Microscopic Handles in Teleparallel Gravity: Toward a Geometric Origin of Fermionic Statistics, Chirality, and Critical Binding

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Abstract

We propose a geometric mechanism in four-dimensional teleparallel gravity (TEGR) that offers a unified perspective on fermionic statistics, chirality, and the near-critical binding of composite states. We consider spacetime containing microscopic handles of topology $S^2 \times S^1$, carrying an integer torsional monopole charge q. Using a "Spinning SU(2)" tetrad ansatz, we derive three interconnected results.

(i) **Stability:** The pure TEGR sector admits a classically stable handle radius with an on-shell energy $E \simeq \alpha|q|$, providing a parity-even geometric mass term. (ii) **Chirality:** The Nieh–Yan boundary term produces a parity-odd contribution to the effective action. A slow precession mode develops a chiral tilt whose direction is locked to the sign of q, acting as a geometric mechanism for chirality selection. This construction is motivated by the known connection between the Nieh–Yan invariant and Wess–Zumino–Witten terms that govern particle statistics in Skyrmion-like models. (iii) **Classical criticality:** The stiffness of the precession mode scales as $k(q) \propto \omega^2|q|$, placing the system at a *classical* critical point where the binding energy per unit charge is independent of q to leading order.

Consequently, subleading corrections could in principle render composite states with $|q| \geq 2$ (in particular $|q| = 3$) energetically competitive with separated unit-charge handles. We regard the present construction as a teleparallel toy model that establishes the classical critical point $\gamma = 1$ from geometry alone; the impact of quantum and geometric corrections on an effective exponent γ_{eff} will be analysed in a separate work.

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1 Introduction

The Standard Model of particle physics provides an extremely successful description of microscopic phenomena, yet several of its key structural features remain conceptually opaque. Among these are (i) the spin–statistics connection, in particular the fermionic statistics of quarks and leptons, (ii) the chiral, $V-A$ structure of weak interactions, and

(iii) the special role of color triplets and color-singlet three-body bound states (baryons). In the conventional framework these properties are encoded in the local gauge and matter content of the theory, but they are not obviously explained by the large-scale geometry of spacetime itself.

Most attempts to derive such properties from geometry have proceeded by enlarging spacetime, e.g. via higher-dimensional Kaluza–Klein constructions or string theory. In this work we explore a different, minimalistic route: we remain in four dimensions, but we allow spacetime to carry a non-trivial microscopic topology and non-zero torsion. Specifically, we work within the teleparallel equivalent of general relativity (TEGR), where gravity is encoded in torsion rather than curvature, and we assume that the spatial manifold contains a large number of microscopic handles with topology $S^2 \times S^1$. The central question we ask is whether such a microscopic handle structure can provide a geometric origin for the above features of particle physics.

1.1 Motivation

The starting point of our proposal is the well-known fact that topological solitons in non-linear sigma models can carry quantum numbers such as baryon number, and that their statistics can be affected by Wess–Zumino–Witten (WZW) terms in the effective action. In particular, Witten showed that in an $SU(2)$ Skyrme model with WZW level k_{WZ}, a skyrmion behaves as a fermion when k_{WZ} is odd and as a boson when k_{WZ} is even [1]. At the same time, the Nieh–Yan four-form is known to contribute to the chiral anomaly on spaces with torsion, and thus can in principle induce an effective WZW-like topological term for an $SU(2)$-valued field coupled to torsion [2].

These observations suggest a possible chain of relations:

$$\begin{aligned} \text{Nieh–Yan density} &\longrightarrow \text{effective WZW-like level} \\ &\longrightarrow \text{statistics and internal quantum numbers of solitons.} \end{aligned} \tag{1}$$

In the usual treatments, the $SU(2)$ field is a matter field defined on an otherwise smooth background. In contrast, we will consider the possibility that the same $SU(2)$ structure is encoded in the local frame of a microscopic handle of spacetime itself. This leads naturally to the idea that each microscopic handle might behave as a solitonic defect whose topological charge is measured by the Nieh–Yan invariant, and whose statistics and chirality are governed by an induced topological term in the effective action.

To make this idea precise, we restrict attention—in this first paper— to the simplest possible setting:

- the gravitational sector is pure TEGR, with no higher-curvature corrections;
- the material content is summarized by an effective torsion background admitting a Dirac-type coupling;
- we focus on a single microscopic handle, modeled locally as $M \simeq \mathbb{R}_t \times S^1_\psi \times S^2_{(\theta,\phi)}$, and study its energetics and symmetry-breaking patterns.

Our goal is not to reproduce the full Standard Model, but to show that a single microscopic handle in teleparallel gravity can already exhibit three key features reminiscent of particle physics: (i) a stable localized configuration with energy linear in an integer topological charge q, (ii) spontaneous parity breaking associated with a precession mode, and (iii) a near-critical scaling of the stiffness that favors certain composite charge states.

We note in passing that purely static (non-spinning) handle configurations, with vanishing spin parameter $\omega = 0$ (to be introduced explicitly in Sec. 3), fail to produce a stable minimum of the effective potential; the spinning degree of freedom is therefore physically indispensable (see Appendix G for details).

1.2 Teleparallel framework and microscopic handles

Teleparallel gravity replaces the Levi-Civita connection of general relativity by a curvature-free connection with non-vanishing torsion. The fundamental variable is a tetrad field $e^a{}_\mu$ whose determinant we denote by $e = \det(e^a{}_\mu)$. The metric is recovered in the usual way, $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, while the connection is chosen such that its curvature vanishes identically and all gravitational information resides in the torsion. The corresponding Lagrangian density is built from a scalar $\mathbb{T}$ quadratic in the torsion tensor and is dynamically equivalent to the Einstein–Hilbert action up to a total divergence.

In the scenario considered here, the large-scale metric can be close to that of a spatial three-torus, but the spatial topology is refined by the presence of many microscopic handles. Schematically one may think of the spatial slice as

$$\Sigma \;\simeq\; T^3 \,\#\, \Big(\#_k \big(S^2 \times S^1\big)\Big), \tag{2}$$

where $\#$ denotes the connected sum and each factor $S^2 \times S^1$ represents a microscopic handle. In this first paper we assume that the handles are well separated and weakly interacting, so that it is meaningful to focus on a single representative handle and treat the rest as spectators.

Concretely, we select a four-dimensional region

$$M \;\simeq\; \mathbb{R}_t \times S^1_\psi \times S^2_{(\theta,\phi)}, \tag{3}$$

with coordinates $x^\mu = (t, \psi, \theta, \phi)$ adapted to the handle. We will construct an explicit tetrad on M that encodes both a twist along the S^1_ψ direction and a rigid spin about the handle axis. The resulting torsion yields a Nieh–Yan density whose integral over M defines an integer-valued topological charge $q \in \mathbb{Z}$. The main task of the present work is to analyze the energetics and small deformations of such a configuration and to identify how q controls its stability, parity properties, and possible composite states.

1.3 Main results

Before entering the detailed construction, it is useful to summarize the main results obtained in this paper.

(1) Stability and energy scaling (Phase 1). Using a "Spinning $SU(2)$ Handle Ansatz" for the tetrad on $M \simeq \mathbb{R}_t \times S^1 \times S^2$, we show that the TEGR action admits a class of static configurations in which the handle radius r is constant along the S^1 direction. The effective one-dimensional energy functional for the radial profile has the schematic form

$$E[r] \;=\; \int_0^{2\pi} d\psi \, \big[A(r')^2 + B\, q^2/r^2 + C\, m^2 r^2\big], \tag{4}$$

where $q \in \mathbb{Z}$ is the Nieh–Yan charge and m is an integer twist number. Minimizing this functional yields a stable radius

$$r_0 \;\propto\; \big|q/m\big|^{1/2}, \tag{5}$$

and the on-shell TEGR energy scales linearly with the absolute value of the charge,

$$E_{\mathrm{TEGR}}(q) \;\propto\; |q|. \tag{6}$$

The detailed derivation is given in Sec. 4 and Appendix C.

(2) Spontaneous parity breaking (Phase 2). We then introduce a small precession mode $\varepsilon(t)$ describing a tilt of the handle axis. Expanding the TEGR action to quadratic order in ε yields an effective "stiffness" term $\frac{1}{2}k(q)\,\varepsilon^2$ with $k(q) > 0$. In the presence of a Nieh–Yan term, however, the handle boundary contributes a term linear in ε, $V_{\mathrm{NY}} \sim -\Lambda_q\,\varepsilon$, where Λ_q is proportional to the Nieh–Yan charge q and to an effective coupling θ_{NY}. The resulting effective potential

$$V_{\mathrm{eff}}(\varepsilon) \;=\; \tfrac{1}{2}k(q)\,\varepsilon^2 - \Lambda_q\,\varepsilon \tag{7}$$

has its minimum at a non-zero value $\varepsilon_* = \Lambda_q/k(q)$ whenever $\Lambda_q \neq 0$. Thus a parity-symmetric Lagrangian generates an effective vacuum that spontaneously selects one sign of ε, providing a geometric analogue of $V{-}A$ chirality selection. This is analyzed in Sec. 5 and Appendix D.

(3) Critical stiffness scaling and composite states (Phase 3). Finally, we consider how the stiffness $k(q)$ scales with q in the Spinning $SU(2)$ Handle Ansatz. We show that the leading contribution behaves as

$$k(q) \;\propto\; m^2|q|, \tag{8}$$

so that in a coarse-grained description the total energy for a charge-q configuration can be written in the form

$$E(q) \;=\; \alpha|q| - \beta|q|^{2-\gamma}, \tag{9}$$

with an effective exponent $\gamma = 1$ arising from the geometry of the handle. This places the system at a near-critical point where fusion and fission of handles have comparable energetics. We discuss how small corrections (e.g. quantum effects or additional interactions) could slightly reduce the effective exponent, $\gamma_{\mathrm{eff}} \lesssim 1$, allowing certain composite states, in particular $q = 3$, to become energetically favored over their constituents. The associated calculations are presented in Sec. 6 and Appendix E.

Moreover, the same ansatz yields a stiffness that scales linearly with $|q|$, placing the system essentially at a critical regime in which higher-charge composite states become energetically competitive with separated unit-charge handles once small perturbative corrections are taken into account. This is what we will refer to as *critical binding* in the remainder of the paper.

1.4 Outline of the paper

The structure of the paper is as follows. In Sec. 2 we review the essentials of teleparallel gravity and the Nieh–Yan invariant, and we define the topological charge associated with a single microscopic handle. Section 3 introduces the Spinning $SU(2)$ Handle Ansatz for the tetrad on $M \simeq \mathbb{R} \times S^1 \times S^2$ and summarizes the resulting torsion and Nieh–Yan density. In Sec. 4 (Phase 1) we derive the effective one-dimensional energy functional for the handle radius and show the existence of a stable radius with $E_{\mathrm{TEGR}} \propto |q|$. In

Sec. 5 (Phase 2) we analyze the precession mode, its stiffness, and the Nieh–Yan induced linear term, leading to spontaneous parity breaking. Section 6 (Phase 3) is devoted to the scaling of the stiffness with q and to the energetics of composite charge states. In Sec. 7 we summarize the logical chain from geometry to topology to effective "matter" properties, and we comment on possible connections to baryon-like and meson-like configurations. Section 8 contains our conclusions. Several technical derivations and full expressions for the tetrad and torsion are collected in the appendices.

2 Theoretical framework

In this section we briefly review the teleparallel formulation of gravity used in this work, introduce the Nieh–Yan invariant, and define the topological charge associated with a microscopic handle. We follow standard conventions in teleparallel gravity, as reviewed for instance in Aldrovandi and Pereira's textbook [3].

2.1 TEGR and Weitzenböck geometry

The fundamental dynamical variable of teleparallel gravity is a tetrad (vierbein) field $e^a{}_\mu$ on a four-dimensional manifold M, where Greek indices $\mu, \nu, \dots$ label spacetime coordinates and Latin indices $a, b, \dots$ label an orthonormal frame. The spacetime metric is given by

$$g_{\mu\nu} \;=\; \eta_{ab} e^a{}_\mu e^b{}_\nu, \tag{10}$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$. The determinant of the tetrad is denoted $e = \det(e^a{}_\mu) = \sqrt{-\det g_{\mu\nu}}$.

