# Artificial Intelligence

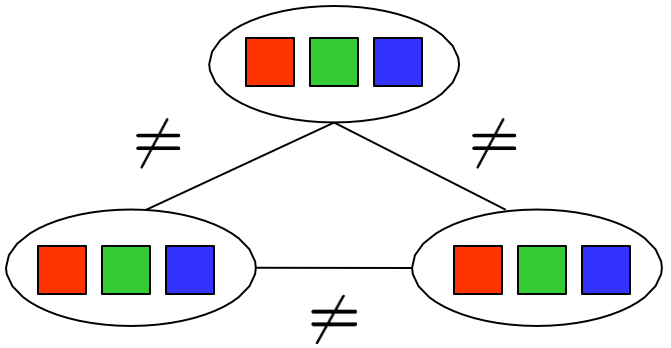
## Constraint Satisfaction Problems II

# Today

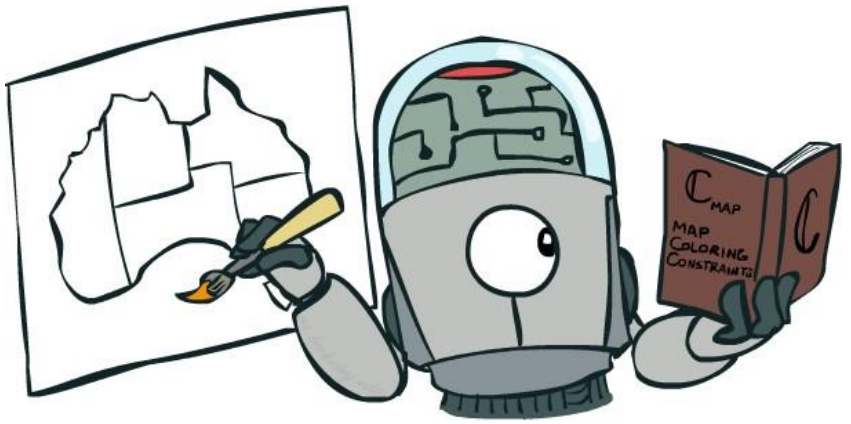
- Efficient Solution of CSPs
- Local Search

# Reminder: CSPs

- CSPs:
  - Variables
  - Domains
  - Constraints
    - Implicit (provide code to compute)
    - Explicit (provide a list of the legal tuples)
    - Unary / Binary / N-ary




- Goals:
  - Here: find any solution
  - Also: find all, find best, etc.



# Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
    return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            result ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure then return result
            remove {var = value} from assignment
    return failure
```

# Improving Backtracking

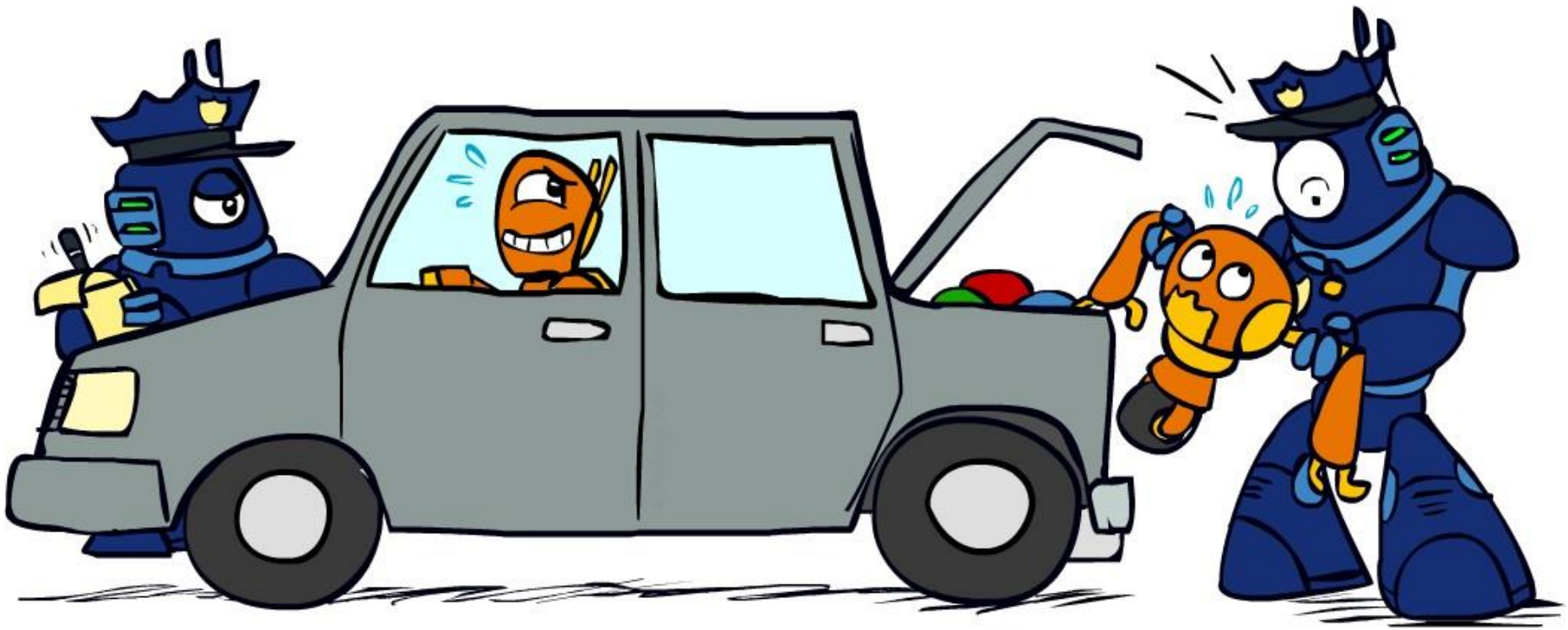
- General-purpose ideas give huge gains in speed
  - … but it's all still NP-hard

- Filtering: Can we detect inevitable failure early?

- Ordering:
  - Which variable should be assigned next? (MRV)
  - In what order should its values be tried? (LCV)

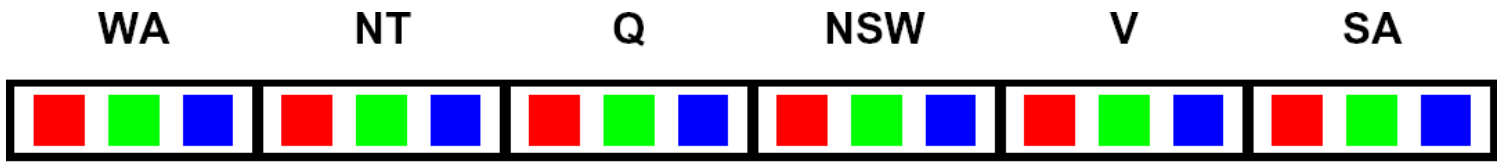
- Structure: Can we exploit the problem structure?

