PROBABILITY CURRENT AND TRAJECTORY REPRESENTATION

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Abstract

A unified form for real and complex wave functions is proposed for the stationary case, and the quantum Hamilton-Jacobi equation is derived in the three-dimensional space. The difficulties which appear in Bohm's theory like the vanishing value of the conjugate momentum in the real wave function case are surmounted. In one dimension, a new form of the general solution of the quantum Hamilton-Jacobi equation leading straightforwardly to the general form of the Schrödinger wave function is proposed. For unbound states, it is shown that the invariance of the reduced action under a dilatation plus a rotation of the wave function in the complex space implies that microstates do not appear. For bound states, it is shown that some freedom subsists and gives rise to the manifestation of microstates not detected by the Schrödinger wave function.

Key words: probability current, quantum Hamilton-Jacobi equation, trajectory representation, microstates.

1 INTRODUCTION

The debate opened by Einstein and Bohr about the interpretation of quantum mechanics is far from being closed. Among all attempts to obtain a deterministic theory, the approach proposed by Bohm [1] is one of the most interesting. The starting point is the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t} \ , \tag{1}$$

which describes the evolution of the wave function of a non-relativistic spinless particle of mass m in a potential V. Bohm writes the wave function in the form

$$\psi(x,y,z,t) = A(x,y,z,t)\ \exp\left(\frac{i}{\hbar}S(x,y,z,t)\right) \ , \tag{2}$$

where $A(x,y,z,t)$ and $S(x,y,z,t)$ are real functions. By substituting (2) in (1), it follows

$$\frac{1}{2m}(\vec{\nabla}S)^2 - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + V = -\frac{\partial S}{\partial t} \ , \tag{3}$$

$$\vec{\nabla} \cdot \left(A^2 \frac{\vec{\nabla} S}{m} \right) = -\frac{\partial A^2}{\partial t} \ . \tag{4}$$

The term proportional to $\hbar^2$ in Eq. (3)

$$V_B \equiv -\frac{\hbar^2}{2m} \frac{\Delta A}{A} \tag{5}$$

is called the Bohm quantum potential. At first sight, setting $\hbar = 0$ in Eq. (3), making V_B vanish, gives the classical Hamilton-Jacobi equation which describes the motion of the particle. S is then identified as the reduced action and V_B is interpreted as describing the quantum effects. However, the classical limit $\hbar \to 0$ is not trivial. In fact, Floyd [2] showed that, in general, when taking the limit $\hbar \to 0$, a residual indeterminacy subsists.

Relation (4) represents the conservation equation of the probability current. Indeed, if one substitutes (2) in the expression of the current

$$\vec{\jmath} = \frac{\hbar}{2mi} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \ , \tag{6}$$

one finds

$$\vec{\jmath} = A^2 \frac{\vec{\nabla} S}{m} \ . \tag{7}$$

This expression is a product of the probability density $|\psi|^2 = A^2$ by $\vec{v} \equiv \vec{\nabla} S/m$ which was recognized by Bohm [1] and de Broglie [3] in his pilot wave theory as the velocity of the particle. In the stationary case, where

$$S(x, y, z, t) = S_0(x, y, z) - Et \ , \tag{8}$$

$$\frac{\partial A}{\partial t}(x, y, z, t) = 0 \ , \tag{9}$$

and the constant E representing the energy of the particle, Floyd [4] showed that in one-dimensional space, the velocity was not given by $m^{-1} \partial S_0/\partial x$ as defined by Bohm [1] and de Broglie [3], but by the expression

$$\frac{dx}{dt} = \frac{\partial S_0/\partial x}{m(1 - \partial V_B/\partial E)} \ . \tag{10}$$

The reduced action S_0 as defined by (2) and (8) cannot be used to define correctly the conjugate momentum as $\vec{\nabla} S_0$. To see this, consider the case in which the wave function is real as it is for the ground state of hydrogenoïd atoms or the one-dimensional harmonic oscillator. In this case, using (2), S_0 is constant and then the conjugate momentum has a vanishing value. Obviously, this conclusion is not satisfactory.

In the one-dimensional case, Floyd [4, 5] surmounted this difficulty by using a trigonometric representation. He showed then that trajectory representation described microstates [4, 6, 7] not detected by the Schrödinger wave function. In a recent paper [8], the same author synthesized the basic ideas of trajectory representation and contrasted them to those of the Copenhagen school and those of Bohmian mechanics.

Recently, by assuming that all quantum systems can be connected by a coordinate transformation, Faraggi and Matone [9, 10] have derived in one dimension the quantum Hamilton-Jacobi equation, which in turn, leads to the Schrödinger equation. In other words, from an equivalence principle, they deduced quantum mechanics. In particular, without using the usual axiomatic interpretation of the wave function, they showed that the tunnel effect and the energy quantization are consequences of the equivalence principle [10, 11]. These authors, together with Bertoldi [12], deduced the higher dimensions version of the quantum Hamilton-Jacobi equation. They showed in one dimension that there are microstates. They deduced also in higher dimensions the existence of a hidden antisymmetric two-tensor field which can play an important role in the search for the quantum origin of fundamental interactions. This connection between the equivalence principle and quantum mechanics gives some hope for a unified description of fundamental interactions. Furthermore, Matone [13] has suggested that gravity interaction can be derived from the quantum potential.