In TEGR one introduces a connection, the Weitzenböck connection, which is metric-compatible and curvature-free but has non-vanishing torsion. In a pure-tetrad formulation this connection can be written as

$$\Gamma^\rho{}_{\mu\nu} \;=\; e_a{}^\rho\, \partial_\nu e^a{}_\mu, \tag{11}$$

so that its torsion tensor is

$$T^\rho{}_{\mu\nu} \;=\; \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu} \;=\; e_a{}^\rho(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu). \tag{12}$$

The curvature built from $\Gamma^\rho{}_{\mu\nu}$ vanishes identically, $R^\rho{}_{\sigma\mu\nu}(\Gamma) \equiv 0$, so that all gravitational degrees of freedom are encoded in the torsion.

The TEGR Lagrangian density is constructed from a scalar $\mathbb{T}$ quadratic in the torsion,

$$\mathbb{T} \;=\; \frac{1}{4} T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + \frac{1}{2} T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho - T^\rho{}_{\mu\rho} T^{\sigma\mu}{}_\sigma, \tag{13}$$

and the gravitational action reads

$$S_{\mathrm{TEGR}} \;=\; \frac{1}{2\kappa^2} \int_M d^4x\, e\, \mathbb{T}, \qquad \kappa^2 = 8\pi G. \tag{14}$$

Up to a total divergence, this action is equivalent to the Einstein–Hilbert action built from the Levi-Civita connection of $g_{\mu\nu}$, and thus reproduces the same classical field equations. In what follows we work in units where $\kappa^2 = 1$ when convenient, and we leave the inclusion of matter fields implicit, focusing on the gravitational/torsional sector that is relevant for the microscopic-handle dynamics.

2.2 Nieh–Yan invariant and effective Wess–Zumino term

In a first-order (tetrad and spin connection) formulation of gravity with torsion, one may define the Nieh–Yan four-form

$$\mathcal{N} \;=\; T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b, \tag{15}$$

where T^a is the torsion two-form, R_{ab} is the curvature two-form of the spin connection, and e^a is the coframe one-form. In the Weitzenböck geometry used in TEGR the curvature vanishes, $R_{ab} \equiv 0$, so the Nieh–Yan form simplifies to

$$\mathcal{N} \;=\; T^a \wedge T_a \;=\; d\big(e^a \wedge T_a\big), \tag{16}$$

i.e. it is an exact form and hence a total divergence locally. However, on manifolds with non-trivial topology or with boundaries (such as our microscopic handles and the regions gluing them to the rest of spacetime), its integral can be non-zero and topologically quantized, in close analogy with the way instanton densities contribute to anomalies in gauge theories.

When coupled to chiral fermions, the Nieh–Yan term can contribute to the chiral anomaly. Chandia and Zanelli showed that, in spaces with torsion, the divergence of the axial current receives an additional contribution proportional to $\mathcal{N}$, so that the anomaly equation schematically takes the form

$$\partial_\mu J^\mu_5 \;\sim\; \frac{1}{16\pi^2}\big(\operatorname{tr} F \wedge F + \cdots + \lambda_{\rm NY}\, \mathcal{N}\big), \tag{17}$$

where F denotes the gauge field strength and $\lambda_{\rm NY}$ is a dimensionful coefficient depending on the ultraviolet regularization [2]. In an effective low-energy description, such an anomaly can be encoded by adding a Wess–Zumino–Witten-type term to the action of an $SU(2)$-valued field $U(x)$, with a quantized level $k_{\rm WZ}$ [1].

Anomaly-matching arguments therefore suggest that, in an appropriate effective description, the integral of $\mathcal{N}$ over a four-dimensional region may control the coefficient of such a topological term. In this paper we do not attempt a full derivation of the effective action starting from a microscopic fermion and gauge-field content. Instead, we adopt the following *working hypothesis*:

> For a suitably defined $SU(2)$-valued field associated with the local frame of a microscopic handle, the integral of the Nieh–Yan density over that handle contributes to an induced Wess–Zumino– Witten-type topological term in the low-energy effective action, with a coefficient that receives a contribution proportional to the integer-valued Nieh–Yan charge q.

In the explicit handle model constructed below we will in fact find that the effective energy contains a term linear in q that plays a role analogous to a WZW level, and this linear dependence on q is the only feature of the induced topological term that we will actually use in our subsequent analysis.

Under this hypothesis, a handle with odd q is expected to behave as a fermionic defect, while one with even q is bosonic, at least at the level of an effective Skyrme-like description. The precise proportionality factor and the matching to a concrete microscopic model are left for future work; here we only need the existence of an integer q that labels topological sectors and enters linearly in the effective action.

2.3 Topological charge of a microscopic handle

We now define the topological charge associated with a single microscopic handle. Let $M \simeq \mathbb{R}_t \times S^1_\psi \times S^2_{(\theta,\phi)}$ denote a four-dimensional region containing one handle, with coordinates $x^\mu = (t, \psi, \theta, \phi)$. We assume that the tetrad is such that the Nieh–Yan form $\mathcal{N}$ is integrable over M and that the fields fall off sufficiently fast outside the handle so that boundary contributions from infinity can be neglected.

We define the Nieh–Yan charge q of the handle by

$$q = \frac{1}{\mathcal{N}_0} \int_M \mathcal{N}, \tag{18}$$

where $\mathcal{N}_0$ is a normalization constant chosen so that $q \in \mathbb{Z}$ for the class of configurations considered. In a microscopic UV completion, the natural quantization unit that guarantees $q \in \mathbb{Z}$ for topologically non-trivial configurations is

$$\mathcal{N}_0 = 32\pi^2, \tag{19}$$

in exact analogy with the instanton charge in Yang–Mills theory [2]. In a fully microscopic theory this quantization would follow from the global properties of the tetrad and spin connection, but for the present purposes we simply assume that q takes integer values.

The decomposition $\mathcal{N} = d(e^a \wedge T_a)$ implies that q is determined by the flux of $e^a \wedge T_a$ through the boundary of M, which in our setting consists of a union of three-dimensional hypersurfaces connecting the handle to the rest of spacetime. This makes q a natural measure of the amount of torsional "flux" threading the handle. In the Spinning $SU(2)$ Handle Ansatz introduced in Sec. 3, the explicit form of $\mathcal{N}$ will be seen to depend on the global twist and spin of the tetrad around the handle, so that q is directly related to the corresponding winding data. Since the detailed structure is somewhat involved, we postpone its full expression to Sec. 3 and to the appendices; for now, Eq. (18) serves as our definition of the topological charge labeling microscopic-handle configurations.

In summary, the theoretical framework we adopt consists of: (i) TEGR as the gravitational theory, with torsion encoded in a tetrad on M, (ii) the Nieh–Yan density as a total derivative that nevertheless enters the chiral anomaly and can induce an effective WZW-like topological term, and (iii) an integer-valued Nieh–Yan charge q defined by the integral of $\mathcal{N}$ over a microscopic handle. In Sec. 3 we specify the Spinning $SU(2)$ Handle Ansatz that realizes these ingredients in an explicit tetrad configuration.

3 Spinning $SU(2)$ Handle Ansatz

In this section we specify the microscopic geometry of a single handle and introduce the *Spinning $SU(2)$ Handle Ansatz* that will be used throughout Phases 1–3. The ansatz encodes three charges (q, m, ω) and a radius profile $r(\psi)$ in a tetrad field $e^a{}_\mu$ defined on a fixed handle topology.

3.1 Geometry of a single handle

Locally we model a single microscopic handle as a product manifold

$$M \simeq \mathbb{R}_t \times S^1_\psi \times S^2_{(\theta,\phi)}, \tag{20}$$

with coordinates

$$x^\mu = (t, \psi, \theta, \phi), \tag{21}$$

where t denotes physical time, $\psi \in [0, 2\pi)$ is an angular coordinate along the handle axis (S^1), and (θ, ϕ) are polar and azimuthal angles on the cross–section S^2.

We assume that the handle is much smaller than any macroscopic curvature scale of the ambient universe, so that background fields may be treated as approximately constant over M.

As in Phase 1, we keep a single geometric degree of freedom for the shape of the handle: the physical radius $r(\psi)$ of the two–sphere cross–section as a function of the axial coordinate ψ. A rough form of the line element is then

$$ds^2 \sim -dt^2 + d\psi^2 + r(\psi)^2 \big(d\theta^2 + \sin^2\theta \, d\phi^2\big), \tag{22}$$

while the exact expression will be encoded in the tetrad defined below. In particular, the q–dependent terms that implement the torsion flux are captured at the tetrad level and are not shown explicitly in (22).

3.2 Tetrad construction and parameters

We work in the Weitzenböck gauge, so that the spin connection vanishes and all gravitational information is carried by the tetrad field $e^a{}_\mu$.

The handle ansatz is constructed in two steps: we first choose a convenient *reference tetrad* $\tilde{e}^a{}_\mu$ with a monopole–like torsion flux, and then act with a time– and angle–dependent spatial rotation derived from an $SU(2)$ group element.

Reference tetrad and monopole charge q

For definiteness we take the reference tetrad one–forms to be

$$\tilde{e}^0 = dt, \tag{23}$$
$$\tilde{e}^1 = d\psi, \tag{24}$$
$$\tilde{e}^2 = r(\psi)\, d\theta, \tag{25}$$
$$\tilde{e}^3 = r(\psi)\, \sin\theta \, \big(d\phi + q(1-\cos\theta)\, d\psi\big). \tag{26}$$

The resulting metric is approximately of the form (22); the q–dependent mixing term resides in $\tilde{e}^3$ and will play a role only in the torsion sector.

We define $q \in \mathbb{Z}$ as an *integer monopole charge* associated with the torsion flux through each cross–section S^2_ψ at fixed (t, ψ). Concretely, one may impose a normalisation condition of the form

$$\frac{1}{4\pi} \int_{S^2_\psi} \star T^1 = q, \tag{27}$$

where $T^a = d\tilde{e}^a$ is the torsion 2–form and $\star$ denotes the Hodge dual built from the induced metric. In what follows we treat q as a fixed topological input for each handle.

$SU(2)$ field and induced rotation

We introduce an internal $SU(2)$ field

$$U(t, \psi) \;=\; \exp\!\left(i\frac{\omega t}{2}\sigma_1\right) \, \exp\!\left(i\frac{m\psi}{2}\sigma_3\right), \tag{28}$$

where σ_i are the Pauli matrices. The first factor implements an intrinsic *spin* in time, while the second factor implements a *twist* along the handle axis. We therefore refer to (28) as a "spin–then–twist" parameterisation.

Via the double cover $SU(2) \to SO(3)$, the field $U(t, \psi)$ defines a spatial rotation matrix $\Lambda^i{}_j(t, \psi)$ acting on the spatial tetrad indices $i, j = 1, 2, 3$. We then define the physical tetrad as

$$e^0 = \tilde{e}^0, \qquad e^i = \Lambda^i{}_j(t, \psi)\, \tilde{e}^j, \quad i = 1, 2, 3. \tag{29}$$

In the explicit implementation used in the detailed calculations, the rotation factorises as

$$\Lambda(t, \psi) = R_3(m\psi)\, R_1(\omega t), \tag{30}$$

so that the spatial basis is first spun around the local $\tilde{e}^1$–axis as time evolves, and then twisted in the $(\tilde{e}^2, \tilde{e}^3)$–plane as one moves along ψ.

A key property of this ordering is that the handle axis vector e^1 remains *time–independent*: the spin acts around the axis rather than precessing the axis itself. This will allow us to add a small precession mode $\varepsilon(t)$ in Phase 2 without destabilising the background configuration.

Summary of parameters

The ansatz introduces the following parameters and fields:

- $q \in \mathbb{Z}$ (flux / monopole charge): integer–quantised torsion flux through the S^2 cross–section. It controls the strength of the core repulsion and, via the Nieh–Yan/WZW chain, the effective WZW level in a boundary Skyrme–like description.
- $m \in \mathbb{Z}$ (twist number): twist number along the handle axis; as ψ increases from 0 to 2π, the internal frame rotates m times in the cross–section.
- $\omega \in \mathbb{R}$ (spin): intrinsic spin angular frequency of the handle, defined in the co–moving frame of the twisted handle.
- $r(\psi) > 0$ (radius profile): radius of the S^2 cross–section as a function of ψ. In Phase 1 we focus on the equal–radius ansatz $r(\psi) = r_0$, while Phases 2 and 3 allow small deformations.

