

# *A Priori* Probability Distribution of the Cosmological Constant

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## **Abstract**

In calculations of the probability distribution for the cosmological constant, it has been previously assumed that the *a priori* probability distribution is essentially constant in the very narrow range that is anthropically allowed. This assumption has recently been challenged. Here we identify large classes of theories in which this assumption is justified.

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## I. INTRODUCTION

In some theories of inflation<sup>1</sup> and of quantum cosmology<sup>2</sup> the observed big bang is just one of an ensemble of expanding regions in which the cosmological constant takes various different values. In such theories there is a probability distribution for the cosmological constant: the probability  $d\mathcal{P}(\rho_V)$  that a scientific society in any of the expanding regions will observe a vacuum energy between  $\rho_V$  and  $\rho_V + d\rho_V$  is given by<sup>3,4,5</sup>

$$d\mathcal{P}(\rho_V) = \mathcal{P}_*(\rho_V)\mathcal{N}(\rho_V)d\rho_V, \quad (1)$$

where  $\mathcal{P}_*(\rho_V)d\rho_V$  is the *a priori* probability that an expanding region will have a vacuum energy between  $\rho_V$  and  $\rho_V + d\rho_V$  (to be precise, weighted with the number of baryons in such regions), and  $\mathcal{N}(\rho_V)$  is proportional to the fraction of baryons that wind up in galaxies. (The constant of proportionality in  $\mathcal{N}(\rho_V)$  is independent of  $\rho_V$ , because once a galaxy is formed the subsequent evolution of its stars, planets, and life is essentially unaffected by the vacuum energy.)

The factor  $\mathcal{N}(\rho_V)$  vanishes except for values of  $\rho_V$  that are very small by the standards of elementary particle physics, because for  $\rho_V$  large and positive there is a repulsive force that prevents the formation of galaxies<sup>6</sup> and hence of stars, while for  $\rho_V$  large and negative the universe recollapses too fast for galaxies or stars to form.<sup>7</sup> The fraction of baryons that form galaxies has been calculated<sup>5</sup> for  $\rho_V > 0$  under reasonable astrophysical assumptions. On

the other hand, we know little about the *a priori* probability distribution  $\mathcal{P}_*(\rho_V)$ . However, the range of values of  $\rho_V$  in which  $\mathcal{N}(\rho_V) \neq 0$  is so narrow compared with the scales of energy density typical of particle physics that it had seemed reasonable in earlier work<sup>4,5</sup> to assume that  $\mathcal{P}_*(\rho_V)$  is constant within this range, so that  $d\mathcal{P}(\rho_V)$  can be calculated as proportional to  $\mathcal{N}(\rho_V)d\rho_V$ . In an interesting recent article,<sup>8</sup> Garriga and Vilenkin have argued that this assumption (which they call “Weinberg’s conjecture”) is generally not valid. This raises the problem of characterizing those theories in which this assumption is valid and those in which it is not.

It is shown in Section II that this assumption is in fact valid for a broad range of theories, in which the different regions are characterized by different values of a scalar field that couples only to itself and gravitation. The deciding factor is how we impose the flatness conditions on the scalar field potential that are needed to ensure that the vacuum energy is now nearly time-independent. If the potential is flat because the scalar field renormalization constant is very large, then the *a priori* probability distribution of the vacuum energy is essentially constant within the anthropically allowed range, for scalar potentials of generic form. It is also essentially constant for a large class of other potentials. Section III is a digression, showing that the same flatness conditions ensure that the vacuum energy has been roughly constant since the end of inflation. Section IV takes up the sharp peaks in the *a priori* probability found in theories of quantum cosmology and eternal

inflation.

