NUMERICAL INTEGRATION

TRAPEZOIDAL RULE
SIMPSON 1/3 RULE
SIMPSON 3/8 RULE

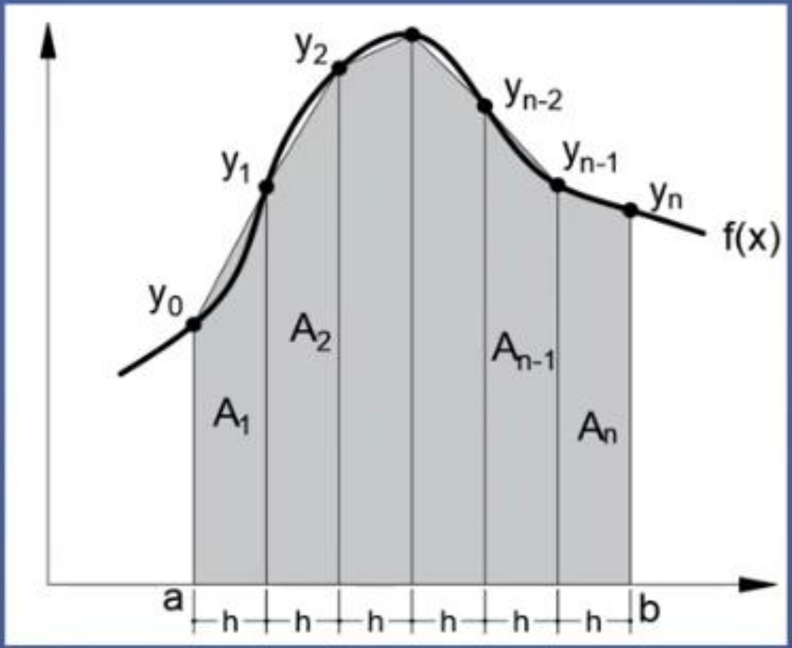
CPS 315

COMPUTATIONAL SCIENCE AND NUMERICAL METHODS

TRAPEZOIDAL RULE

4.3: TRAPEZOIDAL RULE

The trapezoidal rule is considered to be the simplest method of numerical integration. In this method, the value of $\int_a^b f(x)dx$ is approximated by adding the areas of series of trapezoid, as shown below.

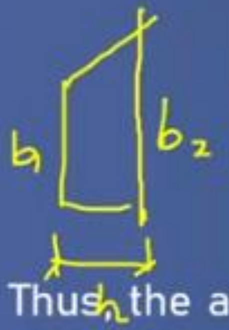


4.3: TRAPEZOIDAL RULE

By definition, the integral can be approximated as:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i$$

From plane geometry, the area of a single trapezoid is equal to:



$$A = \frac{1}{2}h(b_1 + b_2)$$

where h is its altitude and b_1 and b_2 are the measurement of the bases. Thus, the area of each individual trapezoid in the figure above is:

$$A_1 = \frac{1}{2}h(y_0 + y_1)$$
$$A_2 = \frac{1}{2}h(y_1 + y_2)$$
$$\vdots$$
$$A_{n-1} = \frac{1}{2}h(y_{n-2} + y_{n-1})$$
$$A_n = \frac{1}{2}h(y_{n-1} + y_n)$$

4.3: TRAPEZOIDAL RULE

Taking the sum of these areas:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} A_i = \frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \ \ldots + \frac{1}{2}h(y_{n-2} + y_{n-1}) + \frac{1}{2}h(y_{n-1} + y_n)$$

$$\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$$

$$\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + 2(y_1 + y_2 + \cdots + y_{n-1}) + y_n]$$

Let $y_1 + y_2 + \cdots + y_{n-1} = \sum_{i=1}^{n-1} y_i$. Thus, the formula for the trapezoidal rule is:

$$\int_a^b \boldsymbol{f(x)dx} \approx \frac{\boldsymbol{h}}{\boldsymbol{2}}\left[\boldsymbol{y_0} + \boldsymbol{2}\sum_{i=1}^{n-1} \boldsymbol{y_i} + \boldsymbol{y_n}\right]$$

where $\boldsymbol{h} = \frac{b-a}{n}$

4.3: TRAPEZOIDAL RULE

Example:

Determine the value of

$$\int_{1}^{6}\left(9x^3 - 4x + \frac{3}{x}\right)dx$$

a. Using the analytical method.

b. Using the trapezoidal rule (n = 10).

