MTH 232

ELEMENTARY DIFFERENTIAL EQUATION

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Course Title	Elementary Differential Equation
Course Developer	Ajibola Saheed Oluwatoyin National Open University of Nigeria 14/16 Ahmadu Bello Way Victoria Island, Lagos
Course Writer	Ajibola Saheed Oluwatoyin National Open University of Nigeria 14/16 Ahmadu Bello Way Victoria Island, Lagos
Course Editor	Ajibola Saheed Oluwatoyin National Open University of Nigeria Victoria Island, Lagos.
Programme Leaders	Proffessor Adebanjo National Open University of Nigeria Victoria Island, Lagos

NATIONAL OPEN UNIVERSITY OF NIGERIA

National Open University of Nigeria
Headquarters
14/16 Ahmadu Bello Way
Victoria Island
Lagos

Abuja Office
No. 5 Dar es Salaam Street
Off Aminu Kano Crescent
Wuse II, Abuja
Nigeria

e-mail: centralinfo@nou.edu.ng
URL: www.nou.edu.ng

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MODULE 1

UNIT 1 INTRODUCTION TO THE NATURE OF DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

The subject of differential equation constitutes a part of mathematics that plays an important role in understanding physical sciences. In fact, it is the source of most of the ideas and theories which constitute higher analysis. In physics, engineering, chemistry and many other disciplines it has become necessary to build a mathematical model to represent certain problems. These mathematical models often involve the search for an unknown function that satisfies an equation in which derivatives of the unknown function play an important role. Such equations are called **differential equations.** The primary purpose of differential equations is to serve as a tool for studying change in the physical world.

You may recall that if $y = f(x)$ is a given function then its derivation $\frac{dy}{dx}$ can be interpreted as the rate of change of y respect to x . sir Isaac Newton observed that certain important laws of natural sciences can be phrased in terms of equations involving rates of change. The most famous example of such a natural law is Newton's second law of motion. Newton was able to model the motion of a particle by an equation involving an unknown function and one or more of its derivatives.

As early as the 1690s, scientists such as Isaac Newton, Gottfried Leibniz, Jacques Bernoulli, Jean Bernoulli and Christian Huygens were engaged in solving differential equations. Many of the methods which they developed are in use till today. In the eighteenth century the mathematicians Leonhard Euler, Daniel Bernoulli, Joseph Lagrange and others contributed generously to the development of the subject. The pioneering work that led to the development of ordinary differential equations as a branch of modern mathematics is due to Cauchy, **Riemann, Picard, Poincare,** Lyapunov, Birkhoff and others.

Not only are differential equations applied y physicists and engineers, but they are being used more and more in certain biological problems such as the study of animal populations and the study of epidemics. Differential equations have also proved useful in economics and other social sciences. Besides its uses, the theory of differential equations involving the interplay of functions and their derivatives, is interesting in itself.

2.0 OBJECTIVES

In this unit, we introduce the basic concepts and definitions related to differential equations. We also express some of the problems of physical and engineering interest in terms of differential equations in this unit. We shall give the methods of solving differential equations of various types in Units 2 and 3. The physical problems formulated in this unit will be solved in units 2 and 3. The physical problems formulated in unit will be solved in unit 3 after we have learnt the various methods of solving the first order equations.

- Distinguish between the order and degree of a differential equation;
- Define the solution of an ordinary differential equation;
- Identify an initial value problem;
- State and use the conditions for existence and uniqueness of first order ordinary different equations;
- Derive differential equations for some physical problems.

3.0 MAIN CONTENT

3.1 Basic Concepts

In this section we shall define and explain the basic concepts in the theory of differential equations and illustrate them through examples.

In unit 1 of MTH 112 Differentialcalculus, you have learnt that if a relation y = y(x) involving two variables x and y exist then we call x the **independent variable** and y the **dependent variable.**

Further, suppose we are given a relation of the type $f(x,t_1, t_2, \ldots\ldots, t_n) = 0$ involving (n + 1) variables (x and $t_1, t_2, \ldots\ldots\ldots, t_n$,); where the value of x depends on the values of the variables $t_1, t_2, \ldots\ldots, t_n$ are called independent variables and x is called the dependent variable and y is dependent variable. Similarly, if $z = x^2 + y^2 + 2xy$, the x and y are independent variables and z is a dependent variable.

Any equation which gives the relation between the independent and dependent variables and the derivatives of dependent variables is called a **differential equation .**

In general, we have the following definition.

Definition: An equation involving one (or more) dependent variable derivatives with respect to one or more independent variables in called a differential equation.

For example, $\frac{dy}{dx} = \cos x$ …(1)

$$Y = x\frac{dy}{dx} + \frac{a}{d\,y/dx}$$

….(2)

$$y^2 \frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = nzx$$

….(3)

are all differential equations.

In Eqn. (3), $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are partial derivatives of z w.r.t.x and y respectively. The partial derivatives of a function of two variables z = f(x, y) w.r.t to one of the independent variables, can be defined as

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x, y) = \frac{\lim}{\Delta y \to 0}\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

when the limit exist and is independent of the path of approach. $\frac{\partial z}{\partial x}$ is the first order partial derivatives of z w.r.t.x and is obtained by differentiating z w.r.t.x treating y as a constant. It is read as 'del z by del x'. similarly, first order partial derivative of z w.r.t. y is denoted by $\frac{\partial z}{\partial y}$ (or $\frac{\partial f}{\partial y}$ or $f_y(x, y)$, so that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_y(x, y) = \frac{\lim}{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Note that equations of the type

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

are not differential equations. In this equation, if you expand the left hand side then you will find that the left hand side is the same as the right hand side. Such equations are called **identities.** Moreover, a differential equation may have more than one dependent variable. For instance,

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + = y$$

is a differential equation with dependent variable x and y and the independent variable t.

Differential equations are classified into various types. The most obvious classification of differential equations is based on the nature of the dependent variable and its derivatives (or derivatives) in the equation. Accordingly, we divide differential equations into three classes: ordinary, partial and total. The following definitions give these three types of equations.

Definition: A differential equation involving only ordinary derivatives (that is, derivatives with respect to a single independent variable) is called an **ordinary differential equation** (abbreviated as ODE).

Equations

$$\frac{d^2y}{dx^2} + y = x^{2,}$$

$$\left(\frac{dy}{dx}\right)^2 = [\sin(xy) + 2]^2, \text{ and}$$

$$y = x\frac{dy}{dx} + r\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

are all ordinary differentia equations.

The typical form of such equations is

$$g\left[x, y(x), \frac{dy(x)}{dx}, \frac{d^2y(x)}{dx^2}, \ldots\ldots, \frac{d^n y(x)}{dx^n}\right] = 0 \qquad \ldots(4)$$

whenever we talk of Eqn. (4) we assume that g is known real valued function and the unknown to be determined is y. secondly, in an ordinary differential equation, y and its derivatives are evaluated at x.

It may be noted that the equation

$$\left(\frac{dy}{dx}\right)_x = (y)_{x+1}$$

is not a differential equation. This is because y is evaluated at (x + 1) whereas $\frac{dy}{dx}$ is evaluated at x.

Similarly, the equation

$$\frac{dy(x)}{dx} = \int_0^x e^{xs} y(s)ds$$

is not a differential equation since the unknown y is appearing inside an integral. Also, in this case the values of y on the right hand side of the equation depends on the interval 0 to x, whereas, in a differential equation, the unknown y has to be evaluated **only at x.**

Let us now define partial equation.

Definition: Differential equation containing partial derivatives of one (or more) dependent variable with respect to two or more independent variable is called a **partial differential equation. (abbreviated AS PDE)**

The examples of differential equations are

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z = 0,$$

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t^2} + xtu = 0.$$

You may also note that Eqns. (1`) and (2) given earlier are ordinary differential equations, whereas, Eqn. (3) is a partial differential equation.

And now an exercise for you.

Besides ordinary and partial differential equations, namely, total differential equations. Before giving you the definition of total differential equations, we ascribe a meaning to the symbols dx and dy which permit us to manipulate the derivative $\frac{dy}{dx}$ as a quotient of two function y = f(x), we define, the differential of y, by
Dy = f' (x) dx

If u = f(x,y) be a function of two independent variables x and y, then we know that

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

and

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Let Δu be the change in u when both x and y change by the amounts Δx and Δy respectively, so that $\lim_{\Delta x, \Delta y \to 0} \Delta u = du$. Here du is called the **total differential**.

The total differential du of a function u (x, y) is defined as

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad \ldots(5)$$

or

$du = u_x dx + u_y dy$

For instance,

If $\quad u = x^2y - 3y$

then

$$Du = 2xy\,dx + (x^2 - 3)\,dy$$

Now consider the relation u (x, y, z) = c where x, y, z are variables and c is a constant.

Then

$$Du = 0$$

$$\Rightarrow \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz = 0$$

Here, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are known functions of x, y and z, and therefore the above equation can be put in the form

$$P\,dx + Q\,dy + R\,dz = 0$$

Which is called the **total differential equation** in three variables. In this equation any one of the variables x, y, z can be treated as independent and the remaining two are then the dependent variables.

Similarly, if u = u(x, y, z, t) then corresponding total differential equation will be of the form

$$P\,dx + Q\,dy + R\,dz + T\,dt = 0.$$

Remember that a total differential equation always involves three or more variables.

We now give the following definition.

Definition: A total differential equation contains two or more dependent variables together with their derivatives with respect to a single independent variable which may, or may not, exist explicitly into the equation.

For example, equations

$$Yz (1 + 4 xz) dx - xz (1 + 2 xz) dy - xydz = 0,$$

and

$$\frac{xdx + ydy = zdz}{\sqrt{x^2 + y^2 + z^2}} + \frac{zdx - xdz}{x^2 - z^2} + 2ax^2dx + 3by^2dy + 3cz^2dz = 0.$$

are total differential equations.

We shall be dealing with only ordinary differential equations in Modules 1ansd 2 and devote Modules 3 and 4 study total and partial differential equations.

We next consider the concepts of order and degree of a differential equation on the basis of which differential equations can be further classified.

We all know that the **nth derivative** of a dependent variable with respect to one or more independent variables is called a derivative of order n, or simply an nth order derivative.

For example, $\frac{d^2y}{dx^2}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\, \partial y}$ are second order derivatives and $\frac{d^3z}{dx^3}, \frac{\partial^2 z}{\partial x^2 \partial y}$ are third order derivatives.

Definition: The **order** of a differential equation is the order of the highest order derivative appearing in the equation. For instance, the equation

$\frac{d^2y}{dx^2} + y = x^2$ is of **second** order …(6)

(because the highest order derivative is $\frac{d^2y}{dx^2}$, which is of second order), whereas

$(x + y)\left(\frac{dy}{dx}\right)^2 - 1$, is of **first** order …(7)

(highest order derivative is $\frac{dy}{dx}$).

Similarly, equation

$$\left[\frac{d^3y}{dx^3}\right]^2 + 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + x^2\left[\frac{dy}{dx}\right]^3 = 0 \text{ is of \textbf{third} order} \qquad \ldots(8)$$

whereas, $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} = 0$ is of **second** order. …(9)

Note that the order of a differential equation is a positive integer.

Also, if the order of a differential equation is 'n' then it is not necessary that the equation contains some or all lower order derivatives or independent variables explicitly. For instance, equation $\frac{d^4y}{dx^4} = 0$, is a fourth order differential equation.

Definition: The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and any fractional power of the derivatives or negative power. For example Equations. (6) and (9) are of **first** degree and Equations. (7) and (8) are of **second** degree.

Equation

$$y - x\frac{dy}{dx} = r\sqrt{1+\left(\frac{dy}{dx}\right)^3} \qquad \ldots.(10)$$

is of degree **three** for, in order to make the equation free from radicals, we need to square both the sides, so that

$$\left[y\text{-}x\frac{dy}{dx}\right]^2 = r^2\left[1+\left(\frac{dy}{dx}\right)^3\right]$$

since the highest exponent of the highest derivative, that is, $\frac{dy}{dx}$ is three, thus by definition the degree of Equation. (10) is three.

Similarly, Equation. (2), that is,

$$y - x \ \frac{dy}{dx} + \frac{a}{dy/dx} \text{ is of degree two.}$$

This is because we multiplied through by $\frac{dy}{dx}$ to remove negative power of $\frac{dy}{dx}$ and get

$$Y \ \frac{dy}{dx} = x \left[\frac{dy}{dx}\right]^2 + a.$$

You may now try the following exercise.

We now classify the differential equations depending upon the degree of dependent variables and its derivatives into two classes, namely, linear and non-linear.

Definition: When, in an ordinary or partial differential equation, the dependent variables and its derivatives occur to the degree only, and not as higher powers or products, we call the equation **linear.**

The coefficients of a linear equation are therefore either constants or functions of the independent variable or variables. If an ordinary differential equation is not linear, we call it **non-linear.**

For example, the equation

$\frac{d^2y}{dx^2} + y = x^2$, is an ordinary linear differential equation..

However $(x + y)^2 \ \frac{dy}{dx} = 1$ is an ordinary non-linear equation, because of the presence of terms like $y^2 \ \frac{dy}{dx}$ and $2xy\frac{dy}{dx}$.

Similar, equation

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \, \partial y}\right)^2 = 0$, is a non-linear partial differential equation.

Further, if a partial differential equation is not linear, it can be **quasi-linear, semi-linear** or **non-liner.** We will discuss conditions for these classifications in the later part of this course.

You may now try this exercise.

Normally when we encounter an equation, our natural curiosity is to enquire about its solution. But, then it is natural for you to ask as to what exactly is the meaning of a solution of a differential equation. In the next section you will find an answer to this question. There, we also answer many more questions like

i) Under what conditions does the solution of a given ordinary differential equation exists?
ii) If the solution exists, then is it a unique solution?

3.2 Solution of a Differential Equation

You have seen that the general ordinary differential equation of the nth order as given by Equation (4) is

$$g\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$$

using the prime notation for derivatives ($y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}, \ldots.\ Y^n = \frac{d^ny}{dx^n}$) we can rewrite Equation . (4) in the form

$$y^{(n)} = f(x, y, y', y'', \ldots.. y(^{(n-1)}) \qquad \ldots(11)$$

Let us assume that we can solve Eqn. (11) for $y^{(n)}$, that is, Eqn. (11) can be written in the form

$$Y^{(n)} = f(x, y, y', y'', \ldots\ldots., y^{(n-1)}) \qquad \ldots.(12)$$

It is normally a simple task to verify that a given function = $\varnothing(x)$ satisfies an equation like (11) or (12). All that is necessary is to compute the derivatives of y and to show that y = $\varnothing(x)$ and its derivatives, when substituted in the equation, reduce it to an identity in x. if such a function y exists, we call it a solution of the Eqn. (11) or (12).

However, usually we assume that

i) $y = \phi(x)$ is defined on some interval [a, b];
ii) y is n times differentiable on [a, b];
iii) We assume that y has a right derivative at point a and a left derivative at b;
iv) $y = \phi(x)$ can be real valued function or complex valued function (range is a subset of **C**) of x.

We now give the definition of the solution of an ordinary differential equation.

Definition: A real or complex valued function $y = \phi(x)$ defined on an interval I is called a solution or an integral of the differential equation $g(x, y, y', \ldots, y^{(n)}) = 0$ if $\phi(x)$ is n time differentiable and if x, $\phi(x)$, $\phi'(x)$, ….., $\phi^{(n)}(x)$ satisfy this equation for all x in I.

For example, the first-order differential equation

$$\frac{dy}{dx} = 2x - 4x$$

Note: I could represent any interval]a,b[, [a, b]]0, ∞[,]$-\infty, \infty$] and so on

has the solution y = 2x + 1 in the interval $I = \{x : -\infty < x < \infty\}$.
This can be checked by computing y' = 2 and 2y - 4x = 2(1 + 2x) – 4x

In the same way you can check that $y = 1 + 2x + ce^{2x}$, in the interval $-\infty < x < \infty$, is also a solution of this equation for any constant c.

In the above definition you might have noticed that a solution of (11) is real valued or complex valued. In case y is real value it is called a **real solution.** If y is complex valued, it is called **complex solution.** We are usually interested in real solution of Eqn. (11). To help you clarify what we have just said let us take some more examples.

Example 1: Show that for any constant c, the function $y(x) = ce^x$, $x \in R$ is a solution of

$$\frac{dy}{dx} = y, \qquad x \in \mathbf{R} \qquad \ldots(13)$$

Solution: Here I is R itself. For any $x \in \mathbf{R}$, we know that

$$\frac{dy}{dx} = \frac{d}{dx}(ce^x) = c\frac{dy}{dx}(e^x) = ce^x = y$$

which shows that y satisfies equation(13).

Example 2: show that for real constants a and b the functions y (x) = a cos 2x and z(x) = b sin 2x are solutions of the equation bolow;

$$\frac{d^2y}{dx^2} + 4y = 0,\ x \in \mathbf{R} \qquad \ldots(14)$$

solution: We will first show that z (x), x ∈ **R** is a solution of Equation. (14).

Now $\quad \frac{d}{dx}[z(x)] = \boxed{\frac{d}{dx}}(b \sin 2x) = 2b\cos 2x.$

$\therefore \frac{d^2}{dx^2}[z(x)] = \boxed{\frac{d}{dx}}(2b\cos 2x) = -\ 4b \sin 2x = -\ 4z(x).$

thus,

$$\frac{d^2y}{dx^2} + 4z(x) = 0,\ x\square\mathbf{R}.$$

That is, z satisfies Equation (14).

By now you must have understood the meaning of z satisfiying Equation (14)._. It means that Equation (14). holds when y is replaced by z. similarly, you can check that y(x) = a cos 2x is also a solution of Equation (14).

. You may **observe** here that the sum y(x) + z(x) that is, a cos 2x + b sin 2x is again a solution of. Equation (14).

Let us consider another example.

Example 3: Shown that $y(x) = e^{ix}$**,** $x\square\mathbf{R}$ is a solution of

$$\boxed{\frac{d^2y}{dx^2}} + y = 0,\ x\ \square\mathbf{R}$$

Solution: We have,

$$\frac{dy}{dx} = \boxed{\frac{d}{dx}}(e^{ix})\backslash = i\, e^{ix}$$

and

$$\boxed{\frac{d^2y}{dx^2}} = \boxed{\frac{d}{dx}}(ie^{ix}) = i^2 e^{ix} = -e^{ix} = -\, y(x)$$

thus, $\boxed{\frac{d^2y}{dx^2}} + y = 0$

In the examples taken so far, you have seen that the solution(s)differential equation exist. In Example (1) and (2) the solutions were real valued whereas, the solution in Example (3) was a complex valued function. But, there are equations for which real solution does not exist. Suppose that we are looking for real roots of the equation $x^2 + 1 = 0$. We know that it does not exist. Likewise, the equation

$$\left|\frac{dy}{dx}\right| + y^2 + 1 = 0$$

does not admit a real solution.

Similarly, the equation $\sin\left(\frac{dy}{dx}\right) = 2$ does not admit a real solution, because real value of the sin of a real function lies between – 1 and + 1.

You may now try the following exercises.

In the above discussion you must note that a differential equation may have more than one solution. It may even have infinitely many solutions. For instance, each of the functions $y = \sin x$, $y = \sin x + 3$, $y = \sin x - \frac{4}{5}$ is a solution of the differential equation $y' = \cos x$. but from your knowledge of calculus you also know that any solution of the equation is of the form.

$$Y = \sin x + c \qquad \ldots(15)$$

Where c is a constant. If we regard c as arbitrary then relation (15) represents the totality of all solutions of the equation. Thus, we can represents even the infinitely many solutions by a simple formula involving arbitrary constants. Accordingly, we classify various types of solutions of an ordinary differential equation as follows.

Definition: The solution of the nth order differential equation with arbitrary 'n'nconstants is called its **general solution.**

Definition: Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a **particular solution.**

For example, y = a cos 2x + b sin 2x, involving two arbitrary constants a and b, is the general solution of the second order equation $\frac{d^2y}{dx^2} + 4y = 0$ (ref. Example 2) whereas, y = 2 sin 2x + cos 2x is its particular solution (taking a = 1 and b = 2).

In some cases there may be further solutions of given equation which cannot be obtained by assigning a definite value to the constant in the general solution. Such a solution is called a **singular solution** of the equation. For example, the equation

$$y'^2 - xy' + y = 0 \qquad \ldots(16)$$

has the general solution $y = cx - c^2$. A futher solution of Eqn. (16) is $y = \frac{x^2}{4}$. Since this solution cannot be obtained by assigning a definite value to c in the general solution, it is a singular solution of Eqn. (16).

Thus, we have seen the various types of solution of an ordinary differential equation. We have also seen that a solution of a differential equation may or may not exist. Even if a solution exists, it may or may not be unique.

We now try to find the conditions under which the solution of a given ordinary differential equation exists and is unique. Here, we shall confine our attention to the first order ordinary differential equations only. Let us consider the general first order equation.

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(17)$$

In Eqn. (17) we assume that f is known to us. You may be surprised to know that, though this equation looks simple, it is very difficult to get its explicit solution. For clearity, let us look at the following examples.

Example 4: Does the solution y(x) of an ordinary differential equation

$$\frac{dy}{dx} = f(x), \quad \text{where} \quad f(x) = 0 \text{ for } x < 0$$
$$= 1 \text{ for } x \geq 0$$

exist $\forall$ x $\in$ **R**?

Solution: The function defined by

$$Y(x) = \begin{cases} c for x < 0 \\ x + c \text{ for } x \geq 0 \end{cases}$$

Satisfies this equation. at the same time this function has no derivative at x = 0, because of the discontinuity of y(x) at x = 0.

Thus, this differential equation has no valid solution for x = 0.

However, y(x) defined above is the solution of the given differential equation at all points other than x = 0.

Let us look at another example.

Example 5: Does the equation $\boxed{\frac{dy}{dx}} = - e^{-y}x$ have a unique solution?

Solution: Rewrite the above equation in the form

$$\boxed{\frac{dy}{dx}}(e^y) = - x$$

Integrating, we get the solution of given equation as

$$e^y = - \frac{x^2}{2} + A,$$

or $$y = \text{In}\left(-\frac{x^2}{2} + A\right)$$

where A is an arbitrary constant.

You know that in x is defined for positive values of x only. So, the solution of the given differential equation will exists as long as $\left(-\frac{x^2}{2} + A\right) > 0$. Clearly A > 0. Also, for different values of A we get different solutions. Moreover these solutions have different intervals of existence. Thus, the solution of a given differential equation is not unique.

As regards the non-unique solutions, it is obvious that the cause for the non-uniqueness is the arbitrariness of A, (but for A > 0). Thus, we would like to impose some condition on the solution which might determine A. one such condition is to specify the value of y at some point x_0 where x_0 is in the interval of existence of y. such a condition is called **initial condition** and the problem of solving a differential equation together with the initial conditions is called the **initial value problem** (IVP). In other words, initial value problem is the problem in which we look for the solution of a given differential equation which satisfies certain conditions at a single of the independent variable. Thus, the first order initial value problem s

$$\left.\begin{aligned} \frac{dy}{dx} &= f(x,y), \\ y(x_0) &= y_0 \end{aligned}\right\} \qquad \ldots(18)$$

From Example 4 and 5 mainly two questions arise:

1) Under what conditions does an initial value problem of the form (18) have at least one solution?

2) Under what conditions does that problem have a unique solution, that is, only one solution?

The above questions are answered by a theorem, known as **Existence Uniqueness Theorem 1.** We shall now state this theorem for the first order differential equation.

Theorem 1: (Existence – Uniqueness):
If f(x, y) is continuous at all points (x, y) in some rectangle (see Fig. 1.

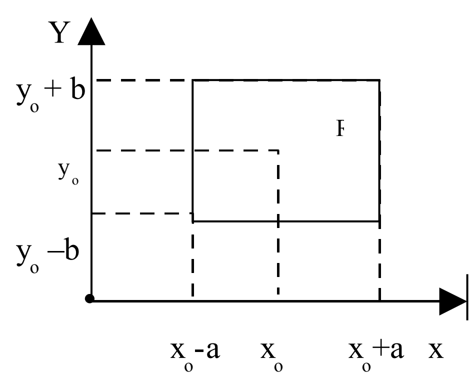

Fig. 1: Rectangle R

R: $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R, say

$|f(x, y)| \leq k \square (x, y)$ in R. …(19)

then the IVP (18) has **at least one solution** y(x) defined for all x the interval $|x - x_0| <$ **h**,

further, if $\frac{\partial f}{\partial y}$ is continuous for all (x, y) in R and bounded say,

$\left|\frac{\partial f}{\partial y}\right| \square$ **M,** $\square (x, y)$ in R …(20)

then the solution y(x) is the **unique solution** for all x in that interval $|x - x_0| < h$,

Note: A function f(x, y) is aid to be **bounded** when (x, y) varies in a region in the xy-plane and if there is a number k such that $|f| \square$ k when (x, y) is in that region. For example.

$F = x^2 + y^2$ is bounded, with K = 2 if $|x| < 1$ and $|y| < 1$.

We shall not be proving this theorem. The proof of this theorem requires familiarity with many other concepts which are beyond the scope of this course. However, we which, we give some remarks which may be helpful for a good understanding of the theorem.

Remark: Since y' = f(, y), the condition (19) implies that $|y'| \leq k$, that is, the slope of any solution curve y(x) in R is at least – k and atmost k. Hence a solution curve which passes through the point (x_0, y) must lie in the shaded region in Fig. 2 bounded by the lines 1_1, and 1_2 whose slopes are – k and k, respectively

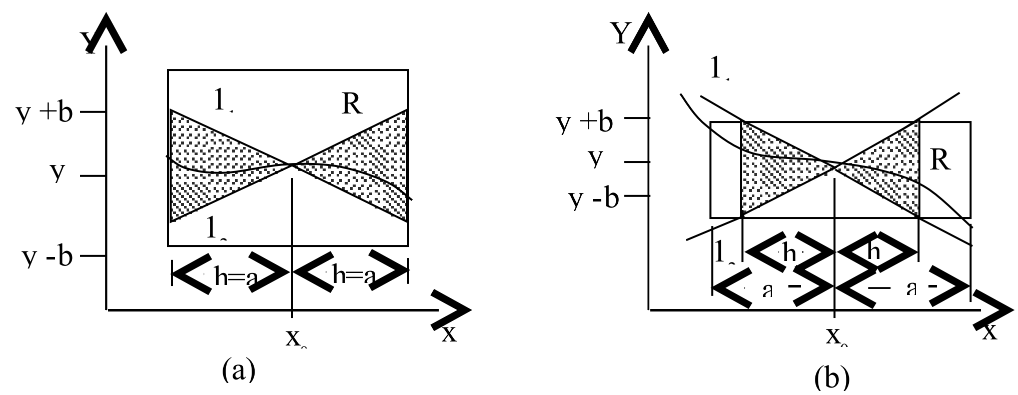

Fig. 2

Now two different cases may arise, depending on the form of R.

i) We may have $\frac{b}{k} \geq a$. Therefore, h = a, which asserts that the solution exists for all x between $x_0 - a$ and $x_0 + a$ (see Fig. 2 (a)).

ii) We may have $\boxed{\frac{b}{k}} < a$. Where, $h = \boxed{\frac{b}{k}}$, and we concluded that the solution exists for all x between $x_0 - \boxed{\frac{b}{k}}$ and $x_0 + \boxed{\frac{b}{k}}$. In this case, for larger or smaller values of x; the solution curve may leave the rectangle R (see fig. 2 (b)). Since we have not assumed anything about f outside R, nothing can be concluded about the solution for those corresponding value of x.

The condition stated in Theorem 1 are **sufficient** but not necessary and can be relaxed. For example, by the mean value theorem of differential calculus, we have (ref. Theorem 1).

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\frac{\partial f}{\partial y}$$

where (x, y_1) and (x, y_2) are assumed to be in R. From condition (20) then it follows that

$$|f(x, y_2) - f(x, y_1)| \leq M|(y_2 - y_1)|$$

condition (20) may be replaced by the condition (21) which is known as a Lipschitx condition, named after the German mathematician, Rudolf Lipschitz (1831 – 1903).

Thus, we can say that for the existence of the solution of the IVP (18), we must have

i) f continuous in T.
ii) f bounded in T.

Further the solution is unique if in addition to (i) and (ii), we have

iii) $\boxed{\frac{\partial f}{\partial y}}$ continuous in T.

iv) $\boxed{\frac{\partial f}{\partial y}}$ bounded in T (or, Lipschitx condition)

However, if the above conditions do not hold, then the IVP (18) may still have either (a) no solution (b) more than one solution (c) a unique solution.
This is because theorem provides only sufficient conditions and not necessary. For instance, consider

$$\frac{dy}{dx} = 3y^{2/3},\ y\,(0) = 0.$$

Here, $f(x, y) = 3y^{2/3}$, $\frac{\partial f}{\partial y} = 3.\frac{2}{3}\ y^{-1/3}$ for $y \neq 0$. $\frac{\partial f}{\partial y}$ does not exists at $y = 0$. so $\frac{\partial f}{\partial y}$ is not bounded but, the solutions $y = x^3$ and $y = 0$ exist.

Let us examine conditions (i) – (iv) for a few differential equations through examples.

Example 6: Examine $\frac{dy}{dx} = y$,with $y\,(0) = 1$ for existence and uniqueness of the solution.

Solution: Here $f(x, y) =y$, $f_y(x, y) = 1$. Also $x_0 = 0$ and $y_0 = 1$.

In this case consider a rectangle T defined by

$T: |x – 0 |<a, |y – 1)<b$
Where a and b are positive numbers.

In any rectangle T (containing the point (0, 1)) the function f(x, y) is continuous and bounded. Hence the solution exists. Further $f_y(x, y)$ is also continuous and bounded in any such rectangle T. Therefore, the solution is unique.

You may verify that $y = e^x$ is a solution of the given equation satisfying the initial condition $y\,(0) = 1$. Hence, it is the unique solution.

However, if the initial condition is changed to $y(0) = 0$ then rectangle T will be of the form

$T: |x – 0| < a, |y – 0| <b$
And in that case $y = 0$ will be the unique solution for all x and y in any rectangle T containing (0, 0).

Example 7: Examine $\boxed{\ }$ = $\sqrt{|y|}$ when y (0) = 0, for existence and uniqueness of solutions.

Solution: Here f(x, y) = $\boxed{\sqrt{|y|}}$, $x_0 = 0$ and $y_0 = 0$. In this case consider the region T with $|x| < a$, $|y| < b$, a and b positive numbers. Function f(x,y) is continuous and bounded in any rectangle T, containing the point (0,0).

Hence solution exists. In order to test the uniqueness of the solution, consider the Lipschitz condition.

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{|\sqrt{|y_2|} - \sqrt{|y_1|}|}{|y_2 - y_1|}$$

for any region containing the line y = 0, Lipschitz condition is violated. Because for $y_1 = 0$ and $y_2 > 0$, we have

$$\frac{|f(x, y_2) - f(x, y_1)|}{|y_2 - y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, (\sqrt{y_2} > 0)$$

and this can be made as large as we please by choosing y_2 sufficiently small, whereas condition (21) requires that the quotient on the left-hand side of (21) does not exceed a fixed constant M.

therefore, the solution is not unique.