## II. SLOWLY ROLLING SCALAR FIELD

One of the possibilities considered by Garriga and Vilenkin is a vacuum energy that depends on a homogeneous scalar field  $\phi(t)$  whose present value is governed by some smooth probability distribution. The vacuum energy is

$$\rho_V = V(\phi) + \frac{1}{2}\dot{\phi}^2, \quad (2)$$

and the scalar field time-dependence is given by

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \quad (3)$$

where  $H(t)$  is the Hubble fractional expansion rate,  $V(\phi)$  is the scalar field potential, dots denote derivatives with respect to time, and primes denote derivatives with respect to  $\phi$ . Following Garriga and Vilenkin,<sup>8</sup> we assume that at present the scalar field energy appears like a cosmological constant because the field  $\phi$  is now nearly constant in time, and that this scalar field energy now dominates the cosmic energy density. For this to make sense it is necessary for the potential  $V(\phi)$  to satisfy certain flatness conditions. In the usual treatment of a slowly rolling scalar, one neglects the inertial term  $\ddot{\phi}$  in Eq. (3) as well as the kinetic energy term  $\dot{\phi}^2/2$  in Eq. (2). With the inertial term neglected, the condition that  $V(\phi)$  should change little in a Hubble time  $1/H$  is that<sup>9</sup>

$$V'^2(\phi) \ll 3H^2|V(\phi)|. \quad (4)$$

With the scalar field energy dominating the total cosmic energy density, the Friedmann equation gives

$$|V(\phi)| \simeq \rho_V \simeq 3H^2/8\pi G , \quad (5)$$

so Eq. (4) requires

$$|V'(\phi)| \ll \sqrt{8\pi G} \rho_V . \quad (6)$$

(The kinetic energy term  $\dot{\phi}^2/2$  in Eq. (2) can be neglected under the slightly weaker condition

$$|V'(\phi)| \ll \sqrt{18H^2 |V(\phi)|} \simeq \sqrt{48\pi G} \rho_V ,$$

which is the flatness condition given by Garriga and Vilenkin.) There is also a bound on the second derivative of the potential, needed in order for the inertial term to be neglected. With the scalar field energy dominating the total cosmic energy density, this condition requires that<sup>9</sup>

$$|V''(\phi)| \ll 8\pi G \rho_V . \quad (7)$$

As Garriga and Vilenkin correctly pointed out, the smallness of the slope of  $V(\phi)$  means that  $\phi$  may vary appreciably even when  $\rho_V \simeq V(\phi)$  is restricted to the very narrow anthropically allowed range of values in which galaxy formation is non-negligible. They concluded that it would be possible for the *a priori* probability  $\mathcal{P}_*(\rho_V)$  to vary appreciably in this range. In particular, Garriga and Vilenkin assumed an *a priori* probability distribution

for  $\phi$  that is constant in the anthropically allowed range, in which case the *a priori* probability distribution for  $\rho_V$  is

$$\mathcal{P}_*(\rho_V) \propto 1/|V'(\phi)| \quad (8)$$

which they said could vary appreciably in the anthropically allowed range.

Though possible, this rapid variation is by no means the generic case. As already mentioned, the second as well as the first derivative of the potential must be small, so that the *a priori* probability density (8) may change little in the anthropically allowed range. It all depends on how the flatness conditions are satisfied. There are two obvious ways that one might try to make the potential sufficiently flat. Potentials of the first type are of the general form

$$V(\phi) = V_1 f(\lambda\phi) , \quad (9)$$

where  $V_1$  is some large energy density, in the range of  $m_W^4$  to  $G^{-2}$ ; the constant  $\lambda$  is very small: and  $f(x)$  is some dimensionless function involving no very large or very small parameters. Potentials of the second type are of the general form

$$V(\phi) = V_1 [1 - \epsilon g(\lambda\phi)] , \quad (10)$$

where  $V_1$  is again some large energy density;  $\lambda$  is here some fixed inverse mass, perhaps of order  $\sqrt{G}$ ; now it is  $\epsilon$  instead of  $\lambda$  that is very small; and  $g(x)$  is some other dimensionless function involving no very large or very small parameters.

