ON THE SELECTION OF TRIADS IN THE TELEPARALLEL GEOMETRY AND BONDI'S RADIATING METRIC

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Abstract

A consistent Hamiltonian formulation of the teleparallel equivalent of general relativity (TEGR) requires the theory to be invariant under the global SO(3) symmetry group, which acts on orthonormal triads in three-dimensional spacelike hypersurfaces. In the TEGR it is possible to make definite statements about the energy of the gravitational field. In this geometrical framework two sets of triads related by a local SO(3) transformation yield different descriptions of the gravitational energy. Here we consider the problem of assigning a unique set of triads to the metric tensor restricted to the three-dimensional hypersurface. The analysis is carried out in the context of Bondi's radiating metric. A simple and original expression for Bondi's news function is obtained, which allows us to carry out numerical calculations and verify that a triad with a specific asymptotic behaviour yields the minimum gravitational energy for a fixed space volume containing the radiating source. This result supports the conjecture that the requirement of a minimum gravitational energy for a given space volume singles out uniquely the correct set of orthonormal triads.

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I. Introduction

General covariance and the notion of absolute parallelism of vector fields in space-time are the two major geometrical concepts of teleparallel theories of gravity. Møller[1] was probably the first to make use of the concept of absolute parallelism to put forward an alternative geometrical framework for general relativity that could satisfactorily address the problem of the definition of the gravitational energy.

The idea of absolute parallelism can be established by considering a space-time vector field $V^\mu(x)$ and a set of orthonormal tetrad fields $e^a{}_\mu(x)$. At the space-time point x^λ the tetrad components of the vector field are given by $V^a(x) = e^a{}_\mu(x) V^\mu(x)$, and at $x^\lambda + dx^\lambda$ by $V^a(x+dx) = e^a{}_\mu(x+dx)V^\mu(x+dx) = V^a(x) + DV^a(x)$, where $DV^a(x) = e^a{}_\mu(\nabla_\lambda V^\mu) dx^\lambda$. The covariant derivative ∇ is constructed out of the connection

$$\Gamma^\lambda_{\mu\nu} = e^{a\lambda}\partial_\mu e_{a\nu} \; . \tag{1}$$

The vector field $V^\mu(x)$ is said to be autoparallel if its tetrad components at distant points coincide. Thus $V^\mu(x)$ is autoparallel if $\nabla_\lambda V^\mu$ vanishes. Therefore connection (1) defines a *condition* for absolute parallelism, or teleparallelism, in space-time. Such connection only makes sense if the tetrad field transforms under the *global* SO(3,1) group.

A gravity theory based on (1) obviously depart from the Riemannian geometry because the curvature tensor constructed out of it vanishes identically. Inspite of this fact, there does exist a theory based on (1) that describes the dynamics of the gravitational field in agreement with Einstein's general relativity. Møller called such theory the "tetrad theory of gravity", but for a long time it has been known as the teleparallel equivalent of general relativity (TEGR)[2, 3]. The Lagrangian density for this alternative description of general relativity is constructed by means of a quadratic combination of the torsion tensor $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu$, which is related to the antisymmetric part of connection (1).

It is possible to establish a theory for the gravitational field directly from (1). However, in order to make contact with more recent analysis, it is instructive to approach the TEGR firstly considering it with a local SO(3,1) symmetry in the Lagrangian context. Eventually we will return to the geometrical framework determined by (1).

Although not extensively investigated in the literature, the TEGR in Lagrangian form has been considered as a viable formulation of the gravitational dynamics[4, 5, 6, 7, 8] inspite of troubles that may spoil the initial value problem[9, 10]. Such problems take place if the Lagrangian density of the TEGR is formulated with a *local* SO(3,1) symmetry, in which case the torsion tensor is defined by $T^a\,_{\mu\nu} = \partial_\mu e^a\,_\nu - \partial_\nu e^a\,_\mu + \omega_\mu\,^a\,_b\, e^b\,_\nu - \omega_\nu\,^a\,_b\, e^b\,_\mu$. The spin connection $\omega_{\mu ab}$ is totally independent of $e^a\,_\mu$ and satisfies the condition of vanishing curvature: $R^a\,_{b\mu\nu}(\omega) \equiv 0$.

In order to solve problems with respect to the initial value problem, the Hamiltonian formulation of the TEGR with the local SO(3,1) symmetry was considered[11]. By working out the constraint algebra one ultimately concludes that in order to formulate consistently the theory in terms of *first class constraints* it is mandatory to break the local SO(3) symmetry of the action in Hamiltonian form. Therefore a consistent Hamiltonian formulation of the theory displays invariance under the global SO(3) group that acts on triads restricted to three-dimensional spacelike hypersurfaces.

The major feature of the Hamiltonian formulation of the TEGR is that the integral form of the Hamiltonian constraint equation can be written as an energy equation of the type $C = H - E = 0$[12, 13, 14]. The reason for this follows from the fact that the Hamiltonian constraint contains a scalar density in the form of a total divergence, whose integral over the whole three-dimensional hypersurface yields the ADM energy[15]. As we will argue, the integral of this scalar density over finite volumes of the three-dimensional hypersurface yields a natural definition for the gravitational field energy. Therefore in the TEGR one can make definite statements about the localizability of the gravitational energy, inspite of claims according to which the latter is not localizable. In fact the very concept of a black hole lends support to the idea that the gravitational energy is localizable, since there is no process by means of which the gravitational mass inside a black hole can be made to vanish. A remarkable application of the energy expression of the TEGR has been made in the evaluation of the irreducible mass of a rotating black hole[16].

The TEGR exhibits two specific properties: the emergence of a possible definition for the gravitational field energy and the global SO(3) symmetry of the theory. We believe that these two features are intimately related. Since the symmetry of the theory is global, triads related by a local SO(3) transformation are inequivalent and a priori we have no means to select the one that actually describes the spacelike hypersurface. We conjecture that the requirement of a minimum gravitational energy for a given space volume is one condition that singles out uniquely the correct set of triads. In this paper we investigate this conjecture in the realistic context of Bondi's radiating metric[17]. This conjecture was already put forward in a previous investigation of Bondi's energy in the framework of the TEGR[18]. An additional, essential requirement for a consistent expression of the gravitational energy is the boundary conditions on the triads. It was noted[12] that the ADM energy is obtained from the energy expression of the TEGR if the asymptotic behaviour of the triads at spacelike infinity is given by

$$e_{(i)j} \approx \eta_{ij} + \frac{1}{2} h_{ij}(\frac{1}{r}) \,, \tag{2}$$

irrespective of any symmetry of the tensor h_{ij}.

A second possible, independent condition for assigning a set of triads to a given three-dimensional metric tensor amounts to requiring a symmetric tensor $h_{ij} = h_{ji}$ in the asymptotic expansion of $e_{(i)j}$. We will prove that this symmetry condition *uniquely* associates $e_{(i)j}$ to the metric tensor of the spacelike hypersurface of Bondi's metric. The unique character of such triads strongly supports this second conjecture.

The two conjectures above are not mutually excluding. In this paper we argue that the set of triads that yield the minimum gravitational energy within a space volume containing the radiating source is the one whose asymptotic behaviour is determined by the symmetry condition $h_{ij} = h_{ji}$. We will show that this fact is indeed verified by analyzing several configurations for the triads. Unfortunately we have not found it possible to prove on general grounds that the "symmetrized" triad yields the minimum energy.

