IB · DP · Maths

5 hours | 31 questions

Exam Questions

Differential Equations

Introduction to First Order Differential Equations / Euler's Method / Separation of Variables / Homogeneous Differential Equations / Integrating Factor / Modelling with Differential Equations / The Logistic Equation

Medium (11 questions)	/87
Hard (10 questions)	/92
Very Hard (10 questions)	/108
Total Marks	**/287**

Scan here to return to the course
or visit savemyexams.com

Medium Questions

1 Consider the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 5x^4 = 3$$

Solve the equation given that $y = 40$ when $x = 2$, giving your answer in the form $y = f(x)$.

(5 marks)

2 (a) Use separation of variables to solve each of the following differential equations for y:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^2}{y^4}$$

(4 marks)

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2 + 1)e^{-y}$

(1 mark)

3 (a) Use separation of variables to solve each of the following differential equations for which satisfies the given boundary condition:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2; \quad y(2) = 1$$

(1 mark)

(b) $(x+3)\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sec y; \quad y(-2) = \dfrac{3\pi}{2}$

(5 marks)

4 (a) At any point in time, the rate of growth of a colony of bacteria is proportional to the current population size. At time $t = 0$ hours, the population size is 5000.

Write a differential equation to model the size of the population of bacteria.

(1 mark)

(b) After 1 hour, the population has grown to 7000.

By first solving the differential equation from part (a), determine the constant of proportionality.

(6 marks)

(c) (i) Show that, according to the model, it will take exactly $\dfrac{\ln 20}{\ln 7 - \ln 5}$ hours (from $t = 0$) for the population of bacteria to grow to $100\ 000$.

(ii) Confirm your answer to part (c)(i) graphically.

(5 marks)

5 (a) After clearing a large forest of malign influences, a wizard introduces a population of 100 unicorns to the forest. According to the wizard's mathemagicians, the population of unicorns in the forest may be modelled by the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.0006\,P(250 - P)$$

where t is the time in years after the unicorns were introduced to the forest.

Show that the population of unicorns at time t years is given by

$$P(t) = \frac{500\mathrm{e}^{0.15t}}{3 + 2\mathrm{e}^{0.15t}}$$

(8 marks)

(b) Find the length of time predicted by the model for the population of unicorns to double in size.

(3 marks)

(c) Determine the maximum size that the model predicts the population of unicorns can grow to.

(2 marks)

6 (a) Show that

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 2x^2$$

is a homogeneous differential equation.

(2 marks)

(b) Using the substitution $v = \dfrac{y}{x}$, show that the solution to the differential equation in part (a) is

$$y = 2x\ln|x| + cx$$

where c is a constant of integration.

(4 marks)

7 (a) Use the substitution $v = \frac{y}{x}$ to show that the differential equation

$$y' = \frac{y^2}{x^2} - \frac{y}{x} + 1$$

may be rewritten in the form

$$v' = \frac{(v-1)^2}{x}$$

(3 marks)

(b) Hence use separation of variables to solve the differential equation in part (a) for which satisfies the boundary condition $y(1) = \frac{2}{3}$. Give your answer in the form $y = f(x)$.

(5 marks)

8 (a) Consider the differential equation

$$y' + 2xy = (4x + 2)e^{x}$$

Explain why it would be appropriate to use an integrating factor in attempting to solve the differential equation.

(2 marks)

(b) Show that the integrating factor for this differential equation is e^{x^2}.

(2 marks)

(c) Hence solve the differential equation.

(5 marks)

9 Use an integrating factor to solve the differential equation

$$(x+3)\frac{dy}{dx} - 4y = (x+3)^6$$

for y which satisfies the boundary condition $y(-2) = 0$.

(7 marks)

10 (a) Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

with the boundary condition $y(1) = 0$.

Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = 2$.

(3 marks)

(b) (i) Explain what method you could use to solve the above differential equation analytically (i.e., exactly).

(ii) The exact solution to the differential equation with the given boundary condition is $y = x \ln x$. Compare your approximation from part (a) to the exact value of the solution at $x = 2$.