Further, it can be checked that the given problem has the following solutins

i) $y = 0 \ \forall \ x$

ii) $$y = \begin{bmatrix} \frac{1}{4}x^2 \text{for } x \geq 0 \\ -\frac{1}{4}x^2 \text{for } x \leq 0 \end{bmatrix}$$

Example 8: Examine $\boxed{\ }$ = f(x, y) = $\begin{cases} y(1-2x) \text{for } x > 0 \\ y(2x-1) \text{for } x < 0 \end{cases}$ with y(1) = 1 for existence and uniqueness of solution.

Solution: Here $x_0 = 1$ and $y_0 = 1$. Rectangle T can be any rectangle containing point (1, 1). You may note that the function is not defined at x = 0. It is discontinuous at x = 0. Thus, at x = 0 the solution does not exist. At all other points the function

$$f(x, y) = \begin{cases} y(1-2x) \text{ for } x > 0 \\ y(2x-1) \text{ for } x < 0 \end{cases}$$

is continuous and bounded in T with (x) = 1. Hence, the solution, exists and is unique for all x other than x = 0. further, you may verify that

$$y = x^{x-x^2} \text{ for } x >= 0$$

and $$y = e^{x^2-x} \text{ for } x < 0$$

is the unique solution of the given problem for all x other than x = 0

you may now try the following exercise.

From the definitions given in page 14, you may have realized that the general solution of a first order differential equation normally contains one arbitrary constant which is called a **parameter.** When this paratmeter is various values, we obtain a one parameter family of curves. Each of these curves is a particular solution or integral curve of the given differential equation, and all of them together constitute its general solution. On the other hand, we expect that the curve of any one-parameter family are integral curves of some first order differential equation. In general we pose a problem: given an n-parameter family of curves, can thus say that differential equations arise from a family of curves. In the next section we shall take up this.

3.3 Family of Curves and Differential Equations

Let us consider a family of straight lines

$$Y = mx + c \qquad \ldots(22)$$

Which is a two-parameter family of curves, parameters being m and c.

It is clear from Eqn. (22) that y can be treated as a function of x, $x \in R$. Differentiating Eqn. (22) w.r.t.x, we have

$$y' = m \qquad \ldots(23)$$

Again, differentiating (23) we get

$$y'' = 0 \qquad \ldots(24)$$

Equation (23) and (24) are differential equations of order one and two respectively. The way in which we have arrived at Eqn. (23) or (24) is clear. We have actually tried to eliminate the parameters, or constants, m and c and the result is Eqn. (23) or (24).

In general, we represent one-parameter family of curves by an equation

$$F(x, y, a) = 0 \qquad \ldots(25)$$

where a is a constant.

In Eqn. (25), let us regard y as a function of x and differentiate if w.r.t.x. Suppose we get

$$G(x, y, y', a) = 0 \qquad \ldots(26)$$

In case, we are able to eliminate the constant a between Eqn. (25) and (26), then we have a relation connecting x, y and y', say

$$h(x, y, y',) = 0 \qquad \ldots(27)$$

Equation (27) is an ODE of order one. In particular, if Eqn. (25) has the form

$$\psi(x, y) = a \qquad \ldots(28)$$

then the elimination of the constant a from Eqn. (28) leads us to the differential equation

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' = 0 \qquad \ldots(29)$$

for example, $x^2 + y^2 = a^2$ …(30)

is the equation of the family of all **concentric circles** with centre at the origin (fig. 3)

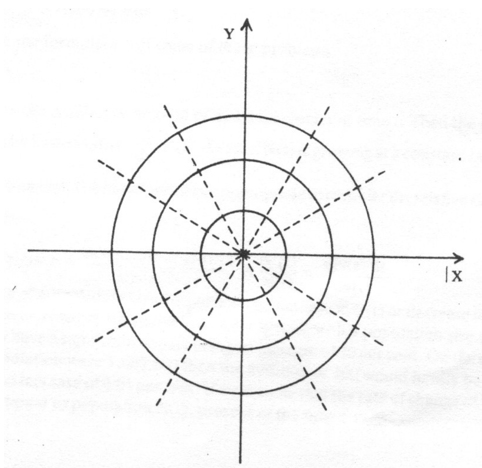

Fig. 3

For different values of a , we get different circles of the family. Differentiating Eqn. (30) with respect to x, we get

$$2x + 2y \boxed{\phantom{\frac{dy}{dx}}} = 0.$$

Or $$x + y \boxed{\phantom{\frac{dy}{dx}}} = 0,$$ as the differential equation of the given family of circles.

Continuing with equation y = mx + c, if we regard only c as an arbitrary constant to be eliminated, then y' = m, represents the required differential equation. Geometrically, for a fixed m, y' = m represents a family or **straight lines**(in the plane) **whose slope is m** (see fig. 4).

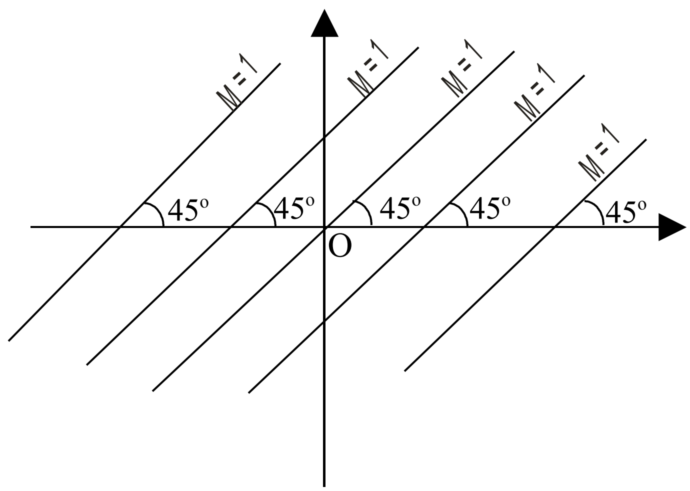

Fig. 4

On the other hand, if we assume that in equation y = mx + c both m and c are constants to be eliminated, then equation y'' = 0 represents the required differential equation. Geometrically, it is the family of the **straight lines in the plane** (see Fig. 5)

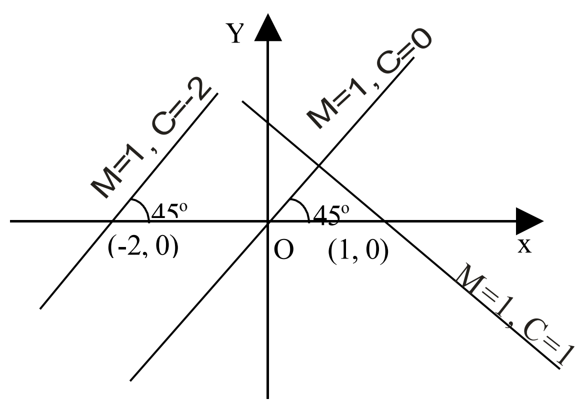

Fig. 5

You may now try the following exercise.

In the introduction of this unit we mentioned that there are many problems of physical and engineering interest which give rise to differential equations. In other words, we can say that some problems have representations through the use of differential equations. In the next section we shall take up such problems.

3.4 Differential Equations Arising From Physical Situations

In this section we shall show that differential equations arise not only out of consideration of families of geometric curves, but an attempt to describe physical problems, in mathematical terms, also result in differential equations.

The initial-value problem

$$\left.\begin{aligned} \frac{dy}{dx} &= ky \\ y(t_0) &= t_0 \end{aligned}\right\} \qquad \ldots(31)$$

where k is a constant of proportionality, occurs in many physical theories involving either **grow** or **decay.** For example, in biology it is often observed that the rate at which certain bacteria grow is proportional to the number of bacterial present at any time. In physics an IVP such as Eqn. (31) provides a model for approximating the remaining amount of a substance that is disintegrating, or decaying, through radioactivity. The differential Eqn. (31) could also determine the temperature of a cooling body. In chemistry, the amount of a substance remaining during certain reactions is also described by Eqn. (31).

Let us now see the formulation of some of these problems.

I: Population Model

Let N(t) = denote the number or amount of a certain species at time t. then the growth of N(t) is given by its derivative $\frac{d}{dt}$ N(t). Thus, if N(t) is growing at a constant rate, N(t) = β, a constant. It is sometimes more appropriate to consider the relative rate of growth defined by

$$\text{relative rate of growth} = \frac{\text{actual rate of growth}}{\text{size of } N(t)} = \frac{N'(t)}{N(t)} = \frac{dN(t)/dt}{N(t)}$$

The relative rate of growth indicates the percentage increase in N(t) or decrease in N(t). For example, an increase of 100 individuals for a species with a population size of 500 would probably have a significant impact being an increase of 20 percent. On the other hand, if the population were 1,000,000 then the addition of 100 would hardly be noticed, being an increase of 0.01 percent. If we assume that the rate of change of N at time t is proportional to population N(t), present at the time t then,

$$\frac{d}{dt}N(t) \propto N(t)$$

which is written as

$$\frac{d}{dt}N(t) = k\,N(t), \qquad \ldots(32)$$

where k is a constant

if N increases with t, then $k > 0$ in Eqn. (32)

If N decreases with t, $k \le 0$ in Eqn. (32).

Normally, we have the knowledge of the population, say N_0, at some initial time t_0. so along with Eqn. (32) we have

$$N(t_0) = N_0. \qquad \ldots(33)$$

Thus, the population N(t) at time t can be found by solving Eqn. (32) with condition (33). We shall reconsider this problem with some modifications later in unit 3.

II: Newton's Law of Cooling

Here we deal with the temperature variations of a hot object kept in a surrounding which is kept at a constant temperature, say T_0. Under certain conditions, a good approximation to the temperature of an object can be obtained by using Newton's law of cooling. Let the temperature of the object be T. If $T \ge T_0$, we know that the object radiates heat to the surrounding resulting in the reduction of its (object's) temperature. Newton's law of cooling states that the rate at which the temperature T(t) changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature T_0 of the surrounding medium. That is

$$T(t) \quad T(t) - T_0$$

or $$T(t) = k\,(T(t) - T_0), \qquad \ldots(34)$$

where k is a constant of proportionality.

Constant $k < 0$, because the temperature of the body is reducing (we have assume that $T(t)$ T_0). We observe that that Eqn. (34) is a differential equation of order one.

III: Radioactive Decay

Many substances are radioactive. This means that the atoms of such a substance break down into atoms of other substances. In Physic, is has been noticed that the radioactive material, at time t, decays at rate proportional to its amount y(t). in other words,

$$y(t) = ky(t) \qquad \ldots(35)$$

where $k < 0$, is a constant. If the mass of the substance at some initial time, say $t = 0$, is A, then y(t) also satisfies the initial condition

$y(0) = A$.

Thus, the physical problem of radioactive decay is modeled by the IVP.

$$y(t) = ky(t),\ Y(0) = A \qquad \ldots(36)$$

where k is a constant.

Remark: I, II and III above indicate situations where differential equations occur naturally. In unit 3 we shall give the methods of solving these equations.

You may now try the following exercise.

4.0 CONCLUSION

We now conclude this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit, we have covered the following points:

1) a) An equation involving one (or more) dependent variables and its derivative w.r. to one or more independent variables is called a **differential equations.**

   b) A differential equation involving only ordinary derivatives is called an **ODE.**

   c) A differential equation involving partial derivatives is called a **partial differential equation (PDE).**

   d) The order of a differential equation is the order of the highest order derivative appearing in the equation.

   e) The **degree** of a differential equation is the highest exponent of the highest order derivative appearing in it after the equation has been expressed in the form free from radicals and fractions of the derivatives.

   f) In a differential equation, when the dependent variable and its derivatives occur in the first degree only, and not as higher powers or products, the equation is said to be **linear.**

   g) If an ordinary differential equation is not linear, it is said to be **non-linear.**

2) a) A real or complex value function = $\phi(x)$ defined on an interval I is called a **solution** of equation
   $g(x, y, y', y'', \ldots, y^{(n)}) = 0$
   if $\phi(x)$ is differentiable n times and if $\phi(x)$, $\phi'(x)$, $\phi''(x)$, ….,(x) satisfy the above equation for all x in I.

   b) The solution of the nth order differential equation which contains n arbitrary constants is called its **general solution.**

   c) Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called **a particular solution** of the differential equation.

d) A solution of a differential equation, which cannot be obtained by assigning definite values to the arbitrary constants in the general solution is called its **singular solution.**

3) a) Conditions on the value of the dependent variable, and its derivatives, at a single value of the independent variable in the interval of existence of the solution are called the **initial conditions.**

b) The problem of solving a differential equation together with the initial conditions is called the **initial value problem.**

4) The **sufficient** conditions for the existence of solution of the first order equation

$$\frac{dy}{dx} = f(x, y), \text{ with } y(x_0) = y_0,$$

in a region T defined by $|x - x_0| < a$ and $|y - y_0| < b$ are

i) f is continuous in T

and

ii) f is bounded in T.

further if the solution exists, then it is unique if, in addition to (i) and (ii), we have

(iii) $\frac{\partial f}{\partial y}$ is continuous in T.

iv) $\frac{\partial f}{\partial y}$ is bounded in T (or, Lipschitz condition is satisfied).

5) The general solution of a first order (nth order) differential equation represents one-parameter (nm-parameter) family of curves.

6) Many physical situation such as population model, Newton's law of cooling, radioactive decay, can be represented by first order differential equations.

6.0 TUTOR MARKED ASSIGNMENT

1. Which of the following are differential equations? Which of the differential equations are ordinary and which are partial?

a) $\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y^3 = 5x + 2$

b) $\frac{dy}{dx} = \int_x^x \sin[xy(s)]ds$

c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

d) $\frac{dy(x)}{dx} = 5x\, y\, (x + 1)$

e) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \int_x^x \sin[xy(s)]ds$

f) $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$

2. Find the order and degree of the following differential equations.

a) $\left(\frac{d^2y}{dx^2}\right)^{2/3} = 1 + 2\frac{dy}{dx}$

b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = x$

c) $\sin\left(\frac{d^2y}{dx^2}\right) + x^2y^2 = 0$

d) $\frac{dy}{dx} + y^3 = 0$

e) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = r\,\frac{d^2y}{dx^2}$

f) $\frac{\partial^4 z}{\partial x^4} + \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = x$

g) $x^2 (dx)^2 + 2xy\, dx\, dy + y^2(dy)^2 - z^2(dz)^2 = 0$

3. Classify the following differential equations into linear and non-linear.

a) $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} - z = 0$

b) $\dfrac{\partial^2 z}{\partial x^2} \dfrac{\partial^2 z}{\partial y^2} - \left(\dfrac{\partial^2 z}{\partial x\, \partial y} \right)^2 = 0$

c) $\dfrac{dy}{dx} = (x + y)^2$

d) $(1 - x^2) \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + n(n + 1)\, y = 0$

e) $(x^2 + y^2)^{3/2} \dfrac{d^2y}{dx^2} + \mu x = 0$

4. Verify that $y = \cos^{-1}\left(-\dfrac{x^2}{2}\right)$, and $2\cos y = -x^2$ are solutions of the equation $\sin y \dfrac{dy}{dx} = x$. Can you state the interval on which y is defined?

5. Verify that $y = \dfrac{1}{x}(\ln y + c)$ is a solution of the equation $\dfrac{dy}{dx} = \dfrac{y^2}{1 - xy}$ for every value of the constant c.

6. Verify that $y = e^{2x}$ and $y = e^{3x}$ are both solutions of the second order equation $y'' - 5y' + 6y = 0$
Can you find any other solution?

7. Examine $\dfrac{dy}{dx} = f(x, y) = \begin{cases} \dfrac{4x^3y}{(x^4 + y^2)}, \text{when } x \text{ and } y \text{ are not both zero} \\ 0, \text{when } x = y = 0 \end{cases}$

With $y(0) = 0$

For existence and uniqueness of the solution.

8. Assuming y to be a function of x, determine the differential equations by Eliminating the arbitrary constant (or constants) indicated in the following problems.

   a) $xy = c$ (arbitrary constant is c)
   b) $y = \cos(ax)$ (arbitrary constant is a).
   c) $y = A\cos(ax)$ (arbitrary constants are A and a).

9. In the following problems derive the differential equation describing the given physical situations.

   a) A culture initially has P_0 number of bacteria. Growth of the bacteria is proportional to the number of bacteria present. What is the number p of bacteria at any time t.
   b) A quantity of a radioactive substance originally weighing x_0 gms decomposes at a rate proportional to the amount present and half the original quantity is left after 2 years. Find the amount x of the substance remaining after t years.

7.0 REFERENCES/FURTHER READINGS

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Indira Gandhi National Open University School Of Sciences Mth-07

UNIT 2 METHODS OF SOLVING EQUATION OF FIRST ORDER AND FIRST DEGREE

CONTENTS

1.0 INTRODUCTION

In unit 1, we introduced the basic concepts and definitions involved in the study of differential equations. We discussed various types of solutions of an ordinary differential equation. We also stated the conditions for the existence and uniqueness of the solution of the first order ordinary differential equation. However, we do not seem to have paid any attention to the methods of finding these solutions. Accordingly, in this unit we shall confine our attention to the same.

In general, it may not be feasible to solve even the apparently simple equation $\frac{dy}{dx} = f(x, y)$ or $g(x, y, \frac{dy}{dx}) = 0$ where f and g are arbitrary functions. This is because no systematic; procedure exists for obtaining its solution for arbitrary forms of f and g. However, there are certain standard types of first order equations for which methods of solution are available. In this unit we shall discuss a few of them with special reference to their applications.

2.0 OBJECTIVES

After studying this unit, you should be able to

- Define separable equations and solve them;

- Define homogeneous equations and solve them;
- Obtain the solution of equations which are reducible to homogenous equations;
- identify exact equations;
- Obtain an integrating factor which may reduce a given differential equation into an exact one and eventually provide its solution.

3.0 MAIN CONTENT

3.1 Separation of Variables

You know that the problem of finding the tangent to a given curve at a point was solved by Leibniz. The search for the solution to the inverse problem of tangents, that is, given the equation of the tangent to a curve at any point to find the equation of the curve led Leibniz to many important developments. A particular mention may be made of the method of separation of variables which was discovered by Leibniz in 1691 by providing that a differential equation of the form

$$\frac{dy}{dx} = X(x)\,Y(y)$$

is integrable quadratures. However, it is John Bernoulli (1694) who is credited with the introduction of the terminology and the process of separation of variables.

In short, it is a method for solving a class of differential equations that arises quite frequently and is defined as follows:

Note: The process of finding the areas of plane regions if called quadrature

Definition: An equation of the form

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(1)$$

is called a **separable equation** or **equation in variable separable form** if f(x, y) can be put in the form

$$f(x, y) = X(x)\,Y(y), \qquad \ldots(2)$$

where X and Y are given functions of x and y respectively.

In other words, Eqn. (1) is a separable equation if f is a product of two functions, one of which is a function of x and the other is a function of y. Here X (x) and Y(y) are real value functions of x and y respectively.

For instance, equation $\frac{dy}{dx} = e^{x+y}$ is a separable equation, since $e^{x+y} = e^x.e^y$ (here $X(x) = e^x$ and $Y(y) = e^y$). The equation $\frac{dy}{dx} = x^2(y^2 + y^3)$ is also a separable equation. But the equation $\frac{dy}{dx} = e^{xy}$ is not a separable equation, because it is not possible to express e^{xy} as a product of two functions in which one is a function of x only and the other is a function of only. Similarly, equation $\frac{dy}{dx} = x + y$ is not a separable equation.

In order to solve Eqn. (1), when it is in variable separable form, we write it as

$$a(y) \frac{dy}{dx} + b(x) = 0 \qquad \ldots(3)$$

where a(y) and b(x) are each functions of only one variable

let us assume that there exist functions A and B such that A'(y) = a(y) and B'(x) = b(x). With this hypothesis, Eqn. (3) can be rewritten as

$$\frac{d}{dx} A(y(x)) + B'(x) = 0 \qquad \ldots(43)$$

[by chain rule $\frac{d}{dx} A(y(x)) = A'(y(x)) \frac{dy}{dx} = a(y(x)) \frac{dy}{dx}$]

Integrating Eqn. (4) with respect to x, we get

$$A(y(x)) + B(x) = c \qquad \ldots(5)$$

Where c is a constant.

Thus, any solution y of (3) is implicitly given by (5).

We now take up a few examples to illustrate this method.

Example 1: Solve $\frac{dy}{dx} = e^{x-y}$

Solution: This equation may be written as

$$\frac{dy}{dx} = e^x e^{-y}$$

or $e^y \dfrac{dy}{dx} = e^x$

or $\dfrac{d}{dx}(e^y) = e^x$

which, on integration, gives $e^y = e^x + c$, where c is a constant.

In case $e^x + c \geq 0$, then $y(x) = \text{In}\,(e^x + c)$.

Example 2: Solve the equation
$(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ with $y(0) = -1$.

Solution: The given equation can be rewritten as

$$\frac{dx}{1+x^2} + \frac{dy}{1+x^2} = 0.$$

Integrating, we get

$\text{Tan}^{-1}x + \tan^{-1}y = c.$

The initial condition that $y = -1$ when $x = 0$ permits us to determine the value of c that must be used to obtain the particular solution desired here. Since $\tan^{-1}0 = 0$ and $\tan^{-1}(-1) = -\dfrac{\pi}{4}$, $c = 0 - \dfrac{\pi}{4}$. Thus, the solution of the initial value problem is $\tan^{-1}x + \tan^{-1}y = -\dfrac{\pi}{4}$.

Let us look at another example.

Example 3: Solve $\dfrac{dy}{dx} = ky - my^2$, where $k > 0$ and $m > 0$ are real constants.

Solution: Let us write the given equation as

$$\frac{1}{ky - my^2}\frac{dy}{dx} = 1.$$

Now we will try do decompose the coefficient of $\dfrac{dy}{dx}$ into partial fractions.

Let $\dfrac{1}{y(k-my)} = \dfrac{A}{y} + \dfrac{B}{k-my}$ …(6)

Where A and B are constants to be determined.

From (6), we get
$1 = A(k - my) + By$
which gives $1 = Ak$ and $1 = \dfrac{Bk}{m}$

or $A = \dfrac{1}{k}$ and $B = \dfrac{m}{k}$

Hence $\dfrac{1}{y(k-my)} = \dfrac{1}{ky} + \dfrac{m}{k}\dfrac{1}{(k-my)}$

Thus the given differential equation can be rewritten as

$$\left(\frac{1}{ky} + \frac{m}{k}.\frac{1}{k-my}\right)\frac{dy}{dx} = 1, \qquad \text{…(7)}$$

for $y \neq 0$ and $k\text{-}my \neq 0$.

In the integration of Eqn. (7) the sign of y and k –my play a important role. We now discuss the following possible cases:

Case 1: y > 0 and k-my > 0 ($0 < y < \dfrac{k}{m}$).

For the case under consideration, Eqn. (7) can be exdpressed as

$$\frac{d}{dx}\left[\left(\frac{1}{k}\text{In}y\right) - \frac{1}{k}\text{In}(k-my)\right] = 1$$

which on integrating, yields

$$\frac{1}{k}\ \text{IN}y - \frac{1}{k}\text{In}\ (k\text{-}my) = x + c,$$

where c is a constant of integration. The above equation can be further express as
$\text{In}\ (y)^{1/k} - \text{In}\ (k - my)^{1/k} = x + c$

or $\left[\dfrac{y}{k-my}\right]^{1/k} = \exp(x + c)$

Case II: y < 0

When $y < 0$ then $k - my > 0$ because $m > 0$,. In this case, Eqn. (7) on integration can be written as

$$- \frac{1}{k} \text{In}(-y) \pm \frac{1}{k} \text{In}(k - my) = x + c$$

or $$\frac{1}{(-y)^{1/k}(k - my)^{\pm 1/k}} = e^{x+c}$$

Cases III: y> 0 and k – my < 0 (y > $\frac{k}{m}$).

In this case Eqn. (7) after integration gives

$$\frac{1}{k} \text{In}(y) - \frac{1}{k} \text{In}(-k + my) = x + c$$

or $$\left[\frac{y}{-k + my}\right]^{1/k} = e^{x+c}$$

You may now try the following exercises.

Many differential equations that are not separable can be reduced to the separable form by a suitable substitution. In the next section we shall study one classof such equations.

3.2 Homogeneous Equations

In this section we shall study equations like

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{3x^2 + y^2}$$

This is an example of homogenous differential equations. In 16921 Leibniz made known to the world the method of solving homogeneous equation differential equations of the first order.

Before we discuss the method of solving a homogeneous equation, we define homogeneuous functions of two variables x and y.

Definition: A real-value function h(x, y) of two variables x and y is called a homogeneous function of degree n, where n is a real number, if we have

$$h(\lambda x, \lambda y) = \lambda^n h(x, y)$$

for $\forall$ x, y and any constant $\lambda > 0$.

For example, $h(x,y) = x^3 + 2x^2y + 3xy^2 + 4y^3$ is homogeneous of degree three because $h(\lambda x, \lambda y) = \lambda^3 h(x, y)$

Also, $h(x,y) = x^2 \cos\left(\frac{y}{x}\right) + (\ln|x| - \ln|y|)\, xy$ is homogeneous function of degree 2 and $\frac{x^2}{x^2 + 2xy + y^2}$ is homogeneous of degree 0.

But, the function $h(x, y) = x^2 + 2xy + 4$ is not homogeneous because $h(\lambda x, \lambda y) \neq \lambda^n (x2 + xy + 4)$ for any value of n.

if h(x, y) is a homogeneous function of degree n, that is, $h(\lambda x, \lambda y) = \lambda^n h(x,y)$, then a useful relation is obtained by letting $\lambda = \frac{1}{x}$. This gives $\frac{1}{x^n} h(x,y) = h\left(1, \frac{y}{x}\right) = \varnothing\left(\frac{y}{x}\right)$ (say) or , $h(x,y) = x^n \varnothing\left(\frac{y}{x}\right)$.

We shall be particularly interested in the case where h(x,y) is **homogeneous of degree 0** that is, if $h(\lambda x, \lambda y) = \lambda^0 h(x,y) = h(x,y)$. We now give the following definition.

Definition: A differential equation

$$y' = f(x,y) \qquad \ldots(8)$$

is called a **homogenous differential equation** when f is a homogeneous function of degree 0.

For instance, the following equations are homogeneous differential equations:

i) $\frac{dy}{dx} = \frac{2y}{x}$,

ii) $\frac{dy}{dx} = \frac{2x + 3y}{4x} = \frac{2 + 3(y/x)}{4}$

iii) $\frac{dy}{dx} = \frac{x^3 + x^2y + y^3}{3x^2y + y^3} = \frac{1 + (y/x) + (y/x)^3}{3(y/x) + (y/x)^3}$

from the above equations, you may have noticed that if an equation can be put in the form

$$\frac{dy}{dx} = f(x, y) = \frac{f_1(x,y)}{f_2(x,y)},$$

where f_1 and f_2 are homogeneous expressions of the same degree in x and y, then f is a homogeneous function of degree 0.

Further, if in Eqn. (8) we let $\lambda = \frac{1}{x}$, the we have

$$y' = f\left(1, \frac{y}{x}\right) = F\left(\frac{y}{x}\right) \qquad \ldots(9)$$

This suggests making the substitution $v = \frac{y}{x}$ to solve this equation. since we seek y as a function of x this substitution means

$$V(x) = \frac{y(x)}{x} \text{ or } y(x) = xv(x)$$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

with this substitution Eqn, (9) reduces to

$$v + x\frac{dv}{dx} = f(v)$$

or $$\frac{dv}{dx} = \frac{F(v) - v}{x} \qquad \ldots(10)$$

which shows that Eqn. (10) is a separable equation in v and x. if we can solve Eqn. (10) for v in term of x, using the technique of Sec. 2.2 then the solution of Eqn. (9) is y = vx and hence we can solve equations of the type (9).

We now illustrate this method with the help of the following examples.

Example 4: Solve $\frac{dy}{dx} = \frac{2y^2 + 3xy}{x^2}$

Solution: You can easily check that the given equation is homogeneous of degree 0. it can be rewritten as

$$\frac{dy}{dx} = 2\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right) \qquad \ldots(11)$$

By making the substitution, $v = \frac{y}{x}$, Eqn. (11) reduces to

$$x\frac{dv}{dx} + v = 2v^2 + 3v$$

or $x\frac{dv}{dx} = 2v^2 + 2v = 2v(v + 1)$

or $\frac{dv}{v(v+1)} = \frac{2dx}{x}$

which is in variable separable form.

Resolving $\frac{1}{v(v+1)}$ into partial fractions, we have

$$\left(\frac{1}{v} - \frac{1}{v+1}\right) dv = \frac{2}{x}\ dx$$

which on integration, gives

In $|v|$ - In$|v+1|$ = Inx^2 + In $|c|$ …(12)

Where c is an arbitrary constant.

From Eqn. (12), we have

$$\frac{v}{v+1} = cx^2$$

replacing v by $\frac{y}{x}$, we get

$$cx^2 = \frac{y/x}{(y/x)+1} = \frac{Y}{X+Y}$$

or $y = \frac{cx^3}{1-cx^2}$,

which is the general solution of the given eqution.

Example 5: Solve $\frac{dy}{dx} = \frac{y^3}{x^3} + \frac{y}{x}$, $x > 0$.

Solution: With the substitution y = vx, we have

$$V + x\frac{dv}{dx} = v^3 + v$$

$$\text{or } \frac{1}{v^3}\frac{dv}{dx} = \frac{1}{x} \qquad \ldots(13)$$

Integration of Eqn. (13) yields

$$-\frac{1}{2v^2} = \text{In}x + \text{In}\,|c|,$$

where c is a real constant. On replacing $v = \frac{y}{x}$, the general solution of the given equation can be expressed as

$$y^2 = -\frac{x^2}{2[\text{In}x + \text{In}\,|\,c\,|]} \text{ or } y^2 = -\frac{x^2}{2}\,\frac{1}{\text{In}(x\,|\,c\,|)}$$

Let us consider another example.

Example 6: Solve $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad \ldots(14)$$

Putting y = vx, in Eqn. (14), we get

$$V + x\frac{dv}{dx} = \frac{1+v^2}{2v},$$

$$\text{or } x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\text{or } \frac{2v}{1-v^2}\,\frac{dv}{dx} = \frac{1}{x}$$

Integrating, we get,

In $|x|\,|(1 - v^2)| = \text{In}\,|c|$, where c is a constant of ingration or
$X(1 - v^2) = c$

On substituting for v, we can write the solution of Eqn. (14) in the form;

$$x^2 - y^2 = cx.$$

How about trying some exercise now?

Sometimes it may happen that a given equation is not homogeneous but cann be reduced to a homogeneous form by considering a transformation of the variables. We now consider such equations.

Equations reducible to homogeneous form

Consider the wquation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'} \qquad \ldots(15)$$

where a, b, c, a', b' and c' are all constants.

Eqn. (15) can be reduced to a homogeneous form by using the substitution
$X = x' + h$ and $y = y' + k$,

Where h and k are constants to be so chosen as to make the given equation homogeneous. In terms of these new variables, Eqn. (15) becomes

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \frac{ax' + by' + (ah + bk + c}{a'x' + b'y' + (a'h + b'k + c')}, \qquad \ldots(16)$$

which will be homogeneous provided h and k are so chosen that

$$\left.\begin{aligned} ah + bk + c = 0 \\ a'h + b'k + c' = 0 \end{aligned}\right\} \qquad \ldots(17)$$

Consequently Eqn. (16) reduces to

$$\frac{dy'}{dx'} = \frac{ax' + by'}{a'x' + b'y'} \qquad \ldots(18)$$

which can be solved by means of the substitution $y' = vx'$.

