CP **7** $\mathbf{M} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$.

(i) Find the determinants of **M** and **N**.

(ii) Find the matrix **MN** and show that det **MN** = det **M** × det **N**.

8 The plane is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix}$.

(i) Draw a diagram to show the image of the unit square under the transformation represented by **M**.

(ii) Describe the effect of the transformation and explain this with reference to the determinant of **M**.

CP **9** The plane is transformed by the matrix $\mathbf{N} = \begin{pmatrix} 5 & -10 \\ -1 & 2 \end{pmatrix}$.

(i) Find the image of the point (p, q).

(ii) Hence show that the whole plane is mapped to a straight line and find the equation of this line.

(iii) Find the determinant of **N** and explain its significance.

PS **10** A matrix **T** maps all points on the line $x + 2y = 1$ to the point (1, 3).

(i) Find the matrix **T** and show that it has determinant of zero.

(ii) Show that **T** maps all points on the plane to the line $y = 3x$.

(iii) Find the coordinates of the point to which all points on the line $x + 2y = 3$ are mapped.

CP **11** The plane is transformed using the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc = 0$.

Prove that the general point P(x, y) maps to P′ on the line $cx - ay = 0$.

12 The point P is mapped to P′ on the line $3y = x$ so that PP′ is parallel to the line $y = 3x$.

(i) Find the equation of the line parallel to $y = 3x$ passing through the point P with coordinates (s, t).

(ii) Find the coordinates of P′, the point where this line meets $3y = x$.

(iii) Find the matrix of the transformation that maps P to P′ and show that the determinant of this matrix is zero.

6.2 The inverse of a matrix

The identity matrix

Whenever you multiply a 2 × 2 matrix **M** by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the product is **M**.

It makes no difference whether you **pre-multiply**, for example,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}$$

or **post-multiply**

$$\begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 3 \end{pmatrix}.$$

ACTIVITY 6.2

(i) Write down the matrix **P** that represents a reflection in the x-axis.

(ii) Find the matrix $\mathbf{P}^2$.

(iii) Comment on your answer.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known as the 2×2 identity matrix.

Identity matrices are often denoted by the letter **I**.

For multiplication of matrices, **I** behaves in the same way as the number 1 when dealing with the multiplication of real numbers.

The transformation represented by the identity matrix maps every points to itself.

Similarly, the 3×3 identity matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Example 6.4

(i) Write down the matrix **A** that represents a rotation of 90° anticlockwise about the origin.

(ii) Write down the matrix **B** that represents a rotation of 90° clockwise about the origin.

(iii) Find the product **AB** and comment on your answer.

Solution

(i) $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(iii) $\mathbf{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

AB represents a rotation of 90° clockwise followed by a rotation of 90° anticlockwise. The result of this is to return to the starting point.

To undo the effect of a rotation through 90° anticlockwise about the origin, you need to carry out a rotation through 90° clockwise about the origin. These two transformations are inverses of each other.

Similarly, the matrices that represent these transformations are inverses of each other.

In Example 6.4, $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the inverse of $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and vice versa.

Finding the inverse of a 2 × 2 matrix

Finding the inverse of 3 × 3 matrices is covered in Section 6.3.

If the product of two square matrices, **M** and **N**, is the identity matrix **I**, then **N** is the inverse of **M**. You can write this as $\mathbf{N} = \mathbf{M}^{-1}$.

Generally, if $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ you need to find an inverse matrix $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

ACTIVITY 6.3

Multiply $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

What do you notice?

Use your result to write down the inverse of the general matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

How does the determinant $|\mathbf{M}|$ relate to the matrix $\mathbf{M}^{-1}$?

You should have found in the activity that the inverse of the matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is given by}$$

$$\mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If the determinant is zero then the inverse matrix does not exist and the matrix is said to be **singular**. If det $\mathbf{M} \neq 0$ the matrix is said to be **non-singular**.

If a matrix is singular, then it maps all points on the plane to a straight line. So an infinite number of points are mapped to the same point on the straight line. It is therefore not possible to find the inverse of the transformation, because an inverse matrix would map a point on that straight line to just one other point, not to an infinite number of them.

A special case is the zero matrix, which maps all points to the origin.