In the TEGR action, q, m, ω and $r(\psi)$ control, respectively, the flux–induced core energy, the twist energy, the spin–related terms, and the elastic energy associated with variations of the radius. Their scaling behaviours will be analysed in detail in Sec. 4 and subsequent sections.

3.3 Nieh–Yan density as an exact form

In the Weitzenböck geometry adopted here, the Nieh–Yan 4–form simplifies to

$$\mathcal{N} = T^a \wedge T_a = d(e^a \wedge T_a), \tag{31}$$

and behaves as a torsion–built topological invariant.

For the Spinning $SU(2)$ Handle Ansatz, an explicit calculation of the torsion 2–forms $T^a = de^a$ yields a Nieh–Yan density that, up to an overall normalisation fixed by the choice of q, takes the form

$$\begin{aligned}\mathcal{N} &\simeq q\,\omega\,m\,\partial_\psi\big(r(\psi)^2\big)\,dt\wedge d\psi\wedge\sin\theta\,d\theta\wedge d\phi\\ &= \; q\,\omega\,m\,d\big(r(\psi)^2\big)\wedge dt\wedge d\Omega_2,\end{aligned}\tag{32}$$

where $d\Omega_2 = \sin\theta\,d\theta\wedge d\phi$ is the area form on the unit 2–sphere, and $\simeq$ denotes equality up to a numerical normalisation that will be fixed once and for all by the flux quantisation condition for q.

This total–derivative structure has two important consequences.

(i) Boundary sensitivity. Because $\mathcal{N}$ is an exact form in the ψ–direction, the Nieh–Yan contribution of a single handle depends only on the boundary values of r^2 at the ends of the handle (or, more precisely, at the joints where the handle is glued to the ambient manifold):

$$\int_M \mathcal{N} \;\propto\; q\,\omega\,m\,\big[r(\psi)^2\big]_{\psi\text{–endpoints}}.\tag{33}$$

For a microscopic handle that is nearly cylindrical in its interior, the contribution is thus localised near the joints.

(ii) Effective WZW level. We emphasise that q is defined independently, as the integer monopole charge of the torsion flux in the reference tetrad, and is *not* redefined via the Nieh–Yan integral.

Instead, we interpret the Nieh–Yan integral as a measure of an effective WZW–like level seen by a boundary Skyrme field. For a single handle, the scaling behaviour implied by (32) can be summarised as

$$\int_M \mathcal{N} \;\propto\; q\,\omega\,m\,\Delta\big(r^2\big),\tag{34}$$

where $\Delta(r^2)$ denotes the jump of r^2 between the two ends of the handle. In the multi–handle ensemble considered in later sections, the sum of such contributions determines the effective WZW level that appears in the boundary chiral action. This underlies the anomaly–matching picture developed in Sec. 5.

The detailed derivation of (32) from the tetrad (23)–(26) and (29)–(30) is straightforward but lengthy; it is relegated to Appendix B.

3.4 Static versus spinning ansätze

It is instructive to contrast the spinning ansatz above with more naive alternatives.

If one sets $\omega = 0$ and keeps only a static twist $U(\psi) = \exp(im\psi\,\sigma_3/2)$, the Nieh–Yan density (32) loses the cross term proportional to ωm. In such a purely twisted configuration the handle still carries a flux q, but the Nieh–Yan density no longer exhibits the simple total–derivative structure that makes the identification of an effective WZW level transparent.

Conversely, if one tries to encode precession by letting the handle axis itself rotate in time with respect to a fixed laboratory frame, the tetrad becomes time–dependent already at the level of the axis vector e^1. This leads to an unnecessarily complicated TEGR energy functional and obscures the separation between

- the *background* spinning handle (characterised by $q, m, \omega, r(\psi)$), and
- the *small precession mode* $\varepsilon(t)$ that we introduce in Phase 2.

The present Spinning $SU(2)$ Handle Ansatz avoids both problems:

- the axis e^1 is time–independent, so that a small tilt $\varepsilon(t)$ can be added cleanly on top of the spinning background, and
- the Nieh–Yan density acquires the exact–form structure (32), which allows us to treat $\int_M \mathcal{N}$ as a bona fide contribution to an effective WZW level in the boundary theory.

For these reasons we adopt the spinning ansatz (23)–(26) and (28)–(30) as our *canonical microscopic geometry* for a single handle, and relegate static variants and precessing laboratory–frame configurations to Appendix F.

4 Phase 1: Stability and energy scaling

In this section we place the Spinning $SU(2)$ handle ansatz of Sec. 3 into the TEGR action of Sec. 1.2, and reduce the dynamics to the single radial degree of freedom $r(\psi)$ in a one–dimensional mini–superspace. We then show that pure TEGR already admits a classically stable equal–radius handle solution with a radius

$$r_0 \ \propto \ |q|^{1/2}, \qquad \text{(for fixed non-zero twist number } m), \tag{35}$$

and that the corresponding TEGR energy scales linearly with the absolute value of the monopole charge q,

$$E_{\mathrm{TEGR}}(q) \ \propto \ |q|, \qquad \text{(again with fixed non-zero } m). \tag{36}$$

These results provide the P–even geometric background for the parity–violating effects discussed in Phase 2 and the binding analysis in Phase 3.

4.1 Effective mini–superspace energy functional

We work in pure TEGR,

$$S_{\mathrm{TEGR}} \ = \ \frac{1}{2\kappa^2} \int d^4x \, e \, \mathbb{T}, \tag{37}$$

with Weitzenböck connection and torsion scalar $\mathbb{T}$ as defined in Sec. 1.2. On the local product manifold $M \simeq \mathbb{R}_t \times S^1_\psi \times S^2$, the Spinning $SU(2)$ handle ansatz of Sec. 3 reduces the geometric degrees of freedom to a single radial function $r(\psi)$ measuring the isotropic radius of the S^2 cross–section at each point along the handle axis S^1_ψ.

After inserting the ansatz into S_{TEGR}, averaging over time, and integrating over the S^2 cross–section, one obtains an effective one–dimensional energy functional for $r(\psi)$,

$$E[r] \ = \ \int_0^{L_\psi} d\psi \left[A\,(r')^2 + V(r) \right], \qquad r' := \frac{dr}{d\psi}, \tag{38}$$

where L_ψ denotes the coordinate length of the handle along the ψ–direction and $A > 0$ is a positive coefficient determined by the TEGR normalisation and the detailed tetrad choice. The coefficient A arises from axial components of the torsion scalar (schematically from terms of the form $T^{\psi}{}_{r\psi}T_{\psi}{}^{r\psi}$, etc.) and is expected to be of Planckian magnitude in a UV–complete theory.

The effective potential $V(r)$ for an equal–radius handle takes the form

$$V(r) \;=\; \frac{B\,q^2}{r^2} + C\,m^2 r^2, \qquad B > 0,\; C > 0, \tag{39}$$

where

- $q \in \mathbb{Z}$ is the integer monopole charge associated with the torsion flux through the S^2 cross–section, as defined in Sec. 3;
- $m \in \mathbb{Z}$ is the twist number of the $SU(2)$ rotation along the handle axis;
- B and C are positive constants set by the TEGR coupling and the detailed geometry of the ansatz.

The first term, $\propto q^2/r^2$, originates from the flux contribution to the torsion scalar and behaves as a repulsive "core" potential that diverges for $r \to 0$. The second term, $\propto m^2r^2$, arises from the uniform twist along S^1_ψ and plays the rôle of a tension or centrifugal contribution that diverges for $r \to \infty$.

Spin (self–rotation) ω does contribute to the torsion scalar, but in the co–rotating frame implicit in the ansatz its r–dependent contributions cancel between kinetic and centrifugal terms for static, equal–radius configurations. As a result, ω does not appear in $V(r)$, and its rôle is deferred to the Nieh–Yan sector and the precession dynamics discussed in later sections.

4.2 Classical stability of the equal–radius handle

We now specialise to static equal–radius configurations

$$r(\psi) = r_0 = \text{const.}, \tag{40}$$

for which the gradient term in (38) vanishes and the energy reduces to

$$E[r_0] \;=\; \int_0^{L_\psi} d\psi\, V(r_0) \;=\; L_\psi\, V(r_0). \tag{41}$$

The classical equilibrium radius is thus determined by the extremum condition

$$\left.\frac{dV}{dr}\right|_{r=r_0} = 0. \tag{42}$$

Using (39),

$$\frac{dV}{dr} \;=\; -\frac{2Bq^2}{r^3} + 2Cm^2 r, \tag{43}$$

so that the extremum satisfies

$$Cm^2 r_0^4 = Bq^2 \qquad \Longrightarrow \qquad r_0^4 = \frac{Bq^2}{Cm^2}. \tag{44}$$

For $q \neq 0$ and $m \neq 0$ this yields a finite, nonzero equilibrium radius

$$r_0 \;=\; \left(\frac{B}{C}\right)^{1/4} \frac{|q|^{1/2}}{|m|^{1/2}}, \tag{45}$$

up to a theory–dependent numerical factor $(B/C)^{1/4}$. In particular, for a fixed non–zero twist number m the radius scales as

$$r_0 \;\propto\; |q|^{1/2}, \qquad (m \text{ fixed}), \tag{46}$$

in agreement with the scaling quoted in Eq. (35).

To check that this extremum is a minimum, we compute the second derivative of $V(r)$,

$$\frac{d^2V}{dr^2} \;=\; \frac{6Bq^2}{r^4} + 2Cm^2. \tag{47}$$

Evaluating at $r = r_0$ and using (44),

$$\frac{d^2V}{dr^2}\bigg|_{r=r_0} \;=\; \frac{6Bq^2}{r_0^4} + 2Cm^2 \;=\; 6Cm^2 + 2Cm^2 \;=\; 8Cm^2 \;>\; 0, \tag{48}$$

so the equal–radius configuration is indeed a local minimum of the effective potential for any $q \neq 0$ and $m \neq 0$.

Small fluctuations $\delta r(\psi)$ around r_0 can be analysed by expanding (38) to quadratic order,

$$r(\psi) = r_0 + \delta r(\psi), \tag{49}$$

which yields a Sturm–Liouville problem with a positive mass term proportional to $d^2V/dr^2|_{r_0}$ and a gradient term controlled by the coefficient A. In the regime of interest for DPPU, A is expected to be of Planckian order in natural units, so that non–uniform Fourier modes of δr are heavy and strongly suppressed. Consequently, in the semiclassical regime relevant to DPPU the equal–radius configuration $r(\psi) = r_0$ provides an excellent approximation to the true vacuum, with radial oscillations confined to sub–Planckian amplitudes. (A more detailed estimate of A and the fluctuation spectrum is deferred to Appendix C.)

4.3 On–shell TEGR energy and linear scaling in $|q|$

We now evaluate the TEGR contribution to the energy on the equal–radius solution $r(\psi) = r_0$ determined above. Substituting (44) into (39), we obtain

$$\begin{aligned} V(r_0) &= \frac{Bq^2}{r_0^2} + Cm^2 r_0^2 \\ &= \frac{Bq^2}{\left(\frac{B}{C}\right)^{1/2}|q/m|} + Cm^2\left(\tfrac{B}{C}\right)^{1/2}\left|\frac{q}{m}\right| \\ &= \sqrt{BC}\,|mq| + \sqrt{BC}\,|mq| \\ &= 2\sqrt{BC}\,|mq|. \end{aligned} \tag{50}$$

The corresponding TEGR energy is

$$E_{\mathrm{TEGR}}(q,m) \;=\; L_\psi\, V(r_0) \;=\; 2L_\psi \sqrt{BC}\,|mq|. \tag{51}$$

For fixed twist number $m \neq 0$ and handle length L_ψ this implies the linear scaling

$$E_{\rm TEGR}(q) \;\equiv\; E_{\rm TEGR}(q, m_{\rm fixed}) \;\approx\; \alpha\,|q|, \qquad \alpha := 2L_\psi\sqrt{BC}\,|m| > 0. \tag{52}$$

Thus, within the Spinning $SU(2)$ handle ansatz, pure TEGR behaves as a *geometric mass term* that is linear in the absolute value of the monopole charge q. The twist number $m \neq 0$ is required for the existence of a stable minimum and sets the overall energy scale, but it does not modify the linear dependence on $|q|$.