(4 marks)

(c) Explain how the accuracy of the approximation in part (a) could be improved.

(1 mark)

11 (a) A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{te^{3t^2} + 1}{4x^2}, \quad t \geq 0$$

At time $t = 0$, $x = \frac{1}{2}$.

By using Euler's method with a step length of 0.1, find an approximate value for x at time $t = 0.3$.

(3 marks)

(b) (i) Solve the differential equation with the given boundary condition to show that

$$x = \frac{1}{2}\sqrt[3]{e^{3t^2} + 6t}$$

(ii) Hence find the percentage error in your approximation for x at time $t = 0.3$.

(5 marks)

Hard Questions

1 Consider the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x^3 = 2\sin x$$

Solve the equation given that $y = 0$ when $x = 0$, giving your answer in the form $y = f(x)$
.

(5 marks)

2 (a) Use separation of variables to solve each of the following differential equations:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^3y^3$$

(4 marks)

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = x(x^2-1)^3e^{3y}$

(5 marks)

3 (a) Use separation of variables to solve each of the following differential equations for y which satisfies the given boundary condition:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos 3x}{y}; \quad y\left(\frac{\pi}{6}\right) = -1$$

(5 marks)

(b) $e^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} = \cos^2 y; \quad y(0) = \frac{\pi}{4}$

(5 marks)

4 (a) After an invasive species of insect has been introduced to a new region, it is estimated that at any point in time the rate of growth of the population of insects in the region will be proportional to the current population size P. At the start of a study of the insects in a particular region, researchers estimate the population size to be 1000 individuals. A week later another population survey is conducted, and the population of insects is found to have increased to 1150.

By first writing and solving an appropriate differential equation, determine how long it will take for the population of insects in the region to increase to 10 000.

(8 marks)

(b) Comment on the validity of the model for large values of t.

(2 marks)

5 (a) Ignoring the advice of her father's professional dragon keepers, Princess Sarff releases her personal menagerie of 800 dragons onto the archipelago known as the Sheep Islands. Sarff believes that the dragons will thrive in such a sheep-rich environment. The chief dragon keeper, however, has studied the sheep population of the islands as well as the appetite of dragons. Based on his research, he believes that the population P of dragons in the islands may be modelled by the logistic equation

$$\frac{dP}{dt} = 0.00025P(160 - P)$$

where t is the time in years after the dragons were introduced to the archipelago.

Use the logistic equation to explain why, according to the model, the dragon population will initially be decreasing.

(2 marks)

(b) By first solving the logistic equation for P, determine the amount of time it will take for the dragon population to shrink to half its original size.

(10 marks)

(c) Determine the long-term trend for the dragon population, using mathematical reasoning to justify your answer.

(3 marks)

6 (a) Consider the differential equation

$$(x^2 + y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy$$

Explain why the substitution $v = \dfrac{y}{x}$ would be an appropriate method to use to solve the differential equation.

(2 marks)

(b) Show that the solution to the differential equation may be expressed in the form

$$y = Ae^{\frac{x^2}{2y^2}}$$

where A is an arbitrary constant.

(5 marks)

(c) Find the precise solution to the differential equation given that $y = \dfrac{1}{2}$ when $x = 1$.

(3 marks)

7 Use the substitution $v = \dfrac{y}{x}$ to solve the differential equation

$$x^2 y' = y^2 + 7xy + 9x^2$$

for y which satisfies the boundary condition $y(1) = -2$. Give your answer in the form $y = f(x)$.

(8 marks)

8 Use an integrating factor to solve the differential equation

$$xy' + 2y = 1 + e^{x^2}$$

(6 marks)

9 (a) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\sec x}{e^{\sqrt{x}}}\right)^2 - \frac{y}{\sqrt{x}}$$

with the boundary condition $y\left(\frac{\pi}{3}\right) = 0$.

Apply Euler's method with a step size of $h = 0.01$ to approximate the solution to the differential equation at $x = \frac{20\pi + 3}{60}$.

(3 marks)

(b) Solve the differential equation analytically, for y which satisfies the given boundary condition.