# Arc Consistency and Beyond



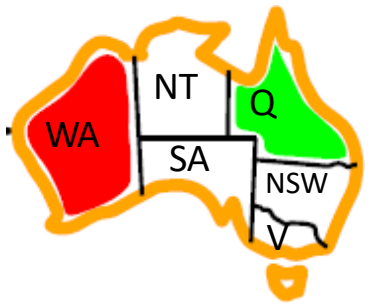
# Filtering: Forward Checking

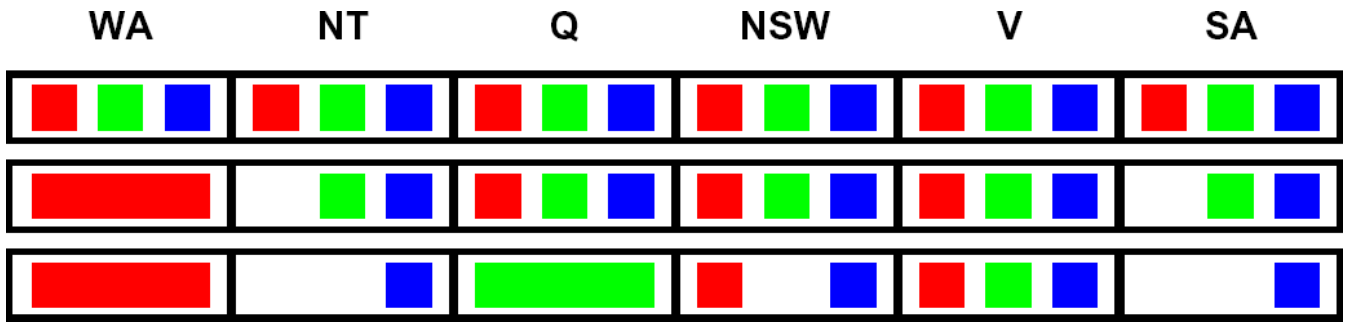
- Filtering: Keep track of domains for unassigned variables and cross off bad options
- Forward checking: Cross off values that violate a constraint when added to the existing assignment

# Filtering: Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

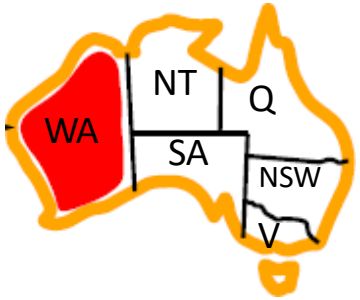


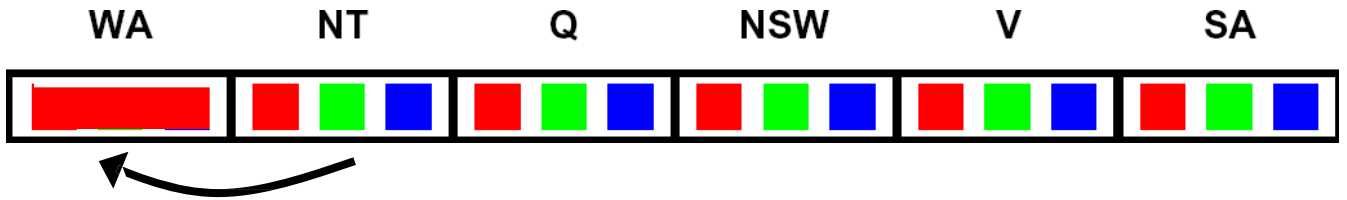




- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation:* reason from constraint to constraint

# Consistency of A Single Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint




- Tail = NT, head = WA
  - If NT = blue: we could assign WA = red
  - If NT = green: we could assign WA = red
  - If NT = red: there is no remaining assignment to WA that we can use
  - Deleting NT = red from the tail makes this arc consistent

# Consistency of A Single Arc

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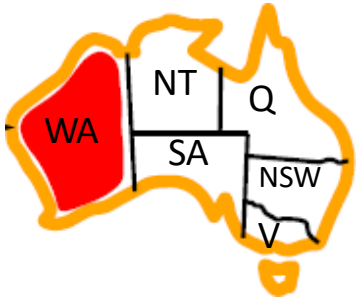
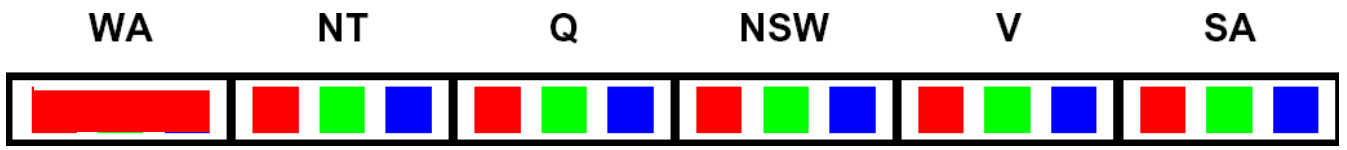
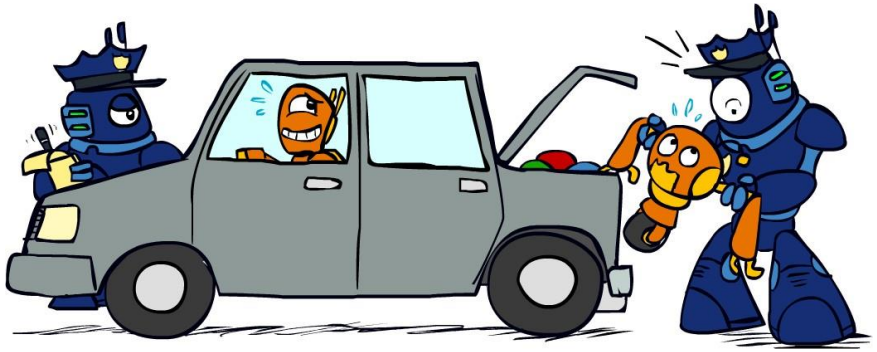


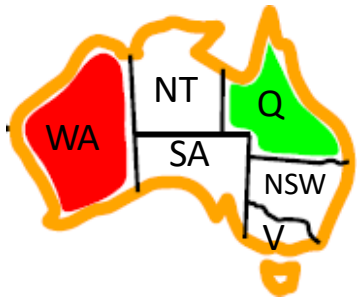



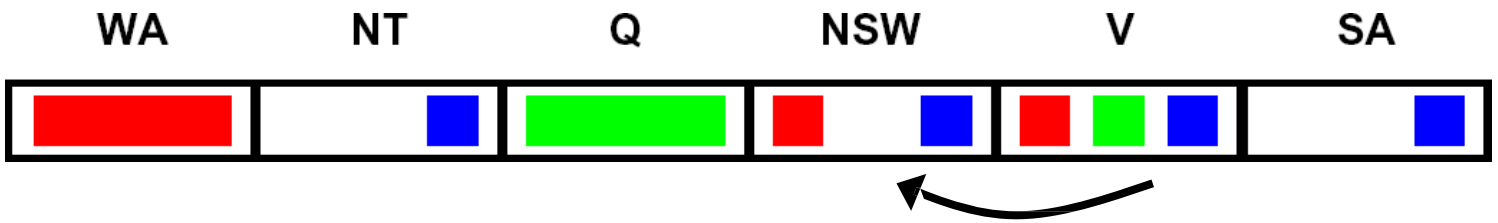
*Delete from the tail!*

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

# Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:




- Arc V to NSW is consistent: for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint

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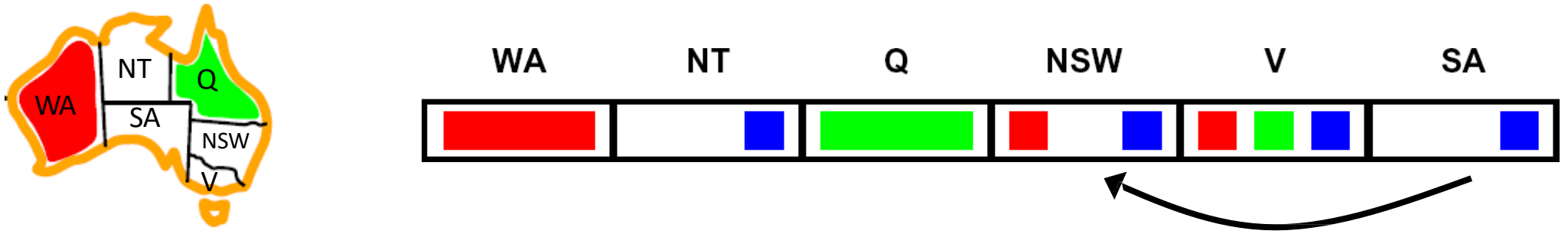
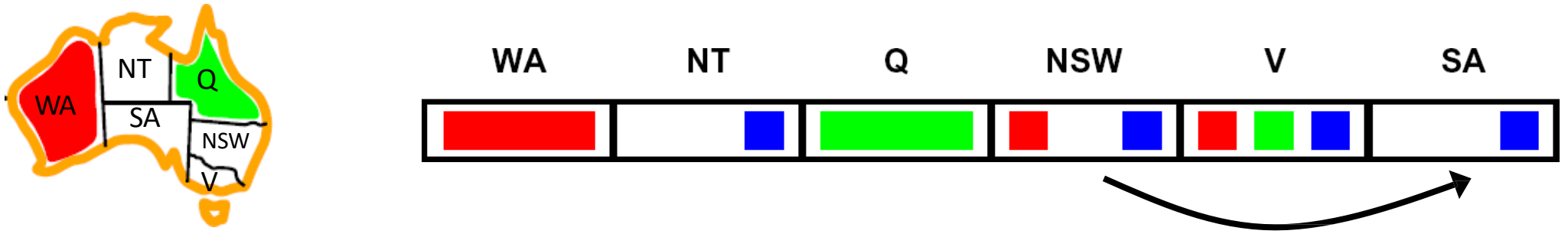



- Arc SA to NSW is consistent: for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint

# Arc Consistency of an Entire CSP

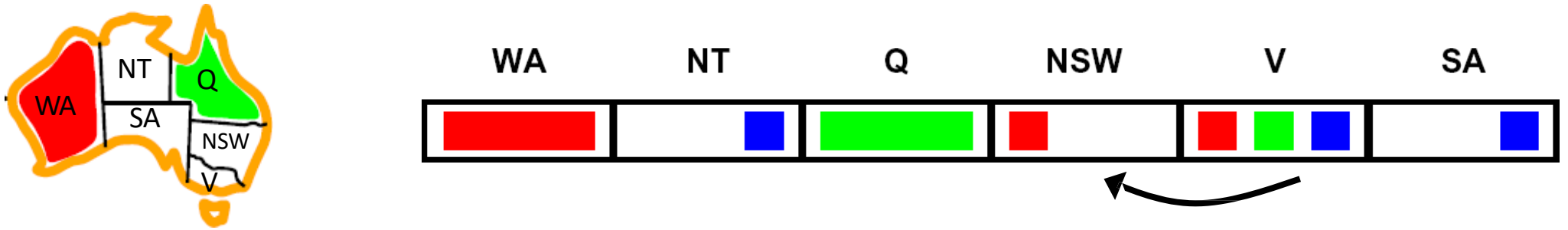
- A simple form of propagation makes sure all arcs are consistent:




- Arc NSW to SA is not consistent: if we assign NSW = blue, there is no valid assignment left for SA
- To make this arc consistent, we delete NSW = blue from the tail

# Arc Consistency of an Entire CSP

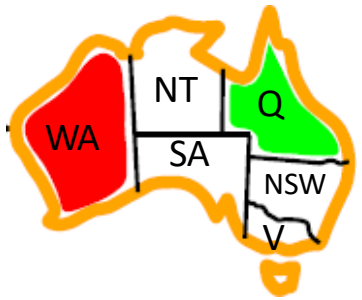
- A simple form of propagation makes sure all arcs are consistent:

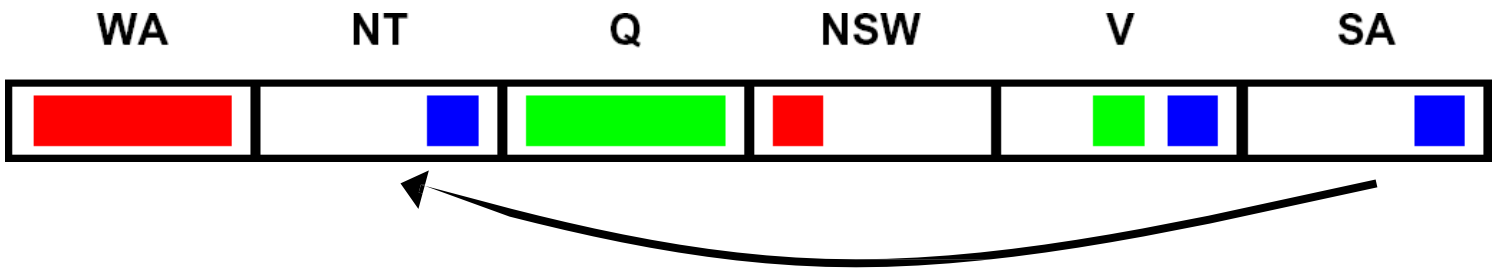



- Remember that arc V to NSW was consistent, when NSW had red and blue in its domain
- After removing blue from NSW, this arc might not be consistent anymore! We need to recheck this arc.
- Important: If X loses a value, neighbors of X need to be rechecked!

# Arc Consistency of an Entire CSP

- A simple form of propagation makes sure all arcs are consistent:

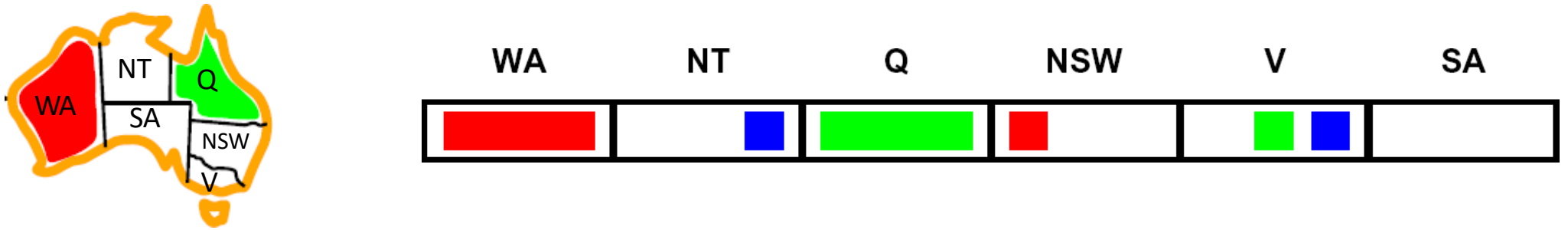



- Arc SA to NT is inconsistent. We make it consistent by deleting from the tail (SA = blue).

# Arc Consistency of an Entire CSP

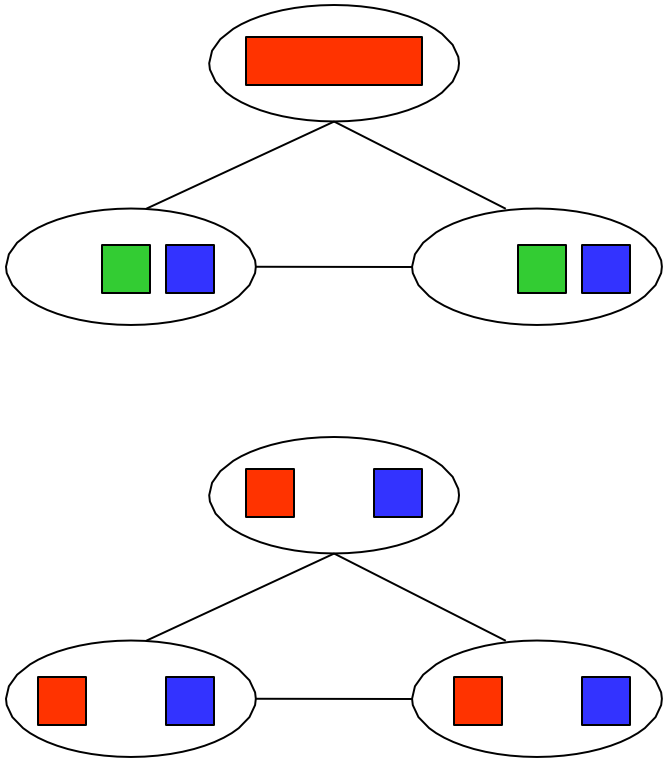
- A simple form of propagation makes sure all arcs are consistent:




- SA has an empty domain, so we detect failure. There is no way to solve this CSP with WA = red and Q = green, so we backtrack.
