The dynamics of zeros of the elliptic solutions to the Schrödinger equation

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Abstract

J. F. van Diejen and H. Puschmann have recently shown that the dynamics of zeros of the n-solitonic solutions to the Schrodinger equation with the reflectionless potential is governed by a rational Ruijsenaars – Schneider system. We use the algebraic-geometrical construction of solutions to the Schrodinger equation to generalize this result to the elliptic case.

1   Introduction

In the recent paper [1] it was noticed that the dynamics of zeros of n-solitonic solutions to the Schrödinger equation with the reflectionless potential is governed by the rational Ruijsenaars – Schneider system with the harmonic term [2]. This result appears to be surprising since the aforementioned dynamics was described long ago, though in a different form. In [3] it was shown that the Bloch solution to the Schrödinger equation

$$(\partial_x^2 - u(x))\psi(x, E) = E\psi(x, E)$$

with the finite-gap potential $u(x)$ is a well-defined function on the hyperelliptic curve

$$y^2 = \prod_{i=1}^{2g+1}(E - E_i).$$

The zeros of this function satisfy the Dubrovin equations [4]:

$$\frac{\partial \widehat{\gamma}_s}{\partial x} = \frac{2y(\gamma_s)}{\prod_{j \neq s}(\widehat{\gamma}_s - \widehat{\gamma}_j)},$$

where by $\widehat{\gamma}$ we denote the projection of a point γ on the E-plane. Notice that these equations contain parameters of the curve. An analog of the Dubrovin equations holds also for degenerate hyperelliptic curves (these are described by the same equation where not all E_i's are distinct) and in particular for fully degenerate hyperelliptic curves which can be thought of as a Riemann sphere with n couples of pairwise identified points. As it was shown in [1] in the latter case parameters of the curve can be excluded from the system. The modified system then is a system of second-order differential equations written solely in terms of zeros of the corresponding function. It coincides with the Ruijsenaars-Schneider system and therefore is Hamiltonian, the expressions for the parameters of the curve being the integrals of motion.

In this paper we exploit the algebraic-geometrical approach developed in [5] to apply these ideas to the case of the reflectionless potentials on a background of finite zone potentials, corresponding to the elliptic curves with self-intersections. The dynamics of zeros of the corresponding solutions resembles the elliptic Ruijsenaars–Schneider system [2]. We show that the system describing these dynamics is Hamiltonian and completely integrable, the angle-type variables being the analogs of the components of the Abel map.

We hope to come up with the general system describing the case of the hyperelliptic curve with arbitrary degree of degeneracy shortly.

2 Algebraic-geometrical data

In this section we present some basic facts from the finite-gap theory.

Consider an elliptic curve Γ, given by the equation

$$y^2 = E^3 - g_2 E - g_3. \tag{1}$$

It's compactified at infinity by one point which we denote by ∞. The only (up to multiplication by constant) holomorphic differential on Γ has the following form: $\omega^h = \dfrac{dE}{y}$.

It defines the map from Γ to the torus $\widehat{\Gamma} = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$, where $2\omega_1$ and $2\omega_2$ are a- and b-periods of ω^h, respectively. This map, given by

$$A \colon P \longmapsto z = \int_\infty^P \omega^h$$

and known as the Abel map, allows us to identify Γ and $\widehat{\Gamma}$.

Corresponding to the torus $\widehat{\Gamma}$ are the standard Weierstrass functions

$$\sigma(z|\omega_1, \omega_2), \quad \zeta(z|\omega_1, \omega_2) = \frac{\sigma'(z|\omega_1, \omega_2)}{\sigma(z|\omega_1, \omega_2)}, \quad \wp(z|\omega_1, \omega_2) = -\zeta'(z|\omega_1, \omega_2)$$

(see [7] for reference). The function $\sigma(z)$ has the following properties:

i) in the neighborhood of zero $\sigma(z) = z + O(z^5)$;

ii) $\sigma(z + 2\omega_j) = e^{2\eta_j(z+\omega_j)}\sigma(z)$, where $\eta_j = \zeta(\omega_j)$.