In order to obtain numerical values for the gravitational energy for a finite three-dimensional volume of Bondi's space-time we need an explicit expression of the news function. However the existing expressions in the literature are not suitable for our purposes. In particular, the expression of the news function given by Hobill[19] (to be presented ahead) is very intricate to the extent of not allowing a computer evaluation of numerical values of the gravitational energy. Therefore we have obtained an original and simpler expression for the news function that: (i) satisfies all necessary regularity conditions related to the axial symmetry of the system, (ii) makes the initial and final states (space-times) described by Bondi's metric to be nonradiative and (iii) yields an expression for the energy density that can be numerically integrated.

In Section II we describe the TEGR in Lagrangian and Hamiltonian formulations, show the emergence of the definition of the gravitional energy and further discuss the troubles that arise in the initial value problem of the theory if it is constructed with a local SO(3,1) symmetry. In section III we present Bondi's radiating metric and three expressions for triads restricted to the three-dimensional spacelike hypersurface. In this section we also prove that the symmetry condition $h_{ij} = h_{ji}$ uniquely associates a set of triads (whose asymptotic behaviour is given by (2)) with the metric tensor for the spacelike section of asymptotically flat space-times. The news function and the related mass aspect that will be needed for the calculations of the gravitational energy are obtained in section IV. In section V we carry out several calculations that lead to the main conclusion regarding the selection of triads.

Notation: spacetime indices $\mu, \nu, ...$ and local Lorentz indices $a, b, ...$ run from 0 to 3. In the 3+1 decomposition latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i, \quad a = (0), (i)$. The tetrad field $e^a\,_\mu$ and the spin connection $\omega_{\mu ab}$ yield the usual definitions of the torsion and curvature tensors: $R^a\,_{b\mu\nu} = \partial_\mu \omega_\nu\,^a\,_b + \omega_\mu\,^a\,_c \omega_\nu\,^c\,_b \,-\, ...$, $T^a\,_{\mu\nu} = \partial_\mu e^a\,_\nu + \omega_\mu\,^a\,_b\, e^b\,_\nu \,-\, ...$. The flat space-time metric is fixed by $\eta_{(0)(0)} = -1$.

II. The Lagrangian and Hamiltonian formulations of the TEGR

The Lagrangian density of the TEGR in empty space-time, displaying a local SO(3,1) symmetry, is given by

$$L(e, \omega, \lambda) \;=\; -ke(\frac{1}{4}T^{abc}T_{abc} \,+\, \frac{1}{2}T^{abc}T_{bac} \,-\, T^aT_a) \,+\, e\lambda^{ab\mu\nu}R_{ab\mu\nu}(\omega)\;. \quad (3)$$

where $k = \frac{1}{16\pi G}$, G is the gravitational constant; $e = det(e^a\,_\mu)$, $\lambda^{ab\mu\nu}$ are Lagrange multipliers and T_a is the trace of the torsion tensor defined by $T_a = T^b\,_{ba}$. The tetrad field $e_{a\mu}$ and the spin connection $\omega_{\mu ab}$ are completely independent field variables. The latter is enforced to satisfy the condition of zero curvature. Therefore this Lagrangian formulation is in no way similar to the usual Palatini formulation, in which the spin connection is related to the tetrad field via field equations. Later on we will introduce the tensor Σ_{abc} defined by

$$\frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a \;\equiv\; T^{abc}\Sigma_{abc}\;.$$

The equivalence of the TEGR with Einstein's general relativity is based on the identity

$$eR(e,\omega) \;=\; eR(e) \,+\, e(\frac{1}{4}T^{abc}T_{abc} \,+\, \frac{1}{2}T^{abc}T_{bac} \,-\, T^aT_a) \,-\, 2\partial_\mu(eT^\mu)\;,$$

which is obtained by just substituting the arbitrary spin connection $\omega_{\mu ab} = {}^o\omega_{\mu ab}(e) + K_{\mu ab}$ in the scalar curvature tensor $R(e,\omega)$ in the left hand side; ${}^o\omega_{\mu ab}(e)$ is the Levi-Civita connection and $K_{\mu ab} = \frac{1}{2}e_a{}^\lambda e_b{}^\nu(T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda})$ is the contorsion tensor. The vanishing of $R^a{}_{b\mu\nu}(\omega)$, which is one of the field equations derived from (3), implies the equivalence of the scalar curvature $R(e)$, constructed out of $e^a{}_\mu$ only, and the quadratic combination of the torsion tensor. It also ensures that the field equation arising from the variation of L with respect to $e^a{}_\mu$ is strictly equivalent to Einstein's equations in tetrad form. Let $\frac{\delta L}{\delta e^{a\mu}} = 0$ denote the field equations satisfied by $e^{a\mu}$. It can be shown by explicit calculations that

$$\frac{\delta L}{\delta e^{a\mu}} \;=\; \frac{1}{2}e\{R_{a\mu}(e) \,-\, \frac{1}{2}e_{a\mu}R(e)\}\;. \tag{4}$$

We refer the reader to Ref. [11] for additional details.

Throughout this section we will be interested in asymptoticaly flat space-times. The Hamiltonian formulation of the TEGR can be successfully implemented if we fix the gauge $\omega_{0ab} = 0$ from the outset, since in this case the constraints (to be shown below) constitute a *first class* set[11]. The condition $\omega_{0ab} = 0$ is achieved by breaking the local Lorentz symmetry of (3). We still make use of the residual time independent gauge symmetry to fix the usual time gauge condition $e_{(k)}{}^0 = e_{(0)i} = 0$. Because of $\omega_{0ab} = 0$, H does not depend on P^{kab}, the momentum canonically conjugated to ω_{kab}. Therefore arbitrary variations of $L = p\dot{q} - H$ with respect to P^{kab} yields $\dot{\omega}_{kab} = 0$. Thus in view of $\omega_{0ab} = 0$, ω_{kab} drops out from our considerations.

As a consequence of the above gauge fixing the canonical action integral obtained from (3) becomes[11]

$$A_{TL} \;=\; \int d^4x\{\Pi^{(j)k}\dot{e}_{(j)k} \;-\; H\}\;, \tag{5}$$

$$H \;=\; NC \;+\; N^iC_i \;+\; \Sigma_{mn}\Pi^{mn} \;+\; \frac{1}{8\pi G}\partial_k(NeT^k) \;+\; \partial_k(\Pi^{jk}N_j)\;. \tag{6}$$

N and N^i are the lapse and shift functions, and $\Sigma_{mn} = -\Sigma_{nm}$ are Lagrange multipliers. The constraints are defined by

$$C \;=\; \partial_j(2keT^j) \;-\; ke\Sigma^{kij}T_{kij} \;-\; \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2)\;, \tag{7a}$$

$$C_k \;=\; -e_{(j)k}\partial_i\Pi^{(j)i} \;-\; \Pi^{(j)i}T_{(j)ik}\;, \tag{7b}$$

where $e = det(e_{(j)k})$, $T^i \;=\; g^{ik}e^{(j)l}T_{(j)lk}$, $T_{(i)jk} = \partial_j e_{(i)k} - \partial_k e_{(i)j}$, and

$$\Sigma^{ijk} = \frac{1}{4}(T^{ijk} + T^{jik} - T^{kij}) + \frac{1}{2}(\eta^{ik}T^j - \eta^{ij}T^k)\;.$$

We remark that (5) and (6) are now invariant under global SO(3) and general coordinate transformations. Therefore the torsion tensor restricted to the three-dimensional spacelike hypersurface is ultimately related to the antisymmetric part of the spatial components of (1). Had we dispensed with the connection $\omega_{\mu ab}$ from the outset we would have arrived at precisely the canonical formulation determined by (5), (6) and (7). Such connection has been considered in previous investigations of teleparallel theories, but in fact it is eventually unnecessary for the establishment of the theory.

