Cauchy, *Course d'analyse* (1821)

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Contents

Preface

I solve some exercises and prove some statements taken from Cauchy's Course d'analyse (1821).

Notation

1. $\bigwedge S$ is the least element in the set S (which I assume to be non-empty and strictly ordered).

2. $\bigvee S$ is the greatest element in the set S (which I assume to be non-empty and strictly ordered).

Chapter 1

Prèliminaries

In the chapter *Prèliminaries*, Cauchy discusses the general notion of quantity. He states the following definition and two propositions:

Definition 1.1. Let $n \in \mathbb{N}$. Given a list of real numbers[1] $a_1,\ a_2,\ \ldots,\ a_n$, a real number m is *medium* (*moyenne*) among the $a_1,\ \ldots,\ a_n$ if and only if

$$\bigwedge\{a_1,\ a_2,\ \ldots,\ a_n\} \leq m \leq \bigvee\{a_1,\ a_2,\ \ldots,\ a_n\}.$$

Theorem 1.2. *Let $n \in \mathbb{N}$. Let $(a_i)_{1 \leq i \leq n}$ and $(b_i)_{1 \leq i \leq n}$ be sequences of real numbers. Then,*

$$\frac{\sum_{i=1}^{n} a_1}{\sum_{i=1}^{n} b_i}$$

is a medium of

$$\frac{a_1}{b_1},\ \ldots,\ \frac{a_n}{b_n}.$$

Theorem 1.3. *Let $a_1,\ a_2,\ \ldots,\ a_n$ be real numbers. Therefore,*

$$\frac{\sum_{i=1}^{n} a_i}{n}$$

is a medium of $a_1,\ a_2,\ \ldots,\ a_n$ and $b_1,\ \ldots,\ b_n$ where $b_i = 1$ (and it is called the arithmetic medium).

One proof (using theorem 1.2) is the following.

Proof. Apply theorem 1.2. ☐

The following is a different proof of theorem 1.2 that does not rely on theorem 1.2.

Proof. Let $a = \bigwedge\{a_1,\ \ldots,\ a_n\}$ and $A = \bigvee\{a_1,\ \ldots,\ a_n\}$. Moreover, let $m = \frac{\sum_{i=1}^{n} a_i}{n}$. I have to show that

$$a \leq m \leq A.$$

[1] Cauchy speaks of "quantities" (*quantité*). I do not claim that Cauchy's quantities coincide exactly with the real numbers.

First, I prove that $a \leq m$. Since $a = \bigwedge \{a_1, \ \ldots, \ a_n\}$, for some $k_1, \ \ldots, \ k_n$, one can rewrite the sequence $a_1, \ \ldots, \ a_n$ as

$$a + k_1, \ \ldots, \ a + k_n.$$

Therefore,

$$\begin{aligned} a &= \frac{\sum_{i=1}^n a}{n} \\ &\leq \frac{\sum_{i=1}^n a}{n} + \frac{\sum_{i=1}^n k_i}{n} \\ &= \frac{\sum_{i=1}^n (a + k_1)}{n} \\ &= \frac{\sum_{i=1}^n a_i}{n} \\ &= m. \end{aligned}$$

Now, I show that $m \leq A$. Similarly to in the previous case, since $A = \bigvee \{a_1, \ \ldots, \ a_n\}$, for some $k_1, \ \ldots, \ k_n$, one can rewrite $a_1, \ \ldots, \ a_n$ as

$$A - k_1, \ \ldots, \ A - k_n.$$

Reasoning as above, one shows that

$$m = \frac{\sum_{i=1}^n a_i}{n} = \frac{\sum_{i=1}^n (A - k_1)}{n} \leq \frac{\sum_{i=1}^n A}{n} = A.$$

$\square$

When $b_i = 1$ for all $1 \leq i \leq n$, I will simply say that

$$\frac{\sum_{i=1}^n a_i}{n}$$

is the *arithmetic mean* of $a_1, \ \ldots, \ a_n$ (i.e., I will not mention the b_i's).