Notice that $\wp(z)$ is an elliptic function with the only (double) pole at $z = 0$ and $\zeta(z)$ has the simple pole at $z = 0$ and satisfies the following monodromy conditions:

$$\zeta(z + 2\omega_1) = \zeta(z) + \eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + \eta_2.$$

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The map $z \mapsto (E = \wp(z), y = \wp'(z))$ is inverse to the Abel map.

Let us fix $n-1$ points $\kappa_1, \ldots, \kappa_{n-1}$ on $\widehat{\Gamma}$.

Proposition. [6] *For generic divisor $D = \gamma_1 + \cdots + \gamma_n$ on the curve $\widehat{\Gamma}$ there exists a unique function $\psi(x, z|D)$ satisfying the following conditions:*

1. It's meromorphic on the curve $\widehat{\Gamma}$ outside the point $z = 0$ and has poles of at most first order at the points γ_i, $i = 1, \ldots, n$.

2. In the neigborhood of $z = 0$ it has a form

$$\psi(x, z) = e^{xz^{-1}}\left(1 + \sum_{s=1}^{\infty} \xi_s(x)z^s\right).$$

3. $\psi(x, \kappa_i) = \psi(x, -\kappa_i)$.

Remark. In general the function $\psi(x, z|D)$ is defined on the curve Γ itself, but here for the sake of brevity we use the identification between Γ and $\widehat{\Gamma}$.

Proof. The uniqueness of such a function follows immediately from the Riemann–Roch Theorem. To show the existence we shall consider the following function

$$\psi(x, z|D) = e^{\zeta(z)x} \frac{\prod_{i=1}^{n} \sigma(z - z_i(x))}{\prod_{s=1}^{n} \sigma(z - \gamma_s) \prod_{i=1}^{n} \sigma(z_i(x))}. \tag{2}$$

The set of conditions $\psi(x, \kappa_i) = \psi(x, -\kappa_i)$ and the constraint $\sum_{i=1}^{n} z_i(x) = x$ (the latter means that ψ is an elliptic function) form the system of n equations on the functions $z_i(x)$. For generic data this system is non-degenerate. Then it has the only solution (up to the permutations) and therefore defines the function $\psi(x, z|D)$ uniquely. $\square$

Corollary. *The above-constructed function $\psi(x, z|D)$ is a solution to the Schrödinger equation*

$$(\partial_x^2 + u(x))\psi(x, z) = \wp(z)\psi(x, z), \tag{3}$$

where $u(x) = -2\sum_{i=1}^{n} \wp(z_i(x))z_i'(x)$.

Proof. Consider a function $\psi_0(x, z) = (\partial_x^2 + u(x) - \wp(z))\psi(x, z)$. It's straightforward to check that the function $\psi + \psi_0$ satisfy all defining properties of the function ψ. The uniqueness of ψ implies that $\psi_0 = 0$. $\square$

3 Main results

Theorem 1. *The zeros of the function $\psi(x, z|D)$ satisfy the following dynamics:*

$$z_i'' = \sum_{k \neq i} z_i' z_k' \frac{\wp'(z_i) + \wp'(z_k)}{\wp(z_i) - \wp(z_k)}, \quad i = 1, \ldots, n. \tag{4}$$

Proof. To obtain these equations one has to divide (3) by $\psi(x, z)$ and compare the residues of the both sides of the obtained equation at the points $z_i(x)$. $\square$

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Remark. Theorem 1 provides us with a wide class of solutions to system (4) coming from the algebraic-geometrical data. The simple "dimensional" argument shows that in fact these are all solutions. We could reverse the whole reasoning starting with the solution to (4) and showing that the corresponding elliptic function (2) solves the Shrödinger equation.