If we assume the asymptotic behaviour

$$e_{(j)k} \approx \eta_{jk} + \frac{1}{2}h_{jk}(\frac{1}{r}) \tag{2}$$

for $r \to \infty$, then in view of the relation

$$\frac{1}{8\pi G}\int d^3x\partial_j(eT^j) \;=\; \frac{1}{16\pi G}\int_S dS_k(\partial_i h_{ik} - \partial_k h_{ii}) \;\equiv\; E_{ADM} \tag{8a}$$

where the surface integral is evaluated for $r \to \infty$, the integral form of the Hamiltonian constraint $C = 0$ may be rewritten as

$$\int d^3x \left\{ ke\Sigma^{kij}T_{kij} + \frac{1}{4ke}(\Pi^{ij}\Pi_{ji} - \frac{1}{2}\Pi^2) \right\} \;=\; E_{ADM} \;. \qquad (8b)$$

The integration is over the whole three-dimensional space. Given that $\partial_j(eT^j)$ is a scalar density, from (8a,b) we define the gravitational energy density enclosed by a volume V of the space as

$$E \;=\; \frac{1}{8\pi G}\int_V d^3x \partial_j(eT^j) \;. \qquad (9)$$

It must be noted that E depends only on the triads $e_{(k)i}$ restricted to a three-dimensional spacelike hypersurface; the inverse quantities $e^{(k)i}$ can be written in terms of $e_{(k)i}$. From the right hand side of equation (4) we observe that the dynamics of the triads does not depend on $\omega_{\mu ab}$. Therefore E given above does not depend on the fixation of any gauge for $\omega_{\mu ab}$. The reference space which defines the zero of gravitational energy has been defined in ref.[16]. We briefly remark that the differences between (9) and Møller's expression for the gravitational energy have been thoroughly discussed in ref.[18].

We make now the important assumption that the general form of the canonical structure of the TEGR is the same for any class of space-times, irrespective of the peculiarities of the latter (for the de Sitter space[20], for example, there is an *additional* term in the Hamiltonian constraint C). Therefore we assert that the integral form of the Hamiltonian constraint equation can be written as $C = H - E = 0$ for any space-time, and that (9) represents the gravitational energy for arbitrary space-times with any topology.

We recall finally that Müller-Hoissen and Nitsch[10] and Kopczyński[9] have shown that in general the theory defined by (3) faces difficulties with respect to the Cauchy problem. They have shown that in general six components of the torsion tensor are not determined from the evolution of the initial data. On the other hand, the constraints of the theory constitute a first class set provided we fix the six quantities $\omega_{0ab} = 0$ *before varying the action*[11]. This condition is mandatory and does not merely represent one particular gauge fixing of the theory. Since the fixing of ω_{0ab} yields a well defined theory with first class constraints, we cannot assert that the field configurations of the latter are gauge equivalent to configurations whose time evolution is not precisely determined. The requirement of local SO(3,1) symmetry plus the addition of $\lambda^{ab\mu\nu}R_{ab\mu\nu}(\omega)$ in (3) has the ultimate effect of discarding the connection.

Constant rotations constitute a basic feature of the teleparallel geometry. According to Møller[1], in the framework of the abolute parallelism tetrad fields, together with the boundary conditions, uniquely determine a *tetrad lattice*, apart from an arbitrary *constant rotation of the tetrads in the lattice*.

III. Bondi's radiating metric and the associated triads.

Bondi's metric describes the asymptotic form of a radiating solution of Einstein's equations. It is not an exact solution; it holds only in the asymptotic region. In terms of radiation coordinates (u, r, θ, ϕ), where u is the retarded time and r is the luminosity distance, Bondi's metric is written as[17]

$$\begin{aligned} ds^2 &= -\Big(\frac{V}{r}e^{2\beta} - U^2\, r^2 e^{2\gamma}\Big)du^2 - 2e^{2\beta}du\, dr - 2U\, r^2\, e^{2\gamma}du\, d\theta \\ &\qquad + r^2\Big(e^{2\gamma}\, d\theta^2 + e^{-2\gamma}\, sin^2\theta\, d\phi^2\Big)\,. \end{aligned} \tag{10}$$

This metric tensor displays axial symmetry and reflection invariance. By requiring $u = constant$, (10) describes null hypersurfaces. Each null radial (light) ray is labelled by particular values of u, θ and ϕ. At spacelike infinity u takes the standard form $u = t - r$. The four quantities appearing in (10), V, U, β and γ are functions of u, r and θ. A more general form of this metric has been given by Sachs[21], who showed that the most general metric tensor describing asymptotically flat gravitational waves depends on six functions of the coordinates.

The functions in (10) have the following asymptotic behaviour:

$$\beta = -\frac{c^2}{4r^2} + ...$$

$$\gamma = \frac{c}{r} + O(\frac{1}{r^3}) + ...$$

$$\frac{V}{r} = 1 - \frac{2M}{r} - \frac{1}{r^2}\Big[\frac{\partial d}{\partial\theta} + d\, cot\theta - \Big(\frac{\partial c}{\partial\theta}\Big)^2 - 4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta - \frac{1}{2}\, c^2\Big(1 + 8cot^2\theta\Big)\Big] + ...$$

$$U \;=\; -\,\frac{1}{r^2}\Big(\frac{\partial c}{\partial\theta}+2c\,cot\theta\Big)+\frac{1}{r^3}\Big(2d+3c\,\frac{\partial c}{\partial\theta}+4c^2\,cot\theta\Big)+...$$

where $M = M(u,\theta)$ and $d = d(u,\theta)$ are the mass aspect and the dipole aspect, respectively. From the function $c(u,\theta)$ we define the *news function* $\frac{\partial c(u,\theta)}{\partial u}$.

The functions U, V, β and γ must satisfy *regularity conditions* along the z axis ($\theta = 0, \pi$). We must require

$$V, \quad \beta, \quad \frac{U}{sin\theta}, \quad \frac{\gamma}{sin^2\theta}$$

to be regular functions of $cos\theta$ for $\theta = 0, \pi$. The regularity conditions will be necessary for the construction of the news function, in section IV.