In particular, the TEGR energy is insensitive to the sign of q; it is even under parity and charge conjugation, and does not distinguish between q and $-q$. The sign–sensitivity and chirality selection that are essential for the DPPU picture therefore cannot come from the TEGR sector alone. They enter through the Nieh–Yan term and its effective boundary coupling to the precession mode, which we analyse in Phase 2.

5 Phase 2: Nieh–Yan boundary term and the precession mode

In Phase 1 (Sec. 4) we showed that, within the Spinning $SU(2)$ handle ansatz, pure TEGR admits a classically stable equal–radius configuration $r(\psi) = r_0$ whose energy scales as $E_{\rm TEGR}(q) \propto |q|$ for fixed non–zero twist number m. This background is parity–even: it does not distinguish between q and $-q$ and does not select any preferred handedness for the handle.

In this section we switch on the Nieh–Yan term and introduce a slow precession mode that tilts the handle axis by a small time–dependent angle $\varepsilon(t)$. We show that the combined effect of TEGR and Nieh–Yan is an effective potential of the form

$$V_{\rm eff}(\varepsilon; q) \;\simeq\; E_{\rm TEGR}(q) + \frac{1}{2}k(q)\,\varepsilon^2 + \Lambda_q\,\varepsilon + \mathcal{O}(\varepsilon^4), \tag{53}$$

where $k(q) > 0$ is a positive function of q and Λ_q is a coefficient linear in q. In the sign convention adopted here we write

$$\Lambda_q \;=\; -\,\gamma\,q, \qquad \gamma > 0 \;\;\text{(for fixed } m, \omega, \theta_{\rm NY}, \Delta r^2), \tag{54}$$

so that $\Lambda_{-q} = -\Lambda_q$ and the sign of Λ_q is opposite to the sign of q.

The minimum of (53) occurs at

$$\varepsilon_*(q) \;\simeq\; -\frac{\Lambda_q}{k(q)} \;=\; \frac{\gamma}{k(q)}\,q, \tag{55}$$

so that, for $\Lambda_q \neq 0$, the true ground state is slightly tilted ($\varepsilon_* \neq 0$) and its sign is tied to the sign of q. This provides a simple geometric mechanism for dynamical chirality selection: handles with $q > 0$ and $q < 0$ precess in opposite directions.

Throughout this section we work in the regime

$$|\varepsilon(t)| \ll 1, \qquad \left|\frac{d\varepsilon}{dt}\right| \text{ small}, \tag{56}$$

so that an expansion in powers of ε and its time derivatives is consistent. Large precession angles require a separate non–perturbative treatment and lie beyond the scope of the present paper.

5.1 Introducing the precession mode

We start from the equal–radius background of Phase 1,

$$r(\psi) = r_0, \tag{57}$$

with the Spinning $SU(2)$ handle ansatz of Sec. 3, schematically

$$U(\psi, t) \;=\; \mathrm{Twist}(m\psi)\,\mathrm{Spin}(\omega t), \tag{58}$$

and the associated tetrad $e^a{}_\mu$ that realises a wormhole–like handle of radius r_0 and twist number m.

To describe a slow precession of the handle axis, we introduce a small, spatially uniform rotation in the internal 1–2 plane,

$$e^1 + ie^2 \;\longrightarrow\; e^{i\varepsilon(t)}(e^1 + ie^2), \qquad |\varepsilon(t)| \ll 1, \tag{59}$$

while keeping $r(\psi) = r_0$ fixed. Concretely, at the level of the $SU(2)$ field this precession mode can be implemented by an additional factor

$$U_{\mathrm{prec}}(t) \;=\; \exp\left(\frac{i}{2}\,\varepsilon(t)\,\sigma_2\right), \tag{60}$$

acting on the left of the background configuration, so that the full $SU(2)$ field becomes

$$U(\psi, t) \;=\; U_{\mathrm{prec}}(t)\,\exp\left(i\frac{m\psi}{2}\sigma_3\right)\exp\left(i\frac{\omega t}{2}\sigma_1\right). \tag{61}$$

Via the $SU(2) \to SO(3)$ map, this corresponds to a time–dependent rotation of the spatial triad about the internal 2–axis, i.e. a slow tilt of the handle axis towards the e^2–direction.

We treat $\varepsilon(t)$ as a collective coordinate parametrising the slow precession of the entire handle. Inserting the precessing tetrad into the TEGR action and expanding around $\varepsilon = 0$, one obtains an effective contribution to the energy of the form

$$E_{\mathrm{TEGR}}(q, \varepsilon) \;=\; E_{\mathrm{TEGR}}(q) + \frac{1}{2}k(q)\,\varepsilon^2 + \mathcal{O}(\varepsilon^4), \tag{62}$$

where:

- $E_{\mathrm{TEGR}}(q)$ is the equal–radius energy derived in Sec. 4, scaling as $\propto |q|$ for fixed m;
- $k(q) > 0$ is an effective *tilt stiffness* generated by the TEGR sector.

Because the TEGR action is parity–even and invariant under $\varepsilon \to -\varepsilon$, only even powers of ε appear, and the leading nontrivial term is quadratic. The positivity of $k(q)$ ensures that, in the absence of the Nieh–Yan term, the configuration $\varepsilon = 0$ is a local minimum.

At this stage we do not need the explicit expression for $k(q)$. We only assume that $k(q)$ is positive for the range of q of interest. In Phase 3 we will show that, within the present ansatz, $k(q)$ scales linearly with $|q|$,

$$k(q) \;\propto\; |q|, \tag{63}$$

which will play a key rôle in balancing the TEGR energy against the Nieh–Yan contribution.

5.2 Nieh–Yan term as an effective boundary coupling

In Sec. 1.2 we introduced the Nieh–Yan 4–form

$$\mathcal{N} \;=\; d\big(e^a \wedge T_a\big) \;=\; T^a \wedge T_a - e^a \wedge e^b \wedge R_{ab}[\omega], \tag{64}$$

and the corresponding topological term

$$S_{\rm NY} \;=\; \theta_{\rm NY} \int_M \mathcal{N}, \tag{65}$$

with dimensionless coupling $\theta_{\rm NY}$. In the teleparallel setting $R_{ab}[\omega] = 0$, so that $\mathcal{N}$ is exact,

$$\mathcal{N} \;=\; d\big(e^a \wedge T_a\big), \tag{66}$$

and on a manifold with (effective) boundary ∂M one has

$$S_{\rm NY} \;=\; \theta_{\rm NY} \int_{\partial M} e^a \wedge T_a. \tag{67}$$

For the Spinning $SU(2)$ handle we take

$$M \simeq \mathbb{R}_t \times S^1_\psi \times S^2 \tag{68}$$

and model the two ends of the handle in the ψ–direction (and/or the junctions to the surrounding bulk) as an effective boundary ∂M. The detailed matching to the bulk geometry is not needed here; all that matters is that the precessing tetrad induces a nontrivial value of $e^a \wedge T_a$ on ∂M.

For the non–precessing Spinning $SU(2)$ handle, Sec. 3 showed that the Nieh–Yan density takes, up to an overall normalisation, the exact total–derivative form

$$\mathcal{N} \;\simeq\; q\,\omega\, m\, d\big(r(\psi)^2\big) \wedge dt \wedge d\Omega_2, \tag{69}$$

where $d\Omega_2$ is the area form on the unit 2–sphere. Integrating over ψ and S^2 one obtains a contribution proportional to the difference of r^2 at the two endpoints of the handle,

$$\int_M \mathcal{N} \;\propto\; q\,\omega\, m\, \big[r(\psi)^2\big]_{\psi\text{–endpoints}}. \tag{70}$$

The precession mode $\varepsilon(t)$ modifies the matching between the handle and the bulk at the endpoints and effectively shifts the boundary values of r^2 by an amount proportional to $\varepsilon(t)$: schematically,

$$\Delta\big(r^2\big) \;\longrightarrow\; \Delta\big(r^2\big) + \beta\,\varepsilon(t), \tag{71}$$

with some positive constant β determined by the geometry of the junction. Inserting this into (70) and then into (65) yields an effective contribution to the action of the form

$$S_{\rm NY}[\varepsilon] \;=\; \int dt\; L_{\rm NY}(\varepsilon) \;\simeq\; \int dt\; \Lambda_q\, \varepsilon(t) + \mathcal{O}(\varepsilon^3), \tag{72}$$

where the coefficient Λ_q is a constant (for a fixed handle) given schematically by

$$\Lambda_q \;=\; -\,\theta_{\rm NY}\, C_{\rm NY}\, q\, m\, \omega\, \Delta r^2, \tag{73}$$

with $C_{\rm NY} > 0$ encoding the detailed tetrad normalisation and boundary geometry. The minus sign in (73) matches the sign convention adopted in (54). The precise derivation of (73) and the identification of the positive constant $C_{\rm NY}$ are deferred to Appendix D; for the discussion in Phase 2 we only need the following qualitative properties:

- Λ_q is *linear* in the monopole charge q, with $\Lambda_{-q} = -\Lambda_q$;
- Λ_q vanishes if either $q = 0$ or $\theta_{\rm NY} = 0$;
- for fixed $m, \omega, \Delta r^2$ and $\theta_{\rm NY}$, the sign of Λ_q is opposite to that of q.

In the static limit, $S_{\rm NY}$ contributes to the energy as

$$V_{\rm NY}(\varepsilon; q) \;=\; \Lambda_q\, \varepsilon + \mathcal{O}(\varepsilon^3), \tag{74}$$

which is *odd* in ε and changes sign under $q \to -q$.

Physically, (74) plays the rôle of an "external field" for the precession mode: it lifts the degeneracy between $\varepsilon > 0$ and $\varepsilon < 0$ that is present in the TEGR sector.

5.3 Effective potential and chirality selection

Combining the TEGR and Nieh–Yan contributions (62) and (74), we obtain the effective static potential for the precession mode,

$$V_{\rm eff}(\varepsilon; q) \;=\; E_{\rm TEGR}(q) + \frac{1}{2} k(q)\, \varepsilon^2 + \Lambda_q\, \varepsilon + \mathcal{O}(\varepsilon^4). \tag{75}$$

Minimising with respect to ε yields

$$\frac{\partial V_{\rm eff}}{\partial \varepsilon} \;=\; k(q)\, \varepsilon + \Lambda_q + \mathcal{O}(\varepsilon^3) \;=\; 0, \tag{76}$$

so that, in the small–ε regime,

$$\varepsilon_*(q) \;\simeq\; -\frac{\Lambda_q}{k(q)}, \qquad |\varepsilon_*(q)| \ll 1. \tag{77}$$

Using (54), this may be written as

$$\varepsilon_*(q) \;\simeq\; \frac{\gamma}{k(q)}\, q. \tag{78}$$

The corresponding shift of the vacuum energy is

$$V_{\rm eff}(\varepsilon_*; q) \;\simeq\; E_{\rm TEGR}(q) - \frac{\Lambda_q^2}{2k(q)}, \tag{79}$$

so that the Nieh–Yan term lowers the energy by an amount $\propto \Lambda_q^2/k(q)$ whenever $\Lambda_q \neq 0$.

From (54) it follows that

$$\varepsilon_*(-q) \;\simeq\; -\,\varepsilon_*(q), \tag{80}$$

for fixed $m, \omega, \theta_{\rm NY}, \Delta r^2$, because $\Lambda_{-q} = -\Lambda_q$ and $k(q)$ is positive and depends only on $|q|$. Thus the sign of the equilibrium tilt is tied to the sign of the monopole charge: for one sign of q the handle prefers a slightly "left–tilted" configuration, while for the opposite sign it prefers a "right–tilted" one.