From now on we shall study system (4). Let us introduce the variables $\xi_i = \ln z_i'$, $i = 1, \ldots, n$. In the variables z_i, ξ_i system (4) has the following form:

$$z_i' = e^{\xi_i},$$
$$\xi_i' = \sum_{k \neq i} e^{\xi_k} \frac{\wp'(z_i) + \wp'(z_k)}{\wp(z_i) - \wp(z_k)}, \quad i = 1, \ldots, n. \tag{5}$$

Proposition. *System (5) is Hamiltonian with respect to the Hamiltonian* $H = \sum_{i=1}^{n} e^{\xi_j}$ *and a 2-form*

$$\omega = \sum_{i=1}^{n} dz_i \wedge d\xi_i - \frac{1}{2} \sum_{i \neq j} \frac{\wp'(z_i) + \wp'(z_j)}{\wp(z_i) - \wp(z_j)} dz_i \wedge dz_j. \tag{6}$$

The proof is a straigtforward calculation.
Note that

$$\omega = \sum_{i=1}^{n} dz_i \wedge d\xi_i - \sum_{j \neq i} \frac{\wp'(z_i)}{\wp(z_i) - \wp(z_j)} dz_i \wedge dz_j =$$
$$= \sum_{i=1}^{n} dz_i \wedge d\xi_i + \sum_{j \neq i} dz_i \wedge \frac{\wp'(z_j) \, dz_j - \wp'(z_i) \, dz_i}{\wp(z_j) - \wp(z_i)} =$$
$$= \sum_{i=1}^{n} dz_i \wedge d\xi_i + \sum_{i \neq j} dz_i \wedge d\Big(\ln(\wp(z_j) - \wp(z_i))\Big) = \sum_{i=1}^{n} dz_i \wedge d\rho_i,$$

where

$$\rho_i = \xi_i + \sum_{j \neq i} \ln(\wp(z_j) - \wp(z_i)).$$

The algebraic-geometrical construction from the previous section provides us with a hint on how the first integrals of system (4) should look like. The constraints $\psi(x, \kappa_s) = \psi(x, -\kappa_s)$ imply the equations

$$\sum_{j=1}^{n} \frac{z_j'}{\wp(\kappa_s) - \wp(z_j)} = 0,$$

which can be rewritten in the following form

$$\sum_{i=1}^{n} z_i' \prod_{j \neq i} (\wp(z_j) - \wp(\kappa_s)) = 0.$$

These considerations motivate

Theorem 2. *The coefficients H_k of the polynomial*

$$L(\lambda|z,z') = \sum_{k=0}^{n-1} H_k(z,z')\lambda^k = \sum_{i=1}^{n} z_i' \prod_{j\neq i} (\wp(z_j) - \lambda) \tag{7}$$

are the integrals of motion of system (4).

Remark. Note that the leading cofficient $H_{n-1}(z,z')$ of L is equal up to the sign to the Hamiltonian $H(z,z')$ of system (4).

The statement of the theorem is clear since we know that all solutions are algebraic-geometrical. However, we would like to present an independent direct proof. It can be found in the Appendix I.

Let us notice that $L(\wp(z_j)) = e^{\rho_j}$. Using this identity we can rewrite the form ω in the following way:

$$\omega = \sum_{i=1}^{n} dz_i \wedge d\rho_i = \sum_{i=1}^{n} dz_i \wedge d\ln L(\wp(z_i)) =$$

$$= \sum_{i=1}^{n} \frac{1}{L(\wp(z_i))} dz_i \wedge d\left(\sum_{s=0}^{n-1} H_s \wp^s(z_i)\right) = \sum_{i=1}^{n}\sum_{s=0}^{n-1} \frac{\wp^s(z_i)}{L(\wp(z_i))} dz_i \wedge dH_s =$$

$$= \sum_{s=0}^{n-1} d\left(\sum_{i=1}^{n} \int^{\wp(z_i)} \frac{E^s\, dE}{L(E)y(E)}\right) \wedge dH_s + \sum_{s,k=0}^{n-1} \left(\sum_{i=1}^{n} \int^{\wp(z_i)} \frac{E^{s+k}\, dE}{L(E)^2 y(E)}\right) dH_k \wedge dH_s =$$

$$= \sum_{s=0}^{n-1} d\left(\sum_{i=1}^{n} \int^{\wp(z_i)} \frac{E^s\, dE}{L(E)y(E)}\right) \wedge dH_s,$$

where the function $y(E)$ is given by (1).

Thus we have proved the following statement.