The application of (9) to Bondi's metric requires transforming it to spherical coordinates (t, r, θ, ϕ) for which $t = constant$ defines a space-like hypersurface. Therefore we carry out a coordinate transformation such that the new timelike coordinate is given by $t = u + r$. We arrive at

$$\begin{aligned} ds^2 \;&=\; -\Big(\frac{V}{r}e^{2\beta}-U^2\,r^2\,e^{2\gamma}\Big)dt^2-2U\,r^2\,e^{2\gamma}dt\,d\theta \\ &+2\Big[e^{2\beta}\Big(\frac{V}{r}-1\Big)-U^2\,r^2\,e^{2\gamma}\Big]dr\,dt \\ &+\Big[e^{2\beta}\Big(2-\frac{V}{r}\Big)+U^2\,r^2\,e^{2\gamma}\Big]dr^2+2U\,r^2\,e^{2\gamma}dr\,d\theta+r^2\Big(e^{2\gamma}d\theta^2+e^{-2\gamma}\,sin^2\theta\,d\phi^2\Big)\,. \end{aligned} \tag{11}$$

Therefore the metric restricted to a three-dimensional spacelike hypersurface is given by

$$\begin{aligned} ds^2 \;&=\; \Big[e^{2\beta}\Big(2-\frac{V}{r}\Big)+U^2\,r^2\,e^{2\gamma}\Big]dr^2+2U\,r^2\,e^{2\gamma}\,dr\,d\theta \\ &+r^2\Big(e^{2\gamma}\,d\theta^2+e^{-2\gamma}\,sin^2\theta\,d\phi^2\Big)\,. \end{aligned} \tag{12}$$

We recall that Goldberg[22] and Papapetrou[23] have already considered Bondi's metric in cartesian coordinates.

The crucial point of the present investigation is the determination, in the framework of the TEGR, of the correct set of triads that lead to (12). If the metric tensor has only diagonal components, such as the metric tensor restricted to the three-dimensional spacelike section of Kerr's space-time, then the *simplest* construction has proven to be the correct one[16]. However, for metric tensors that contain off-diagonal terms, the determination of the unique triad is by no means a trivial procedure. In fact there is an infinity of triads that satisfy the boundary conditions determined by (2) and lead to (12). In Ref. [18] two sets of triads that comply with (2) are presented. They are given by

$$e_{(k)i} = \begin{pmatrix} A\,sin\theta\,cos\phi + B\,cos\theta\,cos\phi & rC\,cos\theta\,cos\phi & -rD\,sin\theta\,sin\phi \\ A\,sin\theta\,sin\phi + B\,cos\theta\,sin\phi & rC\,cos\theta\,sin\phi & rD\,sin\theta\,cos\phi \\ A\,cos\theta - B\,sin\theta & -rC\,sin\theta & 0 \end{pmatrix}\,, \tag{13}$$

where

$$A = e^{\beta}\sqrt{2-\frac{V}{r}}\,, \tag{14a}$$

$$B = r\,U\,e^{\gamma}\,, \tag{14b}$$

$$C = e^{\gamma}\,, \tag{14c}$$

$$D = e^{-\gamma}\,, \tag{14d}$$

and

$$e_{(k)i} = \begin{pmatrix} A'\,sin\theta\,cos\phi & rB'\,cos\theta\,cos\phi + rC'\,sin\theta\,cos\phi & -rD'\,sin\theta\,sin\phi \\ A'\,sin\theta\,sin\phi & rB'\,cos\theta\,sin\phi + rC'\,sin\theta\,sin\phi & rD'\,sin\theta\,cos\phi \\ A'\,cos\theta & -rB'\,sin\theta + rC'\,cos\theta & 0 \end{pmatrix}\,, \tag{15}$$

where

$$A' = \left[e^{2\beta}\left(2-\frac{V}{r}\right) + U^2\,r^2\,e^{2\gamma}\right]^{\frac{1}{2}}\,, \tag{16a}$$

$$B' \;=\; \frac{1}{A'}\, e^{\beta+\gamma}\sqrt{2-\frac{V}{r}}\ , \tag{16b}$$

$$C' \;=\; \frac{1}{A'}\, U\, r\, e^{2\gamma}\ , \tag{16c}$$

$$D' \;=\; e^{-\gamma}\ . \tag{16d}$$

It is easy to see that both (13) and (15) yield the metric tensor (12) through the relation $e_{(i)j}e_{(i)k} = g_{jk}$. They are related by a *local* SO(3) transformation.

We have presented (13) and (15) because they are the *simplest* constructions that satisfy two basic requirements: (**i**) the triads must have the asymptotic behaviour given by (2); (**ii**) by making the physical parameters of the metric vanish we must have $T_{(k)ij} = 0$ everywhere. In the present case if we make $M = d = c = 0$ both (13) and (15) acquire the form

$$e_{(k)i} \;=\; \begin{pmatrix} sin\theta\, cos\phi & r\, cos\theta\, cos\phi & -r\, sin\theta\, sin\phi \\ sin\theta\, sin\phi & r\, cos\theta\, sin\phi & r\, sin\theta\, cos\phi \\ cos\theta & -r\, sin\theta & 0 \end{pmatrix}\ . \tag{17}$$

In cartesian coordinates the expression above can be reduced to the diagonal form $e_{(k)i}(x,y,z) = \delta_{ik}$. The requirement (ii) above is essentialy equivalent to the establishment of reference space triads, as discussed in [16]. A proper definition of gravitational energy requires the notion of reference space triads, which in the present case are given by (17). Note that by a suitable choice of a local SO(3) rotation we can make the flat space triads (17) satisfy the requirement (i), but not (ii).

We proceed now to obtain the triads whose aymptotic expansion is given by (2) with the symmetry condition $h_{ij} = h_{ji}$. The procedure is the following. We consider, for instance, triads (13) and transform it to cartesian coordiates. Then we perform a local, asymptotic transformation

$$\tilde{e}_{(k)i}(t,x,y,z) = \Lambda^{(j)}{}_{(k)}\, e_{(j)i}(t,x,y,z)\ , \tag{18}$$

where $\Lambda^{(j)}{}_{(k)}$ satisfies

$$\Lambda^{(j)}{}_{(k)} \approx \delta^{(j)}{}_{(k)} + \omega^{(j)}{}_{(k)}\ , \tag{19a}$$

$$\omega_{(j)(k)} = -\omega_{(k)(j)} \ , \tag{19b}$$

and $\omega_{(j)(k)} \sim O(\frac{1}{r})$ for $r \to \infty$ ($r = \sqrt{x^2+y^2+z^2}$). Transformation (19) preserves the asymptotic behaviour of the triads. By substituting (19) in (18) we find

$$\tilde{e}_{(k)i} = e_{(k)i} + \omega_{(k)i} \ ,$$

from what follows

$$\tilde{h}_{ki} = h_{ki} + 2\omega_{ki} \ , \tag{20}$$

where h_{ki} is given by the asymptotic expansion of (13) in cartesian coordinates. By requiring

$$\tilde{h}_{ki} = \tilde{h}_{ik} \ , \tag{21}$$

and making use of (19b) we find that

$$\omega_{ki} = \frac{1}{4}(h_{ki} - h_{ik}) \ . \tag{22}$$

Substituting now (22) in (20) we arrive at

$$\tilde{h}_{ki} = \frac{1}{2}(h_{ki} + h_{ik}) \equiv h_{(ki)} \ . \tag{23}$$

Thus we ultimately obtain the symmetrized triads in cartesian coordinates:

$$\tilde{e}_{(k)i} \approx \eta_{ki} + \frac{1}{2}\tilde{h}_{ki} \ . \tag{24}$$

It must be noted that we arrive at a symmetrized triad irrespective of the triads we consider initially. The only requirement is that the unrotated triad must satisfy the asymptotic behaviour (2). Had we considered (15) we would arrive at the same result. In particular, from (22) we observe that if the initial triad is already symmetrized, then no local, asymptotic rotation is necessary.

