Petrunin, *Euclidean plane and its relatives*

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Preface

Exercises from Anton Petrunin's *Euclidean plane and its relatives*.

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Proof. Let $A = 0$, $B = \frac{1}{2}$, and $C = 1$. Therefore, $|A - C|^2 = 1$ and $|A - B|^2 + |B - C|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. This violates the triangular inequality. $\square$

Lemma 1.1. *Suppose that $\delta_1, \ldots, \delta_n$ are metrics. Define $d(A,\ B) = \sum_{i=1}^{n}\ \delta_i(A,\ B)$ for every A and B. Then, d is a metric.*

Proof. Consider the following statement:

(S) $\sum_{i=1}^{n}\ d_i$ is a metric.

I reason by induction. Assume $n = 1$. Then (S) holds. Now, suppose that (S) holds for $n \geq 1$. I show that (S) holds for $n + 1$ too. It is easy to see that $\sum_{i=1}^{n+1}\ d_i$ satisfies the first three properties (positiveness, nullness, and symmetry). Concerning triangle inequality, notice that this property holds for every metrics δ_i, i.e., for every point A, B, and C and for every $1 \leq i \leq n + 1$, $\delta_i(A,\ C) \leq \delta_i(A,\ B) + \delta_i(B,\ C)$. Therefore, $\sum_{i=1}^{n+1}\ \delta_i(A,\ C) \leq \sum_{i=1}^{n+1}\ \delta_i(A,\ B) + \sum_{i=1}^{n+1}\ \delta_i(B,\ C)$. $\square$

Lemma 1.2. *Suppose that $\delta_1, \ldots, \delta_n$ are metrics. Define*

$$d(A,\ B) = \max\{\delta_1(A,\ B),\ \ldots,\ \delta_n(A,\ B)\}$$

for every A and B. Then, d is a metric.

Proof. The only property that requires attention is the triangle inequality. Suppose that, for given points A and C, for a specific $1 \leq i \leq n$,

$$\delta_i(A,\ C) = \max\{\delta_1(A,\ C),\ \ldots,\ \delta_n(A,\ C)\}.$$

Therefore, $d(A,\ C) = \delta_i(A,\ C)$. Since δ_i is a metric, for every point B, $\delta_i(A,\ C) \leq \delta_i(A,\ B) + \delta_i(B,\ C)$. Therefore, $\delta_i(A, C) \leq \max\{\delta_1(A,\ B),\ \ldots,\ \delta_n(A,\ B)\}$. $\square$

Exercise 1.3

Proof. The function d_1 (the Manhattan metrics) is the sum of two metrics, i.e. two applications of the real line metric. Therefore, by Theorem 1.1, d_1 is a metric.

Now consider the function d_2 (the euclidean metrics). To show that it is a metric, the only property that requires attention is the triangle inequality. Let $A = (a_0,\ a_1)$, $B = (b_0,\ b_1)$, and $C = (c_0,\ c_1)$. To show that $d_2(A,\ C) \leq d_2(A,\ B) + d_2(B,\ C)$ means to show the following:

$$\sqrt{(c_0 - a_0)^2 + (c_1 - a_1)^2} \leq \sqrt{(b_0 - a_0)^2 + (b_1 - a_1)^2} + \sqrt{(c_0 - b_0)^2 + (c_1 - cb_1)^2}.$$

Using only high-school algebra, the key idea is to define $x = b_0 - a_0$, $y = b_1 - a_1$, $v = c_0 - b_0$, and $w = c_1 - b_1$. Then, the above inequality becomes the following:

$$\sqrt{(x+v)^2 + (y+w)^2} \leq \sqrt{x^2 + y^2} + \sqrt{v^2 + w^2}.$$

By calculation, one proves the following:

$$0 \leq (xw - yv)^2,$$

which holds because $(xw - yv)^2$ is non-negative.

The function d_∞ (the maximum metrics) is the sum of two metrics (two applications of the real line metrics). Therefore, by Theorem 1.2, d_1 is a metric. □

Exercise 1.4

Proof. By triangle inequality, we have the following:

1. $AB \leq AP + PB$
2. $AB \leq AQ + QB$
3. $PQ \leq PA + AQ$
4. $PQ \leq PB + BQ$.

Adding the left hand sides and the right hand sides, one obtains:

$$2 \cdot AB + 2 \cdot PQ \leq AP + PB + AQ + QB + PA + AQ + PB + BQ.$$

By symmetry, one can collect the distances to the right as follows:

$$2 \cdot AB + 2 \cdot PQ \leq 2 \cdot AP + 2 \cdot PB + 2 \cdot AQ + 2 \cdot BQ.$$

It follows:

$$AB + PQ \leq AP + PB + AQ + BQ.$$

□

Exercise 1.5

Proof. Assume that f preserves distances. Suppose that $f(A) = f(B)$. Therefore, by the nullness property, $d_Y(f(A),\ f(B)) = 0$. Since f preserves distances, $d_X(A,\ B) = 0$. Therefore, by the nullness property, $A = B$. □

Exercise 1.6

Proof. Since f is an isometry, for every x and y in $\mathbb{R}$, $d(f(x),\ f(y)) = |x-y|$. Therefore, the distance of $f(x)$ from $f(0)$ is $|x|$. Therefore, the location of $f(x)$ on the real line is either $f(0)+x$ or $f(0)-x$. Now, I show that the operator ($+$ or $-$) immediately after $f(0)$ is fixed, i.e., it is not the case that, for some non-zero x and y, $f(x) = f(0)+x$ and $f(y) = f(0)-y$. For, suppose, toward a contradiction, that for some non-zero x and y, $f(x) = f(0)+x$ and $f(y) = f(0)-y$. Then,

$$\begin{aligned} d(f(x),\ f(y)) &= |f(0)+x-f(0)-(-y)| \\ &= |x+y| \\ &\neq |x-y|. \end{aligned}$$

This contradicts the assumption that f is an isometry. □

Exercise 1.7

Proof. The definition of d_1 is $|x_A - x_B|{+}|y_A - y_B|$ and the definition of $d\infty$ is $\max(|x_A - x_B|, |y_A - y_B|)$. I am looking for a map from $(\mathbb{R}^2,\ d_1)$ onto $(R^2,\ d_\infty)$ such that

$$|x_A - x_B|{+}|y_A - y_B|{=} \max(|x_{f(A)} - x_{f(B)}|, |y_{f(A)} - y_{f(B)}|).$$

The map $f(x,\ y) = (x+y,\ x-y)$ works because

$$|x|{+}|y|{=} \max(|x+y|,\ |x-y|).$$

□

For what is worth, I explain how I thought of that map f. I supposed $A = (x,\ y)$ and $B = (0,\ 0)$. Then the equality of the two metrics becomes

$$|x|{+}|y|{=} \max(|x+y|, |x-y|).$$

The above equality holds for every real number. In fact, if x and y have the same sign, then $|x|{+}|y|{=}\ |x+y|$ and, otherwise, $|x|{+}|y|{=}\ |x-y|$.