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

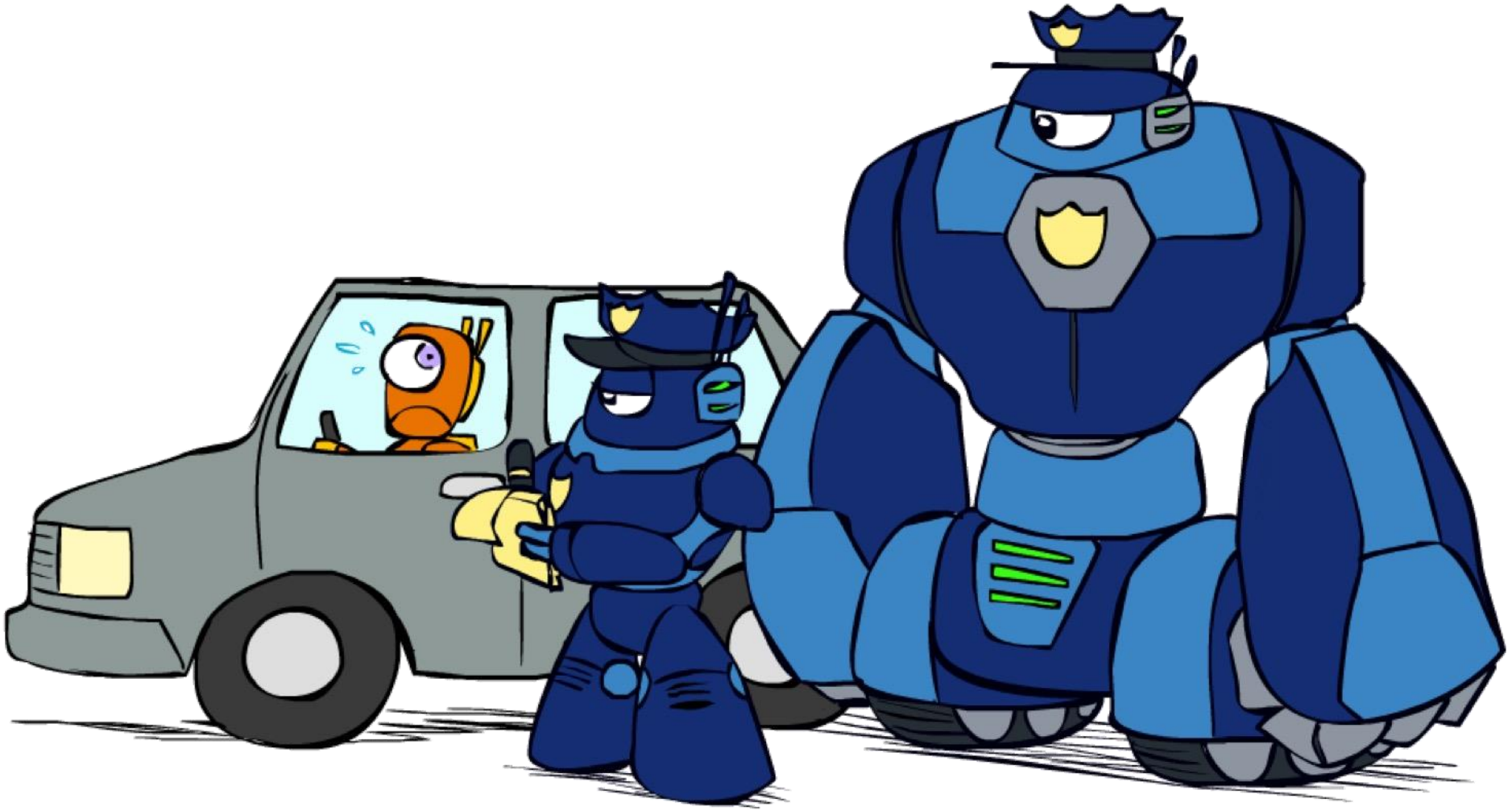
# Limitations of Arc Consistency

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)
- Arc consistency still runs inside a backtracking search!



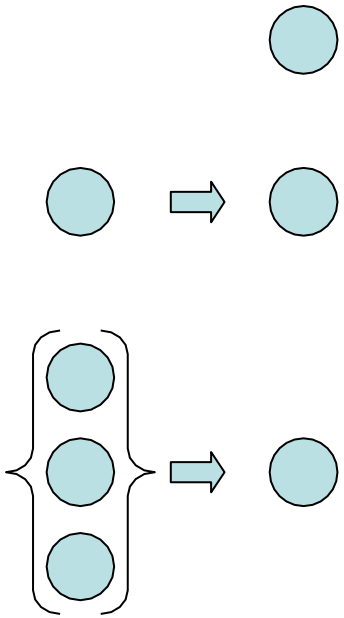
*What went wrong here?*

# K-Consistency

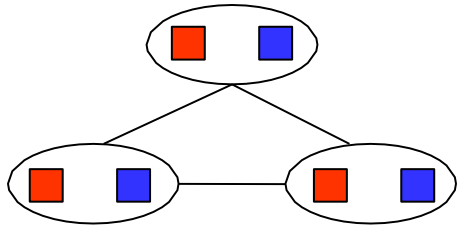


# K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the $k^{th}$ node.



- Higher k more expensive to compute
- (You need to know the k=2 case: arc consistency)



# Strong K-Consistency

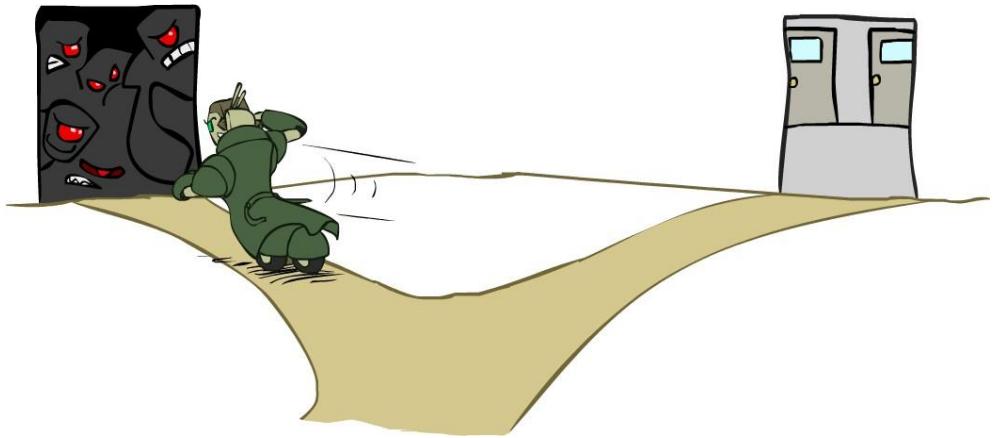
- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

# Ordering

# Ordering: Minimum Remaining Values

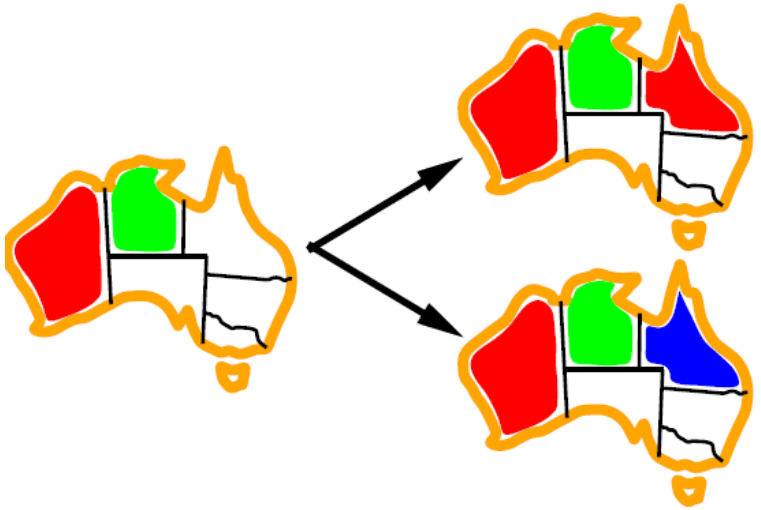
- Variable Ordering: Minimum remaining values (MRV):
  - Choose the variable with the fewest legal left values in its domain

- Why min rather than max?