4.3: TRAPEZOIDAL RULE

Analytical Solution:

From the rules of integration:

$$\int \left(9x^3 - 4x + \frac{3}{x}\right) dx = \frac{9}{4}x^4 - 2x^2 + 3\ln|x| + C$$

Thus:

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx = \left[\frac{9}{4}(6)^4 - 2(6)^2 + 3\ln|6| + C\right] - \left[\frac{9}{4}(1)^4 - 2(1)^2 + 3\ln|1| + C\right]$$

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx = \mathbf{2849.1253}$$

4.3: TRAPEZOIDAL RULE

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx$$

$$f(x) = 9x^3 - 4x + \frac{3}{x}$$

Trapezoidal Rule:

Solve for the value of h:

$$h = \frac{b-a}{n} = \frac{6-1}{10} = 0.50$$

Solve for the values of y_0 to y_{10} (as shown in the table at the right)

	x	y = f(x)	
x_0	1	8	y_0
x_1	1.5	26.375	y_1
x_2	2	65.5	y_2
x_3	2.5	131.825	y_3
x_4	3	232	y_4
x_5	3.5	372.7321	y_5
x_6	4	560.75	y_6
x_7	4.5	802.7917	y_7
x_8	5	1105.6	y_8
x_9	5.5	1475.9205	y_9
x_{10}	6	1920.5	y_{10}

4.3: TRAPEZOIDAL RULE

Solve for $\sum_{i=1}^{n-1} y_i$

$$\sum_{i=1}^{9} y_i = y_1 + y_2 + \cdots + y_9$$

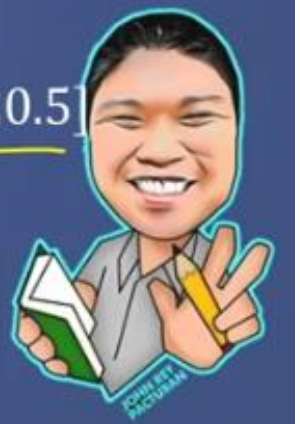
$$\sum_{i=1}^{9} y_i = 26.375 + 65.5 + 131.825 + 232 + 372.7321 + 560.75 + 802.7917 + 1105.6 + 1475.9205$$

$$= \underline{4773.4943}$$

Thus, the approximate value of the integral is:

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \boxed{\frac{h}{2}\left[y_0 + 2\sum_{i=1}^{n-1} y_i + y_n\right]} = \frac{0.5}{2}[8 + 2(\underline{4773.4943}) + \underline{1920.5}]$$

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \mathbf{2868.8722}$$



4.3: TRAPEZOIDAL RULE

For the previous example, the relative error of trapezoidal rule is:

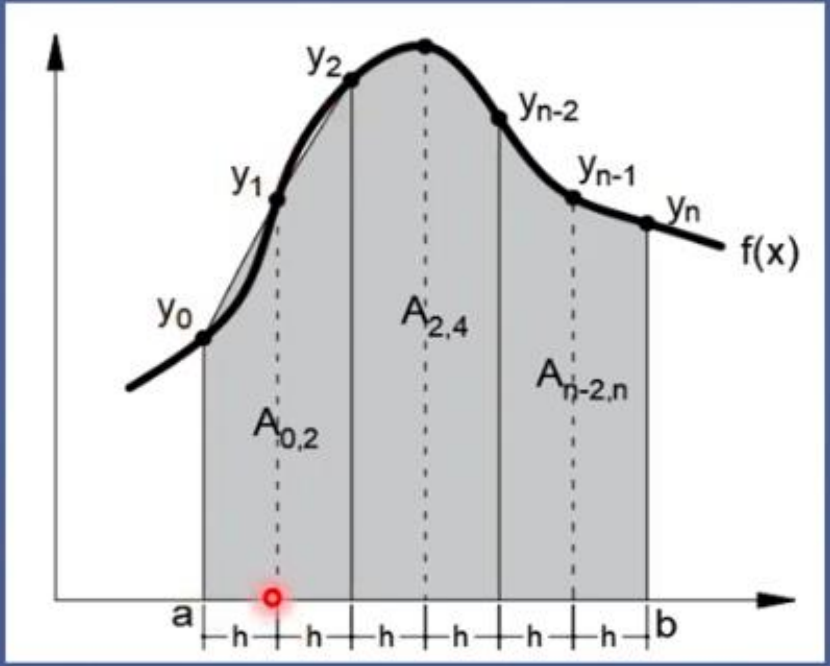
$$\varepsilon = \left|\frac{2868.8722-2849.1253}{2849.1253}\right| = 0.0069 = 0.69\ \%$$