Example 6.5

$\mathbf{A} = \begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}$

(i) Find $\mathbf{A}^{-1}$.

(ii) The point P is mapped to the point Q (5, 2) under the transformation represented by **A**. Find the coordinates of P.

Solution

(i) det $\mathbf{A} = (11 \times 2) - (3 \times 6) = 4$

$$\mathbf{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}$$

(ii)
$$\mathbf{A}^{-1}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} 4 \\ -8 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

A maps P to Q, so $\mathbf{A}^{-1}$ maps Q to P.

The coordinates of P are (1, –2).

As matrix multiplication is generally non-commutative, it is interesting to find out if $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M}$. The next activity investigates this.

ACTIVITY 6.4

(i) In Example 6.5 you found that the inverse of $\mathbf{A} = \begin{pmatrix} 11 & 3 \\ 6 & 2 \end{pmatrix}$ is $\mathbf{A}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -3 \\ -6 & 11 \end{pmatrix}$.

Show that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

(ii) If the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, write down $\mathbf{M}^{-1}$ and show that

$\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

The result $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$ is important as it means that the inverse of a matrix, if it exists, is unique. This is true for all square matrices, not just 2×2 matrices.

?

- How would you reverse the effect of a rotation followed by a reflection?
- How would you write down the inverse of a matrix product **MN** in terms of $\mathbf{M}^{-1}$ and $\mathbf{N}^{-1}$?

The inverse of a product of matrices

Suppose you want to find the inverse of the product **MN**, where **M** and **N** are non-singular matrices. This means that you need to find a matrix **X** such that

$$\mathbf{X}(\mathbf{MN}) = \mathbf{I}.$$

$$\mathbf{X}(\mathbf{MN}) = \mathbf{I} \Rightarrow \mathbf{XMNN}^{-1} = \mathbf{IN}^{-1}$$ Post multiply by $\mathbf{N}^{-1}$

$$\Rightarrow \mathbf{XM} = \mathbf{IN}^{-1}$$ Using $\mathbf{NN}^{-1} = \mathrm{I}$

$$\Rightarrow \mathbf{XMM}^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$$ Post multiply by $\mathbf{M}^{-1}$

$$\Rightarrow \mathbf{X} = \mathbf{N}^{-1}\mathbf{M}^{-1}$$ Using $\mathbf{MM}^{-1} = \mathrm{I}$

So $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$ for matrices **M** and **N** of the same order. This means that when working backwards, you must reverse the second transformation before reversing the first transformation.

Technology note

Investigate how to use a calculator to find the inverse of 2×2 and 3×3 matrices.

Check using a calculator that multiplying a matrix by its inverse gives the identity matrix.

Exercise 6B

1 For the matrix $\begin{pmatrix} 5 & -1 \\ -2 & 0 \end{pmatrix}$

(i) find the image of the point (3, 5)

(ii) find the inverse matrix

(iii) find the point that maps to the image (3, –2).

2 Determine whether the following matrices are singular or non-singular. For those that are non-singular, find the inverse.

(i) $\begin{pmatrix} 6 & 3 \\ -4 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 11 & 3 \\ 3 & 11 \end{pmatrix}$

(iv) $\begin{pmatrix} 11 & 11 \\ 3 & 3 \end{pmatrix}$ (v) $\begin{pmatrix} 2 & -7 \\ 0 & 0 \end{pmatrix}$

(vi) $\begin{pmatrix} -2a & 4a \\ 4b & -8b \end{pmatrix}$ (vii) $\begin{pmatrix} -2 & 4a \\ 4b & -8 \end{pmatrix}$

3 $\mathbf{M} = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 8 & 5 \\ -2 & -1 \end{pmatrix}$.

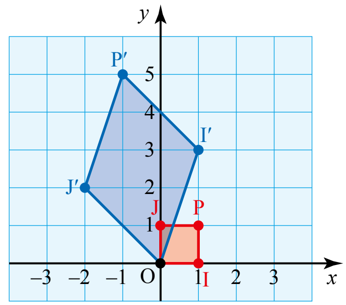
Calculate the following:

(i) $\mathbf{M}^{-1}$ (ii) $\mathbf{N}^{-1}$ (iii) $\mathbf{MN}$

(iv) $\mathbf{NM}$ (v) $(\mathbf{MN})^{-1}$ (vi) $(\mathbf{NM})^{-1}$

(vii) $\mathbf{M}^{-1}\mathbf{N}^{-1}$ (viii) $\mathbf{N}^{-1}\mathbf{M}^{-1}$

4 The diagram shows the unit square OIPJ mapped to the image O I′P′J′ under a transformation represented by a matrix **M**.