- Also called “most constrained variable”
- “Fail-fast” ordering



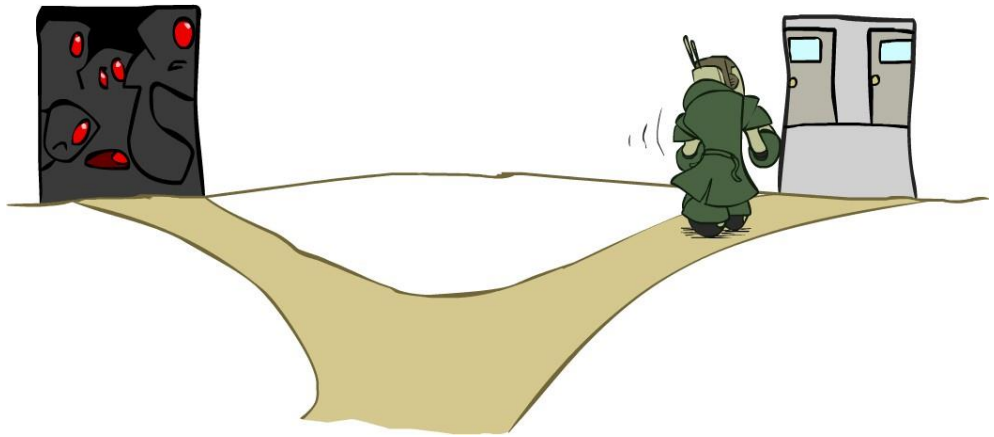
# Ordering: Least Constraining Value

- Value Ordering: Least Constraining Value
  - Given a choice of variable, choose the *least constraining value*
  - I.e., the one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!  (E.g., rerunning filtering)



- Why least rather than most?

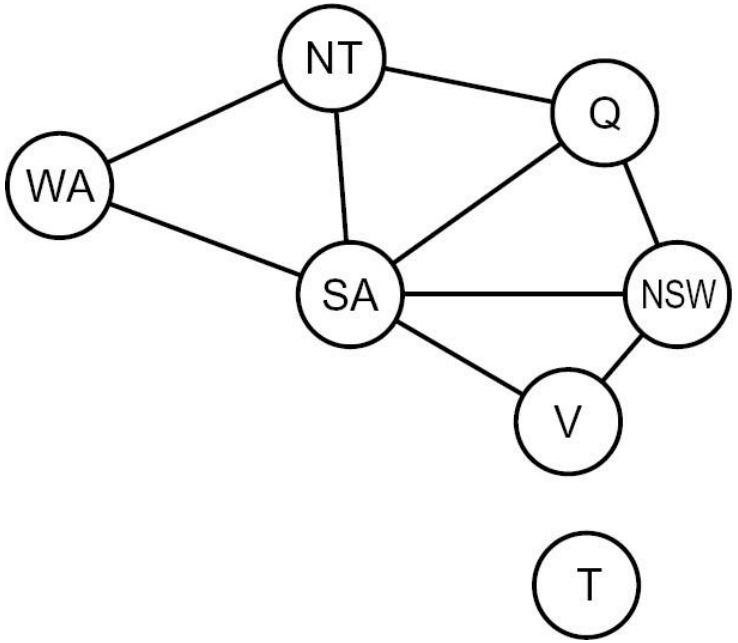
- Combining these ordering ideas makes 1000 queens feasible



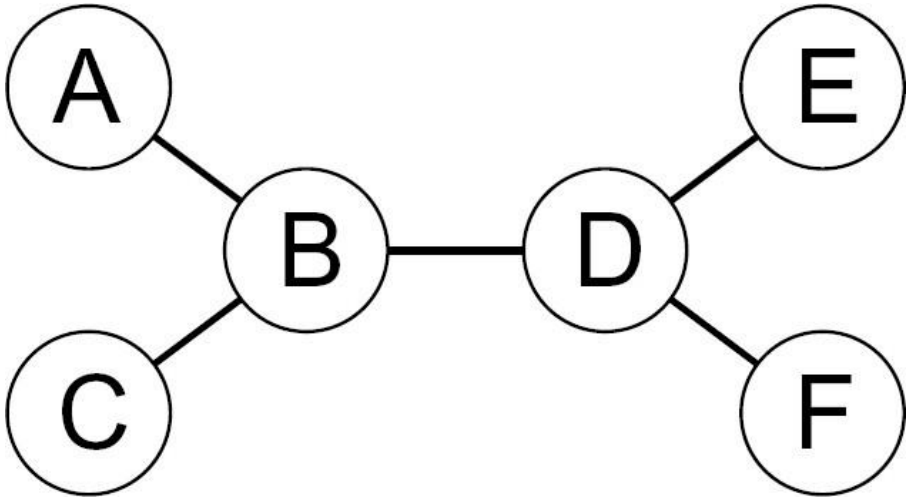
# Structure

# Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact
- Independent subproblems are identifiable as connected components of constraint graph
- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - $2^{80}$ = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



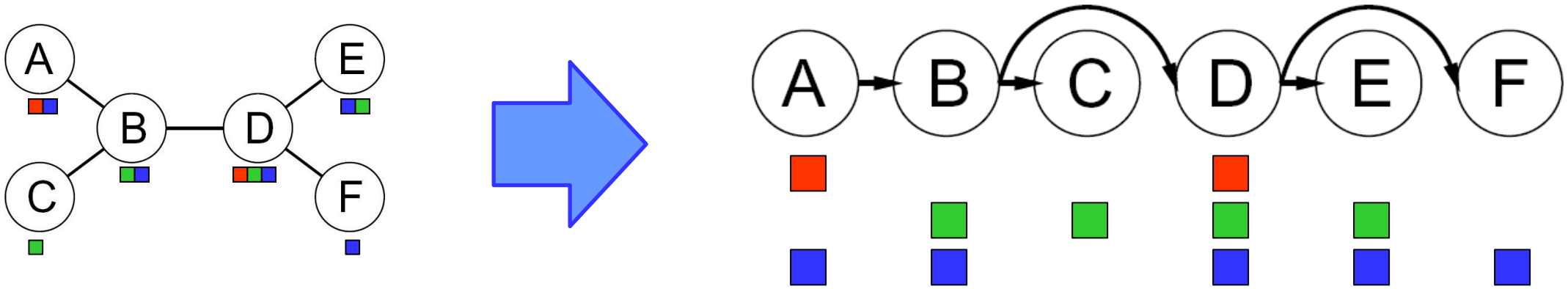
# Tree-Structured CSPs




- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

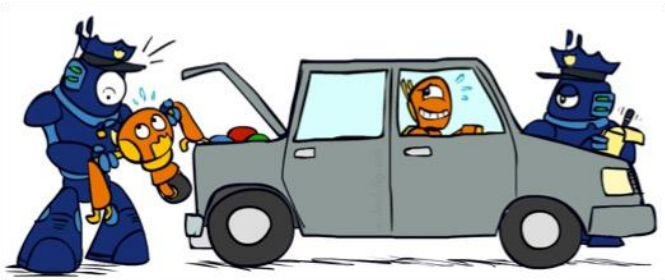
# Tree-Structured CSPs

- Algorithm for tree-structured CSPs:
  - Order: Choose a root variable, order variables so that parents precede children



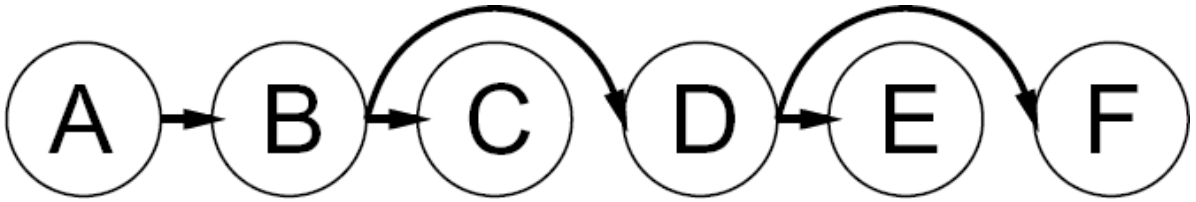

  - Remove backward: For i = n : 2, apply RemoveInconsistent(Parent($X_i$),$X_i$)
  - Assign forward: For i = 1 : n, assign $X_i$ consistently with Parent($X_i$)

- Runtime: $O(n\ d^2)$ (why?)