Theorem 3. *The variables*

$$\varphi_s = \sum_{i=1}^{n} \int^{\wp(z_i)} \frac{E^s\, dE}{L(E)y(E)}, \qquad s = 0,\ldots,n-1$$

and H_s defined by (7) are the action-angle type variables for system (4).

We would like however to rewrite the form ω once again in terms of the zeros of the polynomial $L(\lambda|z,z')$ which we shall denote by $\widehat{\kappa}_j$, $j = 1,\ldots,n-1$. In order to do this we introduce the new variables

$$\chi_j = \sum_{i=1}^{n} \int^{\wp(z_i)} \frac{dE}{(E-\widehat{\kappa}_j)y(E)}, \qquad j = 1,\ldots,n-1.$$

Let us also introduce the variable

$$\chi = \sum_{i=1}^{n} \int^{\wp(z_i)} \frac{dE}{y(E)}.$$

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Theorem 4. *The above-defined form ω admits the following representation*

$$\omega = d\chi \wedge d(\ln H) + \sum_{j=1}^{n-1} d\chi_j \wedge d\widehat{\kappa}_j. \tag{8}$$

Remark. We want to emphasize the fact that the variables $\{\chi, \chi_j, \ j = 1, \dots, n-1\}$ are the degenerate curve analogs of the components of the Abel map. So our Hamiltonian structure fits in the general scheme proposed in [8] and developed in [9].

The proof is a straightforward computation (see Appendix II).

Appendix I

Let us consider the polynomial

$$L(\lambda|z, z') = \sum_{j=1}^{n} z'_j \prod_{i \neq j} (\wp(z_i) - \lambda) = \sum_{k=0}^{n-1} H_k(z, z')\lambda^k.$$

The explicit formulae for the coefficients H_k are

$$H_k = \sum_{|J| = n-k-1} (-1)^k \prod_{j \in J} \wp(z_j) \left(\sum_{k \notin J} z'_k \right),$$

where summation is taken over all subsets $J \subset \{1, \dots, n\}$ of cardinality $n - k - 1$.

We are going to show that the functions H_k are time-independent, i. e. $dH_k/dx = 0$. Indeed,

$$\frac{d(-1)^k H_k(z, z')}{dx} = \sum_J \prod_{j \in J} \wp(z_i) \sum_{k \notin J} z''_k + \sum_J \left(\sum_{s \in J} \frac{\wp'(z_s)}{\wp(z_s)} z'_s \right) \prod_{j \in J} \wp(z_i) \sum_{k \notin J} z''_k =$$

$$= \sum_J \prod_{j \in J} \wp(z_j) \left(\sum_{k \notin J} \sum_{i \neq k} \frac{\wp'(z_k) + \wp'(z_i)}{\wp(z_k) - \wp(z_i)} z'_i z'_k \right) + \sum_J \prod_{j \in J} \wp(z_j) \left(\sum_{k \notin J} \sum_{s \in J} \frac{\wp'(z_s)}{\wp(z_s)} z'_k z'_s \right) =$$

$$= \sum_J \prod_{j \in J} \wp(z_j) \left(\sum_{k \notin J} \sum_{s \in J} \left[\frac{\wp'(z_s)}{\wp(z_s)} + \frac{\wp'(z_k) + \wp'(z_s)}{\wp(z_k) - \wp(z_s)} \right] z'_k z'_s \right) = \sum_{J, k \notin J, s \in J} \alpha(J, k, s),$$

where

$$\alpha(J, k, s) = \prod_{j \in J} \wp(z_j) \left[\frac{\wp'(z_s)}{\wp(z_s)} + \frac{\wp'(z_k) + \wp'(z_s)}{\wp(z_k) - \wp(z_s)} \right] z'_k z'_s.$$

Let us consider the involution on the set of triples $\{J, k \notin J, s \in J\}$ which maps $\{J, k, s\}$ into $\{J', s, k\}$, where $J' = J \cup \{k\} \setminus \{s\}$.