For potentials (9) of the first type, it is always possible to meet all observational conditions by taking  $\lambda$  sufficiently small, provided that the function  $f(x)$  has a simple zero at a point  $x = a$  of order unity, with derivatives at  $a$  of order unity. Because  $V_1$  is so large, the present value of  $\lambda\phi$  must be very close to the assumed zero  $a$  of  $f(x)$ . With  $f'(a)$  and  $f''(a)$  of order unity, the flatness conditions (6) and (7) are both satisfied if

$$|\lambda| \ll \left( \frac{\rho_V}{V_1} \right) \sqrt{8\pi G} . \quad (11)$$

Galaxy formation is only possible for  $|V(\phi)|$  less than an upper bound  $V_{\max}$  of the order of the mass density of the universe at the earliest time of galaxy formation,<sup>6</sup> which in the absence of fine tuning of the cosmological constant is very much less than  $V_1$ . The anthropically allowed range of  $\phi$  is therefore given by

$$\Delta\phi \equiv |\phi - a/\lambda|_{\max} = \frac{V_{\max}}{|\lambda f'(a)V_1|} . \quad (12)$$

The fractional change in the *a priori* probability density  $1/|V'(\phi)|$  in this range is then

$$\left| \frac{V''(\phi)\Delta\phi}{V'(\phi)} \right| = \left| \frac{V_{\max}}{V_1} \right| \left| \frac{f''(a)}{f'^2(a)} \right| , \quad (13)$$

with no dependence on  $\lambda$ . As long as the factor  $f''(a)/f'^2(a)$  is roughly of order unity the fractional variation (13) in the *a priori* probability will be very small, as was assumed in references 4 and 5.

This reasoning applies to potentials of the form

$$V(\phi) = V_1 [1 - (\lambda\phi)^n] ,$$

which, as already noted by Garriga and Vilenkin, lead to an *priori* probability distribution that is nearly constant in the anthropically allowed range. (In this case  $a = 1$  and  $f''(a)/f'^2(a) = (1-n)/n$ .) But this reasoning also applies to the “washboard potential” that was taken as a counterexample by Garriga and Vilenkin, which with no loss of generality can be put in the form:

$$V(\phi) = V_1 [1 + \alpha \lambda \phi + \beta \sin(\lambda \phi)] .$$

The zero point  $a$  is here determined by the condition

$$1 + \alpha a + \beta \sin a = 0 ,$$

and the factor  $f''(a)/f'^2(a)$  in Eq. (13) is

$$\frac{f''(a)}{f'^2(a)} = \frac{-\beta \sin a}{(\alpha + \beta \cos a)^2} .$$

If the flatness condition is satisfied by taking  $\lambda$  small, with  $\alpha$  and  $\beta$  of order unity, as is assumed for potentials of the first kind, then the factor  $f''(a)/f'^2(a)$  in Eq. (13) is of order unity unless  $\alpha$  and  $\beta$  happen to be chosen so that

$$\left| 1 + \alpha \cos^{-1} \left( \frac{-\alpha}{\beta} \right) + \beta \sqrt{1 - \frac{\alpha^2}{\beta^2}} \right| \ll 1 .$$

Of course it would be possible to impose this condition on  $\alpha$  and  $\beta$ , but this is the kind of fine-tuning that would be upset by adding a constant of order  $V_1$  to the potential. Aside from this exception, for all  $\alpha$  and  $\beta$  of order unity the factor  $f''(a)/f'^2(a)$  is of order unity, so the washboard potential also yields



an *a priori* probability distribution for the vacuum energy that is flat in the anthropically allowed range.

In contrast, for potentials (10) of the second kind the flatness conditions are not necessarily satisfied no matter how small we take  $\epsilon$ . Because the present vacuum energy is much less than  $V_1$ , the present value of  $\phi$  must be very close to a value  $\phi_\epsilon$ , satisfying