In this paper, we reproduce in three dimensions some results found in Ref. [12] without appealing to differential geometry. We also give some consequences of the approach presented here about microstates. In section 2, a relationship between the wave function and the reduced action is constructed. The probability current is used in section 3 to establish the quantum Hamilton-Jacobi equation for the complex wave functions and in section 4 for the real wave functions. In section 5, a new form of the general solution of the quantum Hamilton-Jacobi equation in one dimension is proposed. It is shown that there is no trace of microstates for unbound states. However, some freedom subsists for bound states giving rise to the manifestation of microstates. In three dimensions, the hidden antisymmetric two-tensor field is presented. Section 6 is devoted to conclusions.

2 THE FORM OF THE WAVE FUNCTION

Let us begin by the following remark. If one sets

$$\psi(x,y,z,t) = A(x,y,z,t)\ \exp\left(-\frac{i}{\hbar}S(x,y,z,t)\right) \tag{11}$$

and substitutes this expression in the Schrödinger equation, one gets

$$\frac{1}{2m}(\vec{\nabla}S)^2 - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + V = \frac{\partial S}{\partial t}\ , \tag{12}$$

$$\vec{\nabla}\cdot\left(A^2\frac{\vec{\nabla}S}{m}\right) = \frac{\partial A^2}{\partial t}\ . \tag{13}$$

In a non-stationary case ($\partial A/\partial t \neq 0$), by comparing Eqs. (12) and (13) with (3) and (4), it is easy to deduce that expressions (2) and (11) cannot be simultaneously solution of the Schrödinger equation. In the stationary case, the situation is different. In fact, if one replaces successively the two expressions

$$\psi_1 = \exp\left(-\frac{i}{\hbar}Et\right) A\ \exp\left(\frac{i}{\hbar}S_0\right)\ , \tag{14}$$

$$\psi_2 = \exp\left(-\frac{i}{\hbar}Et\right) A\ \exp\left(-\frac{i}{\hbar}S_0\right) \tag{15}$$

in the Schrödinger equation, one gets the same equations

$$\frac{1}{2m}(\vec{\nabla}S_0)^2 - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + V = E \ , \tag{16}$$

$$\vec{\nabla}\cdot(A^2\vec{\nabla}S_0) = 0 \ . \tag{17}$$

This means that if ψ_1 (respectively ψ_2) is solution of the Schrödinger equation, ψ_2 (respectively ψ_1) is also solution. It follows that for the physical wave function in the stationary case we can choose the form

$$\psi(x,y,z,t) = \exp\left(-\frac{i}{\hbar}Et\right)\ \phi(x,y,z) \ , \tag{18}$$

in which $\phi(x,y,z)$ is the linear combination

$$\phi(x,y,z) = A(x,y,z)\left[\alpha\ \exp\left(\frac{i}{\hbar}S_0(x,y,z)\right) + \beta\ \exp\left(-\frac{i}{\hbar}S_0(x,y,z)\right)\right] \ , \tag{19}$$

α and β being complex constants. Note that this form (19) of the wave function has been established with a different method in one dimension [9, 10] and in higher dimensions [12].

Now, if one replaces in the Schrödinger equation ψ by the expressions given in (18) and (19), one finds

$$\left[\frac{1}{2m}(\vec{\nabla}S_0)^2 - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + V - E\right]\left[\alpha\ \exp\left(\frac{i}{\hbar}S_0\right) + \beta\ \exp\left(-\frac{i}{\hbar}S_0\right)\right] - \frac{i\hbar}{2mA^2}\vec{\nabla}\cdot(A^2\vec{\nabla}S_0)\left[\alpha\ \exp\left(\frac{i}{\hbar}S_0\right) - \beta\ \exp\left(-\frac{i}{\hbar}S_0\right)\right] = 0 \ . \tag{20}$$

Before analyzing the content of this equation, let us calculate the probability current. If one replaces (18) in (6), one gets

$$\vec{\jmath} = (|\alpha|^2 - |\beta|^2)A^2\frac{\vec{\nabla}S_0}{m} \ . \tag{21}$$

This form of the current will play a crucial role in the approach which is developed here.

3 THE COMPLEX WAVE FUNCTION

In what follows, one should understand by real wave function, any function which can be written as a product of a constant, which could be complex, with a real function.