Now note that

$$\alpha(J, k, s) + \alpha(J', s, k) = \prod_{j \in J \cap J'} \wp(z_j) z'_k z'_s \Big[\wp'(z_s) + \wp'(z_k) +$$

$$+ \wp(z_s) \frac{\wp'(z_k) + \wp'(z_s)}{\wp(z_k) - \wp(z_s)} + \wp(z_k) \frac{\wp'(z_s) + \wp'(z_k)}{\wp(z_s) - \wp(z_k)} \Big] = 0$$

and therefore the whole sum $\sum_{J, k \notin J, s \in J} \alpha(J, k, s)$ vanishes.

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Appendix II

Consider the 2-form $\omega = \sum_{s=0}^{n-1} d\varphi_s \wedge dH_s$. Recall that $H_s = (-1)^s H \sigma_{n-s-1}(\widehat{\kappa})$, $s = 0, \ldots, n-1$, where $\sigma_{n-s-1}(\widehat{\kappa})$ denotes the coefficient of λ^s in the polynomial $\prod_{i=1}^{n-1}(\lambda + \widehat{\kappa}_i)$. By $\sigma_{n-s-2}^{j}(\widehat{\kappa})$ we denote the coefficient of λ^s in the polynomial $\prod_{i \neq j}(\lambda + \widehat{\kappa}_i)$. Then

$$\omega = \sum_{s=0}^{n-1} d\varphi_s \wedge dH_s = \sum_{s=0}^{n-2} d\varphi_s \wedge (-1)^s d(H\sigma_{n-s-1}(\widehat{\kappa})) + (-1)^{n-1} d\varphi_{n-1} \wedge dH =$$

$$= \sum_{s=0}^{n-1} (-1)^s \sigma_{n-s-1}(\widehat{\kappa}) \, d\varphi_s \wedge dH + \sum_{j=1}^{n-1} \sum_{s=0}^{n-2} (-1)^s H \sigma_{n-s-2}^{j}(\widehat{\kappa}) \, d\varphi_s \wedge d\widehat{\kappa}_j. \quad (9)$$

Now let us notice that

$$\sum_{s=0}^{n-2} (-1)^s H \sigma_{n-s-2}^{j}(\widehat{\kappa}) \, d\varphi_s = \sum_{s=0}^{n-2} \sum_{l=1}^{n} (-1)^s H \sigma_{n-s-2}^{j}(\widehat{\kappa}) \, d \int^{\wp(z_l)} \frac{E^s \, dE}{L(E)y(E)} =$$

$$= \sum_{l=1}^{n} d \int^{\wp(z_l)} \frac{\sum_{s=0}^{n-2}(-1)^s H \sigma_{n-s-2}^{j}(\widehat{\kappa}) E^s}{L(E)y(E)} \, dE - \sum_{l=1}^{n} \int^{\wp(z_l)} d \left(\frac{\sum_{s=0}^{n-2}(-1)^s H \sigma_{n-s-2}^{j}(\widehat{\kappa}) E^s}{L(E)y(E)} \right) dE =$$

$$= \sum_{l=1}^{n} d \int^{\wp(z_l)} \frac{dE}{(E - \widehat{\kappa}_j)y(E)} + \sum_{l=1}^{n} \int^{\wp(z_l)} \frac{dE}{(E - \widehat{\kappa}_j)^2 y(E)} \, d\widehat{\kappa}_j.$$

In the same way one can show that

$$\sum_{s=0}^{n-1} (-1)^s \sigma_{n-s-1}(\widehat{\kappa}) \, d\varphi_s = \sum_{l=1}^{n} d \int^{\wp(z_l)} \frac{dE}{Hy(E)} + \sum_{l=1}^{n} \int^{\wp(z_l)} \frac{dE}{H^2 y(E)} \, dH.$$

Plugging these two formulae into (9) we obtain

$$d\omega = \sum_{j=1}^{n-1} d \left(\sum_{l=1}^{n} \int^{\wp(z_l)} \frac{dE}{(E - \widehat{\kappa}_j)y(E)} \right) \wedge d\widehat{\kappa}_j + d \left(\sum_{l=1}^{n} d \int^{\wp(z_l)} \frac{dE}{Hy(E)} \right) \wedge dH =$$

$$= \sum_{j=1}^{n-1} d\chi_j \wedge d\widehat{\kappa}_j + d\chi \wedge d(\ln H).$$

Acknowledgements

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