$$g(\lambda\phi_\epsilon) = 1/\epsilon . \quad (14)$$

This requires  $\lambda\phi_\epsilon$  to be near a singularity of the function  $g(x)$ , perhaps at infinity, so it is not clear in general that such a potential would have small derivatives at  $\lambda\phi_\epsilon$  for any value of  $\epsilon$ . For instance, for an exponential  $g(x) = \exp(x)$  we have  $\phi_\epsilon = -\ln \epsilon/\lambda$ , and  $V'(\phi_\epsilon)$  approaches an  $\epsilon$ -independent value proportional to  $\lambda$ , which is not small unless we take  $\lambda$  very small, in which case we have a potential of the first kind, for which as we have seen the *a priori* probability density (8) is flat in the anthropically allowed range. The flatness conditions *are* satisfied for small  $\epsilon$  if  $g(x)$  approaches a power  $x^n$  for  $x \rightarrow \infty$ . In this case  $\phi_\epsilon$  goes as  $\epsilon^{-1/n}$ , so  $V'(\phi_\epsilon)$  goes as  $\epsilon^{1/n}$  and  $V''(\phi_\epsilon)$  goes as  $\epsilon^{2/n}$ , both of which can be made as small as we like by taking  $\epsilon$  sufficiently small.

In particular, if the singularity in  $g(x)$  at  $x \rightarrow \infty$  consists only of poles in  $1/x$  of various orders up to  $n$  (as is the case for a polynomial of order  $n$ ) then the anthropically allowed range of  $\phi$  is

$$\left| \phi - \phi_\epsilon \right|_{\max} \approx \frac{V_m}{V_1 \epsilon |g'(\phi_\epsilon)|} \approx \epsilon^{-1/n} \left( \frac{V_m}{V_1} \right) . \quad (15)$$

The flatness conditions make this range much greater than the Planck mass, but the fractional change in the *a priori* probability density (8) in this range is still very small

$$\left| \frac{V''(\phi_\epsilon)}{V'(\phi_\epsilon)} \right| |\phi - \phi_\epsilon|_{\max} \approx \frac{V_m}{V_1} \ll 1. \quad (16)$$

To have a large fractional change in the *a priori* probability distribution in the anthropically allowed range for potentials of the second type that satisfy the flatness conditions, we need a function  $g(x)$  that goes like a power as  $x \rightarrow \infty$ , but has a more complicated singularity at  $x = \infty$  than just poles in  $1/x$ . An example is provided by the washboard potential with  $\alpha$  and  $\beta$  very small and  $\lambda$  fixed, the case considered by Garriga and Vilenkin, for which  $g(x)$  has an essential singularity at  $x = \infty$ .

In summary, the *a priori* probability is flat in the anthropically allowed range for several large classes of potentials, while it seems to be not flat only in exceptional cases.

It remains to consider whether the small parameters  $\lambda$  or  $\epsilon$  in potentials respectively of the first or second kind could arise naturally. Garriga and Vilenkin argued that a term in a potential of what we have called the second kind with an over-all factor  $\epsilon \ll 1$  could be naturally produced by instanton effects. On the other hand, for potentials of type 1 a small parameter  $\lambda$  could be naturally produced by the running of a field-renormalization factor. The field  $\phi$  has a conventional “canonical” normalization, as shown by the fact that the term  $\dot{\phi}^2/2$  in the vacuum energy (2) and the inertial term

$\ddot{\phi}$  in the field equation (3) have coefficients unity. Factors dependent on the ultraviolet cutoff will therefore be associated with external  $\phi$ -lines. In order for the potential  $V(\phi)$  to be expressed in a cut-off independent way in terms of coupling parameters  $g_\mu$  renormalized at a wave-number scale  $\mu$ , the field  $\phi$  must be accompanied with a field-renormalization factor  $Z_\mu^{-1/2}$ , which satisfies a differential equation of the form

$$\mu \frac{dZ_\mu}{d\mu} = \gamma(g_\mu) Z_\mu . \quad (17)$$

At very large distances, the field  $\phi$  will therefore be accompanied with a factor

$$\lambda = Z_0^{-1/2} = \exp \left\{ \frac{1}{2} \int_0^\mu \frac{d\mu'}{\mu'} \gamma(g_{\mu'}) \right\} Z_\mu^{-1/2} . \quad (18)$$

The integral here only has to be reasonably large and negative in order for  $\lambda$  to be extremely small.