In order to show that the wave function (19) cannot be real when $|\alpha| \neq |\beta|$, let us set

$$\alpha = |\alpha|\ \exp(ia) \ , \qquad \beta = |\beta|\ \exp(ib) \ , \tag{22}$$

with a and b real constants. Expression (19) can then be written in the form

$$\phi = A\ \exp\left(i\frac{a+b}{2}\right)\ \left[(|\alpha|+|\beta|)\ \cos\left(\frac{S_0}{\hbar}+\frac{a-b}{2}\right) + i(|\alpha|-|\beta|)\ \sin\left(\frac{S_0}{\hbar}+\frac{a-b}{2}\right)\right] \ . \tag{23}$$

Knowing that S_0 is a function of (x, y, z), this last expression shows clearly that when $|\alpha| \neq |\beta|$, the wave function cannot be brought back to a product of a constant by a real function.

Now, to derive the quantum Hamilton-Jacobi equation, let us use expression (21) for the probability current. The conservation equation, which is a consequence of the Schrödinger equation, can be written as

$$\vec{\nabla} \cdot \left[(|\alpha|^2 - |\beta|^2) A^2 \frac{\vec{\nabla} S_0}{m} \right] = 0 \ . \tag{24}$$

Therefore, for the complex wave functions ($|\alpha| \neq |\beta|$), Eq. (24) turns out to be

$$\vec{\nabla} \cdot (A^2 \vec{\nabla} S_0) = 0 \ . \tag{25}$$

Eq. (20) reduces then to

$$\frac{1}{2m} (\vec{\nabla} S_0)^2 - \frac{\hbar^2}{2m} \frac{\Delta A}{A} + V - E = 0 \ . \tag{26}$$

Although the last two equations have the same form as those established by Bohm [1], they are fundamentally different. In fact, in Bohm's theory, the physical solution of the Schrödinger equation is written in the form (14). In this case, Eqs. (16) and (17) represent the Bohm's equations and then the functions A and S_0 appearing in (16) and (17) are not the same as those of Eqs. (25) and (26). The reason is that in Bohm's theory, Eqs. (16) and (17) are derived by writing the physical solution in the form (14) which does not allow to define correctly the conjugate momentum, while in the present approach Eqs. (25) and (26) are obtained by writing the same physical solution in the form (19).

4 THE REAL WAVE FUNCTION

In the case where $|\alpha| = |\beta|$, and using Eq. (23), the physical wave function defined by (19) becomes

$$\phi = 2|\alpha| A \ \exp\left(i \frac{a+b}{2} \right) \cos\left(\frac{S_0}{\hbar} + \frac{a-b}{2} \right) \ . \tag{27}$$

It is clear that the wave function is real up to a constant phase factor.

Here the vanishing of the probability current is expressed by the fact that $|\alpha| = |\beta|$, and not by $\vec{\nabla} S_0 = \vec{0}$ as in the case of Bohm's approach.

Using (22) with $|\alpha| = |\beta|$, Eq. (20) turns out to be

$$\frac{1}{2m} (\vec{\nabla} S_0)^2 - \frac{\hbar^2}{2m} \frac{\Delta A}{A} + V + \frac{\hbar}{2mA^2} \left[\vec{\nabla} \cdot (A^2 \vec{\nabla} S_0) \right] \ \tan\left(\frac{S_0}{\hbar} + \frac{a-b}{2} \right) = E \ . \tag{28}$$

Comparing with the usual quantum Hamilton-Jacobi equation, (28) contains an additional term proportional to $\hbar$.

At first glance, one may think that for any function $\phi(x, y, z)$ describing a physical state, there is an infinite number of ways to choose the couple (A, S_0) in such a way as to satisfy relation (27). For example, if one chooses S_0 to be constant, Eq. (28) becomes

$$-\frac{\hbar^2}{2m} \frac{\Delta A}{A} + V = E \tag{29}$$

which is exactly the Schrödinger equation. Another possible choice is to take $A = cst.$ and deduce the equation

$$\frac{1}{2m}(\vec{\nabla} S_0)^2 + V + \frac{\hbar}{2m}\Delta S_0 \ \tan\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) = E \tag{30}$$

from which one can reproduce the Schrödinger equation.

Among all these choices, is there any couple (A, S_0) in which S_0 is the good function defining correctly the conjugate momentum by $\vec{\nabla} S_0$? In other words, is there any particular relation between A and S_0 ?

To answer this crucial question, let us analyze the physics content of expression (21) for the probability current. This expression suggests that $\vec{\jmath}$ is a sum of two currents

$$\vec{\jmath} = \vec{\jmath}_+ + \vec{\jmath}_- \ , \tag{31}$$

where $\vec{\jmath}_+ = |\alpha|^2 A^2 \vec{\nabla} S_0/m$ and $\vec{\jmath}_- = -|\beta|^2 A^2 \vec{\nabla} S_0/m$ correspond to the two opposite directions of motion of the particle along the trajectory. The fact that the current has a vanishing value in the case of a real wave function ($|\alpha| = |\beta|$) means that there is an equal probability to have the particle move in one direction or in the other.