If the solution of the Eqn. (18) is of the form
$g(x', y') = 0$,

then the solution of Eqn. (15) is

$g(x – h, y – k) = 0,$

where h and k are obtained by solving the simultaneous Eqns. (17)
Solving Eqns. (17) for h and k, we get,

$$h = \frac{bc' - bc'}{ab' - a'b}, \; k = \frac{a'c - ac'}{ab' - a'b}$$

which are defined except when
$ab' – a'b = 0$ that is, when $\frac{a}{a'} = \frac{b}{b'}$.
If $\frac{a}{a'} = \frac{b}{b'}$, then h and k have either infinite values or are indeterminate. But then the question is what happens if $\frac{a}{a'} = \frac{b}{b'}$?

In such cases, we let $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m}$ (say)

Then Eqn. (15) can be written as

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + ny) + c'} \qquad \ldots(19)$$

On putting ax to by = v, Eqn (19) reduces to

$$\frac{1}{b}\left[\frac{dv}{dx} - a\right] = \frac{v + c}{mv + c'}$$

so that the variables are separated and hence the equation can be solved by the method given in the Sec 2.2.

We now take up some examples to illustrate the above discussion.

Example 7: Solve $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$ …(20)

Solution: Comparing the given equation with Eqn. (15), we have

$a = 1, b = -1, a' = 1, b' = 1.$

$\therefore \dfrac{a}{a'} = 1, \dfrac{b}{b'} = -|1 \text{and} \dfrac{a}{a'} \neq \dfrac{b}{b'}$

Putting x = x' + h and y = y' + k in Eqn. (20), we get

$$\frac{dy'}{dx'} = \frac{y' - x' + k - h + 1}{y'x' + k + h + 5} \qquad \ldots(21)$$

we choose h and k such that

$$\left.\begin{aligned} k - h + 1 = 0 \\ k + h + 5 = 0 \end{aligned}\right\} \qquad \ldots(22)$$

on solving Eqn. (22), we get h = - 2 and k = -3 . with these values of h and k, Eqn. (21) reduces to

$$\frac{dy'}{dx'} = \frac{y' - x'}{y' + x'} \qquad \ldots(23)$$

which is a homogeneous equation.

on putting y' = vx' in Eqn. (23) and simplifying the resulting equation, we get

$$-\frac{1+v}{1+v^2}\frac{dv}{dx'} = \frac{1}{x'}$$

$$\text{or} \left(\frac{1}{1+v^2} + \frac{v}{1+v^2}\right)\frac{dv}{dx'} = -\frac{1}{x'} \qquad \ldots(24)$$

Integration of Eqn. (24) yields

$\text{Tan}^{-1}v + \dfrac{1}{2}$ In $(1+ v^2) = -$ Inx' + c, where c is a constant.

or, $\dfrac{1}{2}$ In $(1 + v^2)\, x'^2 + \tan^{-1}v = c$.

Replacing v by $\dfrac{y'}{x'}$, we have

$\dfrac{1}{2}$ In $(x'^2 + y'^2) + \tan^{-1}\dfrac{y'}{x'} = c$.

substituting x' = x + 2 and y' = y + 3, solution of Eqn. (20) is given by

$$\frac{1}{2} \text{In} \left[(x+2)^2 + (y+3)^2 \right] + \tan^{-1} \left(\frac{y+3}{x+2} \right) = c.$$

Example 8: Solve the differential equation (4x + 6y + 5) dy = (3y + 2x + 5) dx.
Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{3y + 2x + 5}{4x + 6y + 5}$$

$$= \frac{(2x + 3y) + 5}{2(2x + 3y) + 5} \qquad \ldots$$

(250
In this case a = 2, b= 3, a' = 4, b' = 6. Thus,

$\frac{a}{a'} = \frac{b}{b'}$. Therefore, we put 2x + 3y = v, and Eqn. (25) reduces to,

$$\frac{1}{3}\left(\frac{dv}{dx} - 2 \right) = \frac{v+5}{2v+5} \quad \left(\text{here } 2 + 3\frac{dy}{dx} = \frac{dv}{dx} \right)$$

or $\frac{dv}{dx} = \frac{3(v+5)}{2v+5} + 2 = \frac{3v + 15 + 4v + 10}{2v+5} = \frac{7v+25}{2v+5}$

Now variables are separated and we get

$$\frac{2v+5}{7v+25} \frac{dv}{dx} = 1$$

or $\left[\frac{2}{7} - \frac{15}{7(7v+25)} \right] \frac{dv}{dx} = 1.$

Integrating, we get

$\frac{2}{7}$ v **-** $\frac{15}{49}$ In $\left(v + \frac{25}{7} \right)$ **= x + c,** where c is a constant of integration, substituting v = 2x+ 3y, we get

$\frac{2}{7}$ (2x + 3y) - $\frac{15}{49}$ In $\left(2x + 3x + \frac{25}{7} \right) = x + c,$

or, 14(2x + 3y) – 15 In (2x + 3y + $\frac{25}{7}$) = 49 (x + c) or, 42y – 21x – 15 In (14x + 21y + 25) = 49c – 15 In 7 = c_1, say, which is the required solution.

You may now try the following exercise. In each of the equations in this exercise you should first see whether $\frac{a}{a'} = \frac{b}{b'}$ and then decide on the method.

In Unit 1, we defined the total differential of a given function. In the next section we shall make use of this to define and solve exact differential equations.

3.3 Exact Equations

Let us start with a family of curves h(x, y) = c. Then its differential equation can be written in terms of its total differential as

dh = 0, or

$$\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy = 0.$$

For example, the family $x^2y^3 = c$ has $2xy^3dx + 3x^2y^3dy = 0$ as its differential equation. suppose we now consider the reverse situation and begin with the differential equation

a(x, y)dy + b(x,y)dx = 0

If there exists a function h(x,y) such that

$$\frac{\partial h}{\partial x} = b(x, y) \text{ and } \frac{\partial h}{\partial y} = a(x, y),$$

then Eqn. (26) can be written in the form

$$\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}\,dy = 0 \text{ or } dh = 0$$

that is, h(x, y) = constant represents a solution of Eqn. (26).

In this case we call the expression a(x, y)dy + b(x, y)dx an **exact differential** and (26) is called an **exact differential equation.** For instance, equation x^2y^3 dx + x $+3^{y+2}$ dy = 0 is exact, since we have $d\left(\frac{1}{3}x^3y^3\right) = x^2y^{3\,dx + x+3}y^2dy$.

Thus, an exact differential equation is formed by equating an exact differential to zero.

It is sometimes possible to determine exactness and find the function h by merer inspection. Consider, for example, the equations.

$3x^2y^4dx + 4x^3y^3\, dy = 0$
and $xe^{xy}\, dy + (ye^{xy} – 2x)\, dx = 0$.

These two equations can be alternatively written as $d(x^3y^4) = 0$ and $d(e^{xy} – x^2) = 0$, respectively. Thus, the general solution of these equations are give by $x^3y^4 = c$ and $e^{xy} = x^2 + c$, where c is constant.

However except for some cases, this technique of “solution by insight” is clearly impractical. Consequently we seek an answer to the following question: when does a function h(x, y) exist such that Eqn. (26) is exact? An answer to this question is given by the following theorem.

Theorem 1: If the functions a(x, y), b(x, y), $a_x \frac{\partial a}{\partial x}$ and $b_y = \frac{\partial b}{\partial y}$ are continuous functions of x and y, then Eqn. (26), namely,

a(x, y) dy + b(x, y) dx = 0 is exact if and only if

$$\frac{\partial}{\partial y} b(x, y) = \frac{\partial}{\partial x} a(x, y) \qquad \ldots(27)$$

Indeed condition (27) is a necessary and sufficient condition for a function h(x, y) to be such that

$$\frac{\partial}{\partial y} h(x, y) = b(x, y) \text{ and } \frac{\partial}{\partial y} h(x, y) = a(x, y) \qquad \ldots(28)$$

you may note here that if relation (28) is satisfied, then

$$\begin{aligned} d[h(x, y (x))] &= \frac{\partial}{\partial x} h(x,y (x))\, dx + \frac{\partial}{\partial y} h(x,y(x))\, dy \\ &= b(x, y(x))dx + a(x, y(x))\, dy \end{aligned}$$

and hence Eqn. (26) can be rewritten as
$d[h(x, y(x))] = 0$
or that the solution of Eqn. (26) is given by
$h (x, y) = c$,
where c is a constant.

We now give the proof of Theorem 1.
Proof: The condition is necessary

Let the equation

$A(x, y)\dfrac{dy}{dx} + b(x, y) = 0$

Or, $a(x, y)\,dy + b(x, y)\,dx = 0$

Be exact.

Then there exists a function h(x, y) such that

$Dh = b(x, y)\,dx + a(x, y)\,dy$

But $dh = \dfrac{\partial h}{\partial x}dx + \dfrac{\partial h}{\partial y}dy$.

$\therefore \dfrac{\partial h}{\partial x}dx + \dfrac{\partial h}{\partial y}dy = b(x, y)\,dx + a(x, y)\,dy$

Thus, necessarily,

$$\frac{\partial h}{\partial x} = b(x, y) \text{ and } \frac{\partial h}{\partial y} = a(x, y) \qquad \ldots(29)$$

Since a(x, y) and b(x, y) have continuous first order partial derivatives, h possess continuous second order partial derivatives namely, $\dfrac{\partial^2 h}{\partial y\partial x}$ and $\dfrac{\partial^2 h}{\partial y\partial y}$. Refer Unit 6, Block 2 of MTE-07 for second and higher order partial derivatives.

Now,

$$\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = \frac{\partial^2 h}{\partial y\partial x} = \frac{\partial b}{\partial y}(x, y) \qquad \ldots(30)$$

$$\text{and } \frac{\partial}{\partial y}\left(\frac{\partial h}{\partial y}\right) = \frac{\partial^2 h}{\partial x\partial y} = \frac{\partial a}{\partial x}(x, y) \qquad \ldots(31)$$

Since $\dfrac{\partial b}{\partial y}$ and $\dfrac{\partial a}{\partial x}$ are continuous,

$\dfrac{\partial^2 h}{\partial y\partial x} = \dfrac{\partial^2 h}{\partial x\partial y}$ (ref. Unit 6, Block 2 of MTE-07).

There, from Eqn. (30) and (31), we get

$$\frac{\partial a}{\partial x}(x, y) = \frac{\partial b}{\partial y}(x, y)$$

The condition is sufficient: Now suppose that

$$\frac{\partial a}{\partial x}(x, y) = \frac{\partial b}{\partial y}(x, y)$$

and we shall show that a(x, y) dy + b(x, y) dx is an exact differential

Let $\int$ b(x, y) = dx = V, then $\frac{\partial V}{\partial x}$ = b(x, y)

and

$$\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial b}{\partial y} \text{ (x, y)}$$

$$= \frac{\partial a}{\partial x} \text{ (x, y) (using given condition)}$$

$$\therefore \frac{\partial a}{\partial x} \text{ (x, y)} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) \qquad \ldots(32)$$

Integration of Eqn. (32) with respect to x, holding y fixed, yields

$$a(x, y) = \frac{\partial V}{\partial y} + ø\,(y)$$

where ø, a function of y only, is therefore a constant of integration, when y is held fixed.

Thus,

$$a(x, y)\, dy + b(x, y)\, dx = \frac{\partial V}{\partial y} dy + ø(y)\, dy + \frac{\partial V}{\partial x} dx$$

$$= d[V(x, y) + ø(y)\, dy]$$

which establishes that a(x, y) dy + b(x, y) dx is an exact differential implying thereby that, a(x, y) dy + b(x, y)dx = 0 is an exact differential equation. This completes the proof of Theorem 1.

We shall now illustrate this theorem with the help of the following examples.

Example 9: Solve the differential equation
Sin(y) + x cos (y)y' = 0.

Solution: For the case under consideration, a(x, y) = xcos(y) and b(x, y) = sin(y). also

$$\frac{\partial}{\partial x} \text{ a(x, y)} = \cos(y) = \frac{\partial}{\partial y} \text{ b(x, y)}$$

which shows that the given equation is an exact equation.

Therefore, there exists a function h(x, y) = constant such that, $\frac{\partial h}{\partial x}$ = b(x, y) and $\frac{\partial h}{\partial y}$ = a(x, y)

Then we have

= sin y …(33)

and

$\frac{\partial h}{\partial y}$ = x cos y …(34)

Integrating Eqn. (33) with respect to x, treating y as a constant, we get

H(x, y) = x sin y + ø (y) …(35)

Where ø(y) is a constant of integration. Differentiating Eqn. (35) partially w.r.t.y, we get

h(x, y) = x cos y + ø'(y) …(36)

from Eqns. (34) and (36), we get

x cos y = x cos y + ø'(y)

which shows that ø'(y) = 0 ⇒ ø(y) = constant = c_1. Hence from Eqn. (35), we can write

h(x, y) = xsin y + c_1

so the required solution, h(x, y) = constant, is

x sin y + c_1 = c_2, where c_2 is a constant or,

x sin y = c,

where c = $c_2 - c_1$ is a constant.

Example 10: Solve e^x sin y + e^x cosy y' + 2x = 0.

Solution: Comparing with Eqn. (26), we have a(x, y) = e^xcos y and b(x, y) = e^xsin y + 2x. Therefore,

a(x, y) = e^xcos y

and b(x, y) = e^xcos y = a(x, y).

Hence the given equation is exact and can be written in the form dh(x, y(x) = 0 where

$h(x, y) = e^x \cos y + 2x$ …(37)

and $h(x, y) = e^x \cos y$ …(38)

Integrating Eqn. (37) w.r.t.x, we get

$h(x, y) = e^x \sin y + x^2 + ø(y)$ …(39)

Where ø, a function of y only, is a constant of integration
From Eqns. (38) and (39), we get

$h(x, y) = e^x \cos y + ø'(y) = e^x \cos y$

So we have $ø'(y) = 0$ or $ø(y) = c_1$ where c_1 is a constant.
Hence from Eqn. (39) we have the required solution as

$h(x, y) = e^x \sin y + x^2 = - c_1 = c$

where c is a constant.

On the basis of Theorem 1 and Example (9) and (10) we can say that various steps involved in solving an exact differential equation b(x, y) dx + a(x, y) dy = 0 are as follows:

Step 1: Integrate b(x, y) w.r.t.x, regarding y as a constant.

Step 2: Integrate, with respect to y, those terms in a(x, y) which do not involve x.

Step 3: The sum of the two expressions obtained in steps 1 and 2 equated to a constant is the required solution.

We now illustrate these various steps with the help of an example.

Example 11: Solve $(x^2 – 4xy – 2y^2) dx + (y^2 – 2x^2) dy = 0$.

Solution: Here $a(x, y) = y^2 – 4xy – 2x^2$ and $b(x, y) = x^2 – 4xy – 2y^2$

$\therefore \frac{\partial a}{\partial x}$ = - 4y – 4x and $\frac{\partial b}{\partial y}$ = - 4x – 4y

$\therefore \frac{\partial a}{\partial x} = \frac{\partial b}{\partial y}$; hence it is an exact equation.

Step1: Integrating b(x, y) w.r.t.x. regarding y as a constant, we have

$\int (x^2 - 4xy - 2y^2)\, dx = \frac{x^3}{3} - 2x^x y - 2xy^2.$

Step 2: We integrate those terms in a(x, y) w.r.t.y, which do not involve x. there is only one such term namely, y^2.

$\therefore \int y^2\, dy = \frac{y^3}{3}.$

Step 3: The required solution is the sum of expressions obtained from Steps 1 and 2 equated to a constant, that is.

$\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c_1,$

or $x^3 - 6x^2y - 6xy^2 + y^3 = c.$

where c and c_1 are constants.

Note that the test for an exact differential equation and the general procedure for finding the solution can sometimes be simplified. We can pick out those terms of a(x, y) dy + b(x, y) dx = 0 that obviously form an exact differential or can take the form f(u) du. The remaining, expression which is less cumbersome than the original can then be tested and integrated. This is illustrated by the following example.

Example 12: Solve xdx +ydy + $\frac{xdy - ydx}{x^2 + y^2} = 0.$

Solution: Note that the first two terms on the left hand side of the given equation are exact differentials and hence need not be touched. Dividing the numerator and denominator of the last term by x^2, we get

Xdx + ydy + $\frac{d(y/x)}{1 + (y/x)^2} = 0$

Now each term of the above equation is an exact differential. Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \tan^{-1} \frac{y}{x} = c$$

as the required solution with c as a constant.

You may now try the following exercises.

In practice the differential equations of the form a(x, y) dy + b(x, y) dx = 0 are rarely exact, since the condition in Theorem 1 requires a precies balance of the functions a(x, y) and b(x, y). but they can often be transformed into exact equatins on multiplication by a suitable function F(x, y) ≠ 0. This function is then called an integrating factor. The question we, now, must ask is: if

a(x, y(dy + b(x, y) dx = 0

is not exact, then how to find a function

F(x, y) ≠ 0 so that
F(x, y) [a dy + b dx] = 0
Is exact? In the next section we shall give an answer to this question.

3.4 Integrating Factor

We begin with a very simple equation, namely,

$$y' + y = 0 \qquad \ldots(40)$$

In this case a(x, y) = 1 and b(x, y) = y. Here $\frac{\partial}{\partial x}$ a(x, y) = 0

and $\frac{\partial}{\partial y}$ b(x, y) = 1 and hence the given equation is not exact. Let us multiply Eqn. (40) by e_x $^{\text{to get}}$

$$e^x y' + e^x y = 0 \qquad \ldots(41)$$

you may now check that Eqn. (41) is an exact equation. Thus Eqn. (40) is not exact whereas when we multiply EQn. (40) by e^x the resulting equation becomes an exact one.

Here e^x is termed as **integrating factor** for Eqn. (40).

We now give the following definition.

Definition: A factor, which when multiplied with a non-exact differential equation makes it exact, is known as an integrating factor (abbreviated as I.F.).

The term I.F., to solve a differential equation, was first introduced by Fatio de Duillier in 1687.

For a given equation, there may not be a unique integrating factor.

Consider, for example, the equation

$$ydx - xdy = 0 \qquad \ldots(42)$$

You can check that Eqn. (42) is not exact, but when multiplied by $\frac{1}{y^2}$, it becomes

$$\frac{ydx - xdy}{y^2} = 0$$

which is exact. This can now be written as d $\left(\frac{x}{y}\right)$ = 0 and thus has for its solution $\frac{x}{y}$ = c with c being an arbitrary constant.

Further, when Eqn. (42) is multiplied by $\frac{1}{xy}$, it becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

which is given exact and has its solution as Inx –Iny = c.

you may notice that this solution can be transformed into the earlier solution obtained through the I.F. $\frac{1}{y^2}$. Also Eqn. (42) when multiplied by $\frac{1}{x^2}$ reduces to an exact equation $\frac{y}{x^2}$ dx - $\frac{dy}{x}$ = 0 or, - d$\left(\frac{y}{x}\right)$ = 0 with | - $\frac{y}{x}$ = c as its solution.

Thus, we have seen that some of the integrating factors for Eqn. (42) are $\frac{1}{y^2}$, $\frac{1}{xy}$ and $\frac{1}{x^2}$

Now the question arises: Is this the case only with Eqn. (42) or, in general, does an equation of the form a(x, y) dy + b(x, y) dx = 0 have infinitely many integrating factors?

An answer to this question is given in Theorem 2.

Before we give you this theorem, here is an exercise for you.

Theorem 2: The number of integrating factors for the equation
A(x, y) dy + b)x, y) dx = 0
9s infinite

proof: Let g(x, y) be an integrating factor of the given equation. Then, by definition

$$g(x, y)\left[a(x,y)\frac{dy}{dx} + b(x,y)\right] = 0 \qquad \ldots(43)$$

is an exact differential equation.

Therefore, there exist a function h(x, y) such that

$$dh = g(x, y)\left[a(x,y)\frac{dy}{dx} + b(x,y)\right] = f(h)dh = d\left[\int f(h)\ dh\right] \qquad \ldots(44)$$

since the term on the right hand side of Eqn (44) is an exact differential, the term in the left must also be an exact differential. Therefore, g(x, y).f(h) is an integrating factors of the given differential equation.

Since f(h) is an arbitrary function of h, hence the number of integrating factors for equation a(x, y) dy + b(x, y) dx = 0 is infinite.

This fact is, however, of no special assistance in solving the differential equations.

So far, in our discussion we have not paid any attention to the problem of finding the integrating factors. In general, it is quite difficult to obtain an integrating factor for a given equation. However, rules for finding the integrating factors do exist, we shall now take up these rules one by one.

Rules for finding integrating factors.

Rule 1: Integrating factors of obtainable inspection: Sometime integrating fators of a differential equation can be seen at a glance, as in the case of EQn. (42) abve. We give below some more examples in this regard.

Example 13: Solve $(1 + xy)\, ydx + (1 - xy)xdy = 0$, $x>0$, $y > 0$. …(45)

Solution: Rearranging the terms of Eqn. (45), we get

$ydx + xdy + xy^2dx - x^2ydy = 0$

$\Rightarrow\ d(xy) + xy^2dx - x^2ydy = 0$ …(46)

It is immediately seen that multiplication by $\frac{1}{x^2y^2}$ makes Eqn. (46) eact and the equation becomes

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$$

integrating, we get

$$-\frac{1}{xy} + \text{In}x - \text{In}y = \text{In}c,$$

or $x = cy\, e^{1/(xy)}$ (where c is a constant).

Example 14: Solve $(x^4e^x - 2my^2x)\, dx + 2mx^2y\, dy = 0$.

Solution: We can write the given equation as

$x^4e^x\, dx + 2m\, (x+2^{ydy - xy}2^{dx) = 0}$

$$\Rightarrow\ x^4e^x\, dx + 2mx^3y\, d\left(\frac{y}{x}\right) = 0$$

Dividing by x^4, we get

$$dx + 2m\,\frac{y}{x}\, d\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow\ d\left[e^x + m\left(\frac{y}{x}\right)^2\right] = 0$$

thus $\frac{1}{x^4}$ has served the role of an integrating factor in this case.

The required solution is, then, given by

$e^x + m\left(\frac{y}{x}\right)^2 = c$ with c as a constant.

We would like to mention that determination of an integrating factor by inspection is a skill and can be developed through practice only.

At this stage you may try the following exercises by finding an integrating factor through inspection.

Rule II: For a homogeneous equation a(x, y)dy + b(x, y) dx -= 0, when bx + ay $\neq 0$, then $\frac{1}{bx+ay}$ is an integrating factor.

Proof: Consider an equation
a(x, y) dy + b(x, y) dx = 0

$$\text{Now } ady + bdx = \frac{1}{2}\left[(bx+ay)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(bx-ay)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$\therefore \frac{ady+bdx}{ay+bx} = \frac{1}{2}\left[\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{bx-ay}{x+ay}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

since the given equation is homogeneous, therefore a and b are of the same degree in x and y and therefore $\frac{bx-ay}{bx+ay}$ can be written as a function of $\frac{x}{y}$, say $f\left(\frac{x}{y}\right)$.

$$\therefore \frac{ady+bdx}{ay+bx} = \frac{1}{2}\left[\left(\frac{dx}{x}+\frac{dy}{y}\right)+f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[d(\text{In } xy)+f(e^{\text{In}x/y})\ d\left(\text{In}\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[d(\text{In } xy)+dF\left(\text{In}\frac{x}{y}\right)\right]$$

$$= d\left[\frac{1}{2}\ \text{In}xy+\frac{1}{2}F\left(\text{In}\frac{x}{y}\right)\right] \qquad \ldots(47)$$

where $d\,F\left(\frac{x}{y}\right) = f(e^{\text{In}x/y})\,d\left(\text{In}\frac{x}{y}\right)$.

Since right hand side of Eqn. (47) is an exact differential, it shows that $\frac{1}{ay+bx}$ is an integrating factor for the homogeneous equation a(x, y)dy + b(x, y)dx = 0.

We illustrate this rule by the following example.

Example 15: Solve $(x^2y - 2xy^2)\,dx - (x^3 - dx^{2+y)\,dy = 0.}$
Solution: Here the given equation is homogeneous and
$a(x, y) = x^3 + 3x^2y$ and $b(x, y) = x^2y - 2xy^2$
$\therefore bx + ay = x(x^2y - 2xy^2) + y(-x^3 + 3x^2y) = x^2y^2 \neq 0,$
$\therefore \frac{1}{x^2y^2}$ is an integrating factor.

Multiplying the given differential equation by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0,$$

or $\frac{dx}{y} + 3\frac{dy}{y} - 2\frac{dx}{x} - \frac{x}{y^2}\,dy = 0,$

or $\left(\frac{dx}{y} - \frac{x}{y^2}dy\right) + 3\frac{dy}{y} - 2\frac{dx}{x} = 0,$

or $d\left(\frac{x}{y}\right) + d(3\,\text{In}y + 2\,\text{In}x) = 0.$

Therefore, the solution is
$\frac{x}{y} + 2\,\text{In}y - 2\,\text{In}x = c_1$ and c are constants.

Note: In case bx + ay = 0, then $\frac{a}{b} = -\frac{y}{x}$ and the given equation reduces to $\frac{dy}{dx} = \frac{y}{x}$
,
Whose solution is straightaway obtained as x = cy.

You may now try this exercise.

Rule III: when bx – ay ≠ 0 and the different equation a(x, y) dy + b(x, y) dx = 0 can be written in the form $yf_1(x\ y)$ dx + $xf_2(xy)$ dy = 0 then $\frac{1}{bx - ay}$ is an integrating factor.

Proof: If equation a(x, y) dy + b(x, y)dx = 0 can be written in the form $yf_1(xy)dx + xf_2(xy)dy = 0$
Then evidently,

$$\frac{a}{xf_2(xy)} = \frac{b}{yf_1(xy)} = \lambda\text{, say}$$

$\therefore a = \lambda\, xf_2(xy)$ and $b = \lambda\, yf_1(xy)$

$$\text{Also ady + bdx} = \frac{1}{2}\left[(bx + ay)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (bx - ay)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\therefore \frac{ady + bdx}{bx + ay} = \frac{1}{2}\left[\frac{bx + ay}{bx - ay}\left(\frac{dx}{x} + \frac{dy}{y}\right) + \left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f_1 + f_2}{f_1 - f_2} d(\text{In } xy) + d\left(\text{In}\frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[f(xy) d(\text{In } xy) + d\left(\text{In}\frac{x}{y}\right)\right]$$

$$\text{where } f(xy) = \frac{f_1 + f_2}{f_1 - f_2}$$

$$= \frac{1}{2}\left[dF(\text{Inxy}) + d\left(\text{In}\left(\frac{x}{y}\right)\right)\right]$$

$$= d\left[\frac{1}{2}\text{In}\left(\frac{x}{y}\right) + \frac{1}{2}F(\text{In}(xy))\right]$$

where dF (In xy) = f(x, y) d(In xy),
which is an exact differential.

Hence, $\frac{1}{bx - ay}$ is an integrating factor.

We now illustrate this through the following example.

Example 16: Solve $y(xy + 2x^2y^2)\, dx + x(xy - x^2y^2)dy = 0$,

Solution: Here $a = x(xy - x^2y^2)$ and $b = y(xy + 2x^2y^2)$

$\therefore$ bx – ay = xy[xy + x^2y^2 – xy + $2x^2y^2$]
= $3x^3y^3 \neq 0$.

$\therefore \dfrac{1}{3x^3y^3}$ is an I.F.

Multiplying the given equation by $\dfrac{1}{3x^3y^3}$, we get

$$\frac{1}{3x^3y^2}(xy + 2x^2y^2)\,dx + \frac{1}{3x^2y^3}(xy - x^2y^2)\,dy = 0$$

or $\dfrac{dx}{3x^2y} + \dfrac{3dx}{3x} + \dfrac{dy}{3xy^2} - \dfrac{dy}{3y} = 0$

or $\left[\dfrac{dx}{3x^2y} + \dfrac{dy}{3xy^2}\right] + \dfrac{2}{3}\dfrac{dx}{x} - \dfrac{1}{3}\dfrac{dy}{y} = 0$

or $d\left(-\dfrac{1}{3}\dfrac{1}{xy} + \dfrac{2}{3}\,\text{In}x - \dfrac{1}{3}\,\text{In}y\right) = 0.$

Therefore, the solution is

$-\dfrac{1}{3xy} + \dfrac{2}{3}\text{In}x - \dfrac{1}{3}\text{In}y = c_1$ where c_1 is a constant.

or $-\dfrac{1}{xy} + \text{In}x^2 - \text{In}y = 3c_1 = c$ for c being an arbitrary constant.

or $\text{In}\left(\dfrac{x^2}{y}\right) = c + \dfrac{1}{xy}$.

Note: If bx – ay = 0, i.e., $\dfrac{a}{b} = \dfrac{y}{x}$, then given equation will be of the form $\dfrac{dy}{dx} = -\dfrac{y}{x}$ and have a solution xy = c.

Before we go to the next rule here is an exercise for you.

Rule IV: When $\dfrac{1}{a}\left(\dfrac{\partial b}{\partial y} - \dfrac{\partial a}{\partial x}\right)$ is a function of x alone, say f(x), then $e^{\int f(x)\,dx}$ is an I.F. of the equation ady + bdx = 0.

Proof: Consider the equation $e^{\int f(x)\,dx}$ (ady + bdx) = 0 …(48)

Let c = b $e^{\int f(x)\,dx}$ and d = a $e^{\int f(x)\,dx}$

Then Eqn. (48) reduces to c dx + d dy = 0

Now, $\frac{\partial c}{\partial y} = \frac{\partial b}{\partial y} \; e^{\int f(x)\,dx}$

and $\frac{\partial d}{\partial x} = \frac{\partial a}{\partial x} \; e^{\int f(x)\,dx} + a \; e^{\int f(x)\,dx} \; f(x)$

$$= e^{\int f(x)\,dx}\left[\frac{\partial a}{\partial x} + a \; f(x)\right]$$

$$= e^{\int f(x)\,dx}\left[\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right]\left[\text{because}\,\frac{1}{a}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right) = f(x)\right]$$

$$= \frac{\partial b}{\partial y} \; e^{\int f(x)\,dx}$$

$$= \frac{\partial c}{\partial y}$$

therefore, the equation c dx + d dy = 0 is exact.

Hence $e^{\int f(x)\,dx}$ is an I.F. of the equation ady + bdx = 0.

We illustrate this rule with the help of the following example.

Example 17: Solve $(x^2 + y^2)\,dx - 2xydy = 0$.

Solution: Here a = 2xy, b = $x^2 + y^2$

$\therefore \frac{\partial b}{\partial y} = 2y$ and $\frac{\partial a}{\partial x} = -2y$. thus, $\frac{\partial b}{\partial y} \neq \frac{\partial a}{\partial x}$.

Here $\frac{1}{a}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right) = \frac{1}{-2xy}\,(2y + 2y) = -\frac{2}{x}$, which is a function of x alone.

$$\therefore \text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\,\text{In}x} = \frac{1}{x^2}$$

Multiplying the given equation by $\frac{1}{x^2}$, we get

$\frac{1}{x^2}\,(x^2 + y^2)\,dx - \frac{2y}{x}\,dy = 0,$

i.e., $dx\,\frac{y^2}{x^2}\,dx - \frac{2y}{x}\,dy = 0,$

i.e., $dx + d\left(-\frac{y^2}{x}\right) = 0$ …(49)

integrating Eqn. (49), the required solution is obtained as

$x - \frac{y^2}{x} = c$ (a constant)

You may now try this exercise.