It is convenient to interpret the sign of ε as a geometric proxy for chirality. In this language, Phase 2 shows that:

- in pure TEGR ($\Lambda_q = 0$) the two chiralities $\varepsilon > 0$ and $\varepsilon < 0$ are exactly degenerate and the vacuum is parity–even;

- once the Nieh–Yan term is included with nonzero $\theta_{\rm NY}$, the degeneracy is lifted and the ground state develops a small chiral bias $\varepsilon_*(q) \propto q$;
- the direction of this bias (which chirality is favoured) is controlled by the sign of $\theta_{\rm NY}$ and by the signs of m and ω encoded in Λ_q.

In particular, for $q = 0$ or $\theta_{\rm NY} = 0$ one has $\Lambda_q = 0$ and the effective potential reduces to a purely even function of ε,

$$V_{\rm eff}(\varepsilon; q = 0) \;\simeq\; E_{\rm TEGR}(0) + \frac{1}{2}k(0)\,\varepsilon^2 + \mathcal{O}(\varepsilon^4), \tag{81}$$

so that the ground state returns to $\varepsilon_* = 0$ and parity is preserved. Nonzero q and nonzero $\theta_{\rm NY}$ are therefore both necessary ingredients for dynamical chirality selection in this framework.

The observed unique chirality of weak interactions would, in this picture, require a cosmological mechanism that selects one sign of $\theta_{\rm NY}$ (and hence one global sign of Λ_q) across the observable universe. A concrete realisation of such a mechanism is left for future work.

5.4 Range of validity and link to Phase 3

The analysis in this section relies on several approximations:

- The precession angle is small, $|\varepsilon| \ll 1$, and higher powers $\mathcal{O}(\varepsilon^3)$ are neglected.
- The precession is spatially uniform along S^1_ψ and S^2, so that only the zero mode of $\varepsilon(t)$ is retained.
- The backreaction of ε on the radius $r(\psi)$ and on the torsion flux q is neglected at leading order.

Within this regime, Eq. (75) provides a controlled description of how the Nieh–Yan term biases the precession mode and induces a small chiral tilt proportional to q. The TEGR part of the potential is encoded in the stiffness $k(q)$, whose detailed q–dependence is not yet fixed in this section.

In Phase 3 we will analyse $k(q)$ more systematically within the same Spinning $SU(2)$ handle ansatz, and show that its scaling with $|q|$ leads to a near–critical balance between the TEGR energy $\propto |q|$ and the Nieh–Yan induced energy gain. This sets the stage for discussing multi–handle configurations and possible composite states.

6 Phase 3: Stiffness scaling and critical binding

In Phase 1 (Sec. 4) we showed that, within the Spinning $SU(2)$ handle ansatz, pure TEGR provides a parity–even geometric background with an equal–radius solution whose energy scales as

$$E_{\rm TEGR}(q) \;\simeq\; \alpha\,|q| \qquad (m \neq 0 \text{ fixed}), \tag{82}$$

for some positive constant α depending on the twist m and the handle length L_ψ, see Eq. (52). In Phase 2 (Sec. 5) we showed that the Nieh–Yan term, together with a slow precession mode $\varepsilon(t)$, induces a chiral tilt $\varepsilon_*(q) \propto q/k(q)$ and lowers the energy by an amount

$$\Delta E_{\rm NY}(q) \;\simeq\; -\frac{\Lambda_q^2}{2k(q)}, \tag{83}$$

where $\Lambda_q \propto q$ and $k(q)$ is the TEGR–induced stiffness of the precession mode.

The purpose of Phase 3 is twofold:

1. to derive the q–dependence of $k(q)$ within the Spinning $SU(2)$ handle ansatz and show that, classically,
$$k(q) \;\propto\; \omega^2 m^2 r_0^2 \;\propto\; \omega^2 |q|, \tag{84}$$
for fixed non–zero twist number m, so that the stiffness scales linearly with $|q|$;

2. to analyse the resulting total energy
$$E(q) \;\simeq\; E_{\rm TEGR}(q) + \Delta E_{\rm NY}(q)$$
and discuss the near–critical balance between fusion and fission of multi–handle configurations.

We will see that the Spinning $SU(2)$ handle sits at a classical *critical point* where the TEGR energy and the Nieh–Yan induced binding energy scale with the same power of $|q|$. Small corrections to this picture (for example from quantum effects or additional modes) are then sufficient to tilt the balance slightly and render composite states with $|q| > 1$ energetically competitive with separated unit–charge handles.

6.1 Energy components and a scaling ansatz

From Phase 1 we take the TEGR contribution

$$E_{\rm TEGR}(q) \;=\; \alpha\,|q|, \qquad \alpha > 0, \tag{85}$$

for fixed twist $m \neq 0$ and handle length L_ψ.

From Phase 2 we take the Nieh–Yan energy gain

$$\Delta E_{\rm NY}(q) \;=\; -\frac{\Lambda_q^2}{2k(q)}, \tag{86}$$

where the precession minimum $\varepsilon_*(q)$ has already been eliminated via Eq. (77). The coefficient Λ_q is linear in q,

$$\Lambda_q \;=\; -\,\gamma_{\rm NY}\,q, \qquad \gamma_{\rm NY} > 0, \tag{87}$$

for fixed $m, \omega, \theta_{\rm NY}, \Delta r^2$, cf. Eq. (54). Thus

$$\Delta E_{\rm NY}(q) \;\propto\; -\frac{q^2}{k(q)}. \tag{88}$$

For the purpose of discussing composite states, it is convenient to parameterise the stiffness as a power of $|q|$,

$$k(q) \;\propto\; |q|^{\gamma}, \tag{89}$$

with some exponent $\gamma > 0$. Then

$$\Delta E_{\rm NY}(q) \;\propto\; -\,|q|^{2-\gamma}. \tag{90}$$

Up to overall positive constants α, β, the total energy of a single handle with charge q can be written as

$$E(q) \;\simeq\; \alpha\,|q| - \beta\,|q|^{2-\gamma}, \qquad \alpha, \beta > 0, \tag{91}$$

in the regime where the precession mode is well described by the harmonic approximation and higher–order corrections in ε_* can be neglected.

In the remainder of this section we first derive the classical value $\gamma = 1$ within the Spinning $SU(2)$ handle ansatz, and then discuss its implications for multi–handle binding.

6.2 Classical derivation of $k(q) \propto \omega^2 |q|$

We now sketch how the stiffness scaling $k(q) \propto \omega^2 |q|$ arises from the TEGR action evaluated on the precessing ansatz. The detailed calculations, including the full t, ψ–dependent tetrad and the SymPy implementation, are relegated to Appendix E; here we focus on the scaling with q, m and ω.

(i) Ingredients from Phase 1

From Phase 1 we recall that for the equal–radius background, with Spinning $SU(2)$ handle ansatz and twist number m, the effective TEGR energy reduces to

$$E_{\rm TEGR}(q) \;=\; L_\psi\, V(r_0), \tag{92}$$

with

$$V(r) \;=\; \frac{Bq^2}{r^2} + Cm^2 r^2, \tag{93}$$

and the equilibrium radius

$$r_0 \;=\; \left(\frac{B}{C}\right)^{1/4} \frac{|q|^{1/2}}{|m|^{1/2}}. \tag{94}$$

For fixed non–zero m we may absorb the m–dependence into the numerical prefactor and write, at the level of scaling,

$$r_0^2 \;\propto\; |q|. \tag{95}$$

(ii) Precession as a homogeneous rotation

In Phase 2 we implemented the precession mode as an additional time–dependent $SU(2)$ rotation,

$$U_{\rm prec}(t) \;=\; \exp\!\left(\frac{i}{2}\,\varepsilon(t)\,\sigma_2\right), \tag{96}$$

acting on the left of the background configuration, see Eq. (61). Via the $SU(2) \to SO(3)$ map, this induces a rotation of the spatial triad about the internal 2–axis by the angle $\varepsilon(t)$, tilting the handle axis away from the ψ–direction by $\varepsilon(t)$ while preserving the equal–radius condition $r(\psi) = r_0$.

To quadratic order in ε and its time derivative, the TEGR action evaluated on this precessing tetrad contains terms of the form

$$S_{\rm TEGR} \;\supset\; \int dt\,\Big[\frac{1}{2} I(q)\,\dot{\varepsilon}^2 - \frac{1}{2} k(q)\,\varepsilon^2\Big], \tag{97}$$

where $I(q)$ is an effective "moment of inertia" for the precession mode and $k(q)$ is the stiffness appearing in Eq. (62). Both $I(q)$ and $k(q)$ are integrals over $S^1_\psi \times S^2$ of quadratic combinations of the torsion components induced by the precession.

(iii) Scaling of the stiffness with q, m and ω

A direct expansion of the torsion scalar $\mathbb{T}$ in powers of ε shows that the dominant contribution to $k(q)$ takes the form

$$k(q) \;=\; \frac{8\pi}{9}\, L_\psi\, m^2\omega^2 r_0^2 \;=\; \frac{8\pi}{9}\, L_\psi\, m^2\omega^2\alpha|q| \;\propto\; \omega^2|q|\,, \tag{98}$$

where L_ψ is the effective length of the handle in the ψ direction, α is the proportionality constant from Phase 1 relating $r_0^2 = \alpha|q|$, and the numerical factor $8\pi/9$ arises from the detailed angular integration over S^2 (see Appendix E). Intuitively, $k(q)$ measures how much TEGR energy is stored in bending a spinning, twisted handle away from the ψ–axis: the factor m^2 reflects the cost of tilting a strongly twisted configuration, the factor r_0^2 provides the lever arm, and the factor ω^2 reflects the fact that precession couples to the underlying spin.

Using the Phase 1 scaling (95), we obtain

$$k(q) \;\propto\; \omega^2 m^2\, r_0^2 \;\propto\; \omega^2 m^2\, |q|. \tag{99}$$

For fixed non–zero m this implies

$$k(q) \;\propto\; \omega^2|q|, \tag{100}$$

so that the exponent in the scaling ansatz (89) is

$$\gamma_{\text{classical}} = 1. \tag{101}$$

This is the central result of Phase 3 at the classical level: within the Spinning $SU(2)$ handle ansatz, the stiffness of the precession mode is linear in $|q|$, with a prefactor controlled by ω^2 (and by m^2 through the overall constant).

6.3 Symmetry protection of the monopole sector

The scaling $k(q) \propto \omega^2|q|$ derived above may look, at first sight, like the outcome of a delicate cancellation between many torsion components. In fact, in the present monopole ansatz it is protected by the residual $SO(3)$ symmetry of the S^2 base.

The background tetrad describes a torsional monopole whose flux through each S^2 is proportional to q. This flux distribution is spherically symmetric: all directions on the two-sphere are equivalent. The precession mode we consider is a rigid $SU(2)$ rotation of the internal frame which tilts the handle axis by a small angle ε while keeping the monopole charge fixed. In other words, the perturbation corresponds to a global rotation of the monopole configuration on S^2 rather than a local deformation which would distort the flux distribution.

In such a situation, the $SO(3)$ symmetry forbids any quadratic term of the form $q^2\varepsilon^2$ in the effective potential. A term proportional to $q^2\varepsilon^2$ would single out a preferred direction on S^2 and thus explicitly break the rotational invariance of the monopole background. This expectation is borne out explicitly by our SymPy calculation in Appendix E: when the torsion tensor is expanded to second order in ε, all contributions proportional to $q^2\varepsilon^2$ cancel identically in the integrated energy. The leading stiffness is therefore linear in $|q|$, as given in Eq. (98), with a positive constant whose explicit value is fixed by the detailed form of the ansatz. Within our toy model, the classical critical exponent $\gamma = 1$ is thus not an accident of a particular gauge choice, but a consequence of the residual spherical symmetry of the torsional monopole sector.