(i) Find the inverse of **M**.

(ii) Use matrix multiplication to show that $\mathbf{M}^{-1}$ maps OI′P′J′ back to OIPJ.

5 The matrix $\begin{pmatrix} 1-k & 2 \\ -1 & 4-k \end{pmatrix}$ is singular.

Find the possible values of k.

PS **6** Given that $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ and $\mathbf{MN} = \begin{pmatrix} 7 & 2 & -9 & 10 \\ 2 & -1 & -12 & 17 \end{pmatrix}$, find the matrix **N**.

7 Triangle T has vertices at (1, 0), (0, 1) and (−2, 0).

It is transformed to triangle T′ by the matrix $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$.

(i) Find the coordinates of the vertices of T′.

Show the triangles T and T′ on a single diagram.

(ii) Find the ratio of the area of T′ to the area of T.

Comment on your answer in relation to the matrix **M**.

(iii) Find $\mathbf{M}^{-1}$ and verify that this matrix maps the vertices of T′ to the vertices of T.

CP **8** $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a singular matrix.

(i) Show that $\mathbf{M}^2 = (a + d)\mathbf{M}$.

(ii) Find a formula that expresses $\mathbf{M}^n$ in terms of **M**, where n is a positive integer.

Comment on your results.

9 Given that $\mathbf{PQR} = \mathbf{I}$, show algebraically that

(i) $\mathbf{Q} = \mathbf{P}^{-1}\mathbf{R}^{-1}$ (ii) $\mathbf{Q}^{-1} = \mathbf{RP}$.

Given that $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{R} = \begin{pmatrix} 12 & -3 \\ 2 & -1 \end{pmatrix}$

(iii) use part (i) to find the matrix **Q**

(iv) calculate the matrix $\mathbf{Q}^{-1}$

(v) verify that your answer to part (ii) is correct by calculating **RP** and comparing it with your answer to part (iv).

6.3 Finding the inverse of a 3×3 matrix

The determinant of a 3×3 matrix is sometimes denoted $|\mathbf{a}\ \mathbf{b}\ \mathbf{c}|$.

In this section you will find the determinant and inverse of 3×3 matrices using the calculator facility and also using a non-calculator method.

Finding the inverse of a 3×3 matrix using a calculator

ACTIVITY 6.5

Using a calculator, find the determinant and inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{pmatrix}.$$

Still using a calculator, find out which of the following matrices are non-singular and find the inverse in each of these cases.

$$\mathbf{B} = \begin{pmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \\ 2 & 4 & -3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 0 & 3 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & 3 \end{pmatrix}$$

Finding the inverse of a 3×3 matrix without using a calculator

It is also possible to find the determinant and inverse of a 3×3 matrix without using a calculator. This is useful in cases where some of the elements of the matrix are algebraic rather than numerical.

If $\mathbf{M}$ is the 3×3 matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ then the determinant of $\mathbf{M}$ is defined by

$$\det \mathbf{M} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

which is sometimes referred to as the **expansion of the determinant by the first column**.

For example, to find the determinant of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 4 & 0 & 1 \end{pmatrix} \text{ from Activity 6.5:}$$

$$\begin{aligned} \det \mathbf{A} &= 3\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 3(1-0) - 0(-2-0) + 4(-4-1) \\ &= 3 - 20 \\ &= -17 \end{aligned}$$

Notice that you do not really need to calculate $\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix}$ as it is going to be multiplied by zero. Keeping an eye open for helpful zeros can reduce the number of calculations needed.

This is the same answer as you will have obtained earlier using your calculator.