# Tree-Structured CSPs

- Claim 1: After backward pass, all root-to-leaf arcs are consistent
- Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)



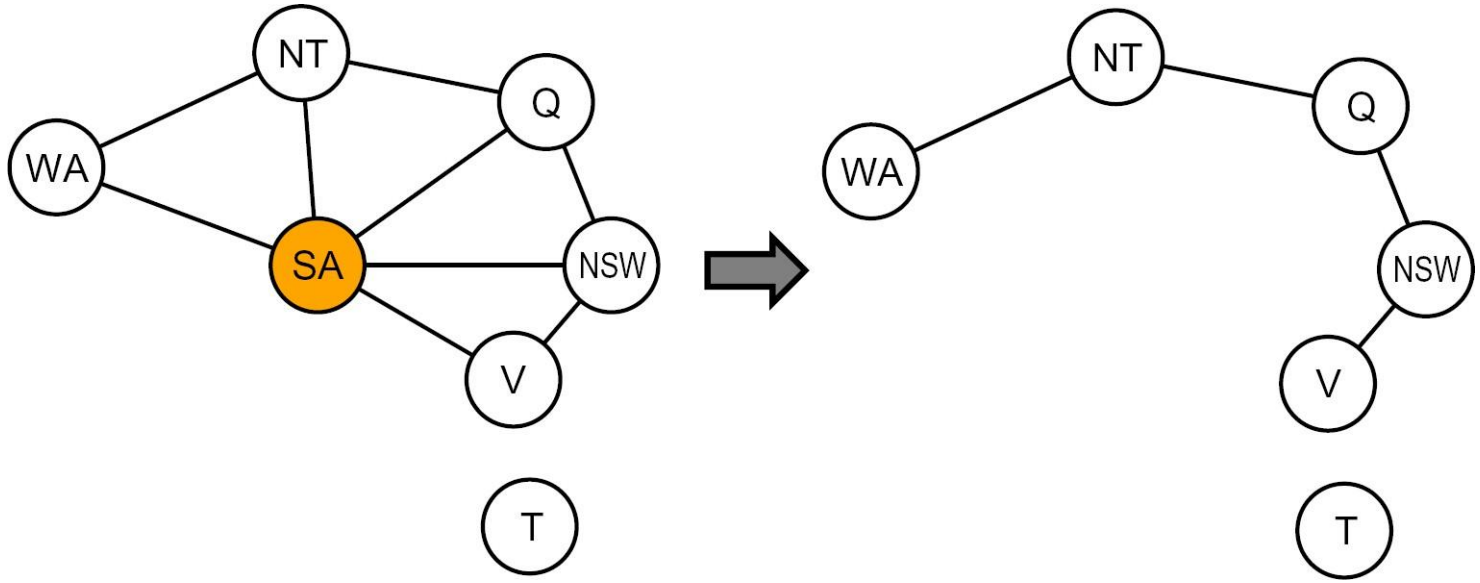

- Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack
- Proof: Induction on position

- Why doesn't this algorithm work with cycles in the constraint graph?

- Note: we'll see this basic idea again with Bayes' nets

# Improving Structure

# Nearly Tree-Structured CSPs




- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime O( $(d^c)$ (n-c) $d^2$ ), very fast for small c

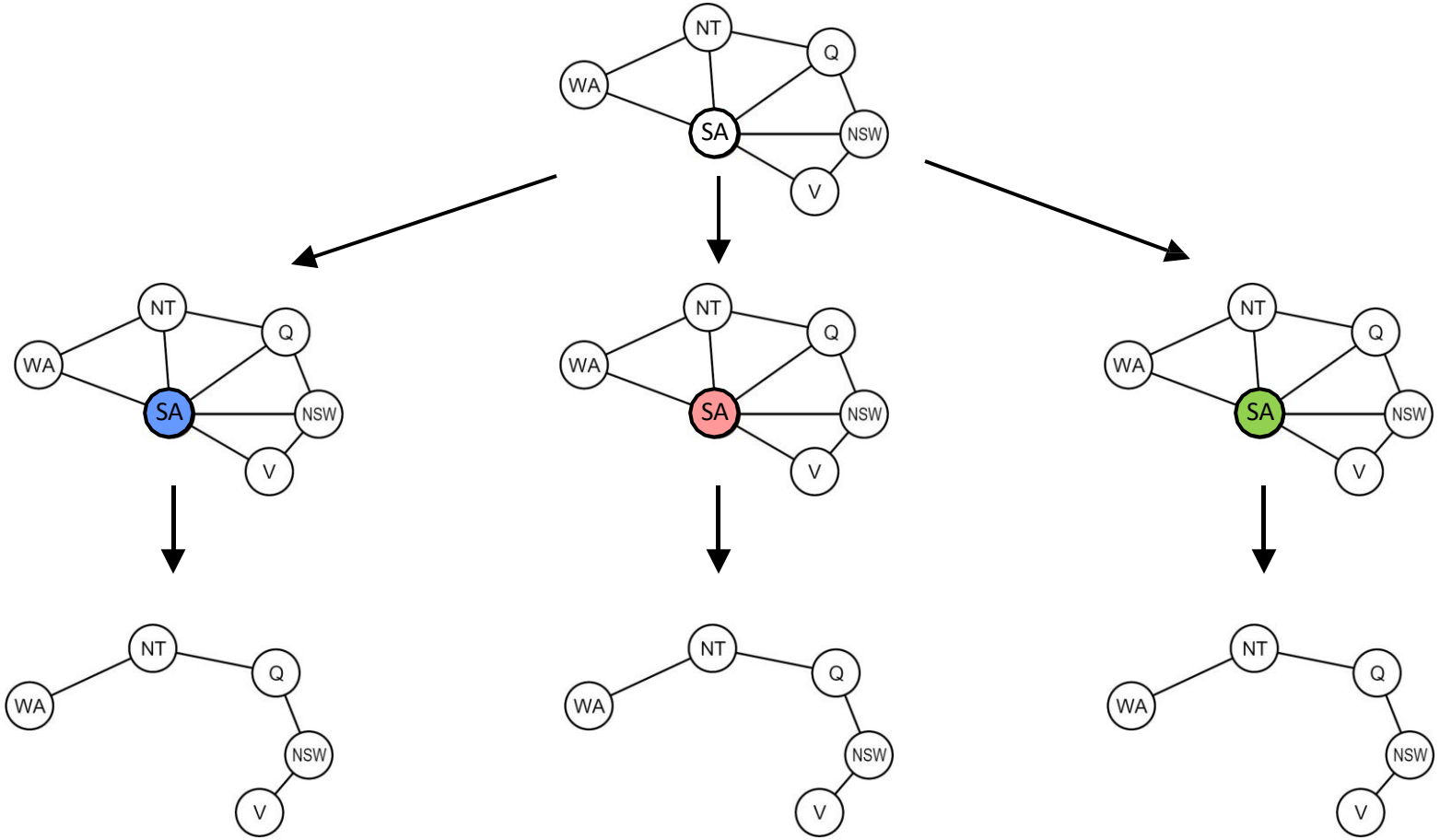
# Cutset Conditioning

Choose a cutset

Instantiate the cutset (all possible ways)

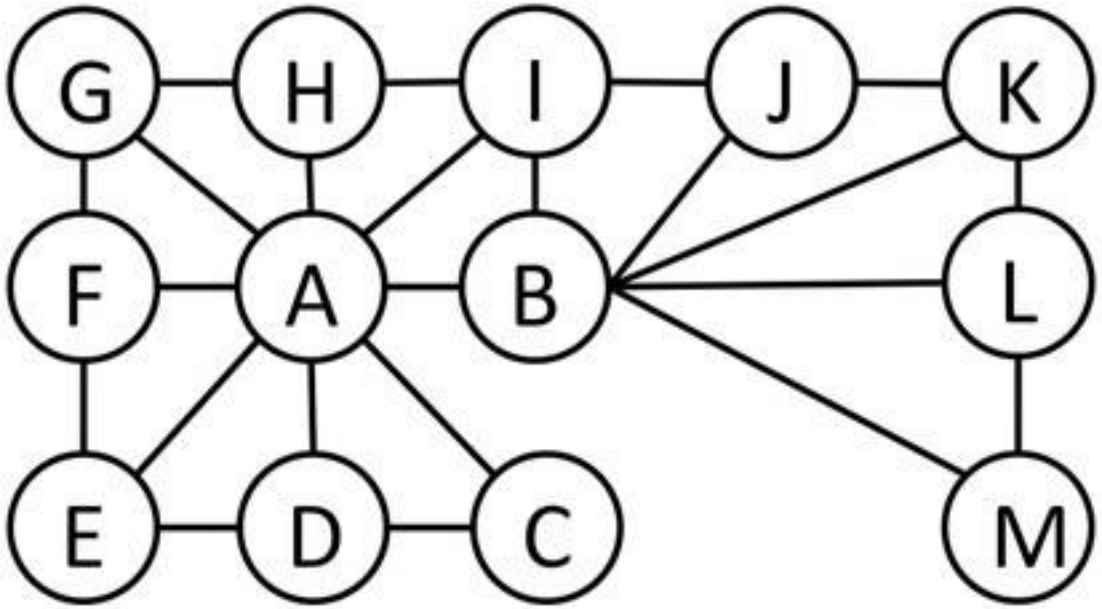
Compute residual CSP for each assignment