(7 marks)

(c) (i) Compare your approximation from part (a) to the exact value of the solution at $x = \frac{20\pi + 3}{60}$.

(ii) Explain how the accuracy of the approximation in part (a) could be improved.

(3 marks)

10 (a) A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

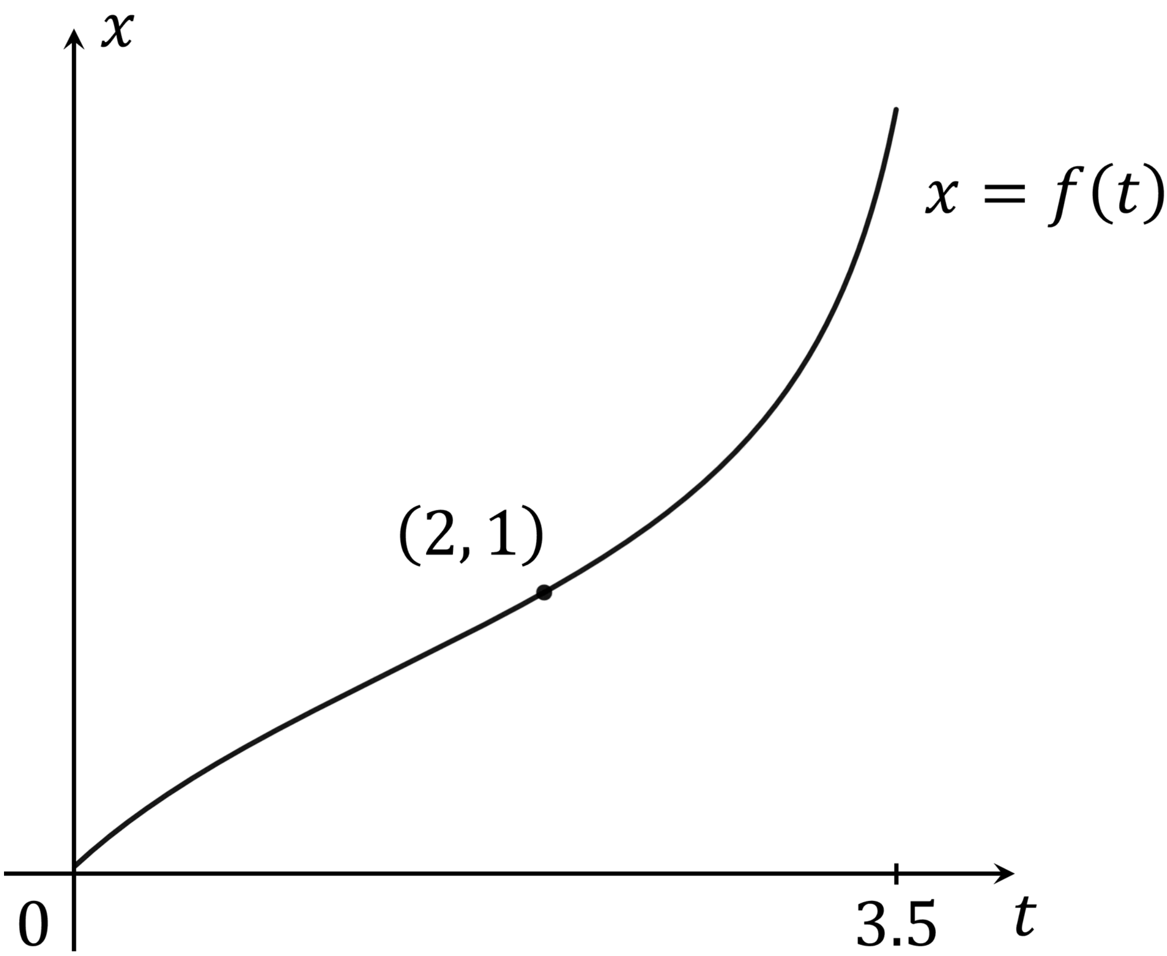
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+\sin(t+1)-\cos(t+1)}, \qquad 0 \le t \le 3.5$$

At time $t=2$, $x=1$.