The 2×2 determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is called the minor of the element a_1. It is obtained by deleting the row and column containing a_1:

$$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$$

Other minors are defined in the same way, for example the minor of a_2 is

$$\begin{vmatrix} \cancel{a_1} & b_1 & c_1 \\ \cancel{a_2} & \cancel{b_2} & \cancel{c_2} \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$

You may have noticed that in the expansions of the determinant, the signs on the minors alternate as shown:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

A minor, together with its correct sign, is known as a **cofactor** and is denoted by the corresponding capital letter; for example, the cofactor of a_3 is A_3. This means that the expansion by the first column, say, can be written as

$$a_1 A_1 + a_2 A_2 + a_3 A_3.$$

Note

As an alternative to using the first column, you could use the **expansion of the determinant by the second column**:

$$\det \mathbf{M} = -b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

or the **expansion of the determinant by the third column**:

$$\det \mathbf{M} = c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

It is fairly easy to show that all three expressions above for det $\mathbf{M}$ simplify to:

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Example 6.6

Find the determinant of the matrix $\mathbf{M} = \begin{pmatrix} 3 & 0 & -4 \\ 7 & 2 & -1 \\ -2 & 1 & 3 \end{pmatrix}$.

To find the determinant you can also expand by rows. So, for example, expanding by the top row would give:

$$3\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - 0\begin{vmatrix} 7 & -1 \\ -2 & 3 \end{vmatrix} + (-4)\begin{vmatrix} 7 & 2 \\ -2 & 1 \end{vmatrix}$$

which also gives the answer −23.

Solution

Expanding by the first column using the expression:

$$\det \mathbf{M} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

gives:

$$\det \mathbf{M} = 3\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - 7\begin{vmatrix} 0 & -4 \\ 1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 0 & -4 \\ 2 & -1 \end{vmatrix}$$

Notice that expanding by the top row would be quicker here as it has a zero element.

$$= 3(6-(-1)) - 7(0-(-4)) - 2(0-(-8))$$
$$= 21 - 28 - 16$$
$$= -23$$

Earlier you saw that the determinant of a 2 × 2 matrix represents the area scale factor of the transformation represented by the matrix. In the case of a 3 × 3 matrix the determinant represents the volume scale factor. For example, the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has determinant 8; this matrix represents

Recall that a minor, together with its correct sign, is known as a cofactor and is denoted by the corresponding capital letter; for example the cofactor of a_3 is A_3.

an enlargement of scale factor 2, centre the origin, so the volume scale factor of the transformation is $2 \times 2 \times 2 = 8$.

For the matrix $\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, the matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ is known as the **adjugate** or **adjoint** of **M**, denoted **adj M**.

The adjugate of **M** is formed by

- replacing each element of **M** by its cofactor;
- then transposing the matrix (i.e. changing rows into columns and columns into rows).

The unique inverse of a 3×3 matrix can be calculated as follows:

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \operatorname{adj} \mathbf{M} = \frac{1}{\det \mathbf{M}} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \quad \det \mathbf{M} \neq 0$$

The steps involved in the method are shown in the following example.

Example 6.7

Find the inverse of the matrix **M** without using a calculator, where

$$\mathbf{M} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & -5 & 2 \\ -3 & 6 & -3 \end{pmatrix}.$$

Solution

Step 1: Find the determinant Δ and check $\Delta \neq 0$.

Expanding by the first column

$$\Delta = 2\begin{vmatrix} -5 & 2 \\ 6 & -3 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 6 & -3 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix}$$

$$= (2 \times 3) - (2 \times -33) - (3 \times 26) = -6$$

Therefore the inverse matrix exists.

Step 2: Evaluate the cofactors.

$$A_1 = \begin{vmatrix} -5 & 2 \\ 6 & -3 \end{vmatrix} = 3 \qquad A_2 = -\begin{vmatrix} 3 & 4 \\ 6 & -3 \end{vmatrix} = 33 \quad A_3 = \begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} = 26$$

$$B_1 = -\begin{vmatrix} 2 & 2 \\ -3 & -3 \end{vmatrix} = 0 \quad B_2 = \begin{vmatrix} 2 & 4 \\ -3 & -3 \end{vmatrix} = 6 \quad B_3 = -\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} = 4$$

$$C_1 = \begin{vmatrix} 2 & -5 \\ -3 & 6 \end{vmatrix} = -3 \quad C_2 = -\begin{vmatrix} 2 & 3 \\ -3 & 6 \end{vmatrix} = -21 \quad C_3 = \begin{vmatrix} 2 & 3 \\ 2 & -5 \end{vmatrix} = -16$$