By transforming (24) into spherical coordinates t, r, θ, ϕ (in which case the triads are no longer symmetrized), we finally arrive at

$$\tilde{e}_{(1)1} \approx (1+\frac{M}{r}) sin\theta\, cos\phi - \frac{f}{2r} cos\theta\, cos\phi \ ,$$

$$\tilde{e}_{(2)1} \approx (1+\frac{M}{r}) sin\theta\, sin\phi - \frac{f}{2r} cos\theta\, sin\phi \ ,$$

$$\tilde{e}_{(3)1} \approx (1+\frac{M}{r}) cos\theta + \frac{f}{2r} sin\theta \ ,$$

$$\tilde{e}_{(1)2} \approx r(1+\frac{c}{r}) cos\theta\, cos\phi - \frac{f}{2} sin\theta\, cos\phi \ ,$$

$$\tilde{e}_{(2)2} \approx r(1+\frac{c}{r}) cos\theta\, sin\phi - \frac{f}{2} sin\theta\, sin\phi \ ,$$

$$\tilde{e}_{(3)2} \approx -r(1+\frac{c}{r}) sin\theta - \frac{f}{2} cos\theta \ ,$$

$$\tilde{e}_{(1)3} \approx -r(1-\frac{c}{r}) sin\theta\, sin\phi \ ,$$

$$\tilde{e}_{(2)3} \approx r(1-\frac{c}{r}) sin\theta\, cos\phi \ ,$$

$$\tilde{e}_{(3)3} \approx 0 \ , \tag{25}$$

where f is given by

$$f = \frac{\partial c}{\partial \theta} + 2c\, cotg\theta \ .$$

We observe that by making $M = c = 0$ triads (25) reduce to (17). We also note that (25), as well as (13) and (15), only make sense in the asymptotic region where Bondi's metric is valid.

In view of (23) and (24) it is now easy to prove the uniqueness of the symmetrized triad. Suppose that the asymptotic behaviour of the metric tensor is given by

$$g_{ij} \approx \eta_{ij} + h^{\star}_{ij}(\frac{1}{r}) \ . \tag{26}$$

On the other hand by making use of (2) it follows from the relation $g_{ij} = e^{(k)}{}_i e_{(k)j}$ and from (23) that

$$g_{ij} \approx \eta_{ij} + \frac{1}{2}(h_{ij} + h_{ji}) = \eta_{ij} + h_{(ij)} \ . \tag{27}$$

Since $h^\star_{ij}$ is unique, by comparing (26) and (27) we are led to conclude that there exists a unique symmetrized triad associated to the spatial section of an asymptotically flat metric tensor.

IV. Construction of the news function and of the mass aspect

In order to establish explicit expressions for the functions $c(u,\theta)$ and $M(u,\theta)$, the suplementary field equations $R_{00} = R_{02} = 0$ for Bondi's metric are considered. In simplified form they are given by[17]

$$\frac{\partial M}{\partial u} = -\left(\frac{\partial c}{\partial u}\right)^2 + \frac{1}{2}\frac{\partial}{\partial u}\left[\frac{\partial^2 c}{\partial \theta^2} + 3\frac{\partial c}{\partial \theta}\, cotg\theta - 2c\right] , \tag{28}$$

$$-3\frac{\partial d}{\partial u} = \frac{\partial M}{\partial \theta} + 3c\,\frac{\partial^2 c}{\partial \theta\, \partial u} + 4c\,\frac{\partial c}{\partial u}\, cotg\theta + \frac{\partial c}{\partial \theta}\frac{\partial c}{\partial u} \ . \tag{29}$$

From (28) we observe that if $\frac{\partial c}{\partial u} = 0$, the mass aspect M does not depend on u.

For a family of null hypersurfaces Bondi's mass is defined by

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\, sin\theta\, d\theta \ . \tag{30}$$

It represents the mass of the system at the retarded time u. Multiplying both sides of (28) by $sin\theta$, integrating in θ and making use of the regularity conditions on the z axis stated in section III, we arrive at

$$\frac{dm}{du} = -\frac{1}{2}\int_0^\pi \left(\frac{\partial c}{\partial u}\right)^2 sin\theta\, d\theta \ , \tag{31}$$

which expresses the loss of mass. This equation was first obtained by Bondi et. al.[17]. Therefore if the news function is nonvanishing, the mass of the system decreases in time. We remark that not only $c(u,\theta)$, but also the news function must be a regular function on the z axis.

The function $c(u,\theta)$ determines not only the loss of mass, but in fact it determines the whole structure of Bondi's metric, since via (28) and (29) it also determines the functions M, d, and all other functions that arise in the process of integration of the field equations. Therefore its construction deserves special attention.

The news function proposed by Bondi et. al.[17] is given by

$$\frac{\partial c}{\partial u} = \sum_{n=0}^{\infty} f_n(u) P_n(\mu) \ ,$$

where $\mu = cos\theta$. $P_n(\mu)$ are the Legendre polynomials and $f_n(u)$ are functions to be determined. These functions become more and more intricate for increasing n, and eventually the above expression cannot be manipulated analytically.

Bonnor and Rotenberg[24], Papapetrou[25] and Hallidy and Janis[26] have attempted at establishing an expression for $c(u,\theta)$, but none of these proposals worked out satisfactorily, either because of the complexity of the structure of $c(u,\theta)$[24], or because the loss of mass does not occur for for a finite number of terms in the expansion in n[25, 26]. Bonnor and Rotenberg's expression describes a radiative period between an initial nonradiative, static state and a final nonradiative, *nonstatic* state. The other approaches mentioned above describe a radiative period between static initial and final states.

Hobill[19] has provided an expression for $c(u,\theta)$ that describes a radiative period between initial and final static states, and that leads to a loss of mass that is exactly equal to the total variation of the mass aspect during the period. Again the expression is not simple. It is given by

$$c(u,\mu) = \frac{[m_b f(\mu) e^{\eta u} + m_a(1-\mu^2)](1+\eta\mu^2)(1-\mu^2)}{f^2(\mu)e^{2\eta u} + (n-1)\mu^2 - n\mu^4 + 1} \ , \qquad (32)$$

where n and η are constants and m_a is identified as one fourth of the total mass loss. It relates to the other constants according to

$$\frac{18n}{n+1} = \eta m_a \ ,$$

$$m_b = \pm\sqrt{\frac{2}{n}}\, m_a \ .$$

$f(\mu)$ is an arbitrary function that must be everywhere regular and positive definite in the interval $-1 \leq \mu \leq 1$. We note that c vanishes over the symmetry axis ($\mu = \pm 1$). The news function obtained from (32) reads

$$\frac{\partial c}{\partial u} = \frac{\eta f(\mu) e^{\eta u}(1-\mu^2)}{[e^{2\eta u} f^2(\mu)(1+n\mu^2)^{-1} + (1-\mu^2)]^2} \Big\{ \frac{-m_b f^2(\mu) e^{2\eta u}}{1+n\mu^2} - \frac{2m_a(1-\mu^2) f(\mu) e^{\eta u}}{1+n\mu^2} + m_b(1-\mu^2) \Big\} . \quad (33)$$

Expressions (32) and (33) lead, via integration of equation (28), to an extremely complicated expression for the mass aspect:

$$M(u,\mu) = M(-\infty) - 4m_a + \Big[\frac{f^2(\mu) e^{2\eta u}}{1+n\mu^2} + (1-\mu^2)\Big]^{-3} \Big[4m_a(1-\mu^2)^3 + \sum_{l=1}^{5} H_l(\mu) e^{l\eta u}\Big] . \quad (34)$$

The intricate expressions for $H_l(\mu)$ are given in the Appendix of Ref. [19]. The problem with (32), (33) and (34) is that they are too complicated to yield numerical values for integrals (to be considered in the next section) that contain these expressions, even by means of computer calculations.