Rule V: When $\frac{1}{b}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right)$ is a function of y alone, say f(y), then $e^{\int f(x)\,dy}$ is an I.F. of the differential equation
a dy + b dx = 0.

The proof of this rule is similar to the proof of Rule IV above and we leave this as an exercise for you (see E 13).

We however illustrate the use of Rule V with the help of following example.

Example 18: Solve $93x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$.

Solution: Here $a = 2x^3y^3 - x^2$ and $b = 3x^2y{+}4^{+3xy}$

$\therefore \frac{\partial b}{\partial y} = 12x^2y^3 + 2x$ and $\frac{\partial a}{\partial x} = 6x^2y^3 - 2x$

Here $\frac{1}{b}\left(\frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}\right) = \frac{1}{3x^2y^4 + 2xy}\ (12x^2y^3 + 2x - 6x^2y^3 + 2x)$

$= \frac{2(3x^2y^3 + 2x}{y(3x^2y^3 + 2x} = \frac{2}{y}$, which is a function of y alone.

Hence, I.F. $= e^{\int -\frac{2}{y}dy|}\,|e^{-2Iny} = y^{-2} = \frac{1}{y^2}$.

Multiplying the given equation by the I.F. $= y^{-2}$, and on rearranging th terms, we get

$(3x^2y^2\,dx + 2x^3y\,dy) + \left(\frac{2x}{y}dx - \frac{x^2}{y^2}dy\right) = 0$

i.e., $d(x^3y^2) + d\left(\frac{x^2}{y}\right) = 0$

Integrating the above equation, we get

$x^3y^2 + \dfrac{x^2}{y} = c$, where c is a constant of integration.

i.e., $x^3y^3 + x^2 = cy$, which is the required solution.

And now an exercise for you.

Rule VI: If the differential equation is of the form $x^{\alpha}y^{\beta}$ (mydx + nxdy) = 0, where α,β, m and n are certain constants, then $x^{km-I-\alpha}y^{kn-I-\beta}$ is an integrating factor, where k can assume any value.

Proof: Multiplying the given equation by I.F., we get

$X^{km-1}\, y^{kn-1}\,(mydx + nxdy) = 0,$
or $km\, x^{km-1}\, y^{kn}\, dx + kn\, x^{km}y^{kn-1}\, dy = 0$
or $(x^{km-1}y^{kn}) = 0$, which is an exact differential.

It may be noted that if the given differential equation is of the form
$x^{\alpha}y^{\beta}(mydx + nxdy) + x^{\alpha 1}y^{\beta 1}(m_1ydx + n_1xdy) = 0$
then also I.F. can be determined.

By Rule VI, $x^{km-1-\alpha}y^{kn-1-\beta}$ will make the first term exact, while $x^{k_1m_1-1-\alpha_1}\, y^{k_1n_1-1-\beta}$ will make the second term exact, where k and k_1 can have any value.

Now these two factors will be identical if

$kn\text{-}1\text{-}\alpha = k_1m_1 - 1\text{-}\square_1$
and $kn\ \text{-}1\text{-}\beta = k_1n_1\text{-}1\text{-}\square_1.$

Values of k and k_1 can be found tp satisfy these two algebraic equations. Then either factor is an integrating factor of the above equation.

We now consider an example to illustrate this rule.

Example 19: Solve $(y^3 - 2yx^2)\, dx + (2xy^2 - x^3)\, dy = 0$.

Solution: On rearranging the term of the given equation, we can write
$Y^2(ydx + 2xdy) - x^2(2ydx + xdy) = 0$ …(50)
For the first term, $\square = 0, \square = 2$, m = 1 and n = 2 and hence its I.F.is $x^{k-1}y^{2k-1-2}$
For the second term $\alpha_1 = 2$, $\beta_1 = 0$, $m_1 = 2$, $n_1 = 1$ and hence for the second term I.F. is

$x^{2k_1-1-2}y^{k_1-1}$

these two integrating factors will be identical if

$$\left.\begin{array}{l} k-1 = 2k_1 -1-2 \\ \text{and } 2k-1-2 = k_1 - 1 \end{array}\right\} \qquad \ldots(51)$$

solving the system fo Eqn. (51) for k and k_1, we get k = 2 and k_1 = 2 and, therefore, integrating factor for Eqn. (50) is
$x^{2-1}\, y^{4-1-2}$, i.e., xy.

Multiplying Eqn. (50) by xy, we get

$xy^3\,(ydx + 2xdy) - x^3y\,(2ydx + xdy) = 0$

$\Rightarrow xy^4dx + x^2y^3dy - (2x^3y^2dx + x^4ydy = 0$

$\Rightarrow \frac{1}{2}\,(2xy^4dx + 4x^2y^3dx) - \frac{1}{2}\,(4x^3y^2dx + 2x^4y^4dy) = 0$

$$\Rightarrow \frac{1}{2}\, d[2x^2y^4] - \frac{1}{2}\, d[x^4y^2] = 0 \qquad \ldots(52)$$

Integrating Eqn. (52), we get the required solution as

$\frac{x^2y^4 - x^4y^2}{2} = c_1$ (a constant)

or $x^2y^2(y^2 - x^2) = 2c_1 = c$ (a constant).

You may now apply your knowledge about these rules and try to solve the following exercises.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have covered the following:

1) An equation = f(x, y) is called a **separable equation** or an **equation with separable variables** if f(x, y) = X(x) Y(y). to solve a separable equation, we can write it as

   A(y) + b(x) = 0

   For some a(y) and b(x). Integrating w.r.t. to x and equating it to a constant, we get its solution.

2) a) A real-valued function h(x, y) of two variables x and y is called a **homogeneous function** of degree n, if

$h(\lambda x, \boxed{\ } y) = \boxed{\ }^n h(x, y)$, where n is a real number and $\boxed{\ }$ is any constant.

b) A differential equation

$$\boxed{\ } = f(x, y)$$

is called a **homogeneous differential equation** of first order when f is a homogeneous function of degree zero.

c) A homogeneous differential equation reduces to separable equation by the substitution y = vx, where v is some function of x.

3) Equations of the form

$\boxed{\ } = \frac{ax + by + c}{a'x + b'y + c'}$ where a, b, c, a', b', c' are constants and $\frac{a}{a'} \neq \frac{b}{b'}$ can be **reduced to homogeneous equations** by the substitution x' = x + h, y' = y + k, where h and k are such that ah + bk + c = 0 and a'h + b'k + c' = 0.

In case $\boxed{\frac{a}{a'} \neq \frac{b}{b'}} = \frac{1}{m}$, say, then substitution ax + by = v reduces this type of equations to separable equations.

4) An exact differential equation s formed by equating an exact differential to zero.

5) The differential equation

A(x, y)dy + b(x, y)dx = 0

6.0 TUTOR MARKED ASSIGNMENT

1. Solve the following equations.

a) $(1 - x)dy - (1 + y)\,dx = 0$

b) $y - x\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0$

c) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

d) $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

c) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

2. Solve the following equations satisfying initial condition indicted alongside.

a) $2xy \dfrac{dy}{dx} = 1 + y^2, y(2) = 3 \forall \ x, y > =$

b) $\dfrac{dy}{dx} = - 4xy, y(0) = y_0 \forall \ y > 0$

c) $\dfrac{dy}{dx} xe^{y- x^2}, y(0) = 0$

d) $y \dfrac{dy}{dx} = g, y (x_0) = y_0$

3. Solve the following equations.

a) $\dfrac{dy}{dx} = \dfrac{y}{x}$ for $x \in]0, \infty [$ and for $x \in] - \infty , 0[$

b) $\dfrac{dy}{dx} = \dfrac{2x + y}{3x + 2y}$

c) $(x \sin \dfrac{y}{x}) dy - (y \sin^{-1} \dfrac{y}{x} - x) dx = 0$

d) $x \dfrac{dy}{dx} = y (Iny - Inx + 1)$

e) $x dy - y dx = \sqrt{x^2 - y^2} dx.$

4. Solve the following equations subject to the indicated initial conditions.

a) $2x^2 \dfrac{dy}{dx} = 3xy + y^2, y(1) = -2$

b) $(x + ye^{y/x}) dx - xe^{y/x} dy = 0, y(1) = 0$

c) $(y^2 + 3xy) dx = (4x^2 + xy) dy, y(1) = 1.$

d) $y^2dx +)x^2 + xy + y^2) dy = 0, y (0) = 1.$

5. Solve the following equations.

a) $\frac{dy}{dx} = \frac{2y - x - 4}{y - 3x + 3}$

b) $(7y - 3x + 3)\frac{dy}{dx} + (3y - 7x + 7) = 0$

c) $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$

d) $(x + y)\,dx + (3x + 3y - 4)\,dy = 0$

6. Prove that the following equations are exact and solve them.

a) $(y\cos(x) + 2x\,e^y) + (\sin(x) + x^2e^y + 2)y' = 0$

b) $y' = -\frac{ax + by}{bx + cy}$ (a,b,c,d are given real constants).

c) $96x + y/x) + (\text{In}x + y)y' = 0,\ x \geq 1.$

7. Determine the values of k for which the equations given below are exact and find the solution for these values of k.

a) $x + kyy' = 0\ (k \neq 0)$
b) $y + kxy' = 0\ (k \neq 0)$
c) $(2y\,e^{2xy} + 2x) + k\,x\,e^{2xy}y' = 0$

8. In each of the following equations verify that the function F(x, y), indicated alongside is an I.F. of the equation:

i. $6xy\,dx + (4y + 9x^2)\,dy = 0;\ F(x, y) = y^2$

ii. $-y^2\,dx + (x^2 + xy)\,dy = 0;\ F(x, y) = \frac{1}{x^2y}$

iii. $(-xy\sin x + 2y\cos x)\,dx + 2x\cos x\,dy = 0;\ F(x, y) = xy.$

9. Solve the following equations.

a) $y(2yx + e^x)\,dx - e^x\,dy = 0$
b) $ydx - xdy + \text{In}x\,dx = 0\ \forall\ x, y > 0.$
c) $(xy - 2y^2)\,dx - (x^2 - 3xy)dy = 0$

10. Solve $(x^4 + y^4)\,dx - xy^3dy = 0.$

11. Solve $y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0.$

12. Solve $(x^2 + y^2 + x)\, dx + xy\, dy = 0$.

13. Prove Rule V above.

14. Solve $(2xy^4e^y + 2xy^3 + y)\, dx + (x^2y^4e^y - x^2y^2 - 3x)\, dy = 0$

15 Solve the following equations.

a) $(x^2 + y^2 + 2x)\, dx + 2y = 0$

b) $x^2y\, dx - (x^3 + y^3)\, dy = 0$

c) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0 \ \forall\ x, y > 0.$

d) $(y^2 + 2x^2y)dy + (2x^3 - xy)dy = 0$

e) $(2x^2y - 3y^4)\, dx + (3x^3 + 2xy^3)\, dy = 0$

16. Solve the following equations.

a) $(x + y)^2 \dfrac{dy}{dx} = a^2$

b) $ydx + dy = 0$

c) $1 + \left(\dfrac{x}{y} - \sin y\right)\dfrac{dy}{dx} = 0$

d) $(3y^2 + 2xy) = (2xy + x^2)\dfrac{dy}{dx} = 0 \ \square\, x > 0, y > 0.$

e) $Y + y_2{}^{+}\left(2xy + \dfrac{y}{1+y}\right)\dfrac{dy}{dx} = 0$

f) $2x^2y^2 + 3x(1 + y^2)\ \square = 0$

g) $\square + \dfrac{2y}{x} = 0,\ y \geq 0$

7.0 REFERENCES/FURTHER READINGS

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Indira Gandhi National Open University School Of Sciences Mth-07

MODULE 2

Unit 1 Linear Differential Equations
Unit 2 Differential Equations of First order but not of first degree

UNIT 1 LINEAR DIFFERENTIAL EQUATION

CONTENTS

1.0 INTRODUCTION

In unit 2, we have discussed methods of solving some first order first degree differential equations, namely,

i) differential equations which could be integrated directly i.e., separable and exact differential equations,

ii) equations which could be reduced to these forms when direct integration is not possible. These includes homogeneous equations, equations reducible to homogeneous form and equations that become exact when multiplied by an I.F.

in this unit, we focus our attention on another very important type of first order first degree differential equations known as **linear equations.** These equations are important because of their wide range of applications, for example, the physical situations we gave in Sec. 1.5 of unit 1 are all governed by linear differential equations. In this unit, we shall solve some of these physical problems.

The problem of integrating a linear differential equation was reduced to quadrature by Leibniz in 1692. In December, 1695, James Bernoulli proposed a solution of a non-linear differential equation of the first order, now known as Bernoulli's equation.

In 1696, Leibniz pointed out Bernoulli's equation may be reduced to a linear differential equation by changing the dependent variable. We shall discuss this equation in the later part of this unit along with some other equations, which may not be of first order or first degree but which can be reduced to linear to linear differential equations.

2.0 OBJECTIVES

After studying this unit, you should be able to

- Identify a linear differential equation;
- Distinguish between homogeneous and non-homogeneous linear differential equation;
- Obtain the general solution of a linear differential equation;
- Obtain the particular integral of a linear equation by the methods of undetermined coefficients and variation of parameters;
- Use general properties of the solutions of homogeneous linear equations for finding their solutions;
- Obtain the solution of Bernoulli's equation;
- Obtain solution to linear equations modeled for certain physical situations.

3.0 MAIN CONTENT

3.1 Classification of First Order Differential Equations

We begin by giving some definitions in this section. You may recall that in Unit 1 we defined the general form of first order differential equation to be

$$g\left(x, y\frac{dy}{dx}\right) = 0$$

and if the equation is of first degree, then it can be expressed as

$$\boxed{\frac{dy}{dx}} = f(x, y)$$

In the above equation if the function f(x, y) be such that it contains dependent variable y in the first degree only, then it is called a linear differential equation. Formally, we have the following definition.

Definition: We say that a differential equation is linear if the dependent variable and all its derivatives appear only in the first degree and also there is no term involving the product of the derivatives or any derivative and the dependent variable.

For example, equation $\frac{dy}{dx} + \frac{2y}{x} = x^3$ and $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x \sin x$ are linear differential equations. However $\frac{dy}{dx} + x^2 = 10$ is not linear equation of the presence of the term $y\frac{dy}{dx}$.

The general form of the linear differential equation of the first order is

$$a(x)\frac{dy}{dx} = b(x)y + c(x) \qquad \ldots(1)$$

where a(x), b(x) and c(x) are continuous real valued functions in some interval $I \subseteq R$.

If c(x) is identically zero, then Eqn. (1) reduces to

$$a(x)\frac{dy}{dx} = b(x)y \qquad \ldots(2)$$

Eqn. (2) is called **a linear homogeneous differential equation.**

When c(x) is zero, Eqn. (1) is called **non-homogeneous** (or **inhomogeneous**) linear differential equation

Note: You may note that the word homogeneous as it is used here has a very different meaning from that used in Sec. 2.3, unit 2.

Any differential equation of order one which is not of type (1) or (2) is called a **non-linear differential equation.**

On dividing Eqn. (1) by a(x) for x s.t $a(x) \neq 0$, it can be put in the more useful form

$$\frac{dy}{dx} + P(x)\, y = Q(x), \qquad \ldots(3)$$

where P a. I Q are functions of x alone or are constants. Consider, for instance, the equation $\frac{dy}{dx} = y$

It is a linear homogeneous equation. Here a(x) = 1 and b(x) = 1. Similarly, $\frac{dy}{dx} = 0$, $\frac{dy}{dx} = e^x y$ are also linear homogeneous equation of order one with

$a(x) = 1$, $b(x) = e^x$ and $c(x) = x$.

Next consider the differential equation $\frac{dy}{dx} = |y|$.

You know that $|y| = y$ for $y \geq 0$ and $|y| = -y$ for $y < 0$. Hence, in order to solve this equation, we will have to square it and the resulting equation is neither of type (1) nor of (2). It is a case of non-linear equation. Similarly, $\left|\frac{dy}{dx}\right| = y$ is a non-linear equation because of the term $\left|\frac{dy}{dx}\right|$. Again $\frac{dy}{dx} = \cos y$ is a non-linear equation (as cos y can be expressed as an infinite series in powers of y).

You may now try this exercise.

You will realize the need for classification of linear differential equation into homogeneous and non-homogeneous equations when we discuss some properties involving the solution of linear homogeneous differential equations. But first let us talk about the general solution of linear non-homogeneous equation of type (1) or (3).

3.2 General Solution of Linear Non-Homogeneous Equation

Consider Eqn. (3) , viz.,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In the discussion that follows, we assume that Eqn. (3) has a solution. You can see that in general, Eqn (3) is not exact. But we will show that we can always find an integrating factor $\mu(x)$, which makes this equation exact – a useful property of linear equations.

Let us suppose that Eqn. (3) is written in the differential form

$$dy + [P(x)y - Q(x)]\,dx = 0 \qquad \ldots(4)$$

Suppose that $\mu(x)$ is an I.F. of Eqn. (4). Then

$\square(x)\,dy + \square(x)\,[P(x)y - Q(x)]\,dx = 0$ …(5)

is an exact differential equation. By Theorem 1 of Unit 2, we know that Eqn. (5) will be an exact differential if

$$\left|\frac{\partial}{\partial x}(\square(x) = \frac{\partial}{\partial y}(\square(x)[P(x)y - Q(x)]\}\right. \qquad \ldots(6)$$

or $\dfrac{d\mu}{dx} = \square P(x)$

This is a separable equation from which we can determine $\square(x)$. we have

$\dfrac{d\mu}{\mu} = P(x)dx$

or In $|\square| = \int P(x)\,dx$ …(7)

so that $\square(x) = e^{\square P(x)dx}$ is an integrating factor for Eqn. (4).

Note that we need not use a constant of integration in relation (7) since Eqn. (5) is unaffected by a constant multiple. Also, you may note that Eqn. (4) is still an exact differential equation even when Q9x) = 0. in fact Q(x) plays no part in determining

$\square(x)$ since we see from (6), that $\square\,\square(x)\,Q(x) = 0$. Thus both

$e^{\square P(x)dx}dy + e^{\square P(x)dx}\,[P(x)y - Q(x)]\,dx$ and

$e^{\square P(x)dx}dy + e^{\square P(x)dx}\,P(x)y\,dx$

are exact differentials.

We, now, write Eqn. (3) in the form

$$e^{\square Pdx}\left(\frac{dy}{dx} + + y\right) = Q\,e^{\square Pdx}$$

This can also be written as

$$\frac{d}{dx}\left(y\,e^{\square Pdx}\right) = Q\,e^{\square Pdx}$$

Integrating the above equation, we get

$y\,e^{\int Pdx}\,\square\,Q\,\square\,dx + \alpha$, where $\square$ is a constant of integration

or $y = \square\,\square\,Q\,\square\,dx + \square\,\square$ …(8)

For initial value problem, the constant in Eqn. (8) can be determined by using initial conditions. Relation (8) gives the general solution of Eqn. (3) and can be used as a formula for obtaining the solution of equation of the form (3). As a matter of advice we may put it that one need not try to learn the formula (8) and apply it mechanically for solving linear equations. Instead, one should use the procedure by which (8) is derived: **multiply by e^{Pdx} and integrate.**

In case of linear homogeneous equation, the general solution can be obtained by putting Q = 0 in Eqn. (8) as

$$y = \alpha\, e^{-\int Pdx}$$

Note that the first term on the right hand side of Eqn. (8) is due to non-homogeneous term Q of Eqn. (3). It is termed as the **particular integral** of the linear non-homogeneous differential equation, that is, particular integral of Eqn. (3) is

$$e^{-\int Pdx} \quad Q\, e^{Pdx}\, dx.$$

The particular integral does not contain any arbitrary constant.

The solution of linear non-homogeneous equation and its corresponding linear homogeneous equation are nicely interrelated. We give the first result, in this direction, in the form of the following theorem:

Theorem 1: In $I \subseteq R$, if y_1 be a solution of linear non-homogeneous differential Eq. (3), that is,

$$+ P(x)y = Q(x)$$

and if z be a solution of corresponding linear homogeneous differential equation

$$+ +(x)y = 0, \qquad \ldots(9)$$

then the function $y = y_1 + z$ is a solution of Eqn. (3) on I.

$$\therefore \quad = \frac{dy_1}{dx} + \frac{dz}{dx} \qquad ..(10)$$

Since y_1 $P(x)y_1 = Q(x)$

$$\frac{dy_1}{dx} + P(x)y_1 = Q(x) \qquad \ldots(11)$$

Also since z is a solution of (9), therefore

$$\frac{dz}{dx} + P(x)z = 0 \qquad \ldots(12)$$

On combining Eqns. (10) – (12), we get

$$= [Q(x) - P(x)y_1] + [-P(x)z]$$
$$= Q(x) - P(x)\,[y_1 + z]$$
$$= Q(x) - yP(x) \qquad \text{as } (y_1 + z = y),$$

i.e., $+ P(x)y = Q(x).$

Hence $y = y_1 + z$ is a solution of Eqn. (3) and this completes the proof of the theorem.

From this theorem, it should be clear that any solution of Eqn. (3) must contain solution of Eqn. (9) (corresponding linear homogeneous equation).

In case, the function Q(x) on the right-hand side of Eqn. (3) is a linear combination of functions, then we can make use of the following theorem:

Theorem 2: Let y_i be a particular solution of

$$+ P(x)y = Q_i(x),$$

where $Q_i(x)$ are continuous functions defined on an interval I for I = 1, 2,…., n. then the function $y_p = y_1 + y_2 + \ldots + y_n$, defined on I, is a particular solution of

$$+ P(x)y = Q(x),$$

where $Q(x) = Q_1(x) + Q_2(x) + \ldots + Q_n(x),\ \forall\, x \in I.$

The proof of this theorem is simple and is left as an exercise for you.

We now take up some examples and illustrate the method of finding the solution of linear non-homogeneous differential equations.

Example 1: Solve $x \;\; + y = x^3$

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{1}{x}\,y = x^2 \qquad \ldots(13)$$

it is a linear equation. Comparing it with Eqn (3), we have

$P = \frac{1}{x}$. So I.F. $= e^{\int P dx} = e^{\int (1/x)\,dx} = e^{\ln x} = x$

Multiplying Eqn. (13) by x, we get

$$x\frac{dy}{dx} + y = x^3$$

i.e., $\frac{d}{dx}(xy) = x^3$, which is exact.

Integrating, we get

$$xy = \frac{x^4}{4} + c,$$

c being a constant, as the required solution.

Example 2: Solve $x\frac{dy}{dx} - ay = x + 1$

Solution: Clearly the given equation is linear and can be written in the form

$$\frac{dy}{dx} - \frac{a}{x}y = \frac{x+1}{x}$$

$$\therefore \text{ I.F.} = e^{\int (-a/x)\,dx} = e^{-a\ln x} = e^{\ln x^{-a}} = \frac{1}{x^a}$$

Multiplying the given equation by $\frac{1}{x^a}$, we get

$$\frac{1}{x^a}\frac{dy}{dx} - \frac{a}{x^{a+1}}\,y = \frac{x+1}{x^{a+1}},$$

i.e., $\frac{d}{dx}\left(\frac{y}{x^a}\right) = \frac{x+1}{x^{a+1}}$

Integrating the above equation w.r.t.x, we get

$$\frac{y}{x^a} = \int \frac{x+1}{x^{a+1}}\,dx + c \text{ (c is constant)}$$

$$= \frac{x^{-a+1}}{-a+1} + \frac{x^{-a}}{-a} + c.$$

Thus $y = \frac{x}{1-a} - \frac{1}{a} + cx^{a}$ is the required solution.

Let us look at another example in which the role of x and y has been interchanged.

Example 3: Solve $y \, \mathrm{Iny} \frac{dx}{dy} + x - \mathrm{Iny} = 0$.

Solution: This equation is of first degree in x and $\frac{dx}{dy}$. Hence it is a linear equation with y as independent variable and x as dependent variable.

Then given equation can be written as

$$\frac{dx}{dy} + \frac{x}{y\mathrm{Iny}} = \frac{1}{y}, \qquad \ldots(14)$$

$$\square \text{I.F.} = e^{\square 1/(y \text{ in } y)\, dy}$$
$$= e^{\mathrm{In}(\mathrm{Iny})}$$
$$= \mathrm{Iny}$$

Multiplying Eqn. (14) by Iny, we get

$$\mathrm{Iny} \square + \frac{1}{y} x = \square \mathrm{Iny},$$

i.e., $\frac{d}{dy}(x\,\mathrm{Iny}) = \square\, \mathrm{Iny}$.

Integrating the above equation w.r.t.y, we get

$$x\mathrm{Iny} = \square \square\, \mathrm{Iny}\, dy + c,$$
$$= \frac{(\mathrm{Iny})^2}{2} + c, \text{ c is a constant,}$$

or $2x\,\mathrm{Iny} = (\mathrm{Iny})^2 + c_1$, is the required solution where $c_1 = 2c$.

let us consider another example.

Example 4: solve the equation $ydx + (3x - xy + 2)\,dy = 0$

Solution: Since the product y dy occurs here, the equation is not linear in dependent variable y. it is, however, linear if we treat variable y as independent variable and x as dependent variable. Therefore, we arrange the terms as
y dx + (3 – y) x dy = - 2 dy,
and write it in the standard form

$$\frac{dx}{dy} + \left(\frac{3}{y} - 1\right)x = -\frac{2}{y}, \text{ for } y \neq 0 \qquad \ldots(15)$$

Now, $\int \left(\frac{3}{y} - 1\right) dy = 3 \text{ In}|y|,$

So that an integrating factor for Eqn. (15) is

$$e^{(3\text{In}|y|-y)} = e^{-y}\, e^{3\text{In}|y|}$$
$$= e^{\text{In}|y|^3 e^{-y}}$$
$$= |y|^3 e^{-y}$$

If follows that for y > 0, y^3e^{-y} is an integrating factor and for y > 0, $-y^3 e^{-y}$ serves as an integrating factor for the given equation. In either case, we are led to the exact equation

$Y^3 e^{-y} dx + y^2 (3 – y e^{-y} x dy = - 2y^2e^{-y} dy,$
i.e., $d(xy^3y^{-y}) = - 2y^2e^{-y}$ dy.

Integrating the above equation w.r.t.y, we get

$$Xy^3e^{-y} = - 2\int y^2 e^{-y} dy$$
$$= 2y^2 e^{-y} + 4y e^{-y} + c \text{ (Integrating by parts)}$$

Thus, we can express the required solution as
$xy^3 = 2y^3 + 4y + 4 + ce^y$, where c is an arbitrary constant.

You may try the following exercises.

We have seen that general solution of a linear non-homogeneous differential Eqn. (3) is given by Eqn. (8), which involves integrals. We remark that an equation y' = f(x, y) is said to be solve when its solution is reduced to the expression of the form $\int h(x)$ dx or $\int ø(y)$ dy for some h(x) and ø(y) even if it is impossible to evaluate these integrals in terms of known functions. Further, the reduction of the solutions from one form to a simpler form may require as much labour as the solving of the equations. In solution (8) of Eqn. (3), $e^{-\int Pdx} \int Q(x)e^{\int Pdx}$ dx is the particular integral of Eqn. (3) and the evaluation of this integral will depend on the form of Q(x). this evaluation may sometimes turn out to be a tedious task. But, there are other methods by which particular integral in some scases can be obtained without carrying rigorous integration. We shall briefly disucss these methods now. As these methods are more helpful for higher order differential equations, we shall discuss them in greater detail in Block 2.

3.2.1 Method of Undetermined Coefficients

This method is applicable when in Eqn. (3), i.e.,

$$\frac{dy}{dx} + P(x)y = Q(x),$$

P(x) is a constant and Q(x) is any of the following forms:

i) an exponential
ii) A polynomial in x
iii) of the form cos βx or sin βx
iv) a linear combination of i), ii) and iii) above.

The general procedure is to assume the particular solution with arbitrary or unknown constants and then determine the constants.

We know that on differentiating functions such as $e^{\alpha x}$ (α constant), x^r (r > 0 is an integer), sin βx or cos βx (β constant), we again obtain an exponential, a polynomial or a function which is a linear combination of sine or cosine function. Hence if the non-homogeneous term Q(x) in Eqn. (3) is in any of the forms (i) – (iv), above, then we can choose the particular integral accordingly to be a suitable combination of the terms n(i) – (iv).

We now take up different cases according to the forms of Q(x).

Case 1: $Q(x) = k\, e^{mx}$, k and m are real constants, that is, Q(x) is an exponential function. In this case, we prove the result in the form of the following theorem.

Theorem 3: If a, k and m are real constants, then a partucular solution of

$$\frac{dy}{dx} + ay = k\, e^{mx}$$

is given by

$$y_p(x) = \begin{cases} \dfrac{k}{(a+m)} e^{mx} \text{if } m \neq -a \\ kxe^{mx} \text{if } m = -a \end{cases}$$

Proof: in this case, since Q(x) is an exponential function, we assume $y_p(x) = re^{mx}$ to be a particular solution of Eqn. (16), where r is some constant to be determined. Now $y_p(x)$ must satisfy Eqn. (16).

Thus, we get
$Rm\ e^{mx} + ar\ e^{mx} = ke^{mx}$
Or $r = \frac{1}{a+m}$ if $m \neq -a$.

Therefore, $y_p(x) = \frac{1}{a+m} e^{mx}$ if $m \neq -a$.
In case m + a = 0, i.e., m = - a, then you may verify that $y_p(x) = kx\ e^{mx}$ satisfies Eqn. (16). The reasoning for this sort solution will be given when we discuss this method in detail in Block 2. However, we illustrate this case by the following example.

Example 5: Solve $y' - y = 2e^x$

Solution: On comparing the given equation with Eqn. (16), we find that a = - 1, k = 2, and m = 1
Also, $m + a = -1 + 1 = 0 \Rightarrow m = -a$.
$\therefore$ By Theorem 3, a particular integral is $2xe^x$.
Further, I.F. = $e^{-\int Pdx} = e^{-\int dx} = e^{-x}$.($\therefore P = -1$)
Therefore, required solution, following relation (8), is
$y = P.I + c\ e^x$,
i.e., $y = 2xe^x + c\ e^x$.
You may now try this exercise.