While this value is very small (since most numerical integration methods give a little to negligible error), there are some circumstances that the obtained value is not accepted. To improve the accuracy of the solution, the student may use larger value of "n" or use the more accurate numerical methods (that will be discussed later).

SIMPSON 1/3 RULE

4.4: SIMPSON'S 1/3 RULE

A more accurate way to approximate the definite integral is by using the Simpson's 1/3 Rule. Here, the three consecutive points are connected by parabolic arc, as shown below.



4.4: SIMPSON'S 1/3 RULE

The area of each segment is:

$$A_{0,2} = \frac{h}{3}[y_0 + 4y_1 + y_2]$$
$$A_{2,4} = \frac{h}{3}[y_2 + 4y_3 + y_4]$$
$$\vdots$$
$$\mathrm{A}_{\mathrm{n-4,n-2}} = \frac{h}{3}[y_{n-4} + 4y_{n-3} + y_{n-2}]$$
$$\mathrm{A}_{\mathrm{n-2,n}} = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$

Taking the sum of these areas:

$$\int_a^b f(x)dx \approx$$
$$\frac{h}{3}[y_0 + 4y_1 + y_2] + \frac{h}{3}[y_2 + 4y_3 + y_4] + \cdots + \frac{h}{3}[y_{n-4} + 4y_{n-3} + y_{n-2}] + \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$$
$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-4} + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n]$$
$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + 4(y_1 + y_3 + \cdots y_{n-3} + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-4} + y_{n-2}) + y_n]$$

4.4: SIMPSON'S 1/3 RULE

Let $y_1 + y_3 + \cdots y_{n-3} + y_{n-1} = \sum_{i=1}^{\frac{n}{2}} y_{2i-1}$

and $y_2 + y_4 + \cdots + y_{n-4} + y_{n-2} = \sum_{i=1}^{\frac{n}{2}-1} y_{2i}$.

Thus, the formula for the Simpson's 1/3 rule is:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[y_0 + 4\sum_{i=1}^{\frac{n}{2}} y_{2i-1} + 2\sum_{i=1}^{\frac{n}{2}-1} y_{2i} + y_n\right]$$

where $h = \frac{b-a}{n}$. Note that for this method, n must be even.

4.4: SIMPSON'S 1/3 RULE

Example:

Determine the value of

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx$$

Using Simpson's 1/3 rule (n = 10). Note that from the previous example, the exact answer of this integral (from the analytic solution) is 2849.1253.

4.4: SIMPSON'S 1/3 RULE

Solution:

Solve for the value of h:

$$h = \frac{b-a}{n} = \frac{6-1}{10} = 0.50$$

Solve for the values of y_0 to y_{10}
(As shown in the table at the right)

	x	y = f(x)	
x_0	1	8	y_0
x_1	1.5	26.375	y_1
x_2	2	65.5	y_2
x_3	2.5	131.825	y_3
x_4	3	232	y_4
x_5	3.5	372.7321	y_5
x_6	4	560.75	y_6
x_7	4.5	802.7917	y_7
x_8	5	1105.6	y_8
x_9	5.5	1475.9205	y_9
x_{10}	6	1920.5	y_{10}

4.4: SIMPSON'S 1/3 RULE

odd even

Solve for $\sum_{i=1}^{\frac{n}{2}} y_{2i-1}$ and $\sum_{i=1}^{\frac{n}{2}-1} y_{2i}$.