Thus, to each direction of motion along the trajectory, it is natural to associate one of the wave functions

$$\phi_1 = A \ \exp\left(\frac{i}{\hbar} S_0\right) \ , \tag{32}$$

$$\phi_2 = A \ \exp\left(-\frac{i}{\hbar} S_0\right) \ , \tag{33}$$

which were combined in Eq. (19) to obtain expression (21) for the current. This means that ϕ_1 and ϕ_2 must be simultaneously solution of the Schrödinger equation. Thus, there is no reason why this should not happen in the particular case $|\alpha| = |\beta|$. Consequently, the couple (A, S_0) must be chosen in such a way as to impose to ϕ_1 and ϕ_2 to be solutions of Schrödinger's equation knowing that expression (27) is also solution. To satisfy this condition, it is sufficient to require that the function

$$\theta(x, y, z) = A \ \sin\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) \tag{34}$$

be a solution of Schrödinger's equation. In fact, if ϕ and θ are solutions, then ϕ_1 and ϕ_2 are also solutions since they are linear combinations of ϕ and θ.

Of course, if one substitutes (27) in the Schrödinger equation, one gets (28). On the other hand, substituting θ by its expression (34) in the Schrödinger equation, one gets

$$\frac{1}{2m}(\vec{\nabla} S_0)^2 - \frac{\hbar^2}{2m}\frac{\Delta A}{A} + V - \frac{\hbar}{2mA^2}\left[\vec{\nabla}\cdot(A^2 \vec{\nabla} S_0)\right] \cot\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) = E \ . \tag{35}$$

It is clear that Eqs. (28) and (35) cannot be simultaneously satisfied unless one has

$$\left[\vec{\nabla}\cdot(A^2 \vec{\nabla} S_0)\right] \ \tan\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) = -\left[\vec{\nabla}\cdot(A^2 \vec{\nabla} S_0)\right] \ \cot\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) \ . \tag{36}$$

This implies that either $\tan^2\left(\frac{S_0}{\hbar}+\frac{a-b}{2}\right)=-1$ which is not possible, or

$$\vec{\nabla}\cdot(A^2\vec{\nabla}S_0)=0\ . \tag{37}$$

In conclusion, the couple (A, S_0) must be chosen in such a way as to satisfy Eq. (37). However, as we will see in the next section, for any physical state ϕ, relations (27) and (37) do not always fix univocally A and S_0. In other words, there are some cases where some freedom subsists in the choice of A and S_0. Eq. (37), imposed by physical considerations, implies that (28) reduces to

$$\frac{1}{2m}(\vec{\nabla}S_0)^2-\frac{\hbar^2}{2m}\frac{\Delta A}{A}+V=E\ . \tag{38}$$

Eqs. (37) and (38) are exactly the same as those obtained for the complex wave functions in the last section.

Thus, for both real and complex wave functions, we obtain the same quantum Hamilton-Jacobi equation (26) or (38), and the functions A and S_0 are related by the same equation (25) or (37).

As shown by Bertoldi-Faraggi-Matone [12], these results can be also obtained within the framework of differential geometry in the following way. From the equivalence principle, which stipulates that physical states are equivalent under coordinate transformations, one can derive the cocycle condition in three dimensions. Then, without appealing to Schrödinger's equation, this condition leads to the quantum Hamilton-Jacobi equation.

5 MICROSTATES, HIDDEN ANTISYMMETRIC TENSOR AND VELOCITY

5.1 The quantum Hamilton-Jacobi equation in one dimension

For both real and complex wave functions, one can integrate (25) or (37) to obtain

$$A=k\left(\frac{\partial S_0}{\partial x}\right)^{-1/2}\ , \tag{39}$$

where k is a constant of integration. Then, by substituting this expression in (26) or (38), we obtain the well-known equation [14]

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial x}\right)^2-\frac{\hbar^2}{4m}\left[\frac{3}{2}\left(\frac{\partial S_0}{\partial x}\right)^{-2}\left(\frac{\partial^2 S_0}{\partial x^2}\right)^2-\left(\frac{\partial S_0}{\partial x}\right)^{-1}\left(\frac{\partial^3 S_0}{\partial x^3}\right)\right]+V=E\ . \tag{40}$$

Of course, this equation is different from the usual one because the function S_0 which appears here is related to the Schrödinger wave function by (19). Note that it is not possible to obtain such an equation for the real wave functions in Bohm's theory.

Before going further, it is interesting to see what the difference is between classical mechanics and quantum mechanics. Setting $\hbar=0$ in (40), we obtain the classical Hamilton-Jacobi equation. It is a differential equation of first-order and its solution $S_0^{cl}=S_0^{cl}(x,E)$ depends on the only one non-additive integration constant E which can be determined by the usual initial conditions. When we take into account the quantum effects ($\hbar\neq 0$), (40) is a differential equation of third-order and its solution $S_0^Q = S_0^Q(x, E, l_1, l_2)$ contains two another non-additive integration constants l_1 and l_2 which can be determined by further initial conditions [4, 15]. Thus, l_1 and l_2 can be considered as hidden variables.