Case II: $Q(x) = \sum_{i=0}^{n} a_i x^i$
That is, Q(x) is a polynomial of degree n. In this Eqn. (3) reduces to

$$\boxed{\frac{dy}{dx}} + ay = \boxed{\sum_{i=0}^{n} a_i x^i} \qquad \ldots(17)$$

If a = 0 in Eqn. (17), then particular solution is
$Y_p(x) = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}$, which follows by direct integration.
If in Eqn. (17), $a \neq 0$, then we assume
$y_p(x) = \sum_{i=0}^{n} P_i x^i$ (Q(x) being a polynomial in this case),

And determine real numbers $P_0, P_1, \ldots, P_n$ so that particular solution $y_p(x)$ satisfies Eqn. (17).
Substituting this value of $y_p(x)$ in Eqn. (17) (with y replaced by $y_p(x)$), we have

$$\sum_{i=1}^{n} iP_i x^{i-1} + \sum_{i=0}^{n} aP_i x^i = \sum_{i=0}^{n} a_i x^i \ (a \neq 0) \qquad \ldots(18)$$

Equating the coefficients of like power of x on both sides of Eqn. (18), we get

$$\left.\begin{array}{l}\text{Coeff. Of } x^i\text{: } (I+1)P_{i+1} + aP_i = a_i \text{ for } I = 0, 1, 2, \ldots, (n-1) \\ \text{Coeff. Of } x^n\text{: } \mathbf{aP_n = a_a}\end{array}\right\} \qquad \ldots(19)$$

Since Q(x) is a polynomial of degree n, thus $a_n \neq 0$ and we can solve EQn. (19) for $P_0\ P_1, \ldots P_n$. from Eqn. (19), we get

$$P_n = a_n/a$$

$$P_{n-1} = \left(a_{n-1} - \frac{na_n}{a}\right)\frac{1}{a},$$

$$P_{n-2} = a_{n-2} - \frac{n-1}{a}\ (a_{n-1} - \frac{n}{a}\ a_n)\ \frac{1}{a}, \text{ and so on.}$$

We illustrate this method with the help of following example.

Example 6: Find the particular solution of $\frac{dy}{dx} + 2y = 2x^2 + 3$.

Solution: We note that, in this case, Q(x) is a polynomial of degree 2. Assume a particular solution of the form

$$y_p(x) = \sum_{i=0}^{n} P_i x^i = P_0 + P_1 x + P_1 x^2.$$

Substitution of $y_P(x)$ in the given equation yields

$$(P_1 + 2P_2x) + 2(P_0 + P_1x + P_2x^2) = 2x^2 + 3. \qquad \ldots(20)$$

Equating the coefficients of like powers of x on both side of Eqn. (20), we get
Coeff. of x^2: $2P_2 = 2$ or $P_2 = 1$.
Coeff. of x^2: $2P_2 = 2$ or $P_2 = 1$.
Coeff. of x^0: $P_1 + 2P_0 = 3$ or $P_0 = 2$.

Hence, required particular solution is
$y_P(x) = x^2 - x + 2$.
And now an exercise for you.
Case III: Q(x) = sin βx or cos βx or a βx + b cos βx

Where β , a and b are real constants.

In all these cases, we assume a particular solution of the form c sin + d cos .

On substituting this solution in the given equation and equating the coefficients of sin and cos on both sides, we determine the constants c and d.

Let us illustrate this case by an example.

Example 7: Find the particular integral of

$$+ y = \cos 3x$$

solution: Here Q(x) = cos 3x.

Hence, any particular solution of the given differential equation must be a combination of sin 3x and cos 3x. let the particular solution be
$y_P(x) = c \cos 3x + d \sin 3x$
On substituting this value of $y_P(x)$ in the given equation, we get
$(-3c \sin 3x + 3x \cos 3x) + (c \cos 3x + d \sin 3x) = \cos 3x$ …(21)
comparing the coefficients of cos 3x and sin 3x on both sides of Eqn. (21), we get

$c + 3d = 1$ and $d – 3c = 0$
or $c = \frac{1}{10}$ and $d = \frac{3}{10}$
Hence, the particular solution is
$y_p(x) = \ (3 \sin 3x + \cos 3x)$
we now take up an example which is a combination of all the three cases discussed above.

Example 8: Compute the general solution of

$$+ y = e^x + x + \sin x \qquad \text{…(22)}$$

Solution: Here $Q(x) = Q_1(x) + Q_2(x) + Q_3(x)$,
With $Q_1(x) = e^x$, $Q_2(x) = x$ and $Q_3(x) = \sin x$.
You may recall Theorem 2; if y_1, y_2 and y_3 are particular solutions of

$$+ y = e^x \qquad \text{…(23)}$$

$$+ y = x \qquad \text{…(24)}$$

and

$$+ y = \sin x \qquad \ldots(25)$$

respectively, then $y_p = y_1 + y_2 + y_3$ is a particular solution of the given equation.
Consider Eqn. (23). Let the particular solution be
$Y_1 = re^x$.
Substituting this in Eqn. (23), we get

$$re^x + re^x = e^x \Rightarrow r = \frac{1}{2}$$

$$\therefore y_1 = \; e^x \qquad \ldots(26)$$

For Eqn. (24) we assume the particular solution as
$y_2 = a_1x + a_0$.
Substituting this in Eqn. (24), we get

$$a_1 + a_1x + a_0 = x \qquad \ldots(27)$$

comparing coefficients of like powers of x on both side of Eqn. (27), we get

$$\left.\begin{aligned} a_0 + a_1 &= 0 \\ a_1 &= 1, \end{aligned}\right\} \quad a_0 = -1,\ a_1 = 1$$

Hence $y_2 = x-1 \qquad \ldots(28)$

In the case of Eqn (25), assume particular solutions as
$y_3 = c \sin x + d \cos x$.
Substituting this in Eqn. (25), we get

$$c \cos x - d \sin x + c \sin x + d \cos x = \sin x \qquad \ldots(29)$$

On equating the coefficients of sin x and cos c on both sides of Eqn. (29), we get

$c - d = 1$

$c + d = 0 \qquad c = \;$ and $d = -$

$$y^3 = \; (\sin x - \cos x) \qquad \ldots(30)$$

Hence, particular solution of Eqn. (22) can be obtained from Eqn. (26), (28) and (30) as

$$y_p(x) = y_1 + y_2 + y_3 = \; e^x + x - 1 + \; (\sin x - \cos x)$$

The solution of homogeneous pat of Eqn. (22), i.e.

$$+ y = 0$$

is given by

$\frac{1}{y}\,\square + 1 = 0$

Integrating the above equation, we get

In y + x = In α, for some constant $\square$,

i.e., $\frac{y}{\alpha} = e^{-x}$

or y = $\square$ e^{-x}

Hence complete solution of Eqn. (22) is given by

y = $\square$ e^{-x} + $\square$ e^{x} + x-1 + $\square$ (sin x – cos x)

How about trying an exercise now?

We thus studied the method of undetermined coefficients for finding the particular integral of the non-homogeneous linear differential EQn. (3). We saw that this method would be applicable only for a certain class of differential equations – those for which P(x) is a constant and Q(x) assumes either of the forms $e^{\alpha x}$, x^r, sin $\square$ or cos $\square$, or their combinations. We shall, now, study a method that carries no such restrictions.

3.2.2 Method of Variation of Parameters

Consider the non-homogeneous linear Eqn. (3), namely,

$\square$ + P(x)y(x) = Q(x).

the homogeneous equation corresponding to the above linear equation is

$\square$ + P(x) y(x) = 0.

Further we know , from Eqn. (8), that the solution $y_h(x)$ of the homogeneous linear equation is given by

$y_h(x) = \square\, e^{-\int P dx}$, ..(31)

where $\square$ is a constant.

In tis method we assume that $\square$, in Eqn. (31), is not a constant but a function of x. that is, we vary $\square$ with x and assume that the resulting function

y(x) = $\square$(x) $e^{-\int P dx}$ …(32)

is a solution of Eqn. (3). That is, we try to determine $\square$(x) such that y given by Eqn. (32) solves Eqn. (3). In other words, we determine a necessary condition on $\square$(x) so that y defined by relation (32), is a solution of Eqn. (3).

On combining Eqns. (3) and (32), we get

$$\frac{d}{dx}[\square(x)\ e^{-\int P(x)dx}] + P(x)\,[\square(x)\ e^{-\int P(x)dx}] = Q(x).$$

i.e., $\square(x) = Q(x)\ \boxed{e^{-\int Pdx}}] + \square'(x)\ e^{-\int P(x)dx} + P(x)\,\square(x)\ e^{-\int P(x)dx} = Q(x),$

i.e., $\square'(x) = Q(x)\ e^{\int P(x)dx}$

Integrating w.r.t.x, we get

$$\square(x) = \square + \int Q(x)\ e^{\int P(x)dx}\,dx \qquad \ldots(33)$$

where $\square$ is a constant of integration.

Substituting the value of $\square(x)$ from eQn. (33) in relation (32), the solution to Eqn. (3) can be expressed as

$$y(x) = \square\, e^{-\int P(x)dx} + e^{-\int P(x)dx}\ \square\, Q(x)\ e^{-\int P(x)dx}\,dx$$

You may note here that the solution obtained above is same as the one given by Eqn. (8) that has been obtained directly. Further, the method of variation of parameter neither simplifies any integration/solution nor provides any other form of the solution for first order first degree differential equation. It only provides an alternative approache to arrive at the general solution in this case. However, as we shall see later in Block 2, this method turns out to be quite powerful in discussing equations of higher order.

Using the method of undetermined coefficients/variation of parameters or otherwise you may now try this exercise.

Now let us discuss some properties of linear homogeneous differential equations, which give us some insight into qualitative theorem rather than|quantitative solutions.

3.3 Properties of the Solution of Linear Homogeneous Differential Equation

In this section we shall discuss certain properties enjoyed by linear homogeneous differential equation. We start with a very important property called **superposition principle.**

Theorem 4: (Superposition Principle)

If y_1 and y_2 are any two solution of the linear homogeneous Eqn. (9), i.e.,

$$\square + P(x)\,y(x) = 0.$$

Then y1 + y2 and cy_1 are also solutions of Eqn. (9), where c is a constant.

Proof: Since y_1 and y_2 are both solutions of Eqn. (9), therefore

$$\frac{dy_1}{dx} + P(x)y_1 = 0 \qquad \ldots(34)$$

and

$$\frac{dy_2}{dx} + P(x)y_2 = 0, \qquad \ldots(35)$$

Let $h(x) = y_1 + y_2$

$$\frac{dh}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx},$$

$= - P(x)y_1 – P(x)y_2$, (using Eqn. (34) and (35))

$= - P(x) (y_1 + y_2)$,

$= - P(x) h(x)$,

i.e., $\frac{dh}{dx} + P(x) h(x) = 0$, which shows that $h(x) = y_1 + y_2$ is indeed a solution of Eqn. (9).

Next, multiplying Eqn. (34) by c (a constant), we get

$$c\frac{dy_1}{dx} + cP(x)y_1 = 0,$$

i.e. $\frac{d}{dx}(cy_1) + P(x). (cy_1) = 0$,

which shows that (cy_1) is also a solution of Eqn. (9).

In many branches of sciences, $y_1 + y_2$ is called superposition of y_1 and y_2 and hence the name superposition principle for Theorem 4.

The conclusions of Theorem 4 can be reframed as – the set of real (or complex) solution of Eqn. (9) forms a ral (or complec) vector space (ref. Block 1, MTE-02, acourse on linear algebra).

Do you think Theorem 4, holds for non-homogeneous linear equation? Consider the non-homogeneous equation $y' = 2x$.

The functions $(1 + x^2)$ and $(2 + x^2)$, $x \in R$ are two solution of

$$y' = 2x \qquad \ldots(36)$$

their sum $(2+x^2) + (2+x^2) = 3 + 2x^2$, $x \in R$ does not satisfy Eqn. (36) since $\frac{d}{dx}(3 + 2x^2) = 4x \neq 2x$, $\forall x \in R$.

Thus, Theorem 4 need not be true for a non-homogeneous linear equation does it work for a non-linear equation? Let us look at

$$y' = - y^2 \qquad \ldots(37)$$

which has solutions $y_1(x) = \frac{1}{(1+x)}$ and $y_2 = \frac{1}{1+2x}$ on an interval $I = [0, \infty[$. This is true because

$y_1'(x) = -\frac{1}{(1+x)^2} = -y_2^1$,

and

$y_2'(x) = -\frac{2.2}{(1+2x)^2} = -y_2^2$,

Let $y = y_1 + y_2$. Here y is well defined on I. Also, by simple computation, we have

$$y'(x) = -\frac{1}{(1+x)^2} - \frac{2.2}{(1+2x)^2} = -\frac{(8x^2+12x+5)}{(1+x)^2(1+2x)^2} \qquad \ldots(38)$$

on the other hand,

$$-y^2(x) = -\frac{(16x^2+24x+9)}{(1+x)^2(1+2x)^2} \qquad \ldots(39)$$

from relations (38) and (39), it is clear that $y = y_1 + y_2$ is not a solution of (37).

In the next exercise we ask you to show an example of a non-linear equation whose solution is y_1 but cy_1 is not a solution, i.e., the later part of Theorem 4 need not be true for a non-linear equation.

Mostly we study the real solutions of Eqn. (1). You may recall that the functions a and b in Eqn. (1) (defined on I) are assume to be valued. The reason for restricting the study to real solutions will be clear from the following theorem.

Theorem 5: If $y = p + iq$ is a complex valued function defined on I, which satisfies Eqn. (2), that is, $a(x)\frac{dy}{dx} = b(x)y(x)$, then the real part p of y and the imaginary part q of y are also solutions of Eqn. (2) on I.
(recall here that a and b are ral valued continuous functions)

proof: By definition $y = p+iq$ and so $y' = p' + iq'$. Since y satisfies Eqn.s (2), we have $a(x)\{p'(x) + iq'(x)\} = b(x)\{p(x) + iq(x)\}$ …(40)

since a and b are real valued, on equating the ral and imaginary parts in Eqn. (4), we get

$a(x)p(x) = b(x)p(x)$,

and

$a(x)q'(x) = b(x)q(x)$,

which show s that p and q are solution of Eqn. (2) on I.

Theorem 5 is also true for higher order linear homogeneous equations which will be discussed in our later blocks and the proof is virtually on the same lines. But the theorem may fail if we replace Eqn. (2) by an arbitrary non-linear equation or a linear non-homogeneous equation. for instance, consider the first order non-linear equation

$yy' = -2x^3$ …(41)

the function $y(x) = ix^2$, $x \in \mathbb{R}$ is a complex valued solution of Eqn. (41), since
$y'(x) = 2ix$
and $y(x)y'(x) = (ix) (ix^2 = -2x^3$.

The real par p of y is the zero function. i.e., $p(x) = 0$. But p is not a solution of Eqn. (41) (since $2x^3 \square 0$ for all $x \in \mathbb{R}$).

The following exercise shows that Theorem 5 may fail in the case of non-homogeneous linear equations.

We shall now be giving another interesting result concerning linear homogeneous equation $a(x) \square = b(x)y(x)$, which can also be written as

$y' = g(x)y$, …(42)

where $g(x) = \frac{b(x)}{a(x)}$, is a real valued continuous function defined on I. Result which we are going to state is a consequence of the uniqueness of solutions of initial value problem for linear equations.

Theorem 6: Let y be a solution of the Eqn. (42) on the interval such that $y(x_1) = 0$ for some x_1 in I. Then $y = 0$ on I.

Proof: Consider the initial value problem
$y' = g(x) y$,
$y(x_1) = 0$

By hypothesis, y is a solution of Eqn. (42). But the function z, defined by z9x) = 0 for all x I), also satisfies Eqn. (42) (because $z'(x) = 0$, $g(x)z(x) = 0$ and $z(x_1) = 0$). By the uniqueness theorem for the initial value problem for linear equation (refer Theorem 1, Unit 1), we have $z = y$ or in other words, $y(x) = 0$ for $x \in I$. This completes the proof. Just as we have seen in the case of Theorem 4 and 5, Theorem 6 may not be true for non-linear or linear non-homogeneous equations. Consider, for instance, the following non-linear differential equation.

$y' = 2\sqrt{y}$, $x \in [0, \square[$ …(43)

Let c>0. we define the function y on $[0, \infty[$ by

$$y(x) = \begin{cases} 0 \text{ if } 0 \le x < c \\ 2(x - c)^2 \text{ if } c \le| x < \infty \end{cases}$$

from the definition of y, we have

$$y'(x) = \begin{cases} 0 \text{ if } 0 \le x < c \\ 2(x - c) = 2\sqrt{y(x)}, \text{if } c \le| x < \infty \end{cases}$$

(Note that y is differentiable at x = c and, in fact, its right as well as left derivative is zero at x = c).

$$y'(x) = \begin{cases} 0 = 2\sqrt{y(x)}, 0 \le| x < c \\ 2(x - c) = 2\sqrt{\ } y(x) \text{if } c \le| x < \infty \end{cases}$$

which shows that y satisfies Eqn (43) for all x > 0. We notice, here, that y vanishes on the interval [0, c[and yet y is a non zero function on $[0, \infty[$; which shows that the conclusions. For example, y(x) = cos x + sin, $x \in R$ is a solution of Theorem 6 may not be true for non-linear equations.

similarly, we can show that Teorem 6 is not valid for linear non-homogeneous equation. for example, y(x) = cos x + sin x, $x \in R$ is a solution of

$y' = y - 2\sin x, x \in R$. …(44)

But, y(x) is zero at many points (like $x = -\frac{\pi}{4}, -\frac{\pi}{4} + 2\pi, \ldots$)

and assume both negative and positive values. Yet y is a non-linear function which solves Eqn. (44).

Did you notice that in Theorem 6 we did not take a general linear homogeneous equation? we only considered linear homogeneous initial value problem. Why?

Well, consider the linear homogeneous equation

$\sin x\, y'(x) = \cos x\, y(x)$ …(45)

The function y(x) = sin x, $x \in R$ is a non-zer solution of Eqn. (45), which vanishes at many points of R (like $x = 0, \pm\pi, \pm 2\pi, \ldots$) and also changes sign.

You may now try this exercise.

In Theorem 4 to 6, we have given some properties of linear homogeneous equations and corresponding initial value problems. But, none of the results stated asserts the existence of solutions of linear equations or corresponding initial value problem. Such results are called qualitative properties of solution of linear equations and their corresponding initial value problems.

Sometimes equations which are not linear can be reduced to the linear form by suitable transformations of the variables.

In the next section we shall take up such equations.

3.4 Equations Reducible to Linear Equations

Let us consider an equation of the form

$$f'(y) \square + P(x), f(y) = Q(x) \qquad \ldots(46)$$

where f'(y) is the differential coefficient of f(y).

an interesting feature of Eqn. (46) is that it is a non-linear differential equation of the first order that can be reduced to the linear form by putting v = f(y). with this substitution Eqn. (46) reduced to

$$\square + P(x)v = Q(x), \left(\square \ \frac{dy}{dx} = f'(y)\frac{dy}{dx}\right),$$

which is a linear equation with v as dependent variable and x as independent variable.

A very important and famous equation of this form, about which we have already mentioned in Sec 3.2, is known as **Bernoulli's Equation,** named after James Bernoulli, who studied it in 1695 for finding its solution. The equation is or the form.

$$\square + Py = Qy'', \qquad \ldots(47)$$

where P an Q are functions of x above and n is neither zero nor one. Dividing Eqn (47) by y^n, we get

$$y^{-n} \square + P\, y^{1-n} = Q \qquad \ldots(48)$$

in the year 1696, Leibniz pointed out that Eqn. (48) can be reduced to a linear equation by takeny^{1-n} as the new dependent variable.

On putting $v = y^{1-n}$ in Eqn. (48), it reduces to

$$\frac{1}{1-n} \square + P\, v = 0. \qquad \ldots(49)$$

which is a linear differential in v and x. Eqn. (49) can now be solve by the known methods.

Note that when n= 0 Eqn. (47)is a linear non-homogeneous equation and when n = 1, Eqn. (47) is a linear homogeneous equation. we now illustrate this method with the help of a few examples.

Example 9: Solve $- \frac{\tan y}{1+x} = (1 + x)e^x \sec y$ …(50)

Solution: Dividing Eqn. (50) by sec y, we get

$\cos y = \frac{\sin y}{1+x} = (1+x)e^x$ …(51)

If we put sin y = f(y), then f'(y) = cos y and hence Eqn. (51) is of the form

$f'(y) - \frac{1}{1+x} f(y) = (1+x+e^x$ …(52)

which is of the type (46). To reduce it to linear from, we put

v = f(y) = sin y

Then Eqn (52) reduces to

$$\frac{dv}{dx} - \frac{1}{1+x} v = (1+x)e^x$$

it is a linear equation with I.F. = $e^{-\int \frac{1}{1+x} dx} = e^{-In(1+x)} = \frac{1}{1+x}$

Multiplying the above equation by I.F., we get

$$\frac{d}{dx}\left(v.\frac{1}{1+x}\right) = \frac{1}{1+x}(1+x)\, e^x.$$

Integrating w.r.t.x, we have

$v \frac{1}{1+x} = e^x + c$, c being a constant

i.e., $v =)1+x)e^x + c(1+x)$.

Substituting sin y for v, the required solution of the given Eqn. (50) is $\sin y = (1+x)e^x + c(1+x)$

Let us look at another example in which n is neither 0 nor 1.

Example 10: Solve $y(axy + e^x)\, dx - e^x\, dy = 0$

Solution: The given equation can be rearranged as

$e^x = e^x y + axy^{2,}$

i.e., $\boxed{\frac{dy}{dx}} - y = axy^{-x}y^2$ …(53)

it is a Bernoulli's equation with n = 2.

To solve it, let $y^{1-2} = v$, i.e., $v = \frac{1}{y}$.

$\therefore \frac{dv}{dx} = -\frac{1}{y^2}\boxed{\frac{dy}{dx}}$

Consequently, Eqn. (53) reduces to

$-\frac{dv}{dx} - v = axe^{-x}$ …(54)

it is a linear equation with I.F. = $e^{\int 1dx} = e^x$

Multiplying both sides of Eqn. (54) by I.F., we get

$\frac{d}{dx}(ve^x) = -ax$

Integrating w.r.t.x, we get

$V\ e^x = -\int ax\,dx + c.$

$\quad = -\frac{ax^2}{2} + c$

Replacing v by $\left(\frac{1}{y}\right)$, the required solution can be expressed as

$E^x = y\left(c - \frac{ax^2}{2}\right)$,

Remark: There are many second or higher order linear equations which can be solved easily by reducing them to linear first order equations by making some transformation of the variables. We shall take up such equations later in Block 2 when we discuss second order equations.

You may now try the following exercises.

You may recall that in Unit 1, we discussed some physical situations expressed in terms of differential equations. In the following section we have attempted to solve some of them.

Applications of Linear Differential Equations

Let us consider the problem discussed in Unit 1 one by one.

I. Population model

You may recall that while studying the equation for population problem we had arrived at the initial value problem. (ref. Eqns. (32) and (33) of unit 1)

$$\left.\begin{aligned} \frac{d}{dt} N(t) &= k\, N(t) \\ N(t_0) &= N_0 \end{aligned}\right\} \qquad \ldots(55)$$

Since k is a constant in Eqn. (55), the first of the equations in (55) is a linear differential equation of order one. From Sec. 3.3 we know that its solution is

$$N(t) = N(t_0) \exp (k(t-t_0)) \qquad \ldots(56)$$

In Eqn. (56), we normally assume that $N(t_0)$ is specified. If k is known then we can find the solution using relation (56). In reality, it is too hard to measure k (which gives the rate of growth). In a particular case, we can actually find the exact value of k if we know the value of N at t_1 ($t_1 \neq t_0$). The details are shown in the following example.

Example 11: Assuming that the rate of growth of a species is proportional to the amount N(t) present at time t, find the value of N(t) given that N(0) = 100 and after one unit of time, the size of the specie has grown to 200.

Solution: In this case $t_0 = 0$, N(0) = 100. the solution of the problem is given by
$N(t) = 100 \exp (kt),\ t \geq 0$
We determine k from the additional condition N(1) = 200 (N(1) = size of population at time t = 1).
Thus $200 = 100 \exp (k) \Rightarrow k = \text{In}2$

Hence the solution is

$N(t) = 100 \exp (t\ \text{In}2) = 100 \exp (\text{In}2^t)$
Or $N(t) = (100)\, 2^t$

In this problem the constant k has been determined from the given date.

In the following exercise we ask you to solve a similar problem.

Let us now discuss the problem of decay of radioactive material.

II. Radioactive Decay

In unit 1, (Ref. Eqn. (35)), we have seen that equation which governs the radioactive decay of a given radioactive material is
$y'(t) = k\, y(t)$ …(57)

Note: Half-life is the time needed for the material to reduce itself to half of its original mass.

Where y(t) is the mass of the radioactive material at time t and k < 0 is a real constant. Eqn. (57) can be used to find the half-life of the radioactive material.

In the following example we consider this problem in detail.

Example 12: A radioactive substance with a mass of 50 gms. Was found to have a mass of 40 gms. After 30 years. Find its half-life.

Solution: The mass y(t) of the material satisfies

$$\left.\begin{array}{l} \frac{d}{dt} y(t) = k\ y(t) \\ y(0) = 50\ \text{gms.}, \\ y(30 = 40\ \text{gms.} \end{array}\right] \qquad ...(58)$$

The solution of the first two equations in Eqns. (58) can be expressed as
$y(t) = 50 \exp(kt)$,
Using the third equation in Eqn. (58), we can write.
$y(30) = 40 = 50 \exp(30k)$,
or $\exp(30k) = 4/5$,
i.e., $k = \frac{1}{30}\ \text{In}\left(\frac{4}{5}\right)$

thus, the mass y(t) satisfies
$$y(t) = 50 \exp\left(\frac{t}{30} \text{In} \frac{4}{5}\right) \qquad …(59)$$

Let t_1 be its half-life, i.e., after time t_1 the mass reduces to $\frac{50}{2}$ = 25 gms.
Then $y(t_1) = 25$ …(60)

We are required to find t_1. using condition (60), Eqn. (59) reduces to

$25 = 50 \exp\left(\frac{t_1}{30} \text{In } 4/5\right)$
or t_1 In (4/5) = 30 In (1/2)
i.e. t_1 = 30 (In (1/2))/In (4/5) …(61)
so after t_1 years (t_1 defined by Eqn. (61)), the mass of the material will be 25 gms.

Let us now deal with the temperature variations of a hot object.

III. Newton's Law of Cooling

The temperature of a hot a hot body kept in a surrounding of constant temperature T_0 has been discussed in Unit 1 and the governing equation of the temperature T of the body is
$T'(t) = k(T(t) - T_0)$ …(62)
(Ref. Eqn. (34) of Unit 1)

we illustrate this by the following example.

Example 13: A rod of temperature 100ºC is kept in a surrounding of temperature 20ºC. If the temperature of the rod was found to be 80ºC after 10 minutes, find the temperature T(t) of the rod.

Solution: We are required to solve
$\frac{d}{dt} T(t) = k(T(t) - 20)$ …(63)

Let us put y(t) = T(t) – 20. Then y'(t) = T'(t) and Eqn. (63) reduces to
$\frac{d}{dt} y(t) = k\, y(t)$ ..(64)

Eqn. (63) is not a linear homogeneous equation whereas Eqn. (64) is which explains the reason for introducing y) Along with Eqn. (64), we have

$$\left.\begin{array}{ll} \text{a)} & y(0) = T(0) - 20 = 100 - 20 = 80^{\circ}\text{C}, \\ \text{b)} & y(10) = T(10) - 20 = 80 - 20 = 60^{\circ}\text{C} \end{array}\right] \quad \text{…(65)}$$

the solution of Eqn. (64), with the condition 65(a), is
$y(t) = 80 \exp(kt)$
with this value of y and condition (65b), we have
$y(10) = 60 = 80 \exp(k.\ 10)$
or, $k = \frac{1}{10}$ In (6/8) = $\frac{1}{10}$ In (3/4)

Hence the value of y is determined by

$$y(t) = 80 \exp\left(\frac{t}{10} \text{In}(.75)\right),$$

and the temperature T is given by

$$T(t) = 80 \exp\left(\frac{t}{10} \text{In}(.75)\right) + 20$$

And now an exercise for you.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have covered the following points:

1) The general form of the linear equation of the first order is $\frac{dy}{dx} + P(x)y = Q(x)$, (see Eqn. (3))
Where P(x) and Q(x) are continuous real-valued functions on some interval $I \subseteq R$.
When Q(x) = 0 it is called **homogeneous linear differential equation** of order one.
When $Q(x) \neq 0$, it is called **non-homogeneous** (or **inhomogeneous**) **linear differential equation** of order one.
I.F. for this equation of $e^{\int P(x)dx}$ and the **general solution** is given by
$y = e^{-\int P(x)dx} . \int Q(x)\ e^{\int P(x)dx}$
Here, $e^{-\int P(x)dx} \int Q(x)\ e^{\int P(x)dx}$ is the particular solution of the equation.
2) The sum of the solution of linear non-homogeneous differential equation of the form (3) and the solution of its corresponding homogeneous equation is again a solution of the equation.
3) If in the differential equation

$$\frac{dy}{dx} + P(x)y = Q(x),$$

P(x) is a constant and Q(x) is any of the forms $e^{\alpha x}$ (α constant), x^r(r > 0, an integer), sin βx or cos βx (β constant) or a linear combination of such functions, then method of undetermined coefficients can be applied to find the particular solution of the equation and the particular integrals for different Q(x) are given by the following table

P(x)	Q(x)	Particular Integral
a(constant	e^{mx} (m constant)	$\begin{cases} \frac{e^{mx}}{m+a} \text{ if } m \neq -a \\ xe^{mx} \text{ if } m = -a \end{cases}$
a	$\sum_{i=0}^{n} a_i x^i$ (i > 0 an integer)	$\begin{cases} \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1} \text{ if } a = 0 \\ \sum_{i=0}^{n} P_i x^i \text{if} a \neq 0 \end{cases}$ with $P_n = \frac{a_n}{a}$, $P_{n-1} = \frac{1}{a}\left(a_{n-1} - \frac{na_n}{a}\right)$, $P_{n-2} = \frac{1}{a}\left[a_{n-2} - \frac{n-1}{a}(a_{n-1} - \frac{n}{a}a_a)\right]$ and so on
a	$\sin \beta x$ or $\cos \beta x$ or A sin βx + B cos βx (β, A, B constants)	A linear combination of sin βx and cos βx

4) Method of variation of parameters for finding the solution of non-homogeneous linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

5) Some properties of the solution of linear homogeneous differential equation

$$\frac{dy}{dx} + P(x)y = 0 \text{ are}$$

a) (Superposition Principle): If y_1 and y_2 are any two solutions of the equation, then $y_1 + y_2$ and cy_1 are also solution of the equation, where c is a real constant.

b) If a complex valued function y = p+iq, defined on 1, is a solution of the equation, then real part of p of y and imaginary part q of y are also solutions of the equation on 1.

c) If y be a solution of the equation on I such that $y(x_1) = 0$ for some x_1 in I, then y = 0 on I.

6) a) Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)\, y^n,$$

where P and Q are functions of x alone and n is neither zer nor one, reduces to a linear equation by the substitution $v = y^{1-n}$.

b) Equations of the type

$$f'(y)\,\frac{dy}{dx} + P(x)\,f(y) = Q(x)$$

reduce to linear equations by the substitution $f(y) = v$.

7) The differential equations governing physical problems such as population model, radioactive decay and Newton's law of cooling have been solved.

