Avigad, Heuele, Nawrocki, Logic and mechanized reasoning

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Contents

Preface

I solve some exercises and prove some statements from Avigad et al., *Logic and mechanized reasoning* (v 0.1). In the appendix, I list the errata that I have found.

Notation

Chapter 1

Introduction

The authors lists three ideas that, it seems, are jointly found for the first time in the work of Ramon Llull (1232?-1316):[1]

1. Symbols can stand for ideas.
2. One can generate complex ideas by combining simpler ones.
3. Mechanical devices can serve as aids to reasoning.

[1]The author spells the monk's last name as "Lull".

Chapter 2

Mathematical background

Key concepts:

1. proof by induction (p. 3)
2. definition by recursion (p. 4)
3. proof by complete induction (p. 5)
4. definition by course-of-values recursion (p. 5).
5. inductive definition (p. 6).

On p. 5, the authors define the following function recursively:

$$f(n,\ k) = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ f(n-1,\ k) + f(n-1,\ k-1) & \text{otherwise} \end{cases}$$

where n and k are natural numbers and $k \leq n$. One more usually write the above function as

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise.} \end{cases}$$

Here $\binom{n}{k}$ indicates the number of ways of choosing k objects out of n without repetition. The equation in the second case, i.e.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

is called *Pascal's identity*. Its intuitive justification is as a follows. Let x be an object among the n-many objects that are given. Then, if you do not choose x, you have to choose k objects from the now $n-1$-many given objects. If you do choose x, then you have to continue by selecting $k-1$ objects from the now $n-1$-many objects. Since every selection of k objects from the given n objects either include or does not include x, then the total number of ways of choosing k objects out of n without repetition is the sum of the ways of selecting k objects from $n-1$ objects (when you do not choose x) and the number of ways of selecting $k-1$ objects from $n-1$ objects (when you choose x).

Theorem 2.1. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof. I reason by induction. The statement is true for $n = 0$. Now, suppose that it holds for $n-1$. I show that it holds for n too. The following equalities hold:

$$\begin{aligned}
\binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} && \text{[by definition]} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} && \text{[by induction]} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-1-k+1)!} \\
&= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{k(k-1)!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)(n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{1}{k} + \frac{1}{(n-k)}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{n-k+k}{k(n-k)}\right] \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)!} \left[\frac{n}{k(n-k)}\right] \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)(n-1-k)!} \\
&= \frac{n!}{k!(n-k)!}
\end{aligned} \tag{2.1}$$

□

Theorem 2.2. *The operation append is associative.*[1]

Proof. Given two lists, l_1 and l_2, I will write $l_1 + l_2$ to indicate *append*(l_1, l_2). I prove that, for every list l_1, l_2, l_3,

$$(l_1 + l_2) + l_3 = l_1 + (l_2 + l_3).\oplus$$

I reason by induction. For the base step, let $l_1 = []$. Therefore,

$$[] + (l_2 + l_3) = l_2 + l_3 = ([] + l_2) + l3.$$

Now, suppose that associativity holds for $l_1 = l$. I prove that it holds for $(a :: l)$, l_2, l_3. I will use the following property from the definition of :::[2]

$$(a :: m) + n = a :: (m + n)$$

where a is an element and m and n are lists. The the proof continues as follow:

$$\begin{aligned}
(a :: l) + (l_2 + l_3) &= a :: (l + (l_2 + l_3)) && \text{[by defin. of ::]} \\
&= a :: ((l + l_2) + l_3) && \text{[by induct. hyp.]} \\
&= (a :: (l + l_2)) + l_3 && \text{[by defin. of ::]} \\
&= ((a :: l) + l_2) + l_3 && \text{[by defin. of ::]}
\end{aligned}$$

□

[1] The authors defined *append* on page 6.
[2] The authors define :: on page 6.

Appendix A

Errata

page	errata	corrige
6	we principles	we apply the principles
7	there is part	there is a part

Table A.1: Errata