### III. SLOW ROLLING IN THE EARLY UNIVERSE

When the cosmic energy density is dominated by the vacuum energy, the flatness conditions (6) and (7) insure that the vacuum energy changes little in a Hubble time. But if the vacuum energy density is nearly time-independent, then from the end of inflation until nearly the present it must have been much smaller than the energy density of matter and radiation, and under these conditions we are not able to neglect the inertial term  $\ddot{\phi}$  in Eq. (3). A separate argument is needed to show that the vacuum energy is

nearly constant at these early times. This is important because, although there is no observational reason to require  $V(\phi)$  to be constant at early times, it must have been less than the energy of radiation at the time of nucleosynthesis in order not to interfere with the successful prediction of light element abundances, and therefore at this time must have been very much less than  $V_1$ , which we have supposed to be at least of order  $m_W^4$ . For potentials (9) of the first kind, this means that  $\phi$  must have been very close to its present value at the time of helium synthesis. Also, if  $\phi$  at the end of inflation were not the same as  $\phi$  at the time of galaxy formation, then a flat *a priori* distribution for the first would not in general imply a flat *a priori* distribution for the second.

At times between the end of inflation and the recent past the expansion rate behaved as  $H = \eta/t$ , where  $\eta = 2/3$  or  $\eta = 1/2$  during the eras of matter or radiation dominance, respectively. During this period, Eq. (3) takes the form

$$\ddot{\phi} + \frac{3\eta}{t}\dot{\phi} = -V'(\phi) , \quad (19)$$

If we tentatively assume that  $\phi$  is nearly constant, then Eq. (19) gives for its rate of change

$$\dot{\phi} \simeq -\frac{t V'(\phi)}{1 + 3\eta} . \quad (20)$$

The change in the vacuum energy from the end of inflation to the present

time  $t_0$  is therefore

$$\Delta V \simeq \int_0^{t_0} V'(\phi) \dot{\phi} dt \simeq -\frac{V'^2(\phi)t_0^2}{2(1+3\eta)} . \quad (21)$$

The present time is roughly given by  $t_0 \approx \eta\sqrt{3/8\pi G\rho_{V0}}$ , so the fractional change in the vacuum energy density since the end of inflation is

$$\left| \frac{\Delta V}{\rho_{V0}} \right| \approx \left( \frac{3\eta^2}{2(1+3\eta)} \right) \left( \frac{V'^2(\phi)}{8\pi G\rho_{V0}^2} \right) , \quad (22)$$

a subscript zero as usual denoting the present instant. The factor  $3\eta^2/2(1+3\eta)$  is of order unity, so the inequality (6) tells us that the change in the vacuum energy during the time since inflation has indeed been much less than its present value.

### III. QUANTUM COSMOLOGY

In some theories of quantum cosmology the wave function of the universe is a superposition of terms, corresponding to universes with different (but time-independent) values for the vacuum energy  $\rho_V$ . It has been argued by Baum<sup>2</sup>, Hawking<sup>2</sup> and Coleman<sup>10</sup> that these terms are weighted with a  $\rho_V$ -dependent factor, that gives an *a priori* probability distribution with an infinite peak at  $\rho_V = 0$ , but this claim has been challenged.<sup>11</sup> As already acknowledged in references 4 and 5, if this peak at  $\rho_V = 0$  is really present, then anthropic considerations are both inapplicable and unnecessary in solving the problem of the cosmological constant.

Garriga and Vilenkin<sup>8</sup> have proposed a different sort of infinite peak, arising from a  $\rho_V$ -dependent rate of nucleation of sub-universes operating over an infinite time. Even granting the existence of such a peak, it is not clear that it really leaves a vanishing normalized probability distribution at all other values of  $\rho_V$ . For instance, the nucleation rate might depend on the population of sub-universes already present, in such a way that the peaks in the probability distribution are kept to a finite size. If  $\mathcal{P}_*(\rho_V) = 0$  except at the peak, then anthropic considerations are irrelevant and the cosmological constant problem is as bad as ever, since there is no known reason why the peak should occur in the very narrow range of  $\rho_V$  that is anthropically allowed. On the other hand, if there is a smooth background in addition to a peak outside the anthropically allowed range of  $\rho_V$  then the peak is irrelevant, because no observers would ever measure such values of  $\rho_V$ . In this case the probability distribution of the cosmological constant can be calculated using the methods of references 4 and 5.

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