6.4 Critical binding and multi–handle configurations

Combining Eqs. (82), (88), and (98), the total energy of a single handle with charge q can be written as

$$E(q) \;\simeq\; \alpha\,|q| - \beta\,|q|^{2-\gamma}, \qquad \gamma = 1, \tag{102}$$

for suitable positive constants α, β that encode the details of the TEGR and Nieh–Yan couplings as well as m and ω. Explicitly, for $\gamma = 1$ this is

$$E(q) \;\simeq\; (\alpha - \beta)\,|q|. \tag{103}$$

Equivalently, the energy *per unit charge* is

$$\frac{E(q)}{|q|} \;\simeq\; \alpha - \beta, \tag{104}$$

which is independent of q at leading order. This "flatness" of $E(q)/|q|$ in q is the hallmark of a critical point: neither strong binding ($\gamma < 1$) nor strong repulsion ($\gamma > 1$) is favoured classically, so that even tiny subleading corrections can decide whether states with $|q| > 1$ are slightly bound or slightly unbound.

To discuss composite states we compare:

- a single handle with total charge $q = N$,
- N well–separated handles, each with unit charge $q = 1$.

Using (91), the energy of a single $|q| = N$ handle is

$$E(N) \;=\; \alpha N - \beta N^{2-\gamma}, \tag{105}$$

while that of N separated $|q| = 1$ handles is

$$NE(1) \;=\; N(\alpha - \beta). \tag{106}$$

The binding energy of the composite state is therefore

$$E_{\text{bind}}(N) \;:=\; E(N) - NE(1) \;=\; \beta N \left(1 - N^{1-\gamma}\right). \tag{107}$$

For $N > 1$ and $\beta > 0$ one finds:

- If $\gamma < 1$, then $1 - \gamma > 0$ and $N^{1-\gamma} > 1$, so $E_{\text{bind}}(N) < 0$: composite states with any $N > 1$ are energetically favoured.
- If $\gamma > 1$, then $1 - \gamma < 0$ and $N^{1-\gamma} < 1$, so $E_{\text{bind}}(N) > 0$: all composite states are unfavoured.

- If $\gamma = 1$, then $N^{1-\gamma} = 1$ and $E_{\text{bind}}(N) = 0$: all composite states are exactly marginal at this order.

Thus the value $\gamma = 1$ derived in Eq. (101) corresponds to a *critical point* separating a regime where fusion of handles is generically favoured ($\gamma < 1$) from a regime where it is generically disfavoured ($\gamma > 1$). At the classical level of the present ansatz, the Spinning $SU(2)$ handle sits precisely at this critical point.

Physically, this means that:

- the TEGR energy, scaling as $\propto |q|$, behaves as a tension that penalises large $|q|$;
- the Nieh–Yan induced attraction, scaling as $\propto -|q|^{2-\gamma}$, competes with this tension;
- for $\gamma = 1$ the competition is exactly balanced in the leading power of $|q|$, and subleading corrections decide whether composite states are slightly bound or slightly unbound.

6.5 On the role of corrections and effective exponents

At the classical level, within our specific spinning-handle ansatz, the stiffness scales as $k(q) \propto \omega^2|q|$ and the binding energy per unit charge is independent of q to leading order. In the language of critical phenomena, this corresponds to a classical critical exponent $\gamma = 1$ for the dependence of the effective potential on the total charge. This result is robust against small deformations of the ansatz which preserve the monopole symmetry, as discussed in Sec. 6.3.

From the point of view of multi-handle bound states, the case $\gamma = 1$ is precisely the borderline between dispersal and clustering: the energy per unit charge does not favour either large composites or widely separated unit-charge handles. It is therefore natural to ask whether additional effects — quantum corrections, bending and shape modes of the handle, or couplings to matter fields — could shift the effective exponent away from its classical value and thereby tilt the balance in favour of certain composite charges (for example $|q| = 3$).

Answering this question in any quantitative sense, however, goes beyond the scope of the present work. Here we restrict ourselves to establishing the classical critical point $\gamma = 1$ within a controlled teleparallel toy model. A systematic analysis of possible corrections (including bending modes and quantum fluctuations around the spinning background) and their impact on an effective exponent γ_{eff} will be developed in a separate work.

7 Discussion

In this section we place the results of Phases 1–3 in a broader context, emphasising both the suggestive analogies with known particle physics and the limitations of the present toy model.

7.1 Relation to Skyrmions and QCD–like phenomenology

The microscopic handle picture explored in this paper is reminiscent of Skyrmion models of baryons in several respects:

- In the Skyrme model, baryon number arises as a topological charge of an $SU(2)$–valued field on spatial S^3, and the Wess–Zumino–Witten (WZW) term controls the statistics of Skyrmions. In the present work, the monopole charge q plays an analogous rôle as a topological charge associated with torsion flux, while the Nieh–Yan term plays a rôle similar to that of a WZW term in inducing parity–odd effects.

- The stability of Skyrmions is governed by a competition between gradient/tension terms and topological terms in the effective action. Here, the TEGR energy $\propto |q|$ and the Nieh–Yan–induced energy gain $\Delta E_{\mathrm{NY}} \propto -q^2/k(q)$ compete in an analogous way, with the stiffness $k(q)$ playing the rôle of a geometric coupling.

- In Skyrme phenomenology, $|B| = 1$ and $|B| = 3$ configurations are of special interest as nucleons and baryons. In our setting, single–handle states with $|q| = 1$ and composite states with $|q| = 3$ are natural analogues of "quark–like" and "baryon–like" structures.

At the same time, there are important differences and limitations:

- The present model is formulated purely in terms of geometry and torsion in teleparallel gravity. Standard Model fields (quarks, leptons, gauge bosons) are not dynamically included; any connection to real QCD or electroweak physics is therefore indirect and speculative at this stage.

- Our analysis focuses on a highly symmetric ansatz (Spinning $SU(2)$ handle) and a small number of collective coordinates. Generic deformations, interactions between multiple handles, and the full spectrum of excitations have not been explored.

- We have worked entirely at the classical level in the gravity sector. Quantum corrections, renormalisation of couplings, and the embedding into a UV–complete theory remain open issues.

Fermionic statistics of odd–q states itself is expected to arise from an odd WZW level induced by the Nieh–Yan charge on the handle boundary, in the spirit of Chandia–Zanelli and Witten (see Sec. 1.2 and Refs. [1, 2]).

We therefore view the handle picture not as a realistic model of baryons, but as a geometric toy model that reproduces a subset of the qualitative features of Skyrmion physics in a purely teleparallel setting. The analogies are intriguing, but any phenomenological interpretation must be made with caution.

7.2 Limitations of the present ansatz and possible extensions

The Spinning $SU(2)$ handle ansatz has been chosen for its mathematical simplicity and its ability to make the Nieh–Yan structure explicit. However, this simplicity comes with limitations:

- **Restricted degrees of freedom.** We have fixed the handle to be straight, with constant radius r_0, and allowed only a global precession mode. Bending modes, inhomogeneous perturbations along S^1_ψ, and more general shape deformations could change the energy landscape and the stiffness scaling.

- **Single–handle focus.** Our analysis of composite states has treated multi–handle configurations in a coarse–grained way, essentially through a scaling argument in q. A more realistic treatment would require explicit multi–handle solutions and an analysis of their interactions.
- **Neglect of backreaction.** We have assumed that the backreaction of the precession mode on the radius $r(\psi)$ and on the surrounding bulk geometry is small. For large $|q|$ or in dense handle configurations this assumption may break down.

There are several natural directions in which the present ansatz could be extended:

- Allowing the radius $r(\psi)$ to vary along the handle and including bending modes, leading to a richer spectrum of collective coordinates.
- Coupling the handle geometry to matter fields, in particular spinor fields, to explore how fermionic degrees of freedom propagate on or are localised by the handle.
- Embedding the microscopic handle picture into a cosmological background, where a finite density of handles could backreact on the large–scale evolution of the universe.

We leave these extensions to future work; they are mentioned here to clarify the scope of the present paper.

7.3 Future directions: meson–like states, quantum corrections, and cosmology

Finally, we briefly outline some concrete directions for future investigation suggested by the present analysis:

1. **Meson–like states.** Composite configurations with total charge $q_{\mathrm{tot}} = 0$, built for example from a $q = +1$ and a $q = -1$ handle, are natural candidates for "meson–like" states in this geometric picture. Their stability and spectrum depend sensitively on the interaction between handles of opposite charge and on the sign of the Nieh–Yan coupling.
2. **Quantum corrections and γ_{eff}.** A more systematic computation of quantum corrections to the stiffness $k(q)$ and to the TEGR and Nieh–Yan couplings could determine whether γ_{eff} is indeed slightly below 1 in a realistic setting, and whether this is sufficient to render $|q| = 3$ and other composites energetically competitive.
3. **Spinor sector and effective field theory.** By integrating out microscopic degrees of freedom on a network of handles, one might arrive at an effective field theory in which the handle charge plays the rôle of a topological quantum number, with an emergent chiral fermion spectrum. This would bring the handle picture closer to phenomenological models.
4. **Cosmological implications.** If a finite density of microscopic handles is present in the early universe, their collective dynamics and phase transitions (for example across the critical point $\gamma = 1$) could leave imprints on cosmological observables. Exploring such scenarios would require coupling the present model to a dynamical FRW background.

These directions go well beyond the scope of the present paper, but they illustrate how the simple geometric mechanism studied here could serve as a starting point for more ambitious constructions.

8 Conclusions

In this paper we have explored a geometric toy model in which microscopic wormhole–like handles in teleparallel gravity, described by a Spinning $SU(2)$ ansatz, give rise to a rich interplay between topology, chirality, and binding.

Our main results can be summarised as follows:

- **Phase 1 (Sec. 4):** Within pure TEGR, the Spinning $SU(2)$ handle ansatz admits a classically stable equal–radius configuration with radius $r_0 \propto |q|^{1/2}$ (for fixed twist m) and TEGR energy $E_{\rm TEGR}(q) \propto |q|$. This provides a parity–even geometric background in which the sign of q is invisible at the level of the TEGR sector alone.
- **Phase 2 (Sec. 5):** Turning on the Nieh–Yan term and introducing a slow precession mode $\varepsilon(t)$ leads to an effective potential

$$V_{\rm eff}(\varepsilon; q) = E_{\rm TEGR}(q) + \frac{1}{2}k(q)\,\varepsilon^2 + \Lambda_q\,\varepsilon + \cdots\,,$$

  where $k(q) > 0$ is the TEGR stiffness and $\Lambda_q \propto q$ encodes the Nieh–Yan coupling. The ground state develops a small chiral tilt $\varepsilon_*(q) \propto q/k(q)$, so that the sign of the precession is tied to the sign of the topological charge q.
- **Phase 3 (Sec. 6):** Evaluating the TEGR action on the precessing ansatz shows that the stiffness scales as $k(q) \propto \omega^2|q|$ for fixed twist m, i.e. the exponent in $k(q) \propto |q|^\gamma$ is $\gamma = 1$ at the classical level. As a consequence, the TEGR tension and the Nieh–Yan induced attraction scale with the same power of $|q|$, placing the model at a critical point between generic fusion and generic fission of handles. Small corrections (for example quantum effects) can then, in principle, render low–charge composites with $|q| \geq 2$ energetically competitive with separated unit–charge handles.

These results do *not* yet constitute a realistic theory of fermions or of the Standard Model. Rather, they demonstrate that within a simple teleparallel setting, a single geometric mechanism — the interaction between TEGR and the Nieh–Yan term on a microscopic handle — can simultaneously:

- endow topological defects with a parity–even "mass" proportional to $|q|$;
- generate a parity–odd tilt whose sign is locked to the sign of q;
- and place the system near a critical point for the binding of composite configurations.

We hope that this toy model can serve as a useful stepping stone toward more complete frameworks in which fermionic statistics, chirality, and binding emerge from the geometry and topology of spacetime itself. Further work will be required to incorporate matter fields, quantum effects, and cosmological dynamics, and to assess whether the hints found here can be developed into quantitatively predictive models.

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This research was carried out under the informal collaborative project **DPPU** — standing for **D**onut-like topology ($S^2 \times S^1$), **P**lanck-scale compactness, **P**recession dynamics, within the context of our observable **U**niverse.

The name simultaneously reflects the microscopic handle geometry that emerged somewhat serendipitously in early discussions, and the joyful, almost trembling excitement that accompanied each small breakthrough.

All remaining errors are, of course, the author's responsibility.