5.2 The solution

In this subsection, we propose a new form of the general solution of the quantum Hamilton-Jacobi equation and show that this solution leads straightforwardly to the form (19) of the wave function. In fact, one can check that the expression

$$S_0 = \hbar \ \arctan\left(\frac{\sigma\theta_1 + \nu\theta_2}{\mu\theta_1 + \gamma\theta_2}\right) + \hbar\lambda \ , \tag{41}$$

is solution of Eq. (40). The set (θ_1, θ_2) represents two real independent solutions of the one dimensional stationary Schrödinger equation $-\hbar^2\phi^{''}/2m + V\phi = E\phi$ and $(\mu, \nu, \sigma, \gamma, \lambda)$ are arbitrary real constants satisfying the condition $\mu\nu \neq \sigma\gamma$. Note that one of the two couples (μ, ν) or (σ, γ) can be absorbed by rescaling the solutions θ_1 and θ_2 since the Schrödinger equation is linear. Therefore, in what follows, we set $\sigma = \gamma = 1$ and interpret μ and ν ($\mu\nu \neq 1$) as integration constants of the second-order differential equation (40) with respect to $\partial S_0/\partial x$ since this derivative depends only on μ and ν and not on λ

$$\frac{\partial S_0}{\partial x} = \frac{\hbar W(\mu\nu - 1)}{A^2} \ . \tag{42}$$

Here, $W = \theta_1\theta_2^{'} - \theta_2\theta_1^{'}$ is the Wronskian and the function A is given by

$$A = \sqrt{(\mu^2+1)\theta_1^2 + (\nu^2+1)\theta_2^2 + 2(\mu+\nu)\theta_1\theta_2} \ . \tag{43}$$

The parameter λ is an additive integration constant arising by integrating $\partial S_0/\partial x$. With these three integration constants (μ, ν, λ), expression (41) with $\sigma = \gamma = 1$, represents the general solution of (40). Note that Eq. (40) and its solution were also investigated by Floyd [6, 7] and by Faraggi-Matone [9, 10].

Let us show now that the solution (41) leads to the general form (19) of the wave function. From Eq. (41), we can deduce

$$\theta_1 + \nu\theta_2 = A \ \sin\left(\frac{S_0}{\hbar} - \lambda\right) \ , \tag{44}$$

$$\mu\theta_1 + \theta_2 = A \ \cos\left(\frac{S_0}{\hbar} - \lambda\right) \ . \tag{45}$$

If we solve this system for θ_1 and θ_2, we easily obtain

$$\theta_1 = A \ \frac{\nu \ \cos\left(\frac{S_0}{\hbar} - \lambda\right) - \sin\left(\frac{S_0}{\hbar} - \lambda\right)}{\mu\nu - 1} \tag{46}$$

$$\theta_2 = A \ \frac{\mu \ \sin\left(\frac{S_0}{\hbar} - \lambda\right) - \cos\left(\frac{S_0}{\hbar} - \lambda\right)}{\mu\nu - 1} \tag{47}$$

The general solution of the Schrödinger equation is given by

$$\phi = C_1\theta_1 + C_2\theta_2 \ . \tag{48}$$

The constants C_1 and C_2 are generally complex and determined by the initial conditions [7]

$$\phi(x_0) = C_1\theta_1(x_0) + C_2\theta_2(x_0) \ , \tag{49}$$

$$\phi^{'}(x_0) = C_1\theta_1^{'}(x_0) + C_2\theta_2^{'}(x_0) \ . \tag{50}$$

If we substitute expressions (46) and (47) in the physical solution (48) of the Schrödinger equation, we obtain

$$\phi = A\left[\alpha \ \exp\left(\frac{i}{\hbar}S_0\right) + \beta \ \exp\left(-\frac{i}{\hbar}S_0\right)\right] \tag{51}$$

in which

$$\alpha = \frac{(\nu C_1 - C_2) + i(C_1 - \mu C_2)}{2(\mu\nu - 1)} \exp\left(-i\lambda\right) \ , \tag{52}$$

$$\beta = \frac{(\nu C_1 - C_2) - i(C_1 - \mu C_2)}{2(\mu\nu - 1)} \exp\left(i\lambda\right) \ . \tag{53}$$

Expression (51) is exactly the same as the one given by Eq. (19) and which we have used to derive the quantum Hamilton-Jacobi equation.

According to (52) and (53), it is clear that for any fixed set (C_1, C_2), one can change arbitrarily the integration constants μ, ν and λ since α and β are also arbitrary. This could then be interpreted as a symmetry of the wave function because S_0, A, α and β defined respectively by (41), (43), (52) and (53) vary with μ, ν and λ, while the wave function (51) remains invariant since it has been constructed from (48). At first sight, this symmetry would give rise to microstates. In the next subsection, we will however show that this conclusion is not correct because in this construction we have two superfluous degrees of freedom.