Therefore we attempted at obtaining a simpler expression for $c(u,\mu)$. We damanded two conditions on $c(u,\mu)$. First, it must satisfy the regularity conditions on the z axis, which guarantees that the system is permanently isolated and leads to a well defined loss of mass. Second, that the initial and final states are nonradiative. However, as we will show, our expression leads to a nonstatic final state. It is given by

$$c(u,\mu) = \frac{a e^{nu}(1-\mu^2) F(\mu)}{e^{2nu}+1} , \quad (35)$$

where n and a are constants (n^{-1} and a have dimension of length) and $F(\mu)$ is a function that must be choosen such that $c\,(1-\mu^2)^{-1} = c\,(sin\theta)^{-2}$ is a regular function. Note that (35) vanishes for $\mu \pm 1$. The news function associated to (35) is given by

$$\frac{\partial c}{\partial u} = \frac{na(1-\mu^2)F(\mu)e^{nu}}{e^{2nu}+1}\Big[1 - \frac{2e^{2nu}}{e^{2nu}+1}\Big] . \quad (36)$$

In the limit $u \to \pm\infty$ we have

$$\lim_{u\to\pm\infty} c = \lim_{u\to\pm\infty} \frac{\partial c}{\partial u} = 0 \; , \tag{37}$$

for any nonvanishing value of n. The property above is a necessary but not sufficient condition for having a static final state. Equation (37) does not determine whether the mass aspect M depends on θ, and so from (29) there is the possibility that d depends on u. The initial state is assumed to be of the Schwarzschild type.

Integrating equation (28) from $-\infty$ to u, considering $c(u,\mu)$ given by (35), we obtain

$$M(u,\mu) = M(-\infty) - n^2a^2(1-\mu^2)^2F^2(\mu)\int_{-\infty}^{u}\left[\frac{e^{nu}}{e^{2nu}+1} - \frac{2e^{3nu}}{(e^{2nu}+1)^2}\right]^2 du$$

$$+\frac{1}{2}\frac{ae^{nu}}{(e^{2nu}+1)}\Big[(1-\mu^2)^2F'' - 8\mu(1-\mu^2)F' + 4(3\mu^2-1)F(\mu)\Big] + \lambda(\mu) \; , \tag{38}$$

where $F' = \frac{dF}{d\mu}$, $F'' = \frac{d^2F}{d\mu^2}$ and $\lambda(\mu)$ is an integrating function that depends only on μ.

The integral in (38) depends on the sign of n. For $n < 0$ we have

$$M(u,\mu) = M(-\infty) + \frac{1}{6}na^2(1-\mu^2)^2F^2(\mu)\left[\frac{3e^{4nu}+1}{(e^{2nu}+1)^3}\right]$$

$$+\frac{1}{2}\frac{ae^{nu}}{(e^{2nu}+1)}\Big[(1-\mu^2)^2F'' - 8\mu(1-\mu^2)F' + 4(3\mu^2-1)F(\mu)\Big] + \lambda(\mu) \; , \tag{39}$$

and for $n > 0$,

$$M(u,\mu) = M(-\infty) + na^2(1-\mu^2)^2F^2(\mu)\left[\frac{1}{6}\frac{3e^{4nu}+1}{(e^{2nu}+1)^3} + \frac{23}{48}\right]$$

$$+\frac{1}{2}\frac{ae^{nu}}{(e^{2nu}+1)}\Big[(1-\mu^2)^2F'' - 8\mu(1-\mu^2)F' + 4(3\mu^2-1)F(\mu)\Big] + \lambda(\mu) \; . \tag{40}$$

By requiring the inital state to be static, expressions (39) and (40) must satisfy

$$\lim_{u\to-\infty} M = M(-\infty) \equiv M_0 \ . \tag{41}$$

Applying the condition above to (39) we find that the function $\lambda(\mu)$ vanishes, and therefore

$$M(u,\mu) = M_0 + \frac{1}{6}na^2(1-\mu^2)^2F^2(\mu)\Big[\frac{3e^{4nu}+1}{(e^{2nu}+1)^3}\Big]$$

$$+\frac{1}{2}\frac{ae^{nu}}{(e^{2nu}+1)}\Big[(1-\mu^2)^2F'' - 8\mu(1-\mu^2)F' + 4(3\mu^2-1)F(\mu)\Big] \ , \tag{42}$$

for $n < 0$. Similarly, applying (41) to expression (40) we find that the integration function $\lambda(\mu)$ is given by

$$\lambda(\mu) = -\Big(\frac{1}{6} + \frac{23}{48}\Big)na^2(1-\mu^2)F^2(\mu) \ .$$

Substituting it back in (40) for $n > 0$ we arrive at

$$M(u,\mu) = M_0 + \frac{1}{6}na^2(1-\mu^2)^2F^2(\mu)\Big[\frac{3e^{4nu}+1}{(e^{2nu}+1)^3} - 1\Big]$$

$$+\frac{1}{2}\frac{ae^{nu}}{(e^{2nu}+1)}\Big[(1-\mu^2)^2F'' - 8\mu(1-\mu^2)F' + 4(3\mu^2-1)F(\mu)\Big] \ . \tag{43}$$

Let us now check the limiting value of (42) and (43) for $u \to \infty$. Considering first (42) we obtain

$$\lim_{u\to\infty} M = M_0 + \frac{1}{6}na^2(1-\mu^2)^2F^2(\mu) \ ,$$

where $n < 0$. It follows from the expression above that the total variation of the mass aspect ΔM_T associated with (42) is given by

$$\Delta M_T = \frac{1}{6}na^2(1-\mu^2)^2F^2(\mu) \ . \tag{44}$$

The mass aspect $M(u,\mu)$ given by (43), for which $n > 0$, leads to a similar expression for ΔM_T:

$$\Delta M_T = -\frac{1}{6} n a^2 (1-\mu^2)^2 F^2(\mu) \ . \tag{45}$$

Hobill[19] obtained a result analogous to (44). However his expression, in the limit $u \to \infty$, does not depend on μ. Consequently the final state in his approach is static. In the present case, the mass aspect will still depend on μ in the limit $u \to \infty$. In view of (29) this fact implies a final nonradiative, nonstatic state, which according to Hobill may describe a even more realistic situation. In any event expressions (35) and (36) describe an isolated physical system whose initial and final states are nonradiative, and whose loss of mass is precisely determined.