6.0 TUTOR MARKED ASSIGNMENT

1. From the following equations, classify which are linear and which are non-linear.
Also state the dependent variable in each case.

a) $\boxed{\ } - y = xy^2$

b) $rdy - 2ydx = (x - 2)\,e^x\,dx.$

c) $\frac{di}{dt} - 6i = 10 \sin 2t$

d) $\boxed{\ } + y = y^2 e^x$

e) $ydx + (xy + x\text{-}3y)\,dy = 0$

f) $(2s - e^{2t})\,ds = 2(se^{2t} - \cos 2t)\,dt$

2. Prove Theorem 2.

3. Solve the following equations:

c) $(x^2 + 1)\boxed{\ } + 2xy = 4x^2$

d) $\boxed{\ } + \frac{2}{x}y = \sin x$

e) $\sec x\boxed{\ } + y = \sin x$

f) $(1 + y^2\,dx = (\tan^{-1}y - x)\,dy$

g) $(2x - 10y^3)\boxed{\ } + y = 0$

4. Solve the following equations.

a) $y' = y + \frac{e^x}{x}$, $e \in [1, \infty[$

b) $y' = y + x + x^{3++}x^5$
c) $y' = y + x \sin x\, e^x + x^5$
d) $y' + 3y = |x|,\ y(0) = 1.$

5. Solve $\frac{dy}{dx} + y = 2ae^x$

6. Solve $\boxed{\frac{dy}{dx}} = y + x^2.$

7. Solve the following differential equations:

a) $\boxed{\frac{dy}{dx}} - y = 6 \cos 2x$

b) $\boxed{\frac{dy}{dx}} + 3y = x^2 + 3e^{2x} + 4 \sin x$

8. Solve the following equations:

a) $y' - 2y = \sin \pi x + \cos \boxed{} x,\ y(1) = 1$
b) $y' - y = \cos 2x + e^x + e^{2x} + x$
c) $y' - 3y = x^2 - \cos 3x + 2$ (Hint: Treat 2 as $2e^{0x}$)
d) $y' + y = -x - x^2,\ y(0) = 0.$
e) $Y' - y = e^x,\ y(0) = -3.$

9. Show that $y_1(x) = -\frac{1}{(1+x)}$ is not a solution of Eqn. (37)

10. Show that the solution $y(x) = e^x + ie^x - (1+x)$ of equation $y' = y + x$, for $x \in I = R$ does not satisfy the hypothesis of Theorem 5.

11. Show that $y(x) = \sin 2x - \cos 2x$, $x \boxed{\in R}$ is a solution of $y' = 2y + 4 \cos 2x$. why does Theorem 6 fail in this case?

12. Solve the following equations:

a) $\boxed{\frac{dy}{dx}} = \frac{e^y}{x^2} - \frac{1}{x}$

b) $\boxed{\frac{dy}{dx}} + xy = x^3y^3$

c) $3e^x \tan y + |(1-e^x) \sec^2 y \boxed{\frac{dy}{dx}} = 0$

13. Find the solutions of

a) $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

b) $\frac{dy}{dx} + y = e^x y^3$

c) $2x \frac{dy}{dx} + y(6y^2 - x - 1) = 0$

14. A culture initially has N_0 number of bacteria. At t = 1 hour, the number of bacteria is measured to be $\left(\frac{3}{2}\right) N_0$. If the rate of growth is proportional to the number of bacteria present, determine the time necessary for the number of bacteria to triple.

15. Suppose that a thermometer having a reading of 70ºF inside a house is placed outside where the air temperature is 10ºF. Three minutes later it is found that the thermometer reading is 25ºF. Find the temperature reading T(t) of the thermometer.

7.0 REFERENCE/FURTHER READINGS

Theoretical Mechanics by Murray. R.Spiegel

Advanced Engineering Mathematics by Kreyzig

Vector Analysis and Mathematical MethodS by S.O. Ajibola

Engineering Mathematics by p d s Verma

Advanced Calculus Schaum’s outline Series by Mc Graw-Hill

Indira Gandhi National Open University School Of Sciences Mth-07

UNIT 2 DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

CONTENTS

1.0 INTRODUCTION

In unit 1, we discussed the nature of differential equations and various types of solutions of differential equations. In Unit 2 and 3, we have given you the methods of solving different types of differential equations of first order and first degree. In this unit we shall consider those differential equations which are of first order but not of first degree.

If we denote $\frac{dy}{dx}$ by P, then the most general form of a differential equation of the first order and nth degree can be expressed in the form

$$P^n + P_1p^{n-1} + P_2p^{n-2} + \ldots..+ P_{n-1}p + P_n = 0, \qquad \ldots(1)$$

Where $P_1, P_2, \ldots.., P_n$ are functions of x and y.

It is difficult to solve Eqn. (1) in its most general form. In this unit we shall consider only those forms of Eqn. (1) which can be easily solved and discuss the methods of solving such equations.

It was Isaac Newton (1642 – 1727), the English mathematician and scientist, who classified differential equations of the first order (then known as fluxional equations) in "Methodus Fluxionum et serierum infinitarum", written around 1671 and published in1736. count Jacopo Riccati (1676-1754), an Italian mathematician, was mainly responsible for introducing the ideal of Newton to Italy. Riccati was destined to play an important part in further advancing the theory of differential equations. In 1712, he reduced an equation of the second order in y to an equation of first order in p. in 1723, he exhibited that under some restricted hypotheses, the particular equation to which the name of Riccati is attached, can be solved.

Later the French mathematician Alexis Claude Clairaut (1713-1765) introduced the idea of differentiating the given differential equations in order to solve them. He applied it to the equations that now bear his name and published the method in 1734. we shall also be discussing the equations introduced by Riccati and Clairaut in this unit.

Clairaut (1713-1765)

2.0 OBJECTIVES

After studying this unit, you should be able to

- Find the solution of the differential equations which can be resolved into rational linear factors of the first degree;
- Obtain the solution of equations solvable for y, x or p;
- Obtain the solution of the differential equations in which x or y is absent;
- Solve the equations which may be homogeneous in x and y;

- Identify and obtain the solution of Clairaut's equation;
- Identify and obtain the solution of Riccati's equation.

3.0 MAIN CONTENT

3.1 Equations which can be factorized

Let us consider the general form of differential equation of the first order and nth degree given by Eqn. (1) namely,

$P^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_{n-1}p + P_n = 0$,
Where $P_1, P_2, \ldots, P_n$ are functions of x and y.

For this equation, we shall consider two possibilities:

a) When the left-hand side of Eqn. (1) can be resolved into rational factors of the first degree.
b) When the left-hand side of Eqn. (1) cannot be factorized.
In this section we shall take up the first possibility.

When Eqn. 91) can be factorized into rational factors of the degree, then it can take the form

$$(p - R_1)(p - R_2)\ldots\ldots(p - R_n) = 0 \qquad \ldots(2)$$

for some $R_1, R_2, \ldots, R_n$, which are functions of x and y.

Eqn. 91) will be satisfied by a value of y that will make any of the factors in Eqn (2) equal to zero. Hence, to obtain the solution of Eqn. (1), we equate each of the factors in Eqn. (2) equal to zero. Thus, we get

$$P - R_1 = 0,\ p - R_2 = 0,\ \ldots,\ p - R_n = 0 \qquad \ldots(3)$$

There are n equations of first degree. Using the methods given in Unit 2 and 3 we can now obtain the solution of the above n equations of first order and first degree.

Let us suppose that the solutions desired for Eqn. (3) are

$$\left.\begin{array}{c} f_1(x, y, c_1) = 0 \\ f_2(x, y, c_2 = 0 \\ ;; \\ f_n(x, y, c_n) = 0 \end{array}\right\} \qquad \ldots(4)$$

where $c_1, c_2, \ldots, c_n$ are the arbitrary constants of integration.

Since each of the constants $c_1, c_2, \ldots, c_n$ can take any one of an infinite number of values, thus these solutions remain general even if

$C_1 = c_2 = \ldots = c_n = c$, say.
In that case, the n solutions will be

$f_1(x, y, c) = 0$
$f_2(x, y, c) = 0$
$f_3(x, y, c) = 0$
….
….
$F_n(x, y, c) = 0$

These n solutions can be left distinct or we can combine them into one equation, namely,
$F_1(x, y, c). f_2(x, y, c) \ldots f_n(x, y, c) =$

The reason of taking all $c_1, c_2, \ldots, c_n$ equal in Eqn. (4) is the fact that Eqn. (2) being of first order, its general solution can contain only one arbitrary constant.

We illustrate this method by the following examples:

Example 1: Solve $p^2 + px + py + xy = 0$

Solution: The given equation is equivalent to
$(p+x)(p+y) = 0$
that is, either
$p+x = 0$ or, $p+y = 0$
In other words,
$\frac{dy}{dx} + x = 0$, or $\frac{dy}{dx} + y = 0$

the solutions of the factors are
$2y = -x^2 + c$
and
$x = -\text{In}\,|y| + c$, for c being an arbitrary sonstant.

Therefore, the general solution of the given equation is
$(2y + x^2 - c).(x + \text{In}\,|y| - c) = 0$.

Let us look at another example.

Example 2: Solve $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0$

Solution: The given equation is equivalent to
$p[p^2(x + 2y) + 3p(x + y) + (y + 2x)] = 0$
$\Rightarrow p[p^2(x + 2y) + p\{(y + 2x) + (x + 2y)\} + (y + 2x)] = 0$
$\Rightarrow p(p + 1) [(x + 2y)p + (y + 2y)] = 0$
its component equations are
$p = 0, p + 1 = 0, (x + 2y)p + (y + 2x) = 0$
Now $p = 0 \Rightarrow \frac{dy}{dx} = 0$, which has the solution
$y = c$ …(5)

Now $p+1 = 0 \Rightarrow \frac{dy}{dx} + 1 = 0$
i.e., $dy + dx = 0$
which has the solution
$y+x =c$ …(6)
Further, $(x + 2y)p + (y + 2x) = 0$
$\Rightarrow (x + 2y)dy + (y + 2x)dx = 0$
$\Rightarrow d(xy + x^2 + y^2) = 0.$
Which has the solution
$xy + {}^2 + y^2 = c$ …(7)

Therefore, the general solution of the given equation, from Eqns. (5), (6) and (7), is
$(y - c). (y+x - c). (xy + x^2 + y^2 - c) = 0.$

You may now try the following exercise.

As you know from algebra, every equation over Q need not have all its roots in Q, i.e., it need not be factorizable in Q.

We now take up those equations of form (1) which cannot be factorized into rational factors of the first degree.

3.2 Equations which cannot be Factorized

in this case, let the form of Eqn. (1) be
$f(x, y, p) = 0$ ….(8)

eqn. (8) is not solvable in its most general form

we shall discuss only those equations of type (8) which possess one or more of the following properties.

i) It may be solvable for y.
ii) It may be solvable for x.
iii) It may solvable for p.
iv) Either it may not contain y or it may not contain x, that is, either x or y is absent from the differential equation.
v) It may be homogeneous in x and y.
vi) It may be of first degree in x and y.
vii) It may be Riccati's equation.

We now discuss these cases one by one.

3.2.1 Equation Solvable for y

Consider an equation

$$xp^2 - yp - y = 0 \qquad \ldots(9)$$

we can write Eqn. (9) in the form

$y(p+1) = xp^2$,

$$\Rightarrow \ y = \frac{xp^2}{p+1}$$

That is, Eqn. (9) can be solvable for y in terms of x and p.

Similarly when Eqn. (8), i.e., f(x, y, p) = 0 is solvable for y, then it can be pt in the form

$$y = F(x, p) \qquad \ldots(10)$$

Differentiating Eqn. (10) w.r. to x, we get an equation of the form

$$P = \varnothing\left(x, p, \frac{dp}{dx}\right) \qquad \ldots(11)$$

Eqn. (11) is in two variables x and p; and we may possibly solve and get a relation of the type

$$\psi\ (x, p, c) = 0 \qquad \ldots(12)$$

for some constant c.

It we now eliminate p between Eqns. (8) and (12), we get a relation involving x, y and c, which is the required solution. In the cases when the elimination of p between Eqns. (8) and (12) is not possible, we then obtain the values of x and y in terms of p as a parameter and these together give us the required solution.

We now illustrate this method with the help of a few examples.
Example 3: The given equation is solvable for y.
Solving it for y, we get

$$y = p + \frac{x}{p} \qquad \ldots(13)$$

Differentiating Eqn. (13) w.r. to x, we get

$$p = \frac{dp}{dx} + \frac{1}{p} + x\left(-\frac{1}{p^2}\right)\frac{dp}{dx}$$

$$\text{i.e., } \left(p - \frac{1}{p}\right)\frac{dx}{dp} + \frac{1}{p^2}\ x = 1 \qquad \ldots(14)$$

this is linear equation of the first order if we consider p as independent variable and x as dependent variable.

We can write Eqn (14) as,

$$\frac{dx}{dp} + \frac{1}{p(p-1)(p+1)}x = \frac{p}{p^2-1} \qquad \ldots(15)$$

For Eqn. (15) $e^{\int \frac{1}{p(p^2-1)}dp}$ is an integrating factor.

Now, $e^{\int \frac{1}{p(p^2-1)}dp} = e^{\int\left[\frac{1}{2(p-1)} + \frac{1}{2(p+1)} - \frac{1}{p}\right]dp}$

$$= e^{\ln\frac{(p^2-1)1/2}{p}} = \frac{(p^2-1)^{1/2}}{p}$$

the, the solution of Eqn. (15) is obtained as

$$x\frac{(p^2-1)^{1/2}}{p} = \int \frac{p}{p^2-1}\frac{(p^2-1)^{1/2}}{p}dp = \int \frac{1}{\sqrt{p^2-1}}dp = c + \cos h^{-1}p,$$

$$\text{or } x = p(c + \cos h^{-1}p)\ (p^2 - 1)^{-1/2} \qquad \ldots(16)$$

you may notice that elimination of p between Eqns. (13) and (16) is not easy. However, by substituting for x from Eqn. (16) in Eqn. (13),
we get

$$y = p + (c + \cosh^{-1}p)\ (p^2 - 1)^{-1/2} \qquad \ldots(17)$$

Eqns. (16) and (17) are two equations for x and y in terms of p. These are the parametric equations of the solution of the given differential equat.

Let us look at another example.

Example 4: Solve $y = 2px + p^4x^2$, $x > 0$.

Solution: The given equation

$y = 2px + p^4x^2$ …(18)

is in itself solvable for y.

differentiating it w.r. to x, we get

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4x^2p^3\frac{dp}{dx}$$

$$\Rightarrow\ p(1 + 2xp^3) + 2x\ (1+2xp^3) = 0,$$

$$(1+2xp^3) + (p+2x\ ) = 0 \qquad ...(19)$$

Eqn. (19) holds when either of the factors $(1+2xp^3)$ or $(p + 2x\ )$ is zero.

First consider the factor

$$P + 2x\ = 0$$

$$\frac{2\,dp}{pdx} + \frac{1}{x} = 0$$

Integrating the above equation w.r.t.x, we get

2 In |p| + In |x| = constant.

$p^2x = c$, (c an arbitrary constant)

or $p = \sqrt{\dfrac{c}{x}}$.

Substituting this value of p in the given Eqn. (18), we get

$$Y = 2\sqrt{cx + c^2}$$

Which is the required solution.

If we consider the factor $1 + 2xp^3 = 0$ in Eqn. (19), then by eliminating p between this factor and given Eqn. (18), we get another solution. This solution will not contain any arbitrary constant and is the singular solution of the given equation.

How about trying an exercise now?

We next consider the case when Eqn. (8) is solvable for x.

3.2.2 Equations Solvable for x

Consider an equation of the form

$P^3 - 4xyp + 8y^2 = 0$ ...(20)

It is difficult to solve Eqn. (20) for y whereas it is easy to solve it for x as a function of y and p and write

$$x = \frac{p^3 + 8y^2}{4yp}.$$

In such cases when equation of the form (8) is solvable for x, and can be put in the form

$x = g(y,p)$ …(21)

then to solve it, we differentiate Eqn. (21) w.r. to y, and get an equation of the form

$$\frac{1}{p} = ø\left(y, p, \frac{dp}{dx}\right)$$

on solving this equation we obtain a relation between p and y in the form

$f(y,p,c) = 0,$ …(22)

where c is an arbitrary constant.

Now, we may eliminate p between Eqn (21) and (22) to obtain the solution or, x and y may be expressed in terms of p as we have done in Sec. 4.3.1.

Remark: Note that when Eqn. (8) is solvable for y, we differentiate it w.r. to x, whereas, when it is solvable for x, we differentiate it w.r. to y.

We illustrate this method by the following examples.

Example 5: Solve $p = \tan\left(x - \frac{p}{1+p^2}\right)$.

Solution: The given equation can be written as

$$X = \tan^{-1}p + \frac{p}{1+p^2} \qquad ...(23)$$

Differentiating Eqn. (23) w.r. y, we get

$$\boxed{\frac{1}{p}} = \frac{p}{1+p^2}\,\frac{dp}{dy} + \frac{(1+p^2) - p(2p)}{(1+p^2)^2}\boxed{\frac{dp}{dy}}$$

$$= \frac{1+p^2+1+p^2-2p^2}{(1+p^2)^2}$$

$$= \frac{2}{(1+p^2)^2} \boxed{}$$

$$\boxed{}\ dy = \frac{2p}{(1+p^2)^2}\ dp$$

…(24)

Note that Eqn. (24) is in variable separable form.
Integrating Eqn. (24), we get

$$y = c - \frac{1}{1+p^2}, \qquad \text{…(25)}$$

c being an arbitrary constant.

It is not possible to eliminate p between Eqns. (23) and (25). Thus, Eqns. (23) and (25) together constitute the solution of the given equation in terms of parameter p.

Let us look at another example.

Example 6: Solve $p^2y + 2px = \forall$ x, y and p > 0

Solution: We can write the given equation in the form

$$X = \frac{y}{2p} - \frac{py}{2} \qquad \text{...(26)}$$

Differentiating Eqn. (26) w.r. to y, we get

$$\boxed{} = \frac{1}{2p} + \frac{y}{2}\left(-\frac{1}{p^2}\right)\frac{dp}{dy} - \frac{p}{2} + \frac{y}{2}\frac{dp}{dy},$$

$$\boxed{}\frac{1}{2p} + \frac{p}{2} + \frac{y}{2}\ \frac{dp}{dy}\ \left(1 + \frac{1}{p^2}\right) = 0$$

$$\boxed{}\ \frac{1+p^2}{2p} + \frac{y}{2}\left(\frac{1+p^2}{p^2}\right)\ \frac{dp}{dy} = 0$$

$$\boxed{}\ \frac{1+p^2}{2p}\ \left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0 \qquad \text{…(27)}$$

In Eqn. (27), we may have

$$\frac{1+p^2}{2p} = 0$$

or $\left(1+\frac{y}{p}\frac{dp}{dy}\right) = 0.$

If we have first factor equals zero, then $p^2 = -1$.

Thus real solution of the given problem is obtained when

$$\frac{1}{y} + \boxed{\ }\boxed{\ } = 0$$

Here variables are separable. Integrating, we get

Iny + Inp = Inc

$\boxed{\ }$ py = c

or $p = \frac{c}{y}$ …(28)

Eliminating p between Eqn.s (26) and (28), we get

$$X = \frac{y^2}{2c} - \frac{c}{y}\frac{y}{2}$$

or $x = \frac{y^2}{2c} - \frac{c}{2}$,

which is the required solution.

Note that you could also have solved Example 6 by taking $y = \frac{2p}{1-p^2}$ and then proceeding as in sec. 4.3.1.

You may now try the following exercise.

We now consider Eqn. (8) with the property that Eqn. (8) may be solvable for p. in that case Eqn. (8) which is of nt degree in p, in general, is reduced to n equations of the first degree and this case has been considered in Section 4.2.

We next take up the case when Eqn. (8) may not contain either independent variable x, or, dependent variable y explicitly.

3.2.3 Equations in which Independent Variable or Dependent Variable is Absent

We shall consider the two cases separately.

Case 1: Equations not containing the independent variable:

When Eqn. (8) does not contain independent variable explicitly then the equation has the form

$$f(y,p) = 0 \qquad \dots(29)$$

For instance, consider the equation

$$y - \frac{1}{\sqrt{1+p^2}} = 0.$$

This equation does not contain x explicitly. Also, it is readily solvable for y, since it can be written in the form

$$y = \frac{1}{\sqrt{1+p^2}} \qquad \dots(30)$$

Eqn. (30) can, now, be solved by the method discussed in Sec. 4.3.1. In case Eqn. (29) is solvable for p, then we can write it in the form

$$p = \frac{dy}{dx} = ø(y) \qquad \dots(31)$$

The integral of Eqn. (31) will, then, give us the solution of Eqn. (29).

To be more clear, let us consider the following example.

Example7: Solve $y = 2p + 3p^2$

Solution: We have

$y = 2p + 3p^2$
which is already in the form y = F(p). following the method discussed in sec. 4.3.1, we differentiate it w.r.t.x, so that

$$p = 2\boxed{\frac{dp}{dx}} + 6p\boxed{\frac{dp}{dx}}$$

or $\dfrac{p}{2+6p} = \boxed{\dfrac{dp}{dx}}$

Here variable are separable and we have

$$dx = \left(\frac{2}{p} + 6\right) dp$$

Integrating, we get

$x = 6p + 2\text{In}\ |p| + c.$ …(33)

c being an arbitrary constant.

Since it is not possible to eliminate p from Eqns. (32) and (33), these equations together yield the required solution in terms of the parameter p.

Let us look at another example

Example 8: Solve $y^2 = a^2 (1 + p^2)$ …(34)

Solution: The given equation is an equation in y and p only. It can be written as

$$P^2 = \frac{y^2}{a^2} - 1$$

Solving for p, we get

$$p = \pm \sqrt{\frac{y^2}{a^2 - 1}}$$

$\therefore$ Either $p = \sqrt{\frac{y^2}{a^2 - 1}}$ or $p = -\sqrt{\frac{y^2}{a^2 - 1}}$,

Now $p \sqrt{-1 + \frac{y^2}{a^2}}$ gives

$$\frac{a}{\sqrt{y^2 - a^2}}\, dy = dx.$$

Integrating the above equation, we get

$A \text{ In } |y + \sqrt{y^2 - a^2}\ | = x + c,$

c being an arbitrary constant.

Similarly, $p = -\sqrt{-1 + \frac{y^2}{a^2}}$, on integration, yields

$a \text{ In } |y + \sqrt{y^2 - a^2}\ | = -x + c$ (c being a constant).

Hence, the general solution of the given equation is

$$[a \text{ In } |y + \sqrt{y^2 - a^2}\ | - x - c]\,[a \text{ In } |y + \sqrt{y^2 - a^2}\ | + x - c] = 0$$

Note that we solve Eqn. (34) for p. You could also have integrated it by solving it for y.

We next consider the equations in which the dependent variable is absent.

Case II: Equations not containing the dependent variable:
In this case Eqn. (8) has the form

$$g(x, p) = 0 \text{ or } x = F(p) \qquad \ldots(35)$$

As in case 1, Eqn. (35) is either solvable for p or solvable for x. if it is solvable for p, then it can be written as

$p = \Psi(x)$
which, on integration, gives the solution of Eqn. (35)

If Eqn. (35) is solvable for x, then it corresponds to the case discussed in Section 4.3.2.

We give below examples to illustrate the theory.

Example 9: Solve $x(1+p^2) = 1$

Solution: The given equation can be written as

$$x = \frac{1}{1+p^2} \qquad \ldots(36)$$

Differentiating Eqn. (36) w.r. to y, we get

$$\boxed{} = \frac{-2p}{(1+p^2)^2}\boxed{},$$

i.e., $dy \frac{-2p^2}{(1+p^2)^2} dp$

i.e., $dy = 2\left[\frac{-1}{1+p^2} + \frac{1}{(1+p^2)^2}\right] dp$

Here variables are separable. Integrating, we get

$$y = -2\tan^{-1}p + 1\int \frac{dp}{(1+p^2)^2} + c \qquad \ldots(37)$$

c is a constant.

Eqns. (36) and (37) together yield the required solution with p as parameter.

Note that problem in example 9 could have also been done by solving it for p. we illustrate this method in the next example.

Example 10: Solve $p^2 - 2xp + 1 = 0$

Solution: The given equation is
$P^2 - 2xp + 1 = 0$
Solving for p, we get

$$p = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \square \sqrt{x^2 - 1}$$

$\square$Either $p = x + \sqrt{x^2-1}$ or $p = x - \sqrt{x^2-1}$
Now $p = x + \sqrt{x^2-1}$, on integration yields

$$y = \frac{x^2}{2} + \frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\,\text{In}\,|x + \sqrt{x^2-1}\,| + c,$$

c being an arbitrary constant.

Similarly, $p = x - \sqrt{x^2-1}$ yields

$$y = \frac{x^2}{2} - \frac{1}{2}\,x\sqrt{x^2-1} + \frac{1}{2}\,\text{In}\,|x + \sqrt{x^2-1}\,| + c,$$

Hence, the general solution of the given equation is
$[x^2 + x\sqrt{x^2-1} - \text{In}\,|x + \sqrt{x^2-1}\,| - 2y + c_1]\,[x^2 - x\sqrt{x^2-1} + \text{In}|x + \sqrt{x^2-1}\,|-2y + c_1]$ $= 0$.
Where $c_1 = 2c$ is an arbitrary constant.

And now some exercise for you.

We next discuss the case when Eqn. (8) may be homogeneous in x and y

3.2.4 Equations Homogeneous in x and y

In this case, Eqn. (8) can be expressed in the form

$$\text{Ø}\left(p, \frac{y}{x}\right) = 0 \qquad \ldots(38)$$

For solving Eqn. (38), we can proceed in two ways. In case Eqn. (38) is solvable for p then it can be expressed as

$$P = \frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \ldots(39)$$

We already know from our knowledge of unit 2 that equations of the type (39) can be solved by using the substitution y = vx.

The second possibility, that is, when Eqn. (38) is solvable for y/x, then it can be put in the form

$$\frac{y}{x} = \Psi(p) \text{ or } y = x\square(p).$$

In this case we can proceed as in Sec. 4.3.1. Differentiating the above equation w.r. to x, we get

$$p = \square(p) + x\square'(p)\square$$

$$\square\ \frac{dx}{x} = \frac{\psi'(p)\,dp}{p - \psi(p)} \qquad \ldots(40)$$

Eqn. (40 is in variable separable form. On integrating, it yields

$$\text{In}|x| = c + \int \frac{\psi'(p)}{p - \psi(p)}\,dp$$
$$= c + ø(p), \text{ say.}$$

The elimination of p between this equation and $y = x\square(p)$ will give us the required solution. But it is not always easy to eliminate p, so it may be retained as the parameter.

To understand the theory, we take an example.

Example 11: Solve $y^2 + xyp - x^2p^2 = 0\ \square x, y, p > 0$.

Solution: The given equation is homogeneous in y and x and it may be written as

$$p^2 - \left(\frac{y}{x}\right) p - \left(\frac{y}{x}\right)^2 = 0 \qquad \ldots(41)$$

Solving Eqn. (41) for p, we get

$$p = \frac{(y/x) \pm \sqrt{(y/x)^2 + 4(y/x)^2}}{2} = (y/x)\left(\frac{1 \pm \sqrt{5}}{2}\right)$$

$$\text{Thus } \square = \square \left(\frac{1 \pm \sqrt{5}}{2}\right) \text{or } \square = \left(\frac{1 \pm \sqrt{5}}{2}\right) \square$$

Let y = vx, then $\square = v + x\,\square$

$$\square\, v + x\frac{dv}{dx} = v\left(\frac{1+\sqrt{5}}{2}\right) \text{ and } v + x\frac{dv}{dx} = \left(\frac{1-\sqrt{5}}{2}\right)v$$

$$\square\, x\frac{dv}{dx} = \left(\frac{\sqrt{5}-1}{2}\right)v \text{ and } x\frac{dv}{dx} = \left(\frac{-1-\sqrt{5}}{2}\right)v$$

Integrating, we get

In xc = In $v^{2/(\sqrt{5}-1)}$ and In xc = In $v^{-2/(\sqrt{5}+1)}$
Or xc = $(y/x)^{2/(\sqrt{5}-1)}$ and xc = $(y/x)^{-2/(\sqrt{5}+1)}$
or y = x $(xc)^{(\sqrt{5}-1)/2}$ and y = $x(xc)^{-(\sqrt{5}+1)/2}$

Hence the general solution is

$[y-x(xc)^{(\sqrt{5}-1)/2}].\ [y-x(xc)^{-(\sqrt{5}+1)/2}] = 0$
i.e., $y^2 - xy\,[(xc)^{(\sqrt{5}-1)/2} + (xc)^{-(\sqrt{5}+1)/2}] + x^2\,(xc)^{-1} = 0$

Now you may try the following exercise.

We next discuss the case when Eqn. (8) may be of first degree in x and y.

3.2.5 Equations of the First Degree in x and y – Clairaut's equation

When Eqn. (8) is of first degree in x and y, it is solvable for x and y both and hence can be put in either of the following forms.

$y = xf_1(p) + f_2(p)$ …(42)
or $x = yg_1(p) + g_2(p)$ …(43)

Hence, we can use the methods discussed in Sec. 4.3.1 and 4.3.2 to solve equations of the for (42) and (43), respectively.

However, if in Eqn. (42), $f_1(p) = p$, then we get one particular form of this equation known as **Clairaut's Equation** and about which we have already mentioned in sec. 4.1.

Thus the Clairaut's equation is of the form

$y = px + f(p)$ …(44)

In Eqn. (44), f(p) is a known function which contains neither x nor y explicitly. also, note, that Eqn. (44) can be non-linear. For instance, $y = px + p^2$ and $y = x + e^p$ are examples of Caliraut equation. But equations $y = xy + p^2$ or $y = xp + yp^2$ are not of the Clairaut's form.

On differentiating Eqn. (44) w.r. to x, we have

$p = p + p'x + f'(p)p'$ …(45)

$p' [x + f'(p)] = 0$

Then either $p' = \quad = 0$ …(46)

Or, $x + f'(p) = 0$ …(47)

The solution of Eqn. (46) is p = c, where c is an arbitrary constant. Thus, we can write the general solution of Eqn. (44) as

$y = cx + f(c)$ …(48)

Note that Eqn. (48) is an equation of a family of straight lines.

Now consider Eqn. (47). Since f(p) and f'(p) are known functions of p, Eqns. (47) and (44) together constitute a set of parametric equations giving x and y in terms of the parameter p.

If we can eliminate p from Eqn. (44) and (47) and if the resulting equation satisfies Eqn. (44), we get another solution of Eqn. (44) (could be an implicit solution). This solution does not contain an arbitrary constant and is a singular solution of Eqn. (44).

We give you some examples to help you understand this method.

Example 12: Solve $(y')^2 + 4xy' - 4y = 0$ …(49)

Solution: With p = y', Eqn. *49) can be written as

$y = px + \frac{1}{4}\,p^2,$ …(50)

which is in the Clairaut's form. Differentiating Eqn. (50) w.r. to x, we get

$p = p + p'x + \frac{p}{2}p'$

$\square\, p'(x + \frac{p}{2}) = 0$

then either p = 0 which gives p = c (a constant) …(51)

or $x + \frac{p}{2} = 0$ …(52)

From Eqns. (50) and (51), we obtain

$y = cx + \frac{c^2}{4}$

as the solution of Eqn. (50). Eliminating p from Eqns. (50) and (52), we get

$y = x(-2x) + \frac{1}{4}(-2x)^2,$

i.e., $y(x) = -x^2,$

which contains no arbitrary constant. Since this value of y satisfies Eqn. (50), it is the singular solution of Eqn. (50).

Let us look at another example.

Example 13: Solve $y = xp + \square$

Solution: If we compare the given equation with Eqn. (44) we notice that in the case $f(p) = \square$ and $f'(p)\ \frac{1}{p^2}$. From Eqn. (48) then the solution is given by

$y = ax + \frac{1}{a}$

where a (≠ 0) is an arbitrary constant.