5.3 Microstates and unbound states

In section 3, we showed that (51) could be written in the form (23). We can therefore write

$$\text{Re}\ \left[\exp\left(-i\frac{a+b}{2}\right)\phi\right] = (|\alpha| + |\beta|)\ A\ \cos\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) \ , \tag{54}$$

$$\text{Im}\ \left[\exp\left(-i\frac{a+b}{2}\right)\phi\right] = (|\alpha| - |\beta|)\ A\ \sin\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) \ . \tag{55}$$

In the unbound states case ($|\alpha| \neq |\beta|$), it follows

$$S_0 = \hbar\ \arctan\left(\frac{|\alpha| + |\beta|}{|\alpha| - |\beta|}\ \frac{\text{Im}\ \left[\exp(-i(a+b)/2)\,\phi\right]}{\text{Re}\ \left[\exp(-i(a+b)/2)\,\phi\right]}\right) + \hbar\frac{b-a}{2} \ . \tag{56}$$

Before analyzing the content of this equation, let us perform the transformation

$$\alpha \to \hat{\alpha} = \Omega\alpha, \qquad \beta \to \hat{\beta} = \Omega\beta \tag{57}$$

implying a dilatation followed by a rotation of the wave function in the complex space

$$\phi \to \hat{\phi} = \Omega\phi \ . \tag{58}$$

The parameter $\Omega = |\Omega| \ \exp(i\omega)$ is an arbitrary complex number and ω a real constant. Setting

$$\hat{\alpha} = |\hat{\alpha}| \ \exp(i\hat{a}) \ , \qquad\qquad \hat{\beta} = |\hat{\beta}| \ \exp(i\hat{b}) \ , \tag{59}$$

we have

$$|\hat{\alpha}| = |\Omega||\alpha| \ , \qquad |\hat{\beta}| = |\Omega||\beta| \qquad \hat{a} = a + \omega \ , \qquad \hat{b} = b + \omega \ . \tag{60}$$

The new wave function must have the form

$$\hat{\phi} = \hat{A}\left[\hat{\alpha} \ \exp\left(\frac{i}{\hbar}\hat{S}_0\right) + \hat{\beta} \ \exp\left(-\frac{i}{\hbar}\hat{S}_0\right)\right] \tag{61}$$

and therefore, as in Eq. (56), the new reduced action is given by

$$\hat{S}_0 = \hbar \ \arctan\left(\frac{|\hat{\alpha}| + |\hat{\beta}|}{|\hat{\alpha}| - |\hat{\beta}|} \ \frac{\mathrm{Im} \ \left[\exp(-i(\hat{a}+\hat{b})/2)\,\hat{\phi}\right]}{\mathrm{Re} \ \left[\exp(-i(\hat{a}+\hat{b})/2)\,\hat{\phi}\right]}\right) + \hbar\frac{\hat{b}-\hat{a}}{2} \ . \tag{62}$$

Substituting Eqs. (58) and (60) in (62), we obtain

$$\hat{S}_0 = S_0 \ . \tag{63}$$

Now, if we substitute (57), (58) and (63) in (61) and then use (51), we deduce that

$$\hat{A} = A \ . \tag{64}$$

In conclusion, the functions A and S_0 are invariant under a dilatation and a rotation of the wave function in the complex space. This invariance of the reduced action allows us to perform a particular transformation for which $\Omega = \alpha^{-1}$ and identify the new wave function to the physical one given by (48)

$$C_1\theta_1 + C_2\theta_2 = A\left[\exp\left(\frac{i}{\hbar}S_0\right) + \eta \ \exp\left(-\frac{i}{\hbar}S_0\right)\right] \ , \tag{65}$$

where $\eta = \beta/\alpha$. Since the additive integration constant λ has no dynamical effects, we set $\lambda = 0$ from now on. Therefore, substituting expressions (41) and (43) for S_0 and A in the right hand side of (65), we obtain

$$C_1\theta_1 + C_2\theta_2 = \left[\mu + i + \eta(\mu - i)\right] \ \theta_1 + \left[1 + i\nu + \eta(1 - i\nu)\right] \ \theta_2 \ . \tag{66}$$

The identification of the coefficients of θ_1 and θ_2 leads to

$$C_1 = \mu + i + \eta(\mu - i) \ , \tag{67}$$

$$C_2 = 1 + i\nu + \eta(1 - i\nu) \ , \tag{68}$$

which are equivalent to Eqs. (52) and (53) if we set $\lambda = 0$, $\alpha = 1$ and $\beta = \eta$. Separating in (67) and (68) the real part from the imaginary part, we obtain four equations which can be solved with respect to $\mathrm{Re}(\eta)$, $\mathrm{Im}(\eta)$, μ and ν. It follows that in this particular choice of Ω, the reduced action S_0 defined by (41) with $\sigma = \gamma = 1$ is entirely determined. Furthermore, since S_0 is invariant, it keeps its value for any choice of Ω. In conclusion, for a given physical state, the initial conditions (49) and (50) fix univocally the reduced action and therefore there is no trace of microstates for unbound states.