V. Evaluation of the gravitational energy

In this section we obtain numerical values for the gravitational energy given by expression (9). The relevance of such expression is that we can apply it to finite spacelike volumes. In the present case we will calculate the gravitational energy inside a large but finite surface of constant radius r_0, centered at the radiating source. Expression (9) will be evaluated as a surface integral in the asymptotic region where the components of Bondi's metric are precisely determined. It reduces to just one integral given by

$$E \ = \ \frac{1}{8\pi} \int_S d\theta \, d\phi \, e T^1 \ , \tag{46}$$

where S is a surface of fixed radius r_0, assumed to be large as compared with the dimension of the source, and the determinant e is given by $e = r^2 \, A \, sin\theta$.

We will consider triads (13), (15) and (25), for which the functions $c(u,\theta)$ and $M(u,\theta)$ are given by (35) and (42), respectively. Thus we take the mass aspect determined by the condition $n < 0$, only. Moreover we will consider four possibilities for the function $F(\mu)$. Therefore we are effectively analyzing twelve distinct sets of triads.

We will follow here the steps of section V of [18]. The energy expression that results from (13) and (15) have already been evaluated. They are given by[18]

$$E_1 = \frac{r_0}{4}\int_0^\pi d\theta\,\left\{sin\theta\Big[e^\gamma + e^{-\gamma} - \frac{2}{A}\Big] + \frac{1}{A}\frac{\partial}{\partial\theta}(Ur\,sin\theta)\right\}\,, \tag{47}$$

and

$$E_2 = \frac{r_0}{4}\int_0^\pi d\theta\,\frac{1}{A}\Big\{sin\theta\Big[e^\gamma A' + e^{-2\gamma}A'B' - 2 + e^{-2\gamma}\frac{\partial A'}{\partial\theta}C' - BC' - Be^{-\gamma}\frac{\partial\gamma}{\partial\theta}\Big]$$

$$-B\,cos\theta\Big[B' - e^{-\gamma}\Big]\Big\}\,, \tag{48}$$

together with definitions (14) and (16). E_1 and E_2 correspond to (13) and (15), respectively. Unfortunately the sign of the first term in the expansion of the function $U(u,r,\theta)$ of the metric tensor (10), in Ref. [18], is changed. Therefore expressions E_1 and E_2 given in the latter reference must be corrected. Minor modifications (such as the modification of some numerical coefficients) are necessary. The correct expressions of E_1 and E_2 in terms of $c(u,\theta)$ and $M(u,\theta)$ are given by

$$E_1 = \frac{1}{2}\int_0^\pi d\theta\,sin\theta\,M - \frac{1}{4r_0}\int_0^\pi d\theta\,sin\theta\Big[\Big(\frac{\partial c}{\partial\theta}\Big)^2 + 4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta + 4c^2\,cot^2\theta\Big]$$

$$+\frac{1}{4r_0}\int_0^\pi d\theta\,M\,\frac{\partial}{\partial\theta}\Big[sin\theta\Big(\frac{\partial c}{\partial\theta} + 2c\,cot\theta\Big)\Big]\,, \tag{49}$$

$$E_2 = \frac{1}{2}\int_0^\pi d\theta\,M\,sin\theta - \frac{1}{4r_0}\int_0^\pi d\theta\,sin\theta\Big[M^2 + \frac{1}{2}\Big(\frac{\partial c}{\partial\theta}\Big)^2 + 4c\Big(\frac{\partial c}{\partial\theta}\Big)cot\theta$$

$$+6c^2cot^2\theta + \Big(\frac{\partial M}{\partial\theta}\Big)\Big(\frac{\partial c}{\partial\theta} + 2c\,cot\theta\Big)\Big] + \frac{1}{4r_0}\int_0^\pi d\theta\,cos\theta\Big[2c\Big(\frac{\partial c}{\partial\theta}\Big) + 4c^2\,cot\theta\Big]\,. \tag{50}$$

It is clear that the total gravitational energy given by both (49) and (50), in the limit $r_0 \to \infty$, which corresponds to the limit $u \to -\infty$, yield the same value, i.e., the total mass:

$$\lim_{r\to\infty} E_1 = \lim_{r\to\infty} E_2 = M(-\infty) \equiv M_0\,. \tag{51}$$

In fact it is proven in [18] that the total gravitational energies calculated out of triads related by a local SO(3) transformation, and that have the asymptotic behaviour given by (2), are the same.

We consider next triads given by (25). The components of the torsion tensor are given by

$$\tilde{T}_{(1)12}=\Big(\frac{c}{r}-\frac{M}{r}+r\partial_1(\frac{c}{r})+\frac{1}{2r}\partial_2 f\Big)cos\theta\,cos\phi-\Big(\frac{1}{2}\partial_1 f+\frac{1}{2r}f+\frac{1}{r}\partial_2 M\Big)sin\theta\,cos\phi\;,$$

$$\tilde{T}_{(1)13}=\Big(\frac{c}{r}+\frac{M}{r}+r\partial_1(\frac{c}{r})\Big)sin\theta\,sin\phi-\frac{1}{2r}f\,cos\theta\,sin\phi\;,$$

$$\tilde{T}_{(1)23}=2c\,cos\theta\,sin\phi+(\partial_2 c-\frac{1}{2}f)sin\theta\,sin\phi\;,$$

$$\tilde{T}_{(2)12}=\Big(\frac{c}{r}-\frac{M}{r}+r\partial_1(\frac{c}{r})+\frac{1}{2r}\partial_2 f\Big)cos\theta\,sin\phi-\Big(\frac{1}{2}\partial_1 f+\frac{1}{2}\frac{f}{r}+\frac{1}{r}\partial_2 M\Big)sin\theta\,sin\phi\;,$$

$$\tilde{T}_{(2)13}=-\Big(\frac{c}{r}+\frac{M}{r}+r\partial_1(\frac{c}{r})\Big)sin\theta\,cos\phi+\frac{1}{2r}f\,cos\theta\,cos\phi\;,$$

$$\tilde{T}_{(2)23}=-2c\,cos\theta\,cos\phi-(\partial_2 c-\frac{1}{2}f)sin\theta\,cos\phi\;,$$

$$\tilde{T}_{(3)12}=\Big(\frac{M}{r}-\frac{c}{r}-r\partial_1(\frac{c}{r})-\frac{1}{2r}\partial_2 f\Big)sin\theta-\Big(\frac{1}{2}\partial_1 f+\frac{1}{2r}f+\frac{1}{r}\partial_2 M\Big)cos\theta\;,$$

$$\tilde{T}_{(3)13}=\tilde{T}_{(3)23}=0\;, \tag{52}$$

where ∂_1 and ∂_2 denote partial derivatives with respect to r and θ, respectively.