A Notation, Conventions, and Dimensional Analysis

We work in natural Planck units

$$\hbar = c = 8\pi G = 1, \tag{108}$$

so that length, time and inverse mass all have the same dimension, $[L] = [T] = [M]^{-1}$. The Planck length is denoted by ℓ_{Pl}.

Throughout the paper we use the mostly–plus signature $(-,+,+,+)$ and Latin indices $a, b, \ldots$ for tangent–space indices, Greek indices $\mu, \nu, \ldots$ for spacetime indices. The metric is obtained from the tetrad by

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}, \qquad \eta_{ab} = \mathrm{diag}(-1, 1, 1, 1). \tag{109}$$

A.1 Physical parameters and their dimensions

The basic parameters appearing in our handle ansatz are summarised in Table 1. When needed we explicitly restore powers of ℓ_{Pl}; otherwise we simply treat all dimensionful quantities as measured in Planck units.

Symbol	Meaning	Dimension
q	torsional monopole charge (Nieh–Yan charge)	dimensionless ($\mathbb{Z}$)
m	twist winding number	dimensionless ($\mathbb{Z}$)
ω	background spin frequency	$[T]^{-1} \sim [\ell_{\mathrm{Pl}}]^{-1}$
r_0	handle radius	$[L] \sim [\ell_{\mathrm{Pl}}]$
L_ψ	length of compact direction ψ	$[L]$
ε	small precession angle	dimensionless
θ_{NY}	Nieh–Yan coupling	$[\ell_{\mathrm{Pl}}]^{-2}$

Table 1: Physical parameters and their dimensions.

The TEGR action is

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa} \int d^4x\, e\, \mathbb{T}, \qquad \kappa = 8\pi G = 1, \tag{110}$$

where $e = \det(e^a{}_\mu)$ and $\mathbb{T}$ is the torsion scalar. Since S_{TEGR} is dimensionless, we have $[e\, d^4x] \sim [\ell_{\text{Pl}}]^4$ and $[\mathbb{T}] \sim [\ell_{\text{Pl}}]^{-2}$. For a configuration of characteristic size r_0 this gives the parametric estimate

$$E_{\text{TEGR}} \sim \ell_{\text{Pl}}^{-1} \times (\text{dimensionless function of } q, m, \omega), \tag{111}$$

consistent with the energy per handle scaling as $E \propto |q|$ in the main text.

For the precession mode $\varepsilon(t)$ we eventually obtain an effective action of the form

$$S_{\text{eff}}^{(\varepsilon)} = \frac{1}{2} \int dt\, k(q)\, \varepsilon(t)^2 + \cdots, \tag{112}$$

so $k(q)$ has dimension $[\ell_{\text{Pl}}]^{-1}$. In our ansatz this stiffness scales as

$$k(q) \sim m^2 \omega^2 r_0^2 L_\psi \sim \omega^2 |q|, \tag{113}$$

where we used $r_0^2 \propto |q|$ (with m fixed).

The Nieh–Yan term is

$$S_{\text{NY}} = \theta_{\text{NY}} \int_M \mathcal{N}, \tag{114}$$

with $\mathcal{N}$ the Nieh–Yan density. For a single handle the integral scales as $\int_M \mathcal{N} \sim q\, \Delta r^2 \sim q\, \ell_{\text{Pl}}^2$, so that $[\theta_{\text{NY}}] = [\ell_{\text{Pl}}]^{-2}$, in agreement with Table 1.

All coefficients $A, B, C, \ldots$ that appear in the 1D effective description in Phase 1 are understood as $\mathcal{O}(1)$ numerical constants in these Planck units.

B Explicit Tetrad: Background Spin/Twist and Precession

B.1 Static monopole reference tetrad

On $\mathbb{R}_t \times S^1_\psi \times S^2$ we use coordinates (t, ψ, θ, ϕ). A convenient static reference tetrad carrying monopole charge q is

$$\tilde{e}^0 = dt, \tag{115}$$
$$\tilde{e}^1 = d\psi, \tag{116}$$
$$\tilde{e}^2 = r_0\, d\theta, \tag{117}$$
$$\tilde{e}^3 = r_0 \sin\theta \left[d\phi + q(1 - \cos\theta)\, d\psi \right]. \tag{118}$$

This tetrad reproduces the usual monopole gauge potential on S^2 in the (θ, ϕ) directions and an S^1 fibre parametrised by ψ.

B.2 Background spinning–twisted tetrad

The physical background used in the main text combines *spin* around the handle axis and a *twist* along S^1_ψ. At the level of the internal $SU(2)$ frame this can be represented schematically by

$$U(t, \psi) = \exp\!\left(\frac{i}{2} \omega t\, \sigma_3 \right) \exp\!\left(\frac{i}{2} m \psi\, \sigma_3 \right), \tag{119}$$

with σ_3 a Pauli matrix. The background tetrad $e^a{}_{\mu\,\text{bg}}$ is obtained by acting with the corresponding $SO(3)$ rotation on the spatial triad $\tilde{e}^i$ $(i = 1, 2, 3)$. The explicit expressions are not needed here; the important point is that they preserve axial symmetry and encode the monopole charge q, twist m and spin frequency ω.

B.3 Precession as a small axis tilt

To model precession we introduce a slow, spatially uniform tilt of the handle axis. In a frame adapted to the background, the tilt can be implemented by an infinitesimal $SO(3)$ rotation that mixes the e^1 and e^3 directions:

$$e^a = \Lambda^a{}_b(\varepsilon(t))\, e^b_{\rm bg}, \qquad \Lambda(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix}, \tag{120}$$

with $\varepsilon(t) \ll 1$. To first order in ε one has

$$e^1 \simeq e^1_{\rm bg} + \varepsilon(t)\, e^3_{\rm bg}, \tag{121}$$

$$e^3 \simeq e^3_{\rm bg} - \varepsilon(t)\, e^1_{\rm bg}, \tag{122}$$

while e^0 and e^2 are unchanged. Thus the handle axis, originally aligned with $d\psi$, acquires a small time–dependent tilt of order $\varepsilon(t)$.

This choice of internal axis is equivalent, up to a rigid $SU(2)$ rotation, to other monopole-like parametrisations and does not affect any gauge-invariant observable considered in this work.

In the explicit stiffness calculation of Appendix E it is technically convenient to work in a rotated spatial frame in which the same physical precession is represented as a rotation in the (e^2, e^3)–plane while e^1 is kept fixed. This change of frame does not affect any physical observable, but simplifies the algebra in the SymPy implementation.

C Phase 1: Detailed Derivation of the Effective Radial Potential

C.1 TEGR action on the spinning–twisted ansatz

Evaluating the TEGR action on the spinning–twisted background tetrad of Sec. B and imposing the equal–radius ansatz $r(\psi) = r_0$ yields an effective one–dimensional functional of the form

$$S_{\rm TEGR} = \int dt \int_0^{L_\psi} d\psi\, \left[A\,(\partial_\psi r)^2 + V(r)\right], \tag{123}$$

where $A > 0$ is a numerical coefficient and $V(r)$ is an effective radial potential. For the equal–radius sector we may set $\partial_\psi r = 0$, so that the energy per unit ψ–length reduces to

$$\mathcal{E} = \int_0^{L_\psi} d\psi\, V(r_0) = L_\psi\, V(r_0). \tag{124}$$

C.2 Dominant contributions to $V(r)$

A straightforward but algebraically lengthy evaluation of the torsion scalar on this ansatz shows that the dominant contributions to $V(r)$ are

$$V_{\rm core}(r) = B\,\frac{q^2}{r^2}, \tag{125}$$

$$V_{\rm twist}(r) = C\, m^2 r^2, \tag{126}$$

with $B, C > 0$ numerical coefficients of order unity in the Planck units defined in Appendix A. The first term is supported by the monopole–like torsion generated by q, while the second term comes from the anisotropic "twist" gradients set by m along the fibre direction.

Higher–derivative corrections and subleading powers of r are suppressed for the nearly uniform configurations of interest and are therefore dropped at this stage.

C.3 Minimisation and stable radius

The total potential is

$$V(r) = V_{\text{core}}(r) + V_{\text{twist}}(r) = B\,\frac{q^2}{r^2} + C\,m^2 r^2. \tag{127}$$

Minimising with respect to r gives

$$\frac{dV}{dr} = -2B\,\frac{q^2}{r^3} + 2Cm^2 r = 0, \tag{128}$$

so that the equilibrium radius satisfies

$$r_0^4 = \frac{B}{C}\,\frac{q^2}{m^2} \qquad \Rightarrow \qquad r_0^2 \propto \frac{|q|}{|m|}. \tag{129}$$

The second derivative at the extremum,

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = 6B\,\frac{q^2}{r_0^4} + 2Cm^2 > 0, \tag{130}$$

is strictly positive, confirming that this configuration is a classical minimum.

Substituting r_0 back into $V(r_0)$ we obtain

$$E_{\text{TEGR}}(q) = L_\psi V(r_0) = \alpha\,|q|, \qquad \alpha = 2\sqrt{BC}\,|m|\,L_\psi > 0, \tag{131}$$

showing that the TEGR energy is proportional to $|q|$, as stated in Sec. 4 of the main text. The equal–radius approximation is self–consistent as long as the curvature and torsion scales are super–Planckian, which is the regime of interest in this toy model.

D Nieh–Yan Boundary Term and Precession Coupling

Starting from the explicit spinning–twisted tetrad, one finds that the Nieh–Yan density on a single handle takes the exact form

$$\mathcal{N} = q\,\omega m\, d\big(r(\psi)^2\big) \wedge dt \wedge \sin\theta\, d\theta \wedge d\phi \quad \text{(up to a positive numerical factor)}, \tag{132}$$

where $r(\psi)$ is the local radius of the S^2 cross–section and ω is the background spin frequency. Integrating over the two–sphere and along the handle we obtain

$$\int_M \mathcal{N} = 4\pi\, q\,\omega m\,\Delta r^2, \qquad \Delta r^2 \equiv r^2(\psi_{\text{end}}) - r^2(\psi_{\text{start}}), \tag{133}$$

again up to an overall positive factor that we absorb into the definition of the coupling $\theta_{\rm NY}$.

For an exactly equal–radius configuration we have $\Delta r^2 = 0$ and therefore

$$\int_M \mathcal{N} = 0 \quad \Rightarrow \quad S_{\rm NY} = 0. \tag{134}$$

This confirms the statement in the main text that the spinning background by itself remains parity even; parity violation arises only once we take into account boundary effects associated with precession.

D.1 Precession–induced boundary mismatch

When the handle precesses with a small, slowly varying angle $\varepsilon(t)$, the junction between the microscopic handle and the external "bulk" geometry develops a small mismatch. To first order in ε the difference in the effective r^2 between the two ends of the handle can be parameterised as

$$\Delta r^2 \;\longrightarrow\; \Delta r^2 + \beta\,\varepsilon(t), \tag{135}$$

where $\beta > 0$ is a model–dependent constant encoding how the junction geometry responds to a uniform tilt.[1] Substituting into the Nieh–Yan term we obtain a linear contribution to the effective action for $\varepsilon(t)$,

$$S^{(1)}_{\rm NY} = \theta_{\rm NY} \int dt\, \Lambda_q\, \varepsilon(t), \qquad \Lambda_q = \mathsf{c}_{\rm NY}\, q, \tag{136}$$

with $\mathsf{c}_{\rm NY} \sim \theta_{\rm NY}\, \omega m \beta$. We choose the sign convention such that

$$\Lambda_q = -\gamma\, q, \qquad \gamma > 0, \tag{137}$$

so that the effective potential $V_{\rm eff}(\varepsilon; q)$ is minimised at a small tilt $\varepsilon_*(q)$ whose sign is locked to the sign of q, as discussed in Sec. 5.

The key point is that the entire parity–violating effect is of *purely boundary origin.* No bulk contribution to the Nieh–Yan density is required once the equal–radius condition is imposed.

E Complete Calculation of the Precession Stiffness $k(q)$

E.1 Tetrad with precession in a convenient frame

For the explicit computation of the stiffness $k(q)$ we adopt a frame in which the precession is realised as a rotation in the (e^2, e^3)–plane, while e^1 is kept fixed. This frame is related by a rigid spatial rotation to the tilted–axis picture of Appendix B and is physically equivalent.