To clarify our point of view, we would like to make the following remark. Eqs. (52) and (53) can be written in the form

$$C_1 = \alpha\ \Big[(\mu + i)\ \exp(i\lambda)\Big] + \beta\ \Big[(\mu - i)\ \exp(-i\lambda)\Big]\ , \tag{69}$$

$$C_2 = \alpha\ \Big[(1 + i\nu)\ \exp(i\lambda)\Big] + \beta\ \Big[(1 - i\nu)\ \exp(-i\lambda)\Big]\ . \tag{70}$$

These last equations show clearly that we can rescale C_1 and C_2 with any complex number by multiplying α and β by the same number without any effect on μ, ν and λ. This rescaling does not affect the wave function up to a multiplicative constant factor. This means that we have two more degrees of freedom. In others words, the fact that Ω can be arbitrarily chosen implies that, among the four real parameters defining the complex numbers α and β in the form (51) of the wave function, we can keep only two parameters free. In this way, for any set (C_1, C_2) of initial conditions, these two free parameters will be fixed for unbound states.

Note that we can also arrive to the same conclusion using another method. In fact, substituting expression (41) for S_0 with $\sigma = \gamma = 1$ in (23), we obtain

$$\begin{aligned}\phi = A\,(|\alpha| + |\beta|)\ \cos\left[\arctan\left(\frac{\theta_1 + \nu\theta_2}{\mu\theta_1 + \theta_2}\right)\right] + \\ iA\,(|\alpha| - |\beta|)\ \sin\left[\arctan\left(\frac{\theta_1 + \nu\theta_2}{\mu\theta_1 + \theta_2}\right)\right]\ ,\end{aligned} \tag{71}$$

where we have discarded the unimportant phase factor $\exp\left(i\frac{a+b}{2}\right)$ and chosen the integration constant λ equal to $(b-a)/2$. Using expression (43) for A, Eq. (71) leads to

$$\phi = \big[\mu(|\alpha| + |\beta|) + i(|\alpha| - |\beta|)\big]\ \theta_1 + \big[|\alpha| + |\beta| + i\nu(|\alpha| - |\beta|)\big]\ \theta_2 \tag{72}$$

Identifying this expression with Eq. (48), we obtain

$$C_1 = \mu(|\alpha| + |\beta|) + i(|\alpha| - |\beta|) \tag{73}$$

$$C_2 = |\alpha| + |\beta| + i\nu(|\alpha| - |\beta|) \tag{74}$$

Separating the real part from the imaginary part in these last two equations, we obtain a system of four equations which can be solved with respect to $|\alpha|$, $|\beta|$, μ and ν. It follows that for a given physical wave function ϕ, these results fix univocally the reduced action.

Note that in this reasoning, we have also eliminated the two superfluous degrees of freedom. The first one has been eliminated by discarding the global phase $\exp\left(i\frac{a+b}{2}\right)$ meaning that we have chosen $\Omega = \exp\left(-i\frac{a+b}{2}\right)$, and the second one in the judicious choice of the value of the integration constant λ.

5.4 Microstates and bound states

For bound states ($|\alpha| = |\beta|$), Eq. (71) reduces to

$$\phi = 2\,|\alpha| A\ \cos\left[\arctan\left(\frac{\theta_1 + \nu\theta_2}{\mu\theta_1 + \theta_2}\right)\right] \tag{75}$$

Using expression (43) for A, this last equation becomes

$$\phi = 2\,|\alpha|(\mu\theta_1 + \theta_2) \ . \tag{76}$$

Identifying now this expression with (48), we obtain

$$C_1 = 2\,|\alpha|\mu \ , \qquad\qquad C_2 = 2\,|\alpha| \tag{77}$$

which leads to

$$\mu = \frac{C_1}{C_2} \ . \tag{78}$$

It is clear that the initial conditions (49) and (50) of the Schrödinger wave function do not allow to fix the value of ν. According to Floyd's proposal [4] for which time parameterization is given by Jacobi's theorem : $t - t_0 = \partial S_0/\partial E$, Eqs. (41) and (77) mean that we obtain, for a given physical state ϕ, a time dependent family of trajectories which can be specified by the different values of ν. Hence, we can assert that ν plays the role of a hidden variable and specifies, for the same state ϕ, the different microstates not detected by the Schrödinger wave function.