Also in this case, equation corresponding to Eqn. (47) is

$0 = x\ \frac{1}{p^2}$

The elimination of p between the above equation and the given equation yields

$y^2 = 4x,$

which is a singular solution of the given equation.

You may, now, try the following exercises.

Finally, we take up in the next solution, another non-linear equation known as Riccati's equation, which we mentioned in Sec. 4.1

3.2.6 Riccati's Equation

Originally, this name was given to the first order differential equation

$$\boxed{} + by^2 = cm^m, \qquad \ldots(53)$$

where b, c and m are constants. This is known as the **special Riccati equation.** Eqn. (53) is solvable in finite terms only if the exponent m is – 2 or, of the form $\frac{-4k}{(2k+1)}$ for some integer k. Riccati merely discussed special cases of this equation without offering any solutions. Now a days Riccati's equation is usually understood by an equation of the form.

$$y' = a(x) + b(x)\, y + c(x)y^2 \qquad \ldots(54)$$

where a, b and c are given functions of x on an interval 1 (of R). Equations $y' = 1 + xy + e^x y^2$ and $y' = x + x^2y + \sin(x)y^2$ are example of Riccati's equations whereas, equations $y' = 1 + y + y^3$, and $y' = 1 + y + 2y^2$ are not of Riccati's type.

It is difficult to obtain a solution of Riccati's Eqn. (54) containing an arbitrary constant. But, the general solution of Eqn. (53) can be obtained if we have the knowledge of a particular solution of Eqn. (53). This can be done as follows:

Let y_1 be a solution of Eqn (53) Then we determine a function v so that y defined bu the relation

$$y = y_1 + \frac{1}{v} \qquad \ldots(55)$$

is a solution of Eqn. (54).
Differentiating Eqn. (55) w.r. to x, we get

$$y' = y_1 + v\left(-\frac{1}{v^2}\right)$$

Since y and y_1 satisfy eqn. (54), we have
$y_1' = a(x) + b(x)y_1 + c(x)y_1^2$

and $y_1' - \dfrac{v'}{v^2} = a(x) + b(x)y + c(x)\, y^2$

subtracting the second equation from the first, we have

$$v'\left(\frac{1}{v^2}\right) = b(x)\,(y_1 - y) + c(x)\,(y_1^2 - y^2)$$

or $v' = b(x)\, v^2(y_1 - y) + c(x)\,(y_1 - y)\,(y_1 + y)v^2$ …(56)

From Eqn. (55), we have

$(y - y_1)\, v = 1$ or $(y_1 - y)\, v = -1$ …(57)

Also, $(y_1 + y)v = (2y_1 + y - y_1)v = 2y_1v + 1\,(y - y_1)v = 2y_1v + 1$ (using Eqn (57))

Now $(y_1^2 - y^2)v^2 = (y_1 - y)v.(y_1 + y)v$

$= (-1)\,(2y_1v + 1) = -1 - 2y_1v$ …(58)

Substituting from Eqn (58) in Eqn (55), we get

$v' = -(b(x) + 2c(x)y_1)v - c(x),$ …(59)

which is a linear (non-homogeneous) equation for determining a function v.

the general solution of Eqn. (59) contains an arbitrary constant and the substitution of this general solution in Eqn. (54 gives us the solution of Eqn. (53) containing an arbitrary constant.

Let us now go through some examples to understand the above theory.

Example 14: Solve $y' = -y + x^2y^2$

Solution: On comparing this equation with Eqn (53) we find that in this case a = 0, b = -1 and c = x^2 (1 = R). The given equation is a Riccati's Equation which has a (particular) solution $y_1 = 0$. by using the substitution

$$y = y_1 + \frac{1}{v} = 0 + \square = \square$$

in Eqn (58), we have $v' = v - x^2$ …(60)

Eqn. (60) is a linear first order equation with

I.F. = $e^{\int -1\,dx} = e^{-x}$

Hence, the general solution of Eqn. (60) is

$v = -e^x \square e^{-x}\, x^2 + dx + Ae^x$

$= (x^2 + 2x + 2) + Ae^x$

and the solution of the given equation is

$$y = \frac{1}{Ae^{x} + x^{2} + 2x + 2},$$

which contains an arbitrary constant.

Let us look at another example.

Example 15: Solve $y' = -1 - x^2 + y^2$.

Solution: By inspection, we see that $y_1(x) = -x$ is a solution of the given equation. Comparing the given equation with Eqn. (53), we get $a = -1 - x^2$, $b = 0$ and c = = 1.

We look for a function v, so that $y = y_1 +$ $= -x +$.

In this case Eqn. (58) reduces to

$v' = 2xv - 1$

$$\frac{dv}{dx} - 2xv = -1$$

The integrating factor for this equation is e^{-x2}, and so it can be written as

$$\frac{d}{dx}[e^{-x2}v] = -e^{-x2}$$

Thus, on integration, we write

$e^{-x2}v = -\int e^{-x2}\,dx + c$

or, $v = e^{x2}[-\int e^{-x2}\,dx + c]$

where c is an arbitrary constant.

So, the required solution is

$$y(x) = -x + \frac{e^{-x2}}{-\int e^{-x2} + c}$$

Now the integral $\int e^{-x2}$ dx cannot be evaluated in terms of elementary functions. When an initial condition is specified, then integral of the form $\int_{x_0}^{x} e^{-t2}\,dt$ can be used.

You may now try the following exercises:

3.3 Bernoulli Equation

Reduction of non-linear equation to linear form.
Here, we shall illustrate that certain non-linear first order differential equations may be reduced to linear form by a suitable change of the dependent variable.

The differential equation
$y' + p(x)y = g(x)y^a$
where 'a' is a real number is call the "Bernoulli Equation".

For a = 0 and a= 1 the equation is linear, and otherwise it is non-linear.

Set $\{y(x)\}^{1-a} = u(x)$ and show that the equation assume the linear form.

$U' + (1 - a)\, p(x)\, u = (1-a)\, g(x)$.

Solve the following Berboulli equation

1) $y' + y = \frac{x}{y}$
2) $y' + xy = xy^{-1}$
3) $3y' + y = *1 - 2x(y^4$
4) (A population model . the logistic law). Matheus's law states that the time rate of change of a population y(t) is proportional to x(t). this holds for many populations as they are not too large. A more refined model is the logistic law given by

   $\frac{dy}{dt} = ay - by^2 \ldots\ldots\ldots (a > 0, b> 0)$
   where the "breaking term" = by^2 has the effect that the population cannot grow indefinitely.

   Solve this Bernoulli equation. What is the limit of y(t) as t as ∞ ? For the united state, vertalst predicted in 1845 the values a = 0.03 and b = 1.6×10^{-4} where t is measured in years and y(t) in millions find the particular solution satisfying y(0) = 5.3 (corresponding to the year 1800) and compare the values of this solution with some actual values.

1800	1830	1860	1890	1920	1950	1980
5 – 3	13	31	63	105	150	230

5) Apply the suitable substitutions, reduce to linear form and solve:

   a) $y' \cos y + x \sin y = 2x$

b) $e^x y^1 + 2e^x y + x = 0$

4.0 CONCLUSION

We end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

in this unit we have covered the following:

1) the general differential equation of first order and nth degree is given by Eqn (1), namely
$p^n + p_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0$
where $P_1, P_2 \ldots\ldots\ldots, P_n$ are functions of x and $p = \boxed{\frac{dy}{dx}}$.

2) If Eqn. (1) can be resolved into rational linear factors of the first order, then Eqn. (1) takes the form
$(p-R_1 (p- R_2) \ldots\ldots (p - R_n) = 0$
for some $R_1, R_2, \ldots\ldots R_n$ which are functions of x and y, and
if $f_1(x, y, c) = 0$, $f_2(x, y, c) = 0 \ldots\ldots$, $f_n(x, y, c) = 0$ are the solutions of $p - R_1 = 0$, $p - R_2 = 0 \ldots\ldots$, $p - R_n = 0$ respectively, then
$f_1(x, y, c) \,.\, f_2(x, y, c) \ldots\ldots f_n(x, y, c) = 0$
is the general solution of Eqn. (1).

3) If Eqn. (1) cannot be factorized into rational linear factors, the
   a) it is said to be solvable for y if we can express it as
   $y = F(x,p)$ (see Eqn. (10)).
   To solve Eqn. (10), differentiate it with respect to x, and it may be possible to solve resulting differential equation in x and p. Elimination of p between the solution of resulting differential equation and EQn. (10) gives the solution of Eqn. (10).

   b) it is said to be solvable for x if we can express it as
   $x = g(x, p)$ (see Eqn. (21)).
   To solve Eqn. (21), differentiate it w.r.t.y and it may be possible to solve the resulting differential equation in y and p. Elimination of p between the solution of the resulting equation and Eqn. (21) gives the solution of Eqn. (21).

4) If Eqn. (1) does not contain independent variable or dependent variable explicitly and can be put in the form

$f(y, p) = 0$ (See Eqn. (29))
or $g(x, p) = 0$ (see Eqn. (35))
then it may either be possible to factorize Eqn. (29) into linear factors or it may be solvable for y.
Similarly, eqn. (35) can either be factorized or it may be solvable for x.

5) If Eqn. (1) is homogeneous in x and y then either substitution y = vx may reduce it to separable equation or it may be put as
$y = x\,(p)$,
which is solvable for y or x.

6) Clairaut's equation is an equation of first order and of any degree if it can be put in the form
$y = xp + f(p)$ (see Eqn. (44))
This equation is solvable for y and its solution is
$y = cx + f(c)$

7) Riccati's equation is an equation of the form
$= a(x) + b(x)y + c(x(y^2$ (see Eqn. (53))
where a(x), b(x) and c(x) are given functions on an interval ! of R.

The general solution of Eqn. (53) can be obtained if we know a particular solution y_1 of Eqn. (53(and then we determine a function v defined by relation

$y = y_1 +$, (see Eqn 54). …(see Eqn 54)
so that y is solution of Eqn. (54).

6.0 TUTOR MARKED ASSIGNMENT

1. Solve the following equations:

a) $p^2y + p(x - y) - x = 0$
b) $p^2 - 5y + 6 = 0$
c) $4y^2p^2 + 2pxy\,(3x+1) + 3x^3 = 0$
d) $\left(\frac{dy}{dx}\right)^3 = ax^4$
e) $x + yp^2 = p(1 + xy)$

2. Solve the following equations:

a) $y = x + a \tan^{-1}p$
b) $x = y + \mathrm{In}p$
c) $p^3 + p = e^y$
d) $y = p \tan p + \mathrm{In} \cos p$

3. Solve the following equations.

a) $p^2 – py + x = 0$
b) $x = y + a\, \mathrm{In}p$
c) $x = y + p^2$
d) $y^2\mathrm{In}y = xyp + p^2$

4. Solve the following equations:

a) $(y')^2 – 4 = 0$
b) $\sin(y') = 0$
c) $(y')^2 + 4y' – x^2 = 0$

5. Obtain the solution of the following equations:

a) $\exp (y' + (1 + x^2)) = 1$
b) $(y')^2 + 2 (x + y)y' + 4xy = 0$
c) $p^2 – (3x +2y)p + 6xy = 0$

6. Solve the following equations.

a) $y = yp^2 + 2px$
b) $x^2p^2 + 4xyp – 8y^2 = 0$

7. Solve the following equations:

a) $y = xp + \frac{a}{p}$ (a $\square$ 0, is a constant)
b) $y = xp + p^2$
c) $y = xp + p – p^2$

8. Solve $e^{4x} (p – 1) + e^{2y} p^2 = 0$

9. Solve $y = x^4p^2 – px$

10. Solve $xy (y – px) = x + py$

11. Which of the following are Riccati's equation, Clairaut's equation or neither.

a) $y = 2xp + y^2p^3$
b) $y' = e^x + e^y + y^2$
c) $y' = (1 + \sin 2x) + \frac{2}{1+x^2} y + e^x y^2$
d) $y = 3px + 6y^2p^2$
e) $y' = \sin x + \sin y$

12. Find a solution, containing an arbitrary constant (given a particular solution), of the following Riccati's equations:

a) $y' = 1 - xt + y^2$ $(y_1(x) = x)$
b) $y' = 2 + 2x + x^2 - y^2$ $(y_1(x) = 1 + x)$
c) $y' = 2x - x^2 - x^2 - x^4 + y + y^2$ $(y_1(x) = x^2)$

13. By eliminating arbitrary constant c from the equation

$$y = \frac{cg(x) + G(x)}{cf(x) + F(x)}$$

obtain the Riccati's equation:

$$(gF - Gf)y' = (gG' - g'D) + (Gf' - gF' + g'F)y + (fF' - fF)y^2.$$

14. Show that, when m = 0, Riccati's equation

$$\boxed{\frac{dy}{dx}} + by^2 = cx^m$$

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MODULE 3

UNIT 1 FAMILIES OF CURVES ORTHOGONAL AND OBLIYUE TRAYECTORIES. APPLICATION TO MECHANICS AND ELECTRICITY

CONTENTS

1.0 INTRODUCTION

In this section we shall learn how to use differential equations for finding curve that intersect given curves at right angles, a task that arises rather often in applications.

If for each fixed real value of c the equation

(1) $F(x, y, c) = 0$

represents a curve in the xy-plane and if for variable c it represents infinitely many curves, then the totality of these curves is called a **one-parameter family of curves,** and c is called the parameter of the family.

2.0 OBJECTIVES

- To find family of curves
- To be able to solve differential equation of family of curves
- One should be able to apply orthogonal trajectories to (i) Electrical field. (2) Mechanical field. (3) Temperature
- Also to be able to find approximate solutions to directions fields iteration.

Example 1: Families of curves

The equation

(2) $F(x, y, c) = x + y + c = 0$
represents a family of parallel straight lines; each line corresponds to precisely one value of the parameter c. The equation

(3) $F(x, y, c) = x^2 + y^2 – x^2 = 0$

Represents a family of concentric circles of radius c with center at the origin.

The general solution of a first-order differential equation involves a parameter c and thus represents a family of curves. This yields a possibility for representing many one-parameter families of curves by first-order differential equations. The practical use of such representations will become obvious from our further considerations.

Example 2: Differential equations of families of curves
By differentiating (2) we see that

$$y' + 1 = 0$$

is the differential equation of that family of straight lines. Similarly, the differential equation of the family (3) is obtained by differentiation, $2x + 2yy' = 0$, that is,

$$y' = x/y.$$

if the equation obtained by differentiating (1) still contains the parameter c, then we have to eliminate c from this equation by using (1). Let us illustrate this by a simple example.

Example 3: eliminate of the parameter of a family
The differential equation of the family of parabolas

(4) $y = cx^2$
is obtained by differentiating (4),

(5) $y' = 2cx,$
and by eliminating x from (5). From (4) we have $c = y/x^2$, and by substituting this into (5) we find the desired result

(6) $y' = 2y/x.$

note that we may also proceed as follows. We solve (4) for c, finding c = y/ x^2, and differentiate

3.0 MAIN CONTENT

3.1 Orthogonal Trajectories

In many engineering and other applications, a family of curves is given, and it is required to find another family whose curves intersect each of the given curves at right angles.[14] The curves of the two families are said to be mutually orthogonal, they form an orthogonal net, and the curves of the family to be obtained are called the **orthogonal trajectories** of the given curves (and conversely); cf. fig. 1.

Let us mention some familiar examples. The meridians on the earth's surface are the orthogonal trajectories of the parallels. On a map the curves of steepest descent are the orthogonal trajectories of the contour lines. In electrostatics the equipotential lines and the lines of electric force are orthogonal trajectories of each other. An illustrative example is shown in Fig. 2. We shall see later that orthogonal trajectories are important in various fields of physics, for example, in pydrodynamics and heat conduction.

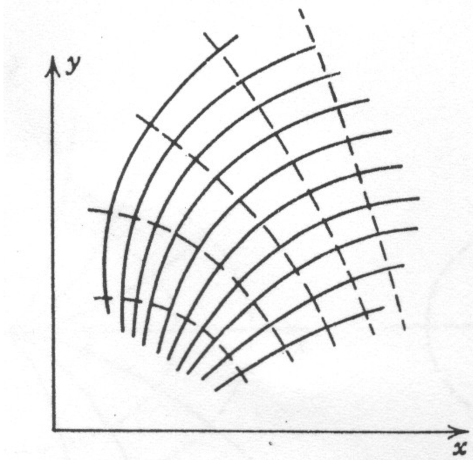
Given a family of curves F(x, y, c) = 0 that can be represented by a differential equation

(7) $y' = f(x, y)$

we may find the corresponding orthogonal trajectories as follows. From (7) we see that a curve of the given family that passes through a point (x_0, y_0) has the slop $f(x_0, y_0)$ at this point. The slop of the orthogonal trajectory through (x_0, y_0) at this point should be the negative reciprocal of (x_0, y_0), that is, $- 1/f(x_0, y_0)$, because this is the condition for the tangents of the two curves at $x_0, y_0)$ to be perpendicular. Consequently, the differential equation of the orthogonal trajectories is

(8) $$\boxed{y' = -\frac{1}{f(x, y)}}$$

and the trajectories are obtained by solving this new differential equation.



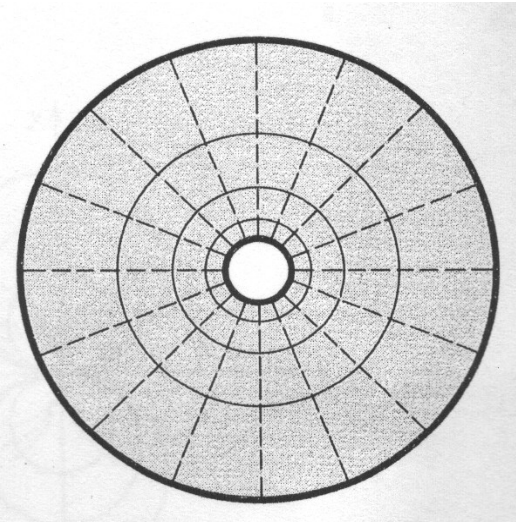


Fig. 1: Curves and their orthogonal trajectories

Fig. 2: Equipotential lines and lines of electric force (dashed) between two concentric cylinders

[14]Remember that the **angle of intersection** of two curves is defined to be the angle between the tangents of the curves at the point of intersection.

Example 4: Orthogonal trajectories
Find the orthogonal trajectories of the parabolas in Example 3.

Solution: From (6) we see that the differential equation (8) of the orthogonal trajectories is

$$y' = -\frac{1}{2y/x} = -\frac{x}{2y}$$

By separating variables and integrating we find that the orthogonal tracjectories are the ellipses

$$\frac{x^2}{2} + y^2 = e^* \qquad \ldots\text{(fig. 3)}$$

Example 5: Orthogonal trajectories
Find the orthogonal trajectories of the circles

9) $x^2 + 2(y - c)y' = 0.$

Solution: we first determine the differential equation of the given family, by differentiating (9) with respect to x we obtain

10) $2x + 2(y - c)y' = 0.$

We must eliminate c. solving (9) fir x, we have

11) $c = \dfrac{x^2 + y^2}{2y}$

By inserting this into (10) and simplifying we get

$$x + \frac{y^2 - x^2}{2y}y' = 0 \text{ or } y' = \frac{2xy}{x^2 - y^2}.$$

From this and (8) we see that the differential equation of the orthogonal trajectories is

$$y' = -\frac{x^2 - y^2}{2xy} \text{ or } 2xyy' - y^2 + x^2 = 0.$$

The orthogonal trajectories obtained by solving this equation (cf. example 1 in sec. 1.4) are the circles (fig. 4)

$$(x - c)^2 + y^2 = c$$

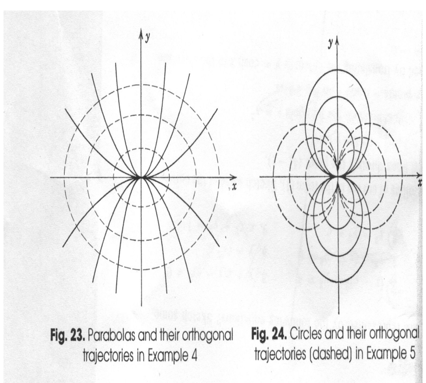


Fig. 3: Parabolas and their orthogonal trajectories in Example 4

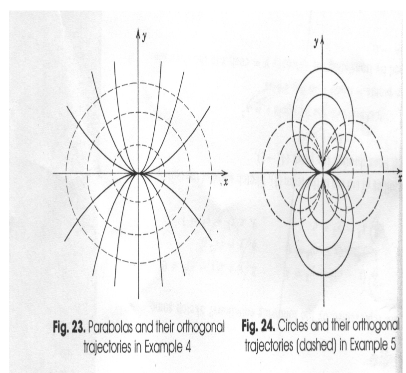


Fig. 4: Circle and their orthogonal trajectories (dashed) in Example 5

In the next section we motivate and discuss two methods of obtaining approximate solutions without actually solving a given differential equation. The first method, called the method of **direction fields,** can relatively easily produce a general picture of the solutions (with limited accuracy) and is of great practical interest. The second method, **Picard's iteration,** is more theoretical; its practical value is limited, since it involved integrations.

SELF ASSESSMENT EXERCISES

i. $4y - x + c = 0$
ii. $(x - c)^2 + y^2 = 4$
iii. $(x - c)^2 + y^2 = c^2/2$
iv. $x^2 - y^2 = c$

Represent the following families of curve in the form (1), stktch some of the curves.

v. All nonvertical straight lines through the point $(4, -1)$.
vi. The catenaries obtained by translating the catenary $y = \cosh x$ in the direction of the straight line $y = - x$.

Using differential equations, find the orthogonal trajectories of the following curves. Graph some of the curves and the trajectories.

1) $y = 2x + c$
2) $y = cx^3$

3.2 applications of Orthogonal Trajectories

3) **(Electric field)** If an electrical current is flowing in a wire along the z-exist, the resulting equipotential lines in the xy-plane are concentric circles about the origin, and the electric lines of force are the orthogonal trajectories of these circles. Find the differential equation of these trajectories and solve it.
4) **(Electric field)** Experiments show that the electric lines of force of two opposite charges of the same strength at) – 1, 0) and (1, 0) are the circles through (- 1, 0) and (1, 0). Show that these circles can be represented by the equation $x^2 + (y - c)^2 = 1 + c^2$. show that the equipotential lines (orthogonal trajectories) are the circles $(x + c^*)^2 + y^2 = c^{*2} - 1$, which are dashed in Fig. 5 on the next page.

Other forms of the differential equations. Isogonal trajectories

5) Show that (8) may be written in the following form and use this result for determining the orthogonal trajectories of the curves $y = \sqrt{x+ c}$.

$$\frac{dx}{dy} = - f(x, y).$$

6) Show that the orthogonal trajectories of a given family $g(x, y) = c$ can be obtained from the differential equation

$$\frac{dy}{dx} = \frac{\partial g/\partial y}{\partial g/\partial x}$$

7) **Isogonal trajectories** of a given family of curves are curves that intersect the given curves at a constant angle θ. Show that at each point the slops m_1 and m_2 of the tangents to the corresponding curves satisfy the relation

$$\frac{m_2 - m_1}{1 + m_1 m_2} = \tan\,\theta = \text{const.}$$

Using this formula, find curves which intersect the circles $x^2 + y^2 = c$ at an angle of 45^o.

8) Using the Cauchy – Riemann equations (Prob. 43), find the orthogonal trajectories of $e^x \cos y = c$.

3.3 Approximate Solutions: Direction Fields, Iteration

In applications, it will often be impossible or not feasible or not necessary to solve differential equation exactly. Indeed, there are various differential equations, even of the first order, for which one cannot obtain formulas for solutions. [15] There are other differential equations for which such formulas can be derived, but they are so complicated that they are practically useless. Finally, since a differential equation is a model of a physical or other system, and in modeling we disregard factors of minor influence in order to keep the model simple, the differential equation will describe the given situation only approximately, and an approximate solution will often be practically as informative as an exact solution.

Approximate solutions of differential equations can be obtained by **numerical methods.** These are discussed in sec 20.1 and 20.2. at present we shall consider the method of direction field, which is a geometric procedure, and then the so-called Picard iteration, which gives formulas for approximate solutions.

3.3.1 Method of Direction Fields

In this method we get a rough picture of all solutions of a given differential equation

1) $$\boxed{y' = f(x, y)}$$

without actually solving the equation. The idea is quite natural and simple, as follows.

We assume that the function f is defined in some region of the xy-plane, so that at each point in that region it has one (and only one) value. The

[15]Reference [All] in Appendix 1 includes more than 1500 important differential equations and their solutions, arranged in systematic order and accompanied by numbers references to original literature.

Solutions of (1) can be plotted as curves in the xy-plane. We do not know the solutions, but we see from (1) that a solution passing through a point (x_0, y_0) must have the slope $f(x_0, y_0)$ at this point. This suggests the following method.

Ist Step (Isoclines). We graph some of the curves in the xy-plane along which f(x, y) is constant. These curves

$$f(x, y) = k = const$$

are called curves of constant slope or **isoclines.** Here the value of k differs from isoclines to isocline. So these are not yet the solution curves of (1), but just auxiliary curves.

2nd Step (Direction field). Along the isocline f(x, y) = k we draw a number of parallel short line segments **(lineal elements)** with slope k, which is the slop of solution curves of (1) at any point of that isocline. This we do for all isoclines which we graphed before. In this way we obtain a field of lineal elements, called the **direction field** of (1).

3rd Step (Approximate solution curves). With the help of the lineal elements we can now easily graph approximation curves to the (unknown) solution curves of the given equation (1) and thus obtain a qualitatively correct picture of these solution curves.

It suffices to illustrate the method by a simple equation that can be solved exactly, so that we get a feeling for the accuracy of the method.

Example 1: Isoclines, direction field

Graph the direction field of the first-order differential equation

2) $y' = xy$

and an approximation to the solution curve through the point (1, 2). Compare with the exact solution.

Solution: The isoclines are the equilateral hyperbolas xy = k together with the coordinate axes. We graph some of them. Then we draw lineal elements by sliding a triangle along a fixed ruler. The result is shown in Fig. 6, which also shows an approximation to the solution curve passing through the point (1, 2).

By separating variables, $y = ce^{x^2/2}$. The initial condition is y(1) = 2. Hence 2 = $ce^{1/2}$, and the exact solution is

$$y = 2e^{(x^2-1)/2}.$$

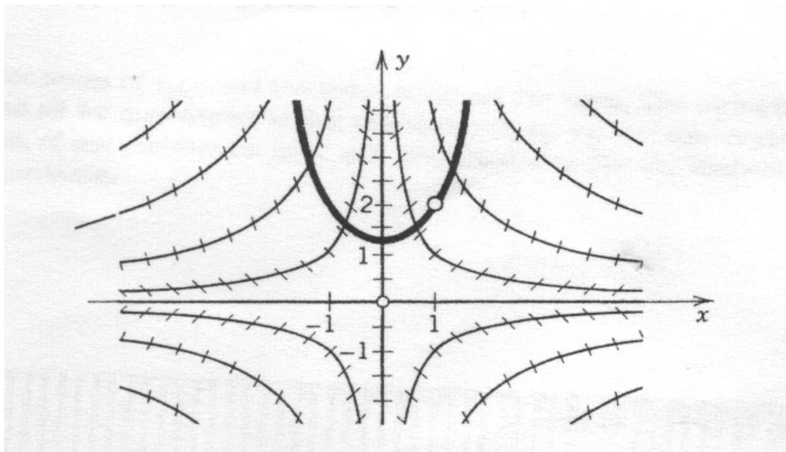


Fig. 27. Direction field of the differential equation (2)

Fig. 7: Direction field of the differential equation (2)

3.3.2 Picard's Iteration method[16]

This method gives approximate solutions of an initial value problem

3) $\boxed{y' = f(x, y), \quad y(x_0) = y_0}$

Which is assume to have a unique solution in some interval on the x-axist containing x_0. Picard's existence and uniqueness theorem, which we shall discuss in the next section. Its practical value is limited because it involves integrations that may be complicated.

The basic idea of Picard's method is very simple. By integration we see that (3) may be written in the form

4)
$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\ dt}$$

Where t denotes the variable of integration. In fact, when $x = x_0$ the integral is zero and $y = y_0$, so that (4) satisfies the initial condition in (3); furthermore, by differentiating (4) we obtain the differential equation in (3).

To find approximations to the solution y(x) of (4) we proceed as follows. We substitute the crude approximation $y = y_0 =$ const on the right; this yields the presumably better approximation

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\ dt .$$

In the next step we substitute the function $y_1(x)$ in the same way to get

$$y_2(x) = y_0 + \int_{x_0}^{x} f[t, y_1(t)]\ dt$$

etc. The nth step of this iteration gives an approximating function

5)
$$\boxed{y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t()]\ dt .}$$

In this way we obtain a sequence of approximations.

$$y_1(x), \qquad y_2(x),\ldots\ldots\ldots.. \qquad y_n(x),\ldots\ldots\ldots,$$

and we shall see in the next section that the conditions under which this sequence converges to the solution y(x) of (3) are relatively general.

[16]EMILE PICARD (1856 – 1941), French mathematician, professor in Paris since 1881, also known for his important contributions to complex analysis (see Sec 14.10 for his famous theorem).

An **iteration method** is a method that yields a sequence of approximations to an (unknown) function, say, y_1, y_2 …., where the nth approximation, y_n, is obtained in the nth step by using one (or several) of the previous approximations, and the operation performed in each step is the same. This is a practical advantage, for instance in programming for numerical work.

In the simplest case, y_n is obtained from y_{n-1}; denoting the operation by T, we may write

$$y_n = T(y_{n-1}).$$

Picard's method is of this type, because (5) may be written

$$Y_n)x) = T(y_{n-1}(x)) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\ dt\,.$$

To illustrate the method, let us apply it to an equation we can readily solve exactly, so that we may compare the approximations with the exact solution. The example to be discussed will also illustrate that the question of the convergence of the method is of practical interest.

Example 2: Picard iteration

Find approximate solution to the initial value problem

$$y' = 1 + y^2, \quad y(0) = 0$$

Solution: In this case, $x_0 = 0$, $f(x, y) = 1 + y^2$, and (5) becomes

$$y_n(x) = \int_0^x [1 + y_{n-1}^2(t)] = x + \int_0^x y_{n-1}^2(t)\ dt.$$

Starting from $y_0 = 0$, we thus obtain (cf. Fig. 8)

$$y_1(x) = x + \int_0^x 0\ dt = x$$

$$y_2(x) = x + \int_0^x t^2 dt = x + \frac{1}{3}x^3$$

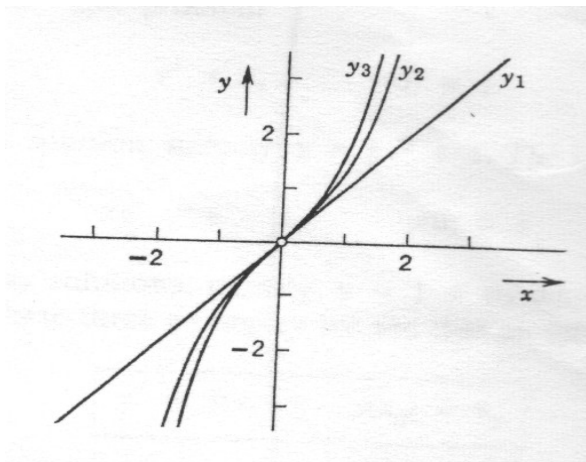
$$y_3(x) = x + \int_0^x \left(t + \frac{t^3}{3}\right)^2 dt = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

etc. of course, we can obtain the exact solution of our present problem by separating variables (see Example 2 in Sec 1.2), finding

6) $$y(x) = \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \ldots \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right).$$

The first three terms of $y_3(x)$ and the series in (6) are the same. The series in (6) converges for $|x| < \pi/2$. This illustrates that the study of convergence is of practical importance.