Solve the residual CSPs (tree structured)



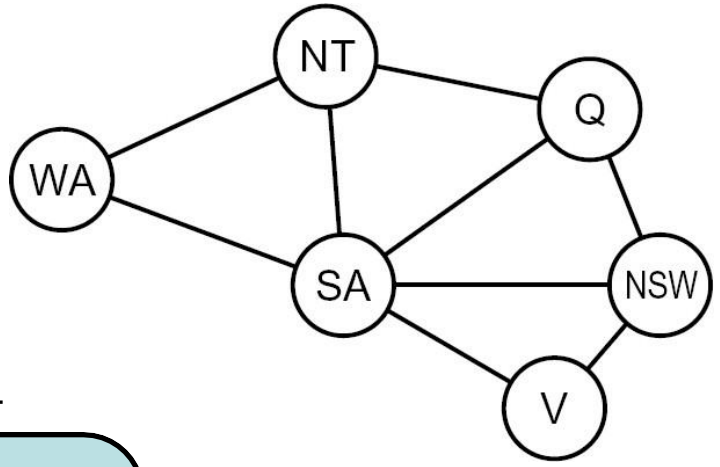
# Cutset Quiz

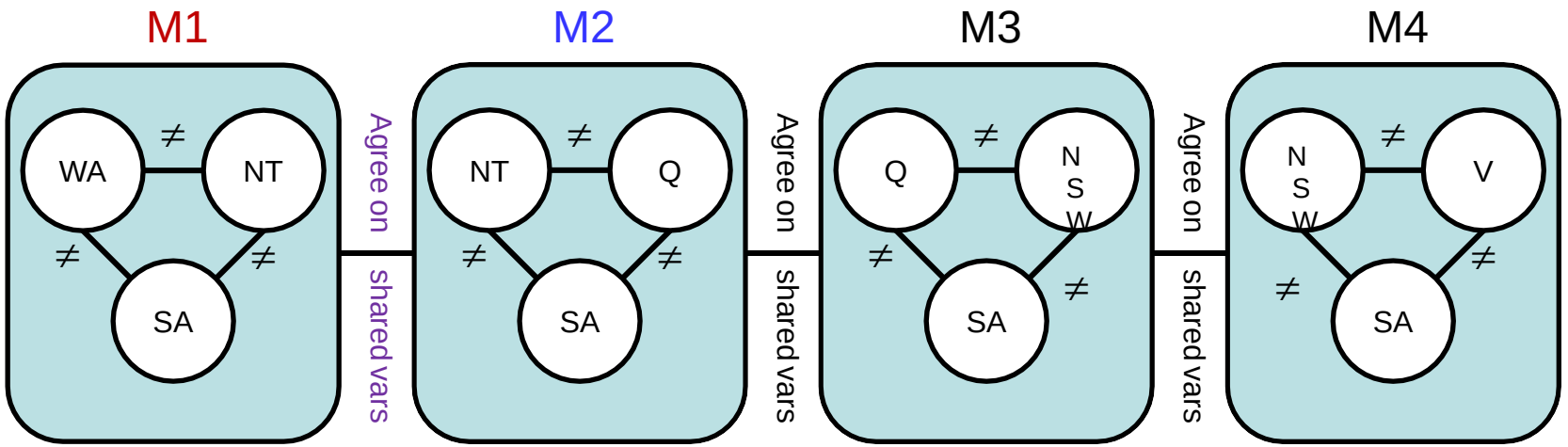
- Find the smallest cutset for the graph below.



# Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
- Each mega-variable encodes part of the original CSP
- Subproblems overlap to ensure consistent solutions



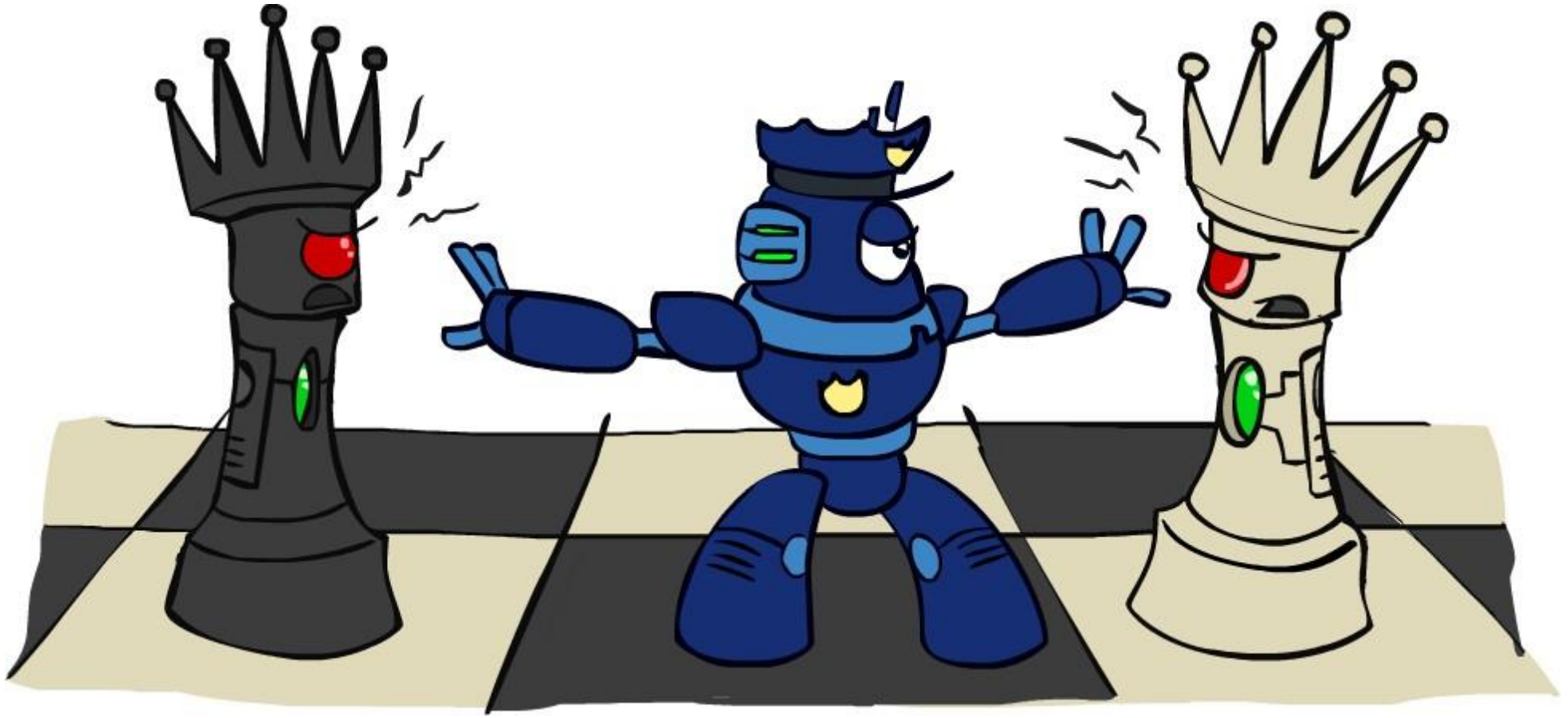




{(WA=r,SA=g,NT=b),
(WA=b,SA=r,NT=g),
…}

{(NT=r,SA=g,Q=b),
(NT=b,SA=g,Q=r),
…}

Agree: (M1,M2) ∈
{((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

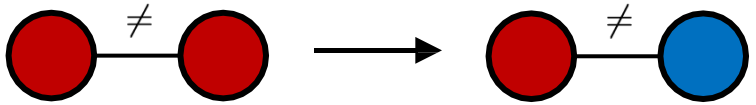
# Iterative Improvement



# Iterative Algorithms for CSPs

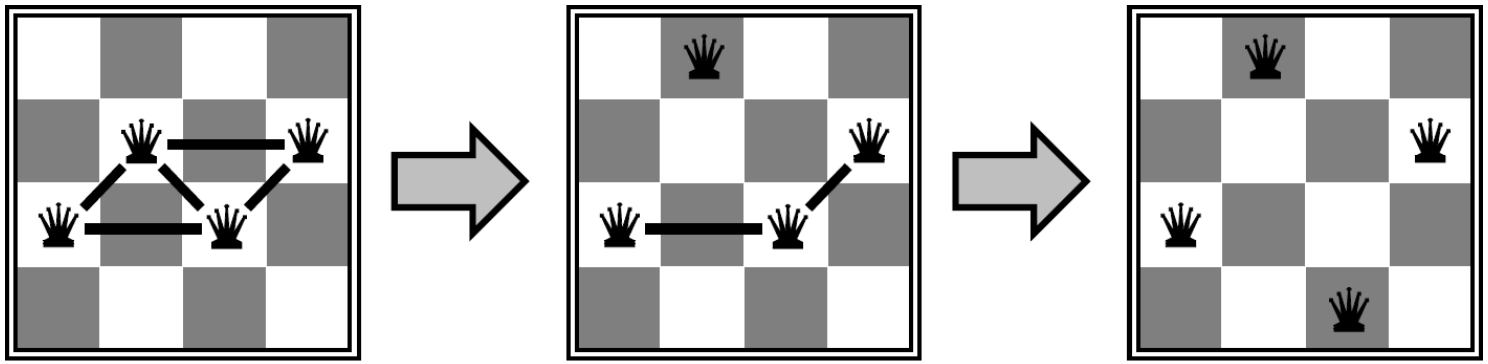
- Local search methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe!  Live on the edge.




- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints

# Example: 4-Queens



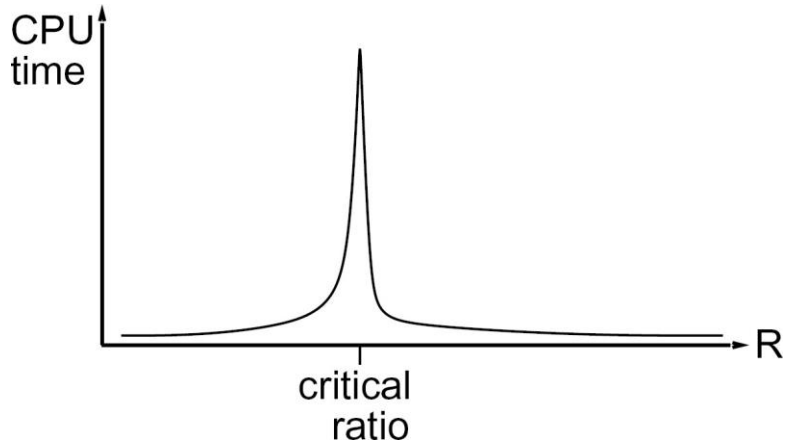

- States: 4 queens in 4 columns ($4^4 = 256$ states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: c(n) = number of attacks

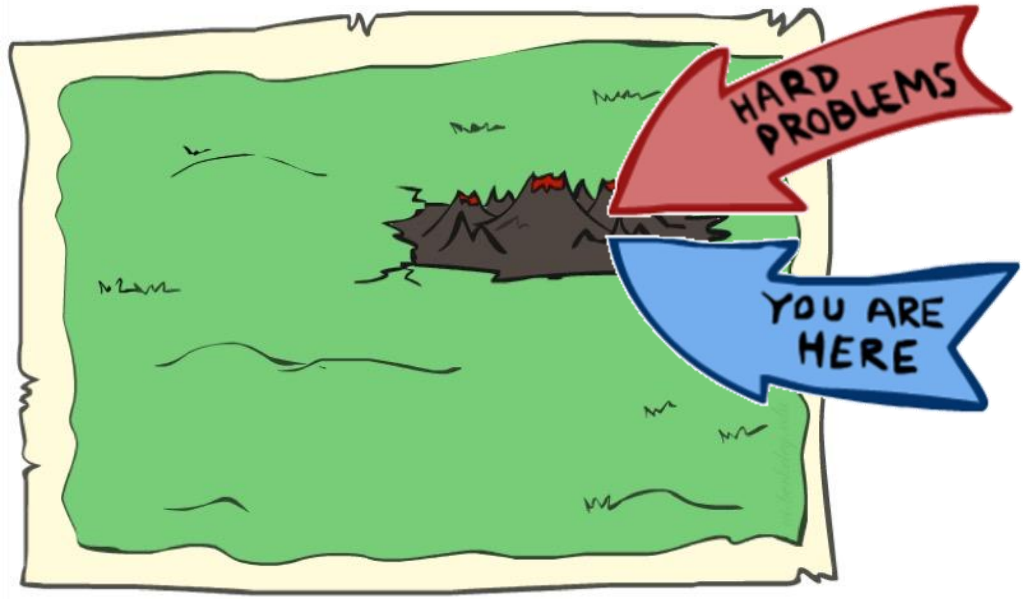
# Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!

- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

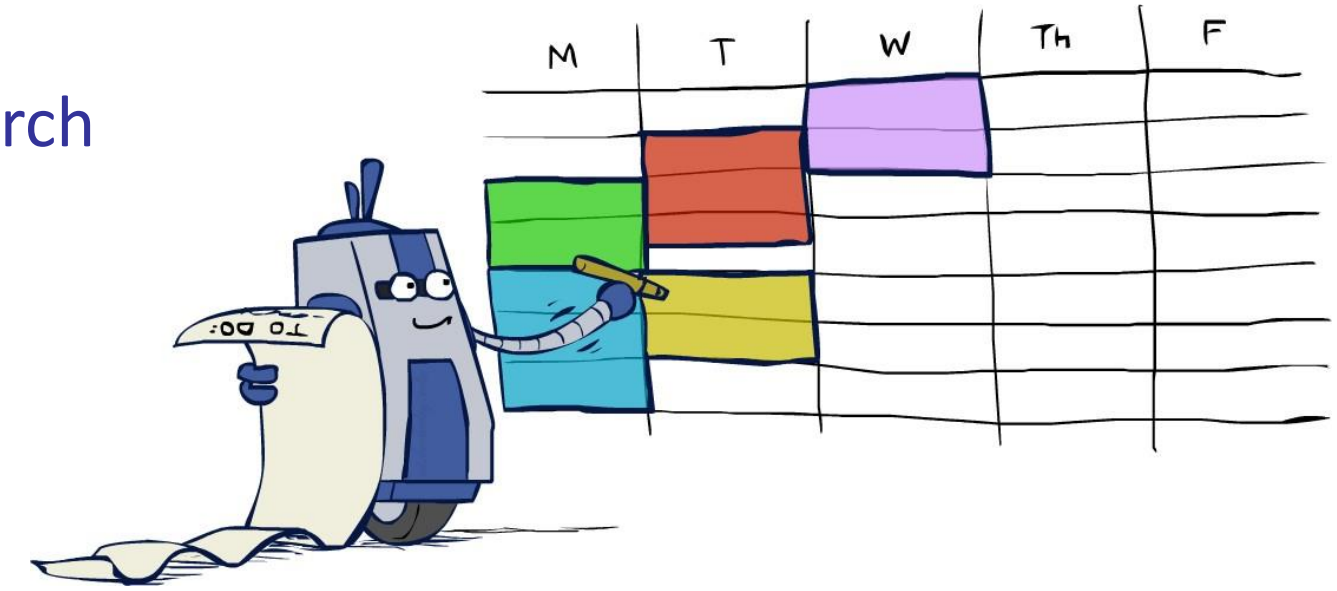
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



# Summary: CSPs

- CSPs are a special kind of search problem:
  - States are partial assignments
  - Goal test defined by constraints
- Basic solution: backtracking search
- Speed-ups:
  - Ordering
  - Filtering
  - Structure




- Iterative min-conflicts are often effective in practice
- For Demos: Visit this Link