By using Euler's method with a step length of 0.25, find an approximate value for x at time $t=3.25$.

(3 marks)

(b) The diagram below shows a graph of the exact solution $x=f(t)$ to the differential equation with the given boundary condition.



Explain using the graph whether the approximation found in part (a) will be an overestimate or an underestimate for the true value of x when $t = 3.25$. Be sure to use mathematical reasoning to justify your answer.

(3 marks)

Very Hard Questions

1 Consider the first-order differential equation

$$\frac{dy}{dx} + \frac{1}{2x} = \sin 3x \cos 3x$$

Solve the equation given that $y = 0$ when $x = \frac{\pi}{2}$, giving your answer in the form $y = f(x)$. .

(5 marks)

2 (a) Use separation of variables to solve each of the following differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^4}{4x^3}$$

(4 marks)

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{y(\pi - x^3)} e^{y^2}$

(5 marks)

3 (a) Solve each of the following differential equations for y which satisfies the given boundary condition, giving your answers in the form $y = f(x)$.

$$\cos \pi x^4 \frac{\mathrm{d}y}{\mathrm{d}x} = \tan \pi x^4 \left(\frac{x}{y}\right)^3; \qquad y(0) = -3$$

(5 marks)

(b) $e^{x^2} \operatorname{cosec} y \dfrac{\mathrm{d}y}{\mathrm{d}x} = x \sin y; \quad y(0) = \dfrac{3\pi}{4}$

(6 marks)

4 (a) As the atoms in a sample of radioactive material undergo radioactive decay, the rate of change of the number of radioactive atoms remaining in the sample at any time t is proportional to the number, N, of radioactive atoms currently remaining. The amount of time, λ, that it takes for half the radioactive atoms in a sample of radioactive material to decay is known as the *half-life* of the material.

Let N_0 be the number of radioactive atoms originally present in a sample.

By first writing and solving an appropriate differential equation, show that the number of radioactive atoms remaining in the sample at any time $t \geq 0$ may be expressed as

$$N(t) = N_0 e^{-\frac{\ln 2}{\lambda} t}$$

(8 marks)

(b) Plutonium-239, a by-product of uranium fission reactors, has a half-life of 24000 years.

For a particular sample of Plutonium-239, determine how long it will take until less than 1% of the original radioactive Plutonium-239 atoms in the sample remain.

(3 marks)

5 (a) Consider the standard logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(a - P)$$

where P is the size of a population at time $t \geq 0$, and where k and a are positive constants. Let the population at time $t = 0$ be denoted by P_0.

Write down the solution to the logistic equation in the case where $P_0 = a$, using mathematical reasoning to justify your answer.

(2 marks)

(b) In the case where $P_0 \neq \alpha$, show that the solution to the logistic equation is

$$P(t) = \frac{aAe^{akt}}{1 + Ae^{akt}}$$

where A is an arbitrary constant.

(8 marks)

(c) In the case where $P_0 \neq \alpha$, write down an expression for A in terms of a and P_0.

(2 marks)

(d) In the case where $P_0 \neq 0$, determine the behaviour of P as t becomes large.

(3 marks)

(e) In the case where $0 < 2P_0 < a$, determine the value of t at which the initial population will have doubled. Your answer should be given explicitly in terms of a, k and P_0.

(4 marks)

6 Solve the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y = \frac{xy^2}{y^2\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{x}{y}\right)}$$

(8 marks)

7 (a) Consider the differential equation

$$x^2 y' = y^2 + 3xy - 8x^2$$

with the boundary condition $y(1) = -3$.

Solve the differential equation for y which satisfies the given boundary condition, giving your answer in the form $y = f(x)$.

(9 marks)

(b) Determine the asymptotic behaviour of the graph of the solution as x becomes large.

(3 marks)

8 Solve the differential equation

$$(4x^2 + 1)y' + y = \frac{1 - x + 4x^2 - 4x^3}{\sqrt{e^{\arctan 2x}}}$$

(7 marks)

9 (a) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{\sqrt{63 + 11x^2 - 2x^4}} - \frac{2xy}{2x^2 + 7}$$

with the boundary condition $y\left(-\frac{3\sqrt{2}}{2}\right) = 1$. Apply Euler's method with a step size of $h = 0.2$ to approximate the solution to the differential equation at $x = \frac{2 - 3\sqrt{2}}{2}$.