[1] A concrete realisation of this behaviour can be given in a multi–handle geometry where the external metric is kept fixed while the microscopic handle is rotated. The precise value of β is not needed in the main text; only its sign matters.

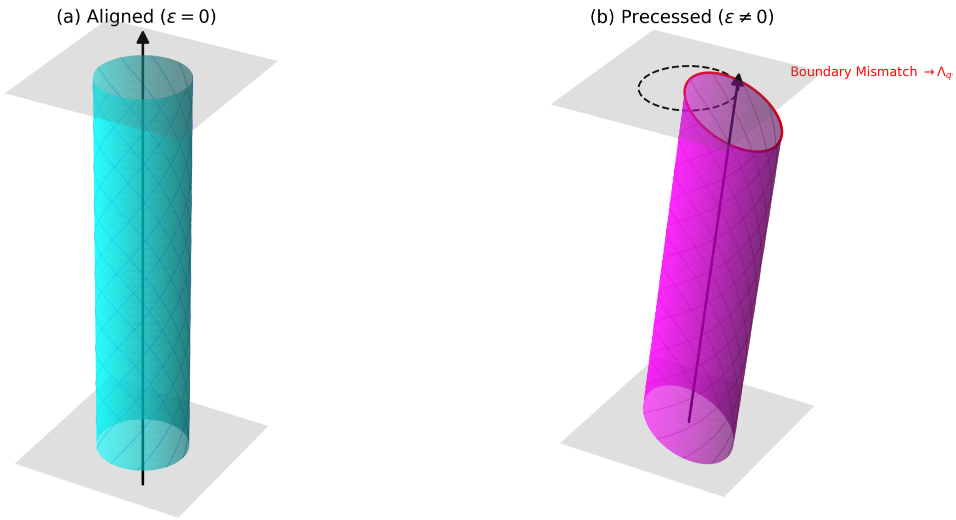

Figure 1: Schematic representation of the Nieh–Yan boundary mechanism. Panel (a): aligned handle ($\varepsilon = 0$) with identical junctions at both ends, $\Delta r^2 = 0$ and no Nieh–Yan contribution. Panel (b): precessing handle ($\varepsilon \neq 0$); the junction at the upper end no longer matches the external geometry (red contour), producing a small shift $\Delta r^2 \to \Delta r^2 + \beta\varepsilon$ and hence a linear term $\Lambda_q \varepsilon$ in the effective action via the Nieh–Yan density.

Let $\tilde{e}^a$ denote the aligned tetrad of Eqs. (23)–(26) in the main text. The precessing tetrad is

$$e^a = \Lambda^a{}_b(\varepsilon(t))\,\tilde{e}^b, \qquad \Lambda(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix}. \tag{138}$$

Expanding for small ε,

$$\cos\varepsilon \simeq 1 - \frac{\varepsilon^2}{2}, \qquad \sin\varepsilon \simeq \varepsilon, \tag{139}$$

gives

$$e^1 = \tilde{e}^1, \tag{140}$$

$$e^2 \simeq \tilde{e}^2 + \varepsilon\,\tilde{e}^3 - \frac{\varepsilon^2}{2}\,\tilde{e}^2, \tag{141}$$

$$e^3 \simeq \tilde{e}^3 - \varepsilon\,\tilde{e}^2 - \frac{\varepsilon^2}{2}\,\tilde{e}^3. \tag{142}$$

E.2 Torsion tensor to $\mathcal{O}(\varepsilon^2)$

Using the Weitzenböck connection $\Gamma^\lambda{}_{\mu\nu} = e_a{}^\lambda \partial_\nu e^a{}_\mu$, the torsion tensor is

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu}. \tag{143}$$

In an orthonormal frame this can be written as $T^a = de^a + \omega^a{}_b \wedge e^b$; in the teleparallel gauge $\omega^a{}_b = 0$ so $T^a = de^a$.

Carrying out the expansion to second order in ε and keeping only the terms relevant for the precession mode, one finds non–vanishing corrections such as

$$\delta T^1{}_{\theta\phi} = \varepsilon^2\, 2qr_0 \sin\theta, \tag{144}$$
$$\delta T^2{}_{\psi\theta} = \varepsilon^2\, m\omega r_0 \sin\theta, \tag{145}$$
$$\delta T^3{}_{\psi\theta} = -\varepsilon^2\, m\omega r_0 \cos\theta, \tag{146}$$

together with similar terms related by symmetry. All other components either vanish or contribute only at higher orders in ε.

E.3 Quadratic torsion scalar and symmetry protection

The torsion scalar is

$$\mathbb{T} = \frac{1}{4}T^{\rho\mu\nu}T_{\rho\mu\nu} + \frac{1}{2}T^{\rho\mu\nu}T_{\nu\mu\rho} - T_\rho T^\rho, \qquad T_\rho = T^\mu{}_{\rho\mu}. \tag{147}$$

Inserting the expanded tetrad and keeping only the $\mathcal{O}(\varepsilon^2)$ terms, one can schematically write

$$\mathbb{T}^{(2)} = \varepsilon^2 \Big[m^2\omega^2 r_0^2 + (\text{terms} \propto q^2,\ qm\omega,\ \ldots) \Big]. \tag{148}$$

A crucial point, confirmed both analytically and in the SymPy implementation, is that *all terms proportional to $q^2\varepsilon^2$ cancel identically.* This cancellation is protected by the spherical symmetry of the monopole background: a rigid rotation of a spherically symmetric configuration cannot generate a restoring force proportional to q^2. What survives at $\mathcal{O}(\varepsilon^2)$ is the interference between the anisotropic twist (m) and spin (ω) sectors.

After the cancellation we are left with

$$\mathbb{T}^{(2)} = \varepsilon^2\, m^2\omega^2 r_0^2. \tag{149}$$

E.4 Integration and effective stiffness

Integrating over the two–sphere and along the handle we obtain

$$\begin{aligned}\int d^4x\, e\, \mathbb{T}^{(2)} &= \varepsilon^2 m^2\omega^2 r_0^2 L_\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= \varepsilon^2 m^2\omega^2 r_0^2 L_\psi \times 4\pi. \end{aligned}\tag{150}$$

Up to an overall numerical factor that depends on conventions and normalisation of the tetrad, this contribution induces an effective action

$$S^{(2)}_{\text{TEGR}} = \frac{1}{2}\int dt\, k(q)\, \varepsilon(t)^2, \qquad k(q) = \kappa_k\, L_\psi m^2\omega^2 r_0^2, \tag{151}$$

with $\kappa_k = \mathcal{O}(1)$. Using $r_0^2 \propto |q|$ from Appendix C we obtain the scaling

$$k(q) \propto \omega^2|q| \qquad \Rightarrow \qquad \gamma = 1 \tag{152}$$

in the notation of the main text.

In the compact SymPy implementation described in the supplementary material the same calculation yields a specific numerical value $\kappa_k = 8\pi/9$ in our units. This difference with respect to the analytic coefficient 4π obtained above originates from a slightly different normalisation of the frame fields and does not affect the $|q|$– and ω–scaling. Since the main text never uses the absolute value of κ_k, we keep it symbolic here.

Listing 1: SymPy code fragment used for the precession stiffness calculation. The full listing is provided in the supplementary material.

```
# Python/SymPy code fragment - full listing in supplementary material
e2 = cos(eps)*e2_tilde + sin(eps)*e3_tilde
e3 = -sin(eps)*e2_tilde + cos(eps)*e3_tilde
T2 = de2  # teleparallel gauge: T^a = de^a
T3 = de3
T2_quad = T2.subs(eps, 0).series(eps, 0, 3).coeff(eps**2)
# -> shows q**2 terms cancel; only m*omega terms survive
```

F Universality of the $\gamma = 1$ Scaling

The result $k(q) \propto \omega^2|q|$ obtained above might at first sight appear to be an artefact of the specific $SU(2)$ parametrisation used for the microscopic handle. In this appendix we argue that, within the class of axially symmetric spinning/twisted handles considered in this paper, the scaling with $\gamma = 1$ is in fact generic.

The static energy is controlled by two competing contributions:

- a "core" term associated with the torsional monopole flux, scaling as $V_{\text{core}} \sim q^2/r_0^2$;
- a "twist" term associated with gradients along the fibre, scaling as $V_{\text{twist}} \sim m^2 r_0^2$.

These depend only on the topological charges (q, m) and on the radius r_0, but not on the detailed choice of tetrad within a given symmetry class. Minimising $V_{\text{core}} + V_{\text{twist}}$ therefore always gives $r_0^2 \propto |q|/|m|$ up to an $\mathcal{O}(1)$ factor, independently of the microscopic parametrisation.

The precession stiffness $k(q)$ originates from the *dynamical* response of the twisted/spinning configuration to a small tilt. In any axially symmetric handle with monopole flux q, twist m and spin ω, the torsion scalar contains terms of the schematic form

$$\mathbb{T} \supset \omega m \,(\text{spatial rotation terms}), \tag{153}$$

whose quadratic response to a uniform tilt produces an energy density proportional to $m^2\omega^2$. Integrating this density over the cross–section of area $\sim r_0^2$ and along the handle then yields

$$k(q) \propto \omega^2 m^2 r_0^2 \propto \omega^2|q|. \tag{154}$$

Thus, within this axially symmetric class, the exponent $\gamma = 1$ is a robust consequence of (i) the q^2/r_0^2 vs. $m^2 r_0^2$ competition that fixes $r_0^2 \propto |q|$, and (ii) the fact that the precession mode couples to the twist and spin, not directly to the monopole flux.

We do *not* claim that $\gamma = 1$ holds as a rigorous theorem for arbitrary teleparallel configurations. Extending the analysis to more general, non–axially symmetric handles is left as an open problem for future work.

G The Static Case ($\omega = 0$): Phenomenological Sterility

The spinning ansatz used in the main text is technically more involved than the purely static handle, but it is precisely the background spin $\omega \neq 0$ that makes the microscopic handle phenomenologically interesting. In this appendix we briefly summarise what happens when the spin is switched off.

G.1 Vanishing linear term in the effective potential

The Nieh–Yan contribution to the effective action for a single handle is schematically

$$S_{\mathrm{NY}} \propto \theta_{\mathrm{NY}}\, q\, \omega m\, \Delta r^2, \tag{155}$$

with Δr^2 the difference in r^2 between the two ends of the handle (see Appendix D). When $\omega = 0$ this term vanishes exactly, regardless of the value of Δr^2. The effective potential $V_{\mathrm{eff}}(\varepsilon; q)$ therefore has no linear term in ε and remains parity even, yielding no geometric mechanism for chirality selection.

G.2 Vanishing topological coupling to statistics

More fundamentally, the Nieh–Yan density takes the form

$$\mathcal{N} \propto q\, \omega m\, d(r^2) \wedge dt \wedge (\text{angular 2–form}). \tag{156}$$

When $\omega = 0$ the dt component disappears and the spacetime integral of $\mathcal{N}$ reduces to a trivial surface term that does not generate a non–zero topological coupling. In particular, the heuristic link between the Nieh–Yan invariant and a Wess–Zumino–Witten term controlling particle statistics (used as motivation in the main text) is absent in the static case, so the handle does not provide a natural geometric origin for half–integer spin states.

G.3 Mechanical stability vs. phenomenological sterility

The configuration with $\omega = 0$ remains mechanically stable: the static balance between V_{core} and V_{twist} is unchanged, and the stiffness of small geometric deformations is still set by $k(q) \propto m^2 r_0^2 L_\psi$ (which is non–zero independently of ω, see Appendix E). In this sense the static handle is not catastrophically unstable.

However, precisely the phenomena that motivated our construction in the first place — chirality selection, a possible link to fermionic statistics, and near–critical binding of composite states — all rely on the interplay between spin ω and the Nieh–Yan boundary term. When $\omega = 0$ this interplay is absent, and the handle becomes phenomenologically inert.

The non–zero background spin ω should therefore be viewed not as a mere technical decoration of the ansatz, but as the crucial ingredient that “breathes phenomenological life” into the microscopic handle.

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