The calculation of (46) out of the components above does not pose any particular problem, except that the calculation is very long. Denoting by $\tilde{E}$ the gravitational energy that follows from (25) and (52), we have

$$\tilde{E}=E_1-\frac{1}{2r}\Big[\int_0^\pi d\theta\,M^2 sin\theta-\frac{1}{4}\int_0^\pi d\theta(\frac{\partial c}{\partial\theta}+2c\,cotg\theta)\frac{\partial c}{\partial\theta}sin\theta$$

$$-\frac{1}{2}\int_0^\pi d\theta\,(\frac{\partial c}{\partial\theta}+2c\,cotg\theta)\,c\,cos\theta$$

$$+\frac{1}{8}\int_0^\pi d\theta(\frac{\partial c}{\partial\theta}+2c\,cotg\theta)^2 sin\theta(3cos^2\theta-1)\Big]\ . \tag{53}$$

In the calculation above we have made use of the regularity condition $c(u,\theta)=0$ for $\theta=0,\pi$. As expected, (53) also satisfies (51).

The comparison of (49), (50) and (53) is crucial for the selection of triads. We are now in a position of obtaining numerical values for these expressions. With this purpose we make use of data based on the work of Saenz and Shapiro[27] for assigning values to M_0 and u. In the latter reference it is discussed a model for stellar colapse of white-dwarves, with or without axial symmetry, rotating or nonrotating. In this model gravitational waves are produced during a burst of $10^{-3}s$. Thus u may be taken to vary from $10^{-10}s$ to $10^{-4}s$. Based on the work of Saenz and Shapiro we consider a white-dwarf whose total mass is $M_0=1.4M_\odot$, where $M_\odot$ is the solar mass. It is believed[28] that the maximum value for a white-dwarf is $1.4M_\odot$. We will use geometrical unities (G=c=1) for the evaluation of the energies. The value of $M_\odot$ in these unities is $M_\odot=1.47664\times10^5cm$. The distance r will be taken to vary from $10^{18}cm$ to $10^{13}cm$. Finally, the parameters a and n assume the values 100 and -0.5, respectively, in proper unities.

We have used the MAPLE V computer package to obtain E_1, E_2 and $\tilde{E}$. For the function $F(\mu)$, which we now denote $F(\theta)$, we take $sin\theta$, $sin^2\theta$, $cos\theta$ and $cos^2\theta$. The resulting values are listed in tables 1-4 for the particular value $u=10^{-4}s$. The numerical values were obtained with a precision of 20 digits.

We have verified that for all these functions, for all distances considered, $\tilde{E}$ is always the minimum energy:

$$\tilde{E}<E_2<E_1\ .$$

Note that since E_1, E_2 and $\tilde{E}$ are calculated at constant $u=t-r$, the radial dependence of these expression is given just by the $\frac{1}{r_0}$ coefficient in (49), (50) and (53). Thus for constant u the intricacy of these expressions reside in the angular dependence. We have further verified numerically that the result above is also obtained for any value of u between $10^{-10}s$ and $10^{-4}s$. The constants a and n must be chosen such that the absolute values of ΔM_T

Table 1: $F(\theta) = sin\theta$

	$r \sim 10^{18}cm$	$r \sim 10^{15}cm$	$r \sim 10^{13}cm$
E_1	20653.912380952**37**8	20653.9123809**52**3	20653.9123**80**9
E_2	20653.912380952**36**1	20653.9123809**35**0	20653.9123**79**2
$\tilde{E}$	20653.912380952**34**4	20653.9123809**17**8	20653.9123**77**4

Table 2: $F(\theta) = sin^2\theta$

	$r \sim 10^{18}cm$	$r \sim 10^{15}cm$	$r \sim 10^{13}cm$
E_1	20656.0287830687**80**9	20656.02878306**87**8	20656.0287**83**0
E_2	20656.0287830687**63**6	20656.02878305**14**9	20656.0287**81**3
$\tilde{E}$	20656.0287830687**46**3	20656.02878303**34**21	20656.0287**69**6

given by expressions (44) and (45) are not greater than M_0, otherwise the total gravitational energy will be negative. In any case, at present we do not know enough about axially-symmetric, nonrotating isolated sources in order to provide realistic values for these constants.

VI. Discussion

The result of the previous section constitutes a strong indication that the set of triads with asymptotic behaviour given by equation (2), and that

Table 3: $F(\theta) = cos\theta$

	$r \sim 10^{18}cm$	$r \sim 10^{15}cm$	$r \sim 10^{13}cm$
E_1	20669.785396825**396**	20669.785396**82**5	20669.78539**68**3
E_2	20669.7853968253**79**	20669.785396**80**8	20669.78539**50**9
$\tilde{E}$	20669.7853968253**62**	20669.785396**79**0	20669.78539**33**6

Table 4: $F(\theta) = cos^2\theta$

	$r \sim 10^{18} cm$	$r \sim 10^{15} cm$	$r \sim 10^{13} cm$
E_1	20671.9017989417**99**	20671.9017989**41**8	20671.90179**89**4
E_2	20671.9017989417**81**	20671.9017989**24**5	20671.90179**72**1
$\tilde{E}$	20671.9017989417**64**	20671.9017989**07**2	20671.90179**54**8

satisfies the symmetry condition $h_{ij} = h_{ji}$, yields the minimum value for the gravitational energy that is computed from expression (9). In addition to the fact that the symmetrized triads are unique (namely, there does not exist a second set of triads that satisfy (2) and the symmetry condition), the present analysis indicates that the correct description of the gravitational field in terms of orthonormal triads, in the realm of the TEGR, is given by (24). The results described above amount to an interesting interplay between the energy properties of the space-time and its tetrad description.

As long as we are interested in the dynamics of the gravitational field only, as described by the metric tensor, it is irrelevant which configuration of triads we adopt. However, if we consider the dynamics of spinor fields, such as the Dirac field, then the correct choice of triads is crucial. Recall that the theory defined by (5) and (6) was established under the asumption of the time gauge condition. The latter, together with (25), establishes the complete set of tetrad fields. And finally, if the detection of the emission of gravitational energy carried by gravitational waves is experimentally feasible, then the whole scheme developed here will play a relevant role.

Bondi's radiating metric is valid only in the asymptotic region. Therefore we can do no better than determining the asymptotic behaviour of the triads. However in the more general case where the metric tensor is valid everywhere, except for singularities, we still have a prescription for assigning a unique set of triads to a given metric tensor restricted to the spacelike section. Such prescription is due to Møller[1], who called it "supplementary conditions". Although he established these conditions by still requiring de Donder relations for the metric tensor, we can dispense with the latter relations and ascribe generality to his proposal. Møller suggested as supplementary conditions for the space-time tetrad field the *weak field* condition

$$e_{a\mu} \approx \eta_{a\mu} + \frac{1}{2} h_{a\mu} \ , \tag{54}$$

with $h_{a\mu}$ satisfying the symmetry condition $h_{a\mu} = h_{\mu a}$. It differs from (2), which is restricted to spacelike sections only, in that the symmetry condition must be verified everywhere. Thus (54) is stronger than (2). Nevertheless we can require the set of triads to satisfy (54) in the general case. Every metric tensor for the spacelike section of a space-time admits a unique set of triads that satisfy a relation similar to (54), which is no longer a boundary condition and therefore it can be applied to space-times with arbitrary topology. We finally mention that the set of triads presented in ref. [16], in the analysis of the irreducible mass of a rotating black hole, satisfies a relation similar to (54), but in the three-dimensional spacelike hypersurface.

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