(3 marks)

(b) Solve the differential equation analytically, for y which satisfies the given boundary condition.

(7 marks)

(c) (i) Compare your approximation from part (a) to the exact value of the solution at $x = \frac{2 - 3\sqrt{2}}{2}$.

(ii) Explain how the accuracy of the approximation in part (a) could be improved.

(3 marks)

10 (a) A particle moves in a straight line, such that its displacement x at time t is described by the differential equation

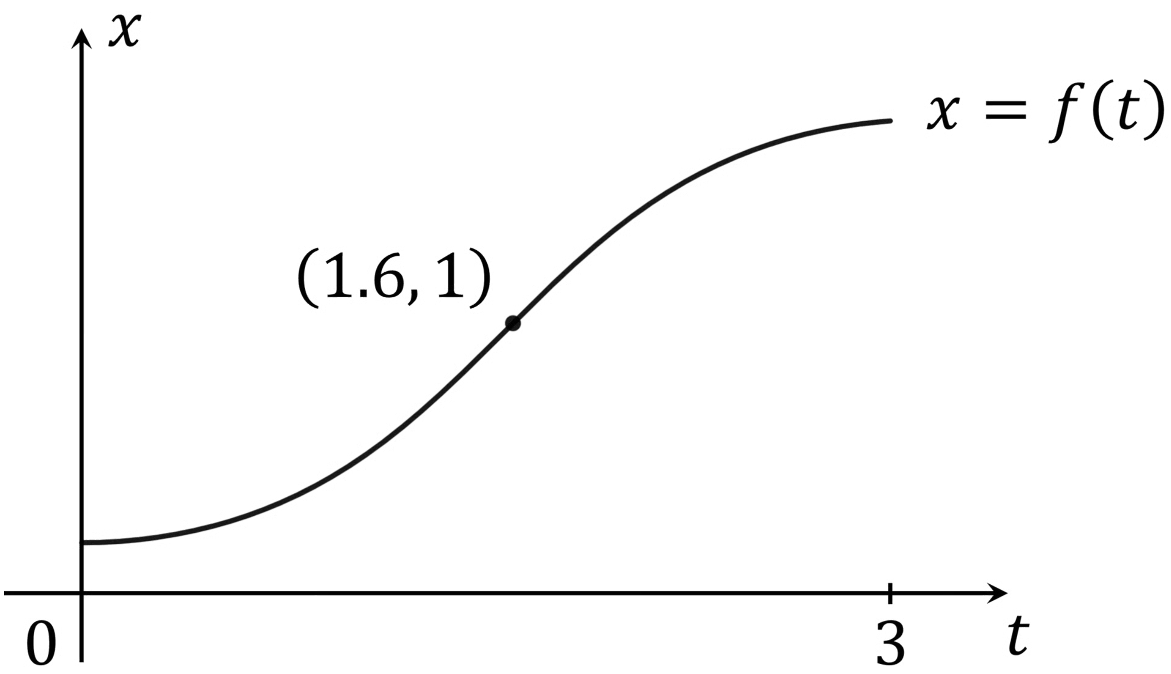
$$\frac{dx}{dt} = \frac{\sin t}{1 + \cos^2 t}, \qquad 0 \leq t \leq 3$$

At time $t = 1.6$, $x = 1$.

By using Euler's method with a step length of 0.04, find an approximate value for x at time $t = 1.8$.

(3 marks)

(b) The diagram below shows a graph of the exact solution $x = f(t)$ to the differential equation with the given boundary condition.



Given that the graph of $x = f(t)$ has exactly one point of inflection, find the exact value of the t-coordinate of the point of inflection.

(7 marks)

(c) Hence determine whether the approximation found in part (a) will be an overestimate or an underestimate for the true value of x when $t = 1.8$. Be sure to use mathematical reasoning to justify your answer.

(3 marks)