle : this is a torus with

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purely imaginary modular parameter equal to $\frac{1}{it}$. So different Klein-bottles are parametrized by t, and may be obtained by identifying the points of the complex plane under the action of the group generated by the two transformations in Eq.(4).

We can now embark upon computing the Klein-bottle amplitude K^Ω of the permutation orbifold $\mathcal{C} \wr \Omega$ in terms of the corresponding amplitude K of $\mathcal{C}$. The general recipe tells us that we have to consider each homomorphism from the fundamental group into the twist group Ω, i.e. each pair $x, y \in \Omega$ that satisfy $x^y = x^{-1}$. Each such homomorphism determines a covering of the Klein-bottle, which is not connected in general, its connected components being in one-to-one correspondence with the orbits $\xi \in \mathcal{O}(x, y)$ of the group generated by x and y. There are two kinds of orbits : those on which the group $\langle x, y^2 \rangle$ generated by x and y^2 acts transitively, the corresponding connected coverings being Klein-bottles again; and those which decompose into two orbits $\xi_\pm$ under the action of $\langle x, y^2 \rangle$, the corresponding coverings being tori. Accordingly, we have $\mathcal{O}(x, y) = \mathcal{O}_-(x, y) \cup \mathcal{O}_+(x, y)$, where $\mathcal{O}_-(x, y)$ contains those orbits whose corresponding covering is a Klein-bottle. There is a simple numerical characterization of these two cases : $\mathcal{O}_-(x, y)$ consists of those orbits $\xi \in \mathcal{O}(x, y)$ which contain an odd number of x-orbits. The uniformizing groups of the above connected components, hence their moduli, may be determined as the point stabilizers of the corresponding orbits. Each homomorphism gives a contribution equal to the product of the partition functions of the connected components of the corresponding covering, and the total Klein-bottle amplitude is the sum of all these contributions divided by the order of the twist group Ω. All in all, we get the result

$$K^\Omega(t) = \frac{1}{|\Omega|} \sum_{x,y \in \Omega} \delta_{x^y, x^{-1}} \prod_{\xi \in \mathcal{O}_-(x,y)} K\left(\frac{\lambda_\xi^2 t}{|\xi|}\right) \prod_{\xi \in \mathcal{O}_+(x,y)} Z\left(\frac{|\xi|}{2\lambda_\xi^2 it} + \frac{\kappa_\xi}{\lambda_\xi}\right) \quad (6)$$

where Z is the torus partition function of the theory. In the above formula, $|\xi|$ stands for the length of the orbit ξ, λ_ξ is the length of the x-orbits contained in ξ, while κ_ξ is the smallest non-negative integer such that $x^{-\kappa_\xi} y^{|\xi|/\lambda_\xi}$ stabilizes the points of $\xi \in \mathcal{O}_+(x, y)$.

Let's now turn to the "orbifold covariance principle". Suppose we have an equality of the form

$$L = R \quad (7)$$

where L and R denote some quantities of the CFT. If such an identity is to hold universally in any CFT, it should obviously hold in any permutation orbifold as well, i.e. Eq.(7) should imply

$$L^\Omega = R^\Omega \quad (8)$$

where we denote by L^Ω (resp. R^Ω) the value of L (resp. R) in the Ω permutation orbifold. As this should hold for an arbitrary permutation group Ω, this gives us an infinite number of highly nonlinear consistency conditions for Eq.(7) to be valid, provided we can express L^Ω and R^Ω in terms of L and R respectively. This is what we call the "orbifold covariance principle", or OCP for short.

In the case at hand, consider the two component quantity

$$L = \left(\begin{array}{c} Z(\tau) \\ K(t) \end{array} \right)$$

which, according to the Pradisi-Sagnotti-Stanev Ansatz, should equal

$$R = \left(\begin{array}{c} \sum_p \chi_p(\tau) \overline{\chi_{\overline{p}}(\tau)} \\ \sum_p \nu_p \chi_p(\frac{1}{it}) \end{array} \right)$$

Note that it is at this point that we restrict ourselves to theories with the charge conjugation invariant. It is now straightforward to verify that the Ansatz $L = R$ indeed satisfies the OCP. This follows from the following general results [12] :

$$Z^{\Omega}(\tau) = \frac{1}{|\Omega|} \sum_{(x,y) \in \Omega^{\{2\}}} \prod_{\xi \in \mathcal{O}(x,y)} Z \left(\frac{\mu_\xi \tau + \kappa_\xi}{\lambda_\xi} \right) \tag{9}$$

$$\chi_{\langle p, \phi \rangle}(\tau) = \frac{1}{|\Omega_p|} \sum_{(x,y) \in \Omega_p^{\{2\}}} \overline{\phi}(x,y) \prod_{\xi \in \mathcal{O}(x,y)} \omega_{p(\xi)}^{-\frac{\kappa_\xi}{\lambda_\xi}} \chi_{p(\xi)} \left(\frac{\mu_\xi \tau + \kappa_\xi}{\lambda_\xi} \right) \tag{10}$$

$$\nu_{\langle p, \phi \rangle} = \frac{1}{|\Omega_p|} \sum_{x, y^2 \in \Omega_p} \delta_{x^y, x^{-1}} \phi(x, y^2) \prod_{\xi \in \mathcal{O}_-(x,y)} \nu_{p(\xi)} \prod_{\xi \in \mathcal{O}_+(x,y)} C_{p(\xi_+)}^{p(\xi_-)} \tag{11}$$

In these formulae $\Omega^{\{2\}}$ denotes the set of commuting pairs of elements of the group Ω, $p : \{1, \ldots, n\} \to \mathcal{I}$ is an n-tuple of primaries (considered as a map), Ω_p is the stabilizer in Ω of the map p under the natural induced action, and ϕ is an irreducible character of the double of the stabilizer Ω_p. For a pair $(x, y) \in \Omega_p^{\{2\}}$, we denote by $\mathcal{O}(x, y)$ the set of orbits on $\{1, \ldots, n\}$ of the group generated by x and y, while for a given orbit $\xi \in \mathcal{O}(x, y)$, λ_ξ denotes the common length of the x-orbits contained in ξ, μ_ξ denotes their number, and κ_ξ is the smallest nonnegative integer such that $y^{\mu_\xi} x^{-\kappa_\xi}$ belongs to the stabilizer of ξ, while $p(\xi)$ denotes the value of the map p on the orbit ξ (on which it is constant because both x and y stabilize p). Finally, C_p^q denotes the charge conjugation matrix, i.e. $C = S^2$, while the notation $\mathcal{O}_\pm(x, y)$ and $\xi_\pm$ has been explained previously in connection with Eq. (6).

With the aid of Eq.(10) and Eq.(6), after performing the required summations, we arrive at the result that $L = R$ implies $L^\Omega = R^\Omega$, confirming the claim. This should be viewed as a strong consistency check of the Pradisi-Sagnotti-Stanev Ansatz.

Of course, the above argument does not exhaust the potential of the OCP, it is just intended to illustrate the application of this powerful tool. As it is possible to apply the OCP even in cases where a formal proof is out of reach for present day techniques, it should be considered as an important tool in the investigation of Conformal Field Theories.

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