$\sum_{i=1}^{\frac{n}{2}} y_{2i-1} = y_1 + y_3 + \cdots + y_9$ = 26.375 + 131.825 + 372.7321 + 802.7917+1475.920 = 2809.6443

$\sum_{i=1}^{\frac{n}{2}-1} y_{2i} = y_2 + y_4 + \cdots + y_8$ = 65.5+232+560.75+1105.6 = 1963.85

Thus, the approximate value of the integral is:

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \frac{h}{3}\left[y_0 + 4\sum_{i=1}^{\frac{n}{2}} y_{2i-1} + 2\sum_{i=1}^{\frac{n}{2}-1} y_{2i} + y_n\right]$$

$$\approx \frac{0.5}{3}[8 + 4(2809.6443) + 2(1963.85) + 1920.5]$$

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \mathbf{2849.1295}$$

SIMPSON 3/8 RULE

4.5: SIMPSON'S 3/8 RULE

The general formula for the Simpson's 3/8 rule is:

$$\int_a^b f(x)dx \approx \frac{3h}{8}\left[y_0 + 3\sum_{i=1}^{n-1} y_i - \sum_{i=1}^{\frac{n}{3}-1} y_{3i} + y_n\right]$$

Where:

$h = \frac{b-a}{n}$ but n must be a multiple of 3.

$\sum_{i=1}^{n-1} y_i = y_1 + y_2 + \ldots + y_{n-2} + y_{n-1}$

$\sum_{i=1}^{\frac{n}{3}-1} y_{3i} = y_3 + y_6 + \ldots + y_{n-6} + y_{n-3}$

4.5: SIMPSON'S 3/8 RULE

Example:

Determine the value of

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx$$

Using Simpson's 3/8 rule (n = 12). Note that from the previous example, the exact answer of this integral (from the analytic solution) is 2849.1253.

4.5: SIMPSON'S 3/8 RULE

Solution:

Solve for the value of h:

$$h = \frac{b-a}{n} = \frac{6-1}{12} = \frac{5}{12} = 0.4167$$

Solve for the values of y_0 to y_{12}
(As shown in the table at the right)

	x	y = f(x)	
x_0	1	8	y_0
x_1	1.4167	22.0395	y_1
x_2	1.8333	49.7614	y_2
x_3	2.25	94.8490	y_3
x_4	2.6667	161.125	y_4
x_5	3.0833	252.4573	y_5
x_6	3.5	372.7321	y_6
x_7	3.9167	525.8441	y_7
x_8	4.3333	715.6923	y_8
x_9	4.75	946.1785	y_9
x_{10}	5.1667	1221.2056	y_{10}
x_{11}	5.5833	1544.6779	y_{11}
x_{12}	6	1920.5	y_{12}

4.5: SIMPSON'S 3/8 RULE

all intermediate values

Solve for $\sum_{i=1}^{n-1} y_i$ and $\sum_{i=1}^{\frac{n}{3}-1} y_{3i}$. y with subscript multiples of 3

$$\sum_{i=1}^{n-1} y_i = y_1 + y_2 + \cdots + y_{10} + y_{11}$$

= 22.0395 + 49.7614 + 94.8490 + 161.125 + 252.4573 + 372.7321 + 525.8441 + 715.6923 + 946.1785 + 1221.2056 + 1544.6779

= 5906.5627

$$\sum_{i=1}^{\frac{n}{3}-1} y_{3i} = y_3 + y_6 + y_9$$ = 94.8490 + 372.7321 + 946.1785

= 1413.7596

4.5: SIMPSON'S 3/8 RULE

Thus, the approximate value of the integral is:

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \frac{3h}{8}\left[y_0 + 3\sum_{i=1}^{n-1} y_i - \sum_{i=1}^{\frac{n}{3}-1} y_{3i} + y_n\right]$$

$$\approx \frac{3}{8}\left(\frac{5}{12}\right)[8 + 3(5906.5627) - (1413.7596) + \underbrace{1920.5}_{y_{12}}]$$

$$\int_1^6 \left(9x^3 - 4x + \frac{3}{x}\right) dx \approx \mathbf{2849.1295}$$