You can evaluate the determinant Δ using these cofactors to check your earlier arithmetic is correct:

2nd column:

$\Delta = 3B_1 - 5B_2 + 6B_3 = (3 \times 0) - (5 \times 6) + (6 \times 4) = -6$

3rd column:

$\Delta = 4C_1 + 2C_2 - 3C_3 = (4 \times -3) + (2 \times -21) - (3 \times -16) = -6$

Step 3: Form the matrix of cofactors and transpose it, then multiply by $\frac{1}{\Delta}$

Multiply by $\frac{1}{\Delta}$.

$$\mathbf{M}^{-1} = \frac{1}{-6}\begin{pmatrix} 3 & 0 & -3 \\ 33 & 6 & -21 \\ 26 & 4 & -16 \end{pmatrix}^T$$

The capital T indicates the matrix is to be transposed.

Matrix of cofactors.

$$= \frac{1}{-6}\begin{pmatrix} 3 & 33 & 26 \\ 0 & 6 & 4 \\ -3 & -21 & -16 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -3 & -33 & -26 \\ 0 & -6 & -4 \\ 3 & 21 & 16 \end{pmatrix}$$

The final matrix could then be simplified and written as

$$\mathbf{M}^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{11}{2} & -\frac{13}{3} \\ 0 & -1 & -\frac{2}{3} \\ \frac{1}{2} & \frac{7}{2} & \frac{8}{3} \end{pmatrix}$$

$$\text{Check: } \mathbf{MM}^{-1} = \begin{pmatrix} 2 & 3 & 4 \\ 2 & -5 & 2 \\ -3 & 6 & -3 \end{pmatrix} \frac{1}{6}\begin{pmatrix} -3 & -33 & -26 \\ 0 & -6 & -4 \\ 3 & 21 & 16 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This adjugate method for finding the inverse of a 3×3 matrix is reasonably straightforward but it is important to check your arithmetic as you go along, as it is very easy to make mistakes. You can use your calculator to check that you have calculated the inverse correctly.

As shown in Example 6.7, you might also multiply the inverse by the original matrix and check that you obtain the 3×3 identity matrix.

Exercise 6C

1 Evaluate these determinants without using a calculator. Check your answers using your calculator.

(i) (a) $\begin{vmatrix} 1 & 1 & 3 \\ -1 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}$

(ii) (a) $\begin{vmatrix} 1 & -5 & -4 \\ 2 & 3 & 3 \\ -2 & 1 & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & -2 \\ -5 & 3 & 1 \\ -4 & 3 & 0 \end{vmatrix}$

(iii) (a) $\begin{vmatrix} 2 & 1 & 2 \\ 3 & 5 & 3 \\ 1 & -1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 5 & 0 \\ 1 & 5 & 0 \\ 2 & 1 & -2 \end{vmatrix}$

What do you notice about the determinants?

2 Find the inverses of the following matrices, if they exist, without using a calculator.

(i) $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 6 \\ 5 & 3 & 11 \\ 7 & 4 & 16 \end{pmatrix}$

(iii) $\begin{pmatrix} 5 & 5 & -5 \\ -9 & 3 & -5 \\ -4 & -6 & 8 \end{pmatrix}$ (iv) $\begin{pmatrix} 6 & 5 & 6 \\ -5 & 2 & -4 \\ -4 & -6 & -5 \end{pmatrix}$

3 Find the inverse of the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 & -2 \\ k & 0 & 4 \\ 2 & -1 & 4 \end{pmatrix}$ where $k \neq 0$.

For what value of k is the matrix $\mathbf{M}$ singular?

PS **4** (i) Investigate the relationship between the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 3 & 1 \\ 2 & 4 & 2 \\ -1 & 3 & 5 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \\ 5 & -1 & 3 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \\ 3 & 5 & -1 \end{pmatrix}$$

(ii) Find det $\mathbf{A}$, det $\mathbf{B}$ and det $\mathbf{C}$ and comment on your answer.

5 Show that $x = 1$ is one root of the equation $\begin{vmatrix} 2 & 2 & x \\ 1 & x & 1 \\ x & 1 & 4 \end{vmatrix} = 0$ and find the other roots.

