

Avigad, Heuele, Nawrocki, Logic and mechanized reasoning

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Preface

I solve some exercises and prove some statements from Avigad et al., *Logic and mechanized reasoning* (v 0.1). In the appendix, I list the errata that I have found.

Notation

Chapter 1

Introduction

The authors lists three ideas that, it seems, are jointly found for the first time in the work of Ramon Llull (1232?-1316):¹

1. Symbols can stand for ideas.
2. One can generate complex ideas by combining simpler ones.
3. Mechanical devices can serve as aids to reasoning.

¹The author spells the monk's last name as "Lull".

Chapter 2

Mathematical background

Key concepts:

1. proof by induction (p. 3)
 2. definition by recursion (p. 4)
 3. proof by complete induction (p. 5)
 4. definition by course-of-values recursion (p. 5).
 5. inductive definition (p. 6).
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2.1 Induction and recursion on the natural numbers

2.2 Complete induction

On p. 5, the authors define the following function recursively:

$$f(n, k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ f(n-1, k) + f(n-1, k-1) & \text{otherwise} \end{cases}$$

where n and k are natural numbers and $k \leq n$. One more usually write the above function as

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise.} \end{cases}$$

Here $\binom{n}{k}$ indicates the number of ways of choosing k objects out of n without repetition. The equation in the second case, i.e.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

is called *Pascal's identity*. Its intuitive justification is as follows. Let x be an object among the n -many objects that are given. Then, if you do not choose x , you have to choose k objects from the now $n-1$ -many given objects. If you do choose x , then you have to continue by selecting $k-1$

objects from the now $n - 1$ -many objects. Since every selection of k objects from the given n objects either include or does not include x , then the total number of ways of choosing k objects out of n without repetition is the sum of the ways of selecting k objects from $n - 1$ objects (when you do not choose x) and the number of ways of selecting $k - 1$ objects from $n - 1$ objects (when you choose x).

Theorem 2.1. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof. I reason by induction. The statement is true for $n = 0$. Now, suppose that it holds for $n - 1$. I show that it holds for n too. The following equalities hold:

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} && \text{[by definition]} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} && \text{[by induction]} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-k+1)!} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{k(k-1)!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{1}{k} + \frac{1}{(n-k)} \right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{n-k+k}{k(n-k)} \right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{n}{k(n-k)} \right] \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)(n-1-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned}$$

□

2.3 Generalized induction and recursion

Given two lists ℓ and m , I write

$$\ell + m$$

as a shortcut for

$$\text{append}(\ell, m).$$

Theorem 2.2. *The operation `append` is associative.*¹

Proof. Given two lists, l_1 and l_2 , I will write $l_1 + l_2$ to indicate `append`(l_1, l_2). I prove that, for every list l_1, l_2, l_3 ,

$$(l_1 + l_2) + l_3 = l_1 + (l_2 + l_3).$$

¹ The authors define `append` on page 6.

I reason by induction. For the base step, let $l_1 = []$. Therefore,

$$[] + (l_2 + l_3) = l_2 + l_3 = ([] + l_2) + l_3.$$

Now, suppose that associativity holds for $l_1 = l$. I prove that it holds for $(a :: l)$, l_2 , l_3 . I will use the following property from the definition of $::$ ²

$$(a :: m) + n = a :: (m + n)$$

where a is an element and m and n are lists. The the proof continues as follow:

$$\begin{aligned} (a :: l) + (l_2 + l_3) &= a :: (l + (l_2 + l_3)) && \text{[by defin. of ::]} \\ &= a :: ((l + l_2) + l_3) && \text{[by induct. hyp.]} \\ &= (a :: (l + l_2)) + l_3 && \text{[by defin. of ::]} \\ &= ((a :: l) + l_2) + l_3 && \text{[by defin. of ::]} \end{aligned}$$

□

Theorem 2.3. *For every element a and list ℓ ,*

$$a :: \ell = [a] + \ell.$$

Proof. For the base case, observe

$$a :: [] = [a] = [a] + [].$$

For the inductive hypothesis, assume

$$a :: \ell = [a] + \ell.$$

For the inductive step, let b be an element:

$$\begin{aligned} a :: (b :: \ell) &= a :: ([b] + \ell) && \text{[by induct. hyp.]} \\ &= (a :: [b]) + \ell && \text{[by defin. of +]} \\ &= ([a] + [b]) + \ell && \text{[by induct. hyp.]} \\ &= [a] + ([b] + \ell) && \text{[by assoc. of +]} \end{aligned}$$

□

Theorem 2.4. *For every list ℓ , $\ell + [] = \ell$.*

Proof. For the base step, observe

$$[] + [] = [].$$

For the induction hypothesis, assume $\ell + [] = \ell$. For the inductive step, observe

$$\begin{aligned} (a :: \ell) + [] &= ([a] + \ell) + [] && \text{[by theorem 2.3]} \\ &= [a] + (\ell + []) && \text{[by assoc. of +]} \\ &= [a] + \ell && \text{[by induct. hyp.]} \\ &= a :: \ell && \text{[by theorem 2.3]} \end{aligned}$$

□

² The authors define $::$ on page 6.