Note that we can also explain these results in the following manner. Substituting in (27) the function A by its expression (39), we obtain

$$\phi = D\ \left(\frac{\partial S_0}{\partial x}\right)^{-1/2}\ \cos\left(\frac{S_0}{\hbar} + \frac{a-b}{2}\right) \ , \tag{79}$$

where we have discarded the unimportant phase factor $\exp\left(i\frac{a+b}{2}\right)$ and set $D = 2|\alpha|k$. Integrating this differential equation gives

$$S_0 = \hbar\ \arctan\left(\frac{1}{\hbar}\int\frac{dx}{(\phi/D)^2} + H\right) + \hbar\frac{b-a}{2} \ , \tag{80}$$

where we have written explicitly the integration constant H which arises by calculating the integral appearing in the right hand side of this last equation. The presence of this arbitrary constant H explains the fact that for any physical state ϕ, some freedom subsists in the choice of the reduced action S_0, giving rise to the existence of microstates for bound states.

In conclusion, the Schrödinger wave function is not an exhaustive description of non-relativistic systems. The quantum Hamilton-Jacobi equation is more fundamental. Hence, we confirm the finding of Floyd [7] who also showed that trajectory representation described microstates for bound states and not for unbound states. Finally, we would like to add that the problem of microstates was also investigated by Carroll [16] and by Faraggi-Matone-Bertoldi [10, 12].

5.5 The hidden antisymmetric two-tensor

In three dimensions, the continuity equation (25) or (37) indicates that we can write

$$A^2\vec{\nabla}S_0 = \vec{\nabla}\times\vec{B} \ , \tag{81}$$

where $\vec{B}$ is a vector field for which we can associate the two-tensor

$$F^{ij} = \partial^i B^j - \partial^j B^i \tag{82}$$

such that

$$A^2 \partial_i S_0 = \frac{1}{2} \epsilon_{ijk} F^{jk} \ , \tag{83}$$

i, j and k represent the indices (x, y, z) and ϵ_{ijk} is the usual Levi-Civita antisymmetric tensor. The continuity equation takes the form

$$\epsilon_{ijk} \partial^i F^{jk} = 0 \ , \tag{84}$$

with F^{ij} being a hidden antisymmetric two-tensor field which may play, as suggested by Bertoldi-Faraggi-Matone [12], an important role in the understanding of the quantum origin of fundamental interactions.

5.6 Velocity

The last point which we will discuss in this section concerns the velocity. At first sight, according to the form (19) of the wave function, one may think that we can associate in any point of the trajectory two values for the velocity. In fact, the two components $\exp\left(+\frac{i}{\hbar} S_0\right)$ and $\exp\left(-\frac{i}{\hbar} S_0\right)$ describe propagation of waves in opposite directions. In this way, and interpreting the probability current as a sum of two currents (Eq. (31)), we can associate a motion of the particle to each of these components and define then at any point of the trajectory two velocities by using the hydrodynamic approach [1, 3]

$$\vec{v}_+ = \frac{\vec{\jmath}_+}{\mathcal{A}^2} \ , \qquad \vec{v}_- = \frac{\vec{\jmath}_-}{\mathcal{A}^2} \tag{85}$$

where $\mathcal{A}$ is the amplitude of the wave function

$$\mathcal{A} = A \ \left[|\alpha|^2 + |\beta|^2 + 2\, |\alpha||\beta| \cos\left(\frac{2\, S_0}{\hbar} + a - b \right) \right]^{1/2} \tag{86}$$

However, in the one-dimensional case with a constant potential, Floyd showed [17] that it was possible for two waves propagating in opposite directions, to synthesize a single wave propagating only in one direction. This result although obtained for a particular case, indicates that the hydrodynamic approach is not appropriate to define correctly the particle velocity. This is different from the ideas developed by Brown and Hiley [18] who consider that the particle velocity is the same to the current velocity and assert that the use of the classical canonical theory to define the momentum is totally unnecessary. Without appealing to the hydrodynamic approach, Floyd showed [4] from the quantum Hamilton-Jacobi equation and Jacobi's theorem ($\partial S_0 / \partial E = t - t_0$), that in one dimension the velocity is given by relation (10). In the same spirit, Carroll showed in Ref. [16] that the current velocity could not be identified with the particle velocity.

6 CONCLUSION

In the three-dimensional space, it is shown in this paper that the wave function, whether real or complex, has the unified form (19) which leads to the same quantum Hamilton-Jacobi equation (26) or (38), and the functions A and S_0 are related by the same continuity equation. The problem of the vanishing value of the conjugate momentum for real wave functions appearing in Bohm's theory is solved by the fact that the reality of the wave function is not expressed by $S_0 = cst.$ but by $|\alpha| = |\beta|$. Let us insist on the fact that the quantum

Hamilton-Jacobi equation obtained here is fundamentally different from the usual one because the reduced action S_0 is related to the wave function by (19).

We have also proposed in one dimension a new form of the general solution of the quantum Hamilton-Jacobi equation and shown in the bound state case that there are microstates not detected by the Schrödinger wave function. In three dimensions, we have seen that there is a hidden antisymmetric two-tensor field underlying quantum mechanics, recently introduced by Bertoldi-Faraggi-Matone.

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