6 Find the values of x for which the matrix $\begin{pmatrix} 3 & -1 & 1 \\ 2 & x & 4 \\ x & 1 & 3 \end{pmatrix}$ is singular.

7 Given that the matrix $\mathbf{M} = \begin{pmatrix} k & 2 & 1 \\ 0 & -k & 2 \\ 2k & 1 & 3 \end{pmatrix}$ has determinant greater than 5, find the range of possible values for k.

CP **8** (i) $\mathbf{P}$ and $\mathbf{Q}$ are non-singular matrices. Prove that $(\mathbf{PQ})^{-1} = \mathbf{Q}^{-1}\mathbf{P}^{-1}$.

(ii) Find the inverses of the matrices $\mathbf{P} = \begin{pmatrix} 0 & 3 & -1 \\ -2 & 2 & 2 \\ -3 & 0 & 1 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 4 & -3 & 2 \end{pmatrix}$.

Using the result from part (i), find $(\mathbf{PQ})^{-1}$.

9 (i) Prove that $\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, where k is a constant.

(ii) Explain in terms of volumes, why multiplying all the elements in the first column by a constant k multiplies the value of the determinant by k.

(iii) What would happen if you multiplied a different column by k?

10 Given that $\begin{vmatrix} 1 & 2 & 3 \\ 6 & 4 & 5 \\ 7 & 5 & 1 \end{vmatrix} = 43$, write down the values of the determinants:

(i) $\begin{vmatrix} 10 & 2 & 3 \\ 60 & 4 & 5 \\ 70 & 5 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 4 & 10 & -21 \\ 24 & 20 & -35 \\ 28 & 25 & -7 \end{vmatrix}$

(iii) $\begin{vmatrix} x & 4 & 3y \\ 6x & 8 & 5y \\ 7x & 10 & y \end{vmatrix}$

(iv) $\begin{vmatrix} x^4 & \frac{1}{x} & 12y \\ 6x^4 & \frac{2}{x} & 20y \\ 7x^4 & \frac{5}{2x} & 4y \end{vmatrix}$

KEY POINTS

1 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the determinant of $\mathbf{M}$, written det $\mathbf{M}$ or $|\mathbf{M}|$, is given by det $\mathbf{M} = ad - bc$.

2 The determinant of a 2×2 matrix represents the area scale factor of the transformation.

3 If $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{M}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

4 The determinant of a 3×3 matrix $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ is given by

$$\det \mathbf{M} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

5 For a 3×3 matrix $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ the minor of an element is formed by crossing out the row and column containing that element and finding the determinant of the resulting 2×2 matrix.

6 A minor, together with its correct sign, given by the matrix $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ is known as a cofactor and is denoted by the corresponding capital letter; for example the cofactor of a_3 is A_3.

7 The inverse of a 3×3 matrix $\mathbf{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ can be found using a calculator or using the formula

$$\mathbf{M}^{-1} = \frac{1}{\det \mathbf{M}} \operatorname{adj} \mathbf{M} = \frac{1}{\det \mathbf{M}}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}, \ \Delta \neq 0.$$

The matrix $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$ is the adjoint or adjugate matrix, denoted adj $\mathbf{M}$, formed by replacing each element of $\mathbf{M}$ by its cofactor and then transposing (i.e. changing rows into columns and columns into rows).

8 $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$

9 A matrix is singular if the determinant is zero. If the determinant is non-zero the matrix is said to be non-singular.

10 If the determinant of a matrix is zero, all points are mapped to either a straight line (in two dimensions) or to a plane (three dimensions).

11 If $\mathbf{A}$ is a non-singular matrix, $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

LEARNING OUTCOMES

Now that you have finished this chapter, you should be able to

- find the determinant of 2×2 and 3×3 matrices using the notation det $\mathbf{M}$
- recall the meaning of the terms
  - singular
  - non-singular, as applied to square matrices
- recall how the area scale factor of a transformation is related to the determinant of the corresponding 2×2 matrix
- understand the significance of a zero determinant in terms of transformations
- recognise the identity matrix
- find the inverses
  - of non-singular 2×2 matrices
  - of non-singular 3×3 matrices
- understand that for non-singular matrices $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and this can be extended to the product of more than two matrices
- understand the relationship between the transformations represented by $\mathbf{A}$ and $\mathbf{A}^{-1}$.