Theorem 2.5. *For every list ℓ and element a , $\text{appendl}(\ell, a) = \ell + [a]$.*

Proof. I reason by induction. For the base case,

$$\text{appendl}([], a) = [a] = [] + [a].$$

Now, as the induction hypothesis, suppose that $\text{appendl}(\ell, a) = \ell + [a]$. Then, let b to be an element and consider the following equalities:

$$\begin{aligned} \text{appendl}(b :: \ell, a) &= b :: \text{appendl}(\ell, a) \quad [\text{by defin. of } \text{appendl}] \\ &= b :: (\ell + [a]) \quad [\text{by induct. hyp.}] \\ &= (b :: \ell) + [a] \quad [\text{by defin. of } +] \end{aligned}$$

□

Theorem 2.6. *For every list ℓ and m ,*

$$\text{reverse}(\ell + m) = \text{reverse}(m) + \text{reverse}(\ell).$$

Proof. I reason by induction. Let $\ell = []$. Therefore

$$\text{reverse}([] + m) = \text{reverse}(m) = \text{reverse}(m) + \text{reverse}(\ell).$$

Now, as the inductive step, suppose that, for l and m ,

$$\text{reverse}(\ell + m) = \text{reverse}(m) + \text{reverse}(\ell).$$

Let a be an element. The following equalities hold:

$$\begin{aligned} \text{reverse}((a :: l) + m) &= \text{reverse}(a :: (\ell + m)) \quad [\text{by defin. of } +] \\ &= \text{appendl}(\text{reverse}(\ell + m), a) \quad [\text{by defin. of } \text{reverse}] \\ &= \text{append}(\text{reverse}(m) + \text{reverse}(\ell), a) \quad [\text{by induct. hyp.}] \\ &= (\text{reverse}(m) + \text{reverse}(\ell)) + [a] \quad [\text{by theorem 2.5}] \\ &= \text{reverse}(m) + (\text{reverse}(\ell) + [a]) \quad [\text{by assoc. of } +] \\ &= \text{reverse}(m) + \text{appendl}(a, \text{reverse}(\ell)) \quad [\text{by theorem 2.5}] \\ &= \text{reverse}(m) + \text{reverse}(a :: \ell) \quad [\text{by defin. of } \text{reverse}] \end{aligned}$$

□

Theorem 2.7. *For every list ℓ , $\text{reverse}(\text{reverse}(\ell)) = \ell$.*

Proof. I reason by induction. For the base step, observe:

$$\text{reverse}(\text{reverse}([])) = \text{reverse}([]) = [].$$

For the induction hypothesis, assume that $\text{reverse}(\text{reverse}(\ell)) = \ell$. For the inductive step, observe:

$$\begin{aligned} \text{reverse}(\text{reverse}(a :: \ell)) &= \text{reverse}(\text{appendl}(\text{reverse}(\ell), a)) \quad [\text{by defin. of } \text{reverse}] \\ &= \text{reverse}(\text{reverse}(\ell) + [a]) \quad [\text{by theorem 2.5}] \\ &= \text{reverse}([a] + \text{reverse}(\text{reverse}(\ell))) \quad [\text{by theorem 2.6}] \\ &= \text{reverse}([a]) + \ell \quad [\text{by induct. hyp.}] \\ &= [a] + \ell \quad [\text{by property of } \text{reverse}] \\ &= a :: \ell \quad [\text{by defin. of } ::] \end{aligned}$$

□

Theorem 2.8. *For every list ℓ , $\text{reverse}(\ell) = \text{reverse}'(\ell)$.*

Proof. For the base case, observe

$$\text{reverse}([]) = [] = \text{reverseAux}([], []) = \text{reverse}'([]).$$

For the inductive hypothesis, assume

$$\text{reverse}(\ell) = \text{reverse}'(\ell)/$$

For the inductive step, observe

$$\begin{aligned} \text{reverse}(a :: \ell) &= \text{reverse}(\ell) + \text{reverse}([a]) && \text{[by theorem 2.6]} \\ &= \text{reverse}(\ell) + [a] && \text{[by property of } \text{reverse}] \\ &= \text{reverseAux}(\ell, a :: []) && \text{[by defin. of } \text{reverseAux}] \\ &= \text{reverseAux}(a :: \ell, []) && \text{[by defin. of } \text{reverseAux}] \\ &= \text{reverse}'(a :: \ell) && \text{[by defin. of } \text{reverse}'] \end{aligned}$$

□

Appendix A

Errata

page	errata	corrigé
6	we principles	we apply the principles
7	there is part	there is a part

Table A.1: Errata