The next section, the last of Chap. 1, concerns the problems of **existence** and **uniqueness** of solutions of first-order differential equations. These problems are of greater relevance to engineering applications than one would at first be inclined to believe. This is so because modeling involves the discarding of minor factors, and in more complicated situations it is often difficult to see whether some physical factor will have a minor or major effect, so that one may not be sure whether a model is faithful and does have a solution, or a unique solution, even though the physical system can be expected to behave reasonably. The matter becomes even more crycial in connection with numerical methods: make sure that the solution exists before you try to compute it.



ig. 28. Approximate solutions in Example

Fig 8: Approximate solutions in Example 2

SELF ASSESSMENT EXERCISES

Direction fields

In each case draw a good direction field. Plot several approximate solution curves. Then solve the equation analytically and compare, to get a feeling for the accuracy of the present method.

i. $y' = - y/x$
ii. $y' = - x/y$
iii. $y' = x + y$

iv. $4yy' + x = 0$

v. **(Verhulst population model)** Draw the direction field of the differential equation in Prob. 54 of Sec. 1.7, with a = 0.03 and b = 1.6. 10^{-4} and use it to discuss the general behavior of solutions corresponding to initial greater and smaller than 187.5

vi. Apply Picard's method to $y' = y$, $y(0) = 1$, and show that the successive approximations tend to $y = e^x$, the exact solution.

vii. In Prob. 12, compute the values y_1, $y_2(1)$, $y_3(1)$ and compare them with the exact value $y(1) = e = 2.718\ldots\ldots$.

Apply Picard's method to the following initial value problems. Determine also the exact solution. Compare.

viii. $y' = xy$, $y(0) = 1$

ix. $y' = 2y$, $y(0) = 1$

x. $y' - xy = 1$, $y(0) = 1$.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

Applications are included at various places. The unit part entirely devoted to applications are 3.3.1 and 3.3.2. on separable and linear equation respectively. And are applied to electrical circults and on orthogonal trajectories, that is curves that intersect given curves at right angles.

Direction field (3.3.1) help in sketching families of solutions curves, for instance, in order to gain an impression of their general behaviour.

Picard's iteration method gives approximate solutions of initial value problems by iteration.

6.0 TUTOR MARKED ASSIGNMENT

In each case draw a good direction field. Plot several approximate solution curves then solve the equation analytically and compare, to get a feeling for the accuracy of the present method;

1) $y' = -\dfrac{y}{x}$

2) $y' = -\frac{x}{y}$

3) $y' - x + y$

4) $4yy' + x = 0$

5) draw the direction field of the differential equation $\frac{dy}{dt} = ay - by^2$ a>0, b> 0. with a = 0.03 and b= 1.6×10^{-1} and use it to discuss the general behaviour of solutions corresponding to initial conditions greater and smaller than 187.5.

6) apply Picard's method to $\frac{dy}{dx} = y$, y(0) = 1 and show that the successive approximations tends to $y = e^x$, the exact solutions.

7) In the i.e. $\frac{dy}{dx} + 3y = e^{2x} + 6$, compute the values $y_1(1)$, $y_2(1)$, $y_3(1)$ and compute them with the exact value y(1) = e = 2.718.

Apply Picard's method to the following initial value problems. Determine also the exact solution.
Compare

8) a) $\frac{dy}{dx} = y, \quad y(0) = 1$

9) b) $\frac{dy}{dx} = 2y, \quad y(0) = 1.$

10) c) $\frac{dy}{dx} - xy = 1, \quad y(0) = 1.$

7.0 REFERENCES/FURTHER READINGS

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Indira Gandhi National Open University School Of Sciences Mth-07.

MODULE 4

UNIT 1 HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

CONTENTS

1.0 INTRODUCTION

In Unit 1 we discussed the basic concepts related to ordinary differential equations. Further in the introduction to Block 2, we have mentioned that the governing differential equations in many physical or biological problems are not necessarily of first order. Besides the differential equations arrived at, in discussing the above said models may be linear or non-linear. Even among linear differential equations, the coefficients of the differentials may be constants or a function of an independent variable. In this unit we classify the general linear differential equations into two broad categories:

i) homogeneous and non-homogeneous
ii) equations with constant coefficients and variable coefficients.

For a general linear differential equation with variable coefficients, we shall state the conditions under which a unique solution can be found. Further, we shall learn methods of finding the complete solutions of homogeneous differential equations with constant coefficients.

2.0 OBJECTIVES

After reading this unit, you should be able to

- Identify linear differential equations with constant as well as with variable coefficients.
- Identify homogeneous and non-homogeneous linear differential equations.
- Describe the conditions under which a unique solution of a linear differential equation exists.
- Write the complete primitive of a given differential equation when its various independent integrals are known.
- Classify solutions of non-homogeneous equations into complementary function and particular integral.
- Obtain a solution for a homogeneous linear differential equation with constant coefficients.

3.0 MAIN CONTENT

3.1 General Equation

We begin our discussion by considering the most general linear differential equation which is of the form

$$a_0 (x) \frac{d^n y}{dx^n} + a_1 (x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} (x) \frac{dy}{dx} + a_n (x) y = b (x) \qquad \ldots(1)$$

For, $a_0(x) = 0$, the differential equation is of nth order. The coefficients $a_0(x)$, $a_1(x), \ldots, a_n(x)$ are functions of independent variable x. Eqn. (1) is called **general linear differential equation of nth order with variable coefficients.**

In case coefficients $a_0(x), a_1(x), \ldots a_n(x)$ are all constants and do not depend on x, then Eqn. (1) will be termed as **general linear differential equation of nth order with constant coefficients.** For example, equation $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + y = x^2$ is a third order linear differential equation with constant coefficients.

Further, the right hand side of Eqn. (1), i.e., b(x) may assume one of the following forms:

i) $b(x) = 0$
ii) $b(x) = \text{constant}$
iii) b9x) a function of x.

when $b(x) = 0$, Eqn. (1) is classified as the **general homogeneous linear differential equation.** This is also known as the **reduced equation** of Eqn. (1). For example, equation

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$$

is a third order linear differential equation. But if $b(x)$ in Eqn. (1) is a constant or a function of x, then Eqn. (1) is called **general non-homogeneous linear differential equation.**

Equation $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + 3y = x^2 + 1$ is a linear non-homogeneous equation of 4^{th} order with constant coefficients; where equation $x^3\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = 2$ is a second order non-homogeneous linear differential equation with variable coefficient.

Now suppose that we are required to find the solution of Eqn. (1) on some interval 1 which also satisfy, at some point $x_0 = \in 1$ the conditions,

$$y(x_0) = y_0,\ y'(x_0) =, \ldots,\ y^{(n-1)}(x_0) = y_0^{(n-1)} \qquad \ldots(2)$$

Note: Depending on the context, i could represent $]a, b[$ $]0, \infty[$, $]-\infty, \infty[$ and so on.

Where y_0, y_0', …., $y_0^{(n-1)}$ are arbitrary constants, then Eqns. (1) and (2) together constitute an **initial-value problem (IVP).** The values $y(x_0) = y_0$ $y'(x_0) = y_0'$,….., $y_0^{(n-1)}(x_0) = y_0^{(n-1)}$ are called **initial conditions.**

In the case of a linear second order equation, we can interpret geometrically a solution to the initial value problem

$$a_{n-2}(x)\frac{d^2y}{dx^2} + a_{n-1}(x)\boxed{\frac{dy}{dx}} + a_n(x)\,y = b(x)$$

$$y(x_0) = y_0,\ y'(_0) = \boxed{y_0'}$$

as a function defined on I whose graph passes through (x_0, y_0) such that the slope of the curve at the point is the number $\boxed{}$.

You may **note** here that an equation of the form (1) may not always have a solution. Moreover, even if its solution exists it may not be unique.

Let us now study the conditions under which the solution of Eqn. (1), if it exists shall be unique.

3.1.1 Conditions for the Existence of a unique solution

We may write the general non-homogeneous linear differential Eqn. (1) in the form

$$L(y) = [a_0(x)\, D^{n+} {}^{+a}{}_1(x)\, D^{n-1} + \ldots + a_{n-1}(x)\, D + a_n(x)]\, y = b(x) \qquad \ldots(3)$$

Where $D = \dfrac{dy}{dx}$, $D^2 = \dfrac{d^2}{dx^2}$, …, $D^n = \dfrac{d^n}{dx^n}$.

The expression in the parentheses in Eqn. (3) is termed as a **symbolic polynomial** or **operator polynomial** or simply a **differential operator.**
Thus we have herein introduced linear differential operator L of order n given by the expression

$$L = a_0(x)\, D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)\, D + a_n(x) \qquad \ldots(4)$$

In unit 8, we shall learn, in more details, about the differential operators and their properties.

We now choose an interval I = $[\alpha, \beta]$ for α, β real and assume that the coefficients $a_0(x)$, $a_1(x)$, …., $a_n(x)$ and the function are continuous one-valued functions of x throughout the interval and that $a_0(x)$ does not vanish at any point of the interval.

We know that the complete solution of Eqn. (3) shall involve arbitrary constants whose number is equal to the order of the highest derivative involved in it, i.e., n in this case. In order to obtain a unique solution of Eqn. (3), it is necessary to specify n initial conditions in terms of constant values of

$$y, \boxed{\frac{dy}{dx}}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}$$

at any point x_0 of the interval $[\alpha, \beta]$.

We now state a theorem which gives the conditions whose fulfillment guarantee the existence and uniqueness of the solution of Eqn. (3).

Theorem 1: If the functions $a_0(x)$, $a_1(x)$, …., $a_n(x)$ and b(x) are continuous function of x in the interval $[\alpha, \beta]$ and $a_0(x)$ does not vanish at any point of that interval, then the initial Eqn. (3) admits of a unique solution of the form y = f(x), which together with its first (n-1_ derivatives, is continuous in $[\alpha, \beta]$ and satisfies the following initial conditions:

$$y(x_0) = y_0 \quad \left(\frac{dy}{dx}\right)_{x=x_0} = \boxed{\;}, \ldots, \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_{x=x_0} = \boxed{y_0^{(n-1)}},$$

where x_0 is a point of the interval $[\alpha, \beta]$.

We shall not be proving this theorem here as it is beyond the scope of the present course. However, if the functions $a_0(x)$, $a_1(x)$,…., $a_n(x)$ are constants, we shall give the solution of the corresponding equation in Sec. 5.4 when b(x) = 0 and in units 6, 7 and 8 when $b(x) \neq 0$.

We now illustrate this theorem with the help of a few examples.

Example 1: Show that $y = 3e^{2x} + e^{-2x} - 3x$ is a unique solution of the initial value problem

$y'' - 4y = 12x$
$y(0) = 4 \; h'(0) = 1.$

Solution: We have $y = 3e^{2x} + e^{-2x} - 3x$, therefore,

$y' = 6e^{2x} - 2e^{-2x} \; 3$ and $y'' = 12e^{2x} + 4e^{-2x}$

Now $y'' = 4y = 12e^{2x} + 4e^{-2x} - 4(3e^{2x} + e^{-2x} - 3x)$
$= 12e^{2x} + 4e^{-2x} - 12e^{2x} - 4e^{-2x} + 12x$
$= 12x$

Also, $y(0) = 3e^{2.0} + e^{-2.0} - 3.0 = 4$
$Y'(0) = 6e^{2.0} - 2e^{-2.0} - 3 = 1.$

Thus, $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the given initial value problem. Moreover, the given differential equation is linear and the coefficients as well as b(x) = 12x are continuous on any interval containing x = 0. we conclude from Theorem 1 that the given function is the unique solution of the given initial vlue problem.

Remember that both the requirements in Theorem 1, that is $a_i(x)$, i = 0, 1, …., n be continuous and $a_0(x)$ $\boxed{\;}$ 0 for every x in some interval say I are important.

Specifically, if $a_0(x) = 0$ for some x in the interval then the solution of a liner initial value problem may not be unique or may not even exist.

We now illustrate this through an example.

Example 2: Obtain the value of c for which the function

$$y = cx^2 + x + 3$$

is a unique solution of the initial value problem

$$x^2y'' - 2xy' + 2y = 6,$$
$$y(0) = 3,\ y'(0) = 1$$

On the interval]- □, □[.

Solution: Since $y' = 2cx + 1$ and $y'' = 2c$, it follows that

$$\begin{aligned} X^2y'' - 2xy' + 2y &= x^2(2c) - 2x(2cx + 1) + 2(cx^2 + x + 3) \\ &= 2cx^2 - 4cx^2 - 2x + 2cx^2 + 2x + 6 \\ &= 6 \end{aligned}$$

Also, $y(0) = c.(0)^2 + 0 + 3 = 3$
and $y'(0) = 2c.0 + 1 = 1$

Thus, $y = cx^2 + x + 3$ is a solution of the given problem for all values of c in the given interval. The problem does not have a unique solution. In this case although the given equation is liner and its coefficients and $b(x) = 6$ are continuous everywhere but the coefficient of y'' i.e., x^2 is zero at $x = 0$.

You may now try the following exercise.

You might be familiar with the liner dependence and independence of a set of functions on an interval. Before we study some elementary properties of the solution of linear differential equations, ;et us recall these two concepts which are basic to the study of linear differential equations.

3.1.2 Linear Dependence and Independence

We begin with the following two definitions.

Definition: A set of function $y_1(x)$, $y_2(x)$,…., $y_n(x)$ is said to be **linearly independent** on an interval I if there exist constants. C_1, c_2, …., $c_n(x)$ not all zero, such that

$$C_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x) = 0$$

For every x in the interval.

Definition: A set of functions y1(x), $y_2(x)$,…., $y_n(x)$ is said to be **linear independent** on an interval I, if it is not linearly dependent on the interval.

In other words, a set of functions is linearly independent on an interval if the only constant for which

$$c_1y_1(x) + c_2y_2(x) + ….. + c_ny_n(x) = 0,$$

For every x in the interval, are $c_1 = c_2 = …. = c_n = 0$.

It is easy to understand these definitions in the case of two functions $y_1(x)$ and $y_2(x)$. if the functions are linearly dependent on an interval, then there exists constants c_1 and c_2, both are not zero, such that for every x in the interval

$$c_1y_1(x) + c_2y_2(x) = 0$$

Since $c_1 \square 0$, it follows that

$$y_1(x) = -\frac{c_2}{c_1}\ y_1(x),$$

That is, **if two functions are linearly dependent, then one is a constant multiple of the other.** Conversely, if $y_1(x) = c_2y_2(x)$ for some constant c_2, then

$$(-1)\ y_1(x) + c_2y_2(x) = 0$$

for every x on some interval. Hence the functions are linearly dependent, at least one of the constants (namely, $c_1 = -1$) is not zero. We thus conclude that **two functions are linearly independent when neither is a constant multiple of the other** on an interval.

Functions, $y_1(x) = \sin 2x$ and $y_2(x) = \sin x \cos x$ are linearly dependent on the interval $]-\square,\square[$ since $c_1 \sin 2x + c_2 \sin x \cos x = 0$ is satisfied for every real x with

$$C_1 = \frac{1}{2} \text{ and } c_2 = -1.$$

In the consideration of linear dependence or linear independence, the interval on which the functions are defined is important. We now illustrate it through an example.

Example3: Show that the function $y_1(x) = x$ and $y_2(x) = |x|$ are

i) linearly independent on the interval] -□, □[.
ii) linearly dependent on the interval]0, □[.

Solution:

(i) it is clear that in the interval] -□, □[neither function is a constant multiple of the other (see Fig. 1)

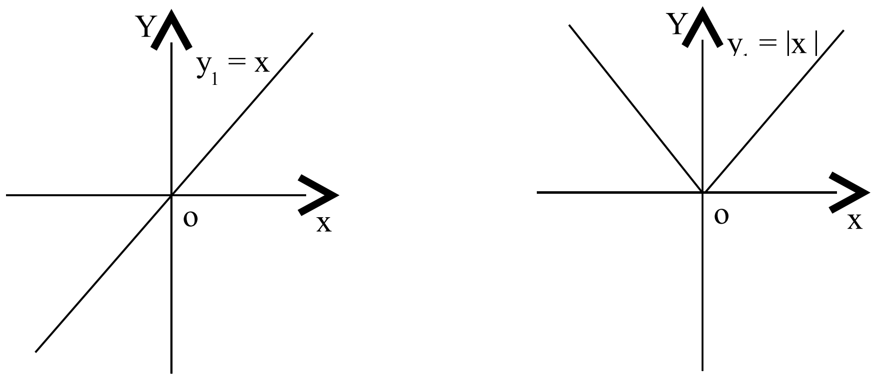

Fig. 1

Thus in order to have $c_1y_1(x) + c_2y_2(x) = 0$ for every real x, we must have $c_1 = 0$ and $c_2 = 0$.

(ii) For $y_1(x) = x$ and $y_2(x) = |x|$ in the interval]0, □[
$c_1x + c_2 |x| = c_1x + c_2x = 0$
is satisfied for any non zero choice of c_1 and c_2 for which $c_1 = - c_2$.
Thus $y_1(x)$ ands $y_2(x)$ are linearly dependent on the interval]0, □[.

You may try the following exercises:

The procedure given for examining the linear dependence or independence of a set of functions appears to be quite involved. We, therefore, outline below sufficient condition of examining the linear independence of a set of n functions.

Suppose that $y_1(x)$, $y_2(x)$,…., $y_n(x)$ are n functions on an interval I possessing derivative upto (n – I)th order. If the determinant.

$$W(y_1(x), y_2(x), \ldots., y_n(x)) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ . & . & \cdots & . \\ . & . & \cdots & . \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Is not zero for at least one point in the interval I, then the functions $y_1(x), \ldots., y_n(x)$ are linearly independent on the interval.

This provides a **sufficient condition** for the linear independence of n functions on an interval. The determinant $W(y_1(x), y_2(x),\ldots., y_n(x))$ is called the **Wronskian** of the functions. It is named after a Polish mathematician Josef Maria Hosene Wornski (1778 – 1853).

The functions $y_1(x) = \sin^2 x$ and $y_2(x) = 1\text{-}\cos 2x$, for instance are linearly dependent on] □, □[because

$$\begin{vmatrix} \text{sun}^2 x & 1-\cos 2x \\ 2\sin x \text{ cox} & 2\sin 2x \end{vmatrix} = 2\sin^2 x \sin 2x - 2\sin x \cos x + 2\sin x \cos x\ 2x$$

$$= \sin 2x\,[2\sin^2 x - 1 + \cos 2x]$$
$$= \sin 2x\,[2\sin^2 x - 1 + \cos^2 x - \sin^2 x]$$
$$= \sin 2x\,[\sin^2 x + \cos^2 x - 1]$$
$$= 0$$

in example 3 we saw that $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on] - □, □[. However, we cannot compute the Wronskian as y_2 is not differentiable at $x = 0$.

Remember that in the above condition the non-vanishing of the Wronskian at a point in the interval provides only a sufficient condition. In other words, if $W(y_1 y_2,\ldots., y_n) = 0$ for x in an interval, it does necessarily mean that the functions are linearly dependent on the interval. We leave it for you to verify it yourself.

With above background in mind we are now set to study the elementary properties of the solutions of linear differential equations.

3.2 Elementary Properties of the Solutions

The general homogeneous linear differential equation corresponding to Eqn. (3) is

$$L(y) = [a_0(x)D^n + a_1(x)D^{n-1} + \ldots + a_{n-1}(x)]\ y = 0$$

i.e., $L(y) = \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ y = 0$ …(5)

We can clearly think of the form of the solutions of linear differential equations by making use of the following elementary theorems:

Theorem 2: If $y = y_1$ is a solution of Eqn. (5) on an interval 1, then $y = cy_1$ is also its solution on I, where c is any arbitrary constant.

Proof: We know that

$D^r (cy^1) = cD^r y^1$

Also, $L(cy^1) = \sum_{r=0}^{n} a_r(x)\ D^{n-r}\ (cy_1)$

$= c \sum_{s=0}^{n} a_s(x)\ D^{n-s}\ y_1$

$= c\ L(y_1).$

$= 0\ (\therefore L(y_1) = 0)$

Thus, if $y = y_1$ is a solution of Eqn (5), so $y = cy_1$. for instance, the function $y = x^2$ is a solution of the homogeneous linear equation.

$X^2y'' - 3xy' + 4y = 0$ on $]0, \square[$.

Hence $y = cx^2$ is also solutions. For various of c, we see that $y = 3x^2$, $y = ex^2$, $y = 0$ … are all solutions of the equation on the given interval.

Have you notice that **a homogeneous linear differential equation always possesses the trivial solution y = 0?** If not, you can check it now.

Now let us look at another property of the solutions of linear differential equations.

Theorem 3: If $y = y_1, y_2, \ldots, y_m$ are m solutions of homogeneous linear differential Eqn. (5) on an interval 1, then $y = c_1y_1 + c_2y_2 + \ldots. + c_my_m$ is also a solution of Eqn. (5) on 1, where $c_1, \ldots., c_m$ are arbitrary constants.

Proof: If y_1 $(i = I, \ldots, m)$ are solutions of Eqn. (5) then

$L(y_1 = 0$ (for $i = 1, 2, \ldots., m$) …(6)

We know that

$D^r [c_1y_1) + c_2y_2 + \ldots + c_my_m]$

$= D^r (c_1y_1) + D^r (c_2y_2) + \ldots + D^r (c_my_m)$

$= c_1\ D^r (y_1) + c_2\ D^r (y_2) + \ldots + c_m\ D^r (y_m)$

now, $L(c_1y_1 + c_2y_2 + \ldots + c_my_m)$

$$= \sum_{r=0}^{n} a_r(x)\, D^{n-r}(c_1y_1 + c_2y_2 + \ldots + c_my_m)$$
$$= c_1 \sum_{r=0}^{n} a_r(x)\, D^{n-r}\, y_1 + c_2 \sum_{r=0}^{n} a_r(x)\, D^{n-r}\, y_2 + \ldots + c_m \sum_{r=0}^{n} a_r(x)\, D^{n-r}\, y_m$$
$$= c_1\, L(y_1) + c_2\, L(y_2) + \ldots + c_m\, L(y_m) \text{ (using Eqn. (5))}$$
$$= c_1.0 + c_2.0 + \ldots + c_m.0 \text{ (using Eqn. (6))}$$
$$= 0$$

Hence if $y_1, y_2 \ldots. y_m$ are solutions of Eqn. (5), then $y = c_1y_1 + c_2y_2 + \ldots + c_my_m$ is also a solution of Eqn. (5).
Theorem 3 is known as the **superposition principle.**

Let us now consider an example.

Example 4: Show that if $y_1 = x^2$ and $y_2 = x^2$ Inx are both solutions of the equation $x^3y'' = 2xy' + 4y = 0$ on the interval $]0, \infty[$. Then $c_1x^2 + c_2x^2$ Inx is also a solution of the equation on the given interval.

Solution: We have $y = c_1x^2 + c_2x^2$ Inx
Now $y' = 2c_1x + 2c_2x \text{ Inx} + c_2x$
$y'' = 2c_1 + 2c_2x \text{ Inx} + 3c_2$
$y'' = \dfrac{2c_2}{x}$

therefore, $x^3y'' - 2xy' + 4y$
$$= x^3\left(\frac{2c_2}{x}\right) - 2x\,(2c_1x + 2c_2x \text{ Inx} + c_2x) + 4c_1x^2 + 4c_2x^2 \text{ Inx}$$
$$= 2c_2x^2 - 4c_1x^2 - 4c_2x^2 \text{ Inx} - 2c_2x^2 + 4c_1x^2 + 4c_2x^2 \text{ Inx}$$
$$= 0$$

Thus, $y = c_1x^2 + c_2x^2$ Inx is also a solution of the equation on the interval.

Theorem 2 and 3 represent properties that non-linear differential equations, in general, do not possess. This will become more clear to you after you have done the following exercises.
Let us now consider the following definition which involves a linear combination of solutions.

Definition: Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of homogeneous linear differential Eqn. (5) of degree n on an interval I. Then

$$y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x),$$

where c_1, i = 1, 2, …, n are arbitrary constants is defined to be the **general solution** or the **complete primitive** of Eqn. (5) on I.

The above definition automatically generates our interest in knowing when n solutions, y_1, y_2, …, y_n of the **homogeneous** differential Eqn. (5) are linearly independent. Surprisingly, the nonvanishing of the Wronskian of a set of n such solutions on an interval I is both **necessary and sufficient** for linear independence.
That is,

If y_1, y_2, …, y_n) be n solutions of homogeneous linear nth order differential EQn. (5) on an interval I, then the set of solutions is linearly independent on I if and only if

$$W(y_1, y_2, \ldots, y_n) \neq 0$$

For every x in the interval. Such a set y_1, …., y_n of n linearly independent solutions of Eqn. (5) on I is said to be a **fundamental set of solutions** on the interval.

For instance, the second order equation y'' – 9y = 0 possesses two solutions

$$y_1 = e^{3x} \text{ and } y_2 = 3^{-3x}$$

Since $W(3^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$

For every value of x, y_1 and y_2 form a fundamental set of solution on]- ∞ , ∞ [. The general solution of the differential equation on the interval is

$$y = c_1 e^{3x} + c^2 e^{-3x}$$

so far we have discussed the properties pertaining to the solution of homogeneous linear equations. We now turn our attention to the **non-homogeneous linear equation.** To this effect, we consider a theorem due to D' Alembert (1762 – 1765) which defines the general solution of a non-homogeneous linear equation.

Theorem 4: If $y = Y_0(x)$ is any solution of the non-homogeneous linear differential Eqn. (3) on an interval I and if $y = Y(x)$ is te complete primitive of the corresponding homogeneous linear differential Eqn. (5) on the interval, then

$$Y_0(x) + Y(x)$$

Is the general solution of Eqn. (3) on the given interval.

Proof: Since $Y_0(x)$ is a solution of Eqn. (3),
$\therefore L[Y_0(x)] = b(x)$ ...(7)

also, $Y(x)$ is the complete primitive of Eqn. (5),
$\square L[Y_0(x)] = 0$ …(8)

further, in Theorem 2 and 3 above, we have seen that the operator Dr and linear differential operator L are distributive. Thus, using relations (7) and (8), we get

$$\begin{aligned} L(y) &= L[Y_0(x) + Y(x)], \\ &= L[Y_0(x)] + L[Y(x)] \\ &= B(x) + 0 \end{aligned}$$

Thus, $y = Y_0(x) + Y(x)$ is a solution of Eqn. (3).

Since $y(x) = Y_0(x) + Y(x)$ involves n arbitrary constants (due to presence of n arbitrary constants in $Y(x)$), it is, therefore, the general solution of Eqn. (3).

If $Y(x)$ is chosen as to satisfy the condition (2) and if $Y_0(x)$, for some point x_0 of the interval I, is such that

$$Y_0(x_0) = 0 = \left(\frac{dY_0}{dx}\right)_{x=x_0} = \left(\frac{d^2Y_0}{dx^2}\right)_{x=x_0} = \ldots = \left(\frac{d^{n-1}Y_0}{dx^{n-1}}\right)_{x=x_0},$$

Which is possible provided that $b(x)$ is not identically zero, then the solution

$y = Y_0(x) + Y(x)$ …(9)
also satisfies the conditions.

$$y(x_0) = y_0, \left(\frac{dy}{dx}\right)_{x=x_0} = y_0', \ldots, \left(\frac{d^{n-1}y}{dx^{n-1}}\right)_{x=x_0} = y_0^{n-1}$$

we usually refer the solution of Eqn. (5) in the form (9) as the **general solution** of the non-homogeneous linear differential Eqn. (3) and it consist of two parts:

i) The complete primitive of Eqn. (5) (the corresponding homogeneous part of Eqn. (3)) in the form

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \ldots. + c_ny_n(x).$$

Which contains n arbitrary constants. The solution y = Y(x) of Eqn. (5) is known as **complementary function** of Eqn (3). We denote the complementary function
Y)x) by $y_c(x)$.

ii) Any solution $y = Y_0(x)$ of Eqn. (3), (which cannot be obtained by assigning any particular value to the arbitrary constants in $y_c(x)$ is known as **particular integral** of Eqn. (3). We denote $Y_0(x)$ by $y_p(x)$.
Thus, we may write Eqn. (9) in the form $y(x) = y_p(x) + y_c(x)$.

You may then ask the natural question – how to find the solution y(x) of Eqn. (3)?

In the next section we give you the methods of finding the complementary function $y_c(x)$ of the given linear equation with constant coefficients. Since the complementary function refer to the solution of the homogeneous equation corresponding o the given equation, we consider the general nth order homogeneous linear differential equation with constant coefficients.

You may recall that in Sec. 5.2, we had mentioned that if the coefficients of y and its derivatives in Eqn (1) are constants and $a_0 \neq 0$, then Eqn (1) is termed as linear differential equation of nth order with constant coefficients. Further, we had mentioned that if the right hand side of Eqn. (1) is zero, then it will be classified as homogeneous linear differential equation for this reason

Eqn. (1) i.e. the function b(x) is also called non-homogeneous term of Eqn. (1). Thus, the general nth order homogeneous linear differential equation with constant coefficients may be expressed as

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad \ldots(10)$$

where the coefficients $a_1, a_2, \ldots, a_n$ are constants.

we would like to mention here that in writing Eqn. (10) we have taken the coefficient of $\frac{d^n y}{dx^n}$ as unity. Even if it is not so, dividing throughout by the coefficient of $\frac{d^n y}{dx^n}$ (which is also assumed to be constant), the equation can be reduced to the form (10).

Let us now discuss the methods of solving EQn. (10).

3.3 Method of Solving Homogeneous Equation with Constant Coefficients

The method of solving Eqn. (10) was given in the year 1739 by Leonhard Euler (1707 – 1783) who was born in Basel, Switzerland and was one of the most distinguished mathematicians of the eighteenth century.

The method is as follows:

Assume that $y = e^{mx}$ is a solution of Eqn. (10). On replacing y and its derivatives upto order n by e^{mx} and $m^n e^{mx}$ in Eqn. (10), we get

$$(m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n) e^{mx} = 0 \qquad \ldots(11)$$

since $e^{mx} \neq 0$ for real values of x, Eqn. (11) is satisfied if

$$m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0 \qquad \ldots(12)$$

Eqn. (12) is called an **auxiliary equation or characteristic equation** corresponding to differential Eqn (10).

You might have observed that an auxiliary equation of a homogeneous or non-homogeneous linear differential operator on replacing D by some finite constant m and equating it to zero.

You may wonder why we assumed the solution of Eqn. (10) in the exponential form. This is because we know that the linear first order equation $\frac{dy}{dx} + ay = 0$. Where a is a constant, has the exponential solution $y = c_1 e^{-ax}$ on $]-\infty, \infty[$. Therefore, it is natural to determine whether exponential solution exist on $]-\infty, \infty[$. For higher order equations of the form (10).

In the discussion to follow, you will be surprised to see that all solutions of Eqn. (10) are exponential functions or constructed out of exponential functions.

Let us now consider the following examples:

Example 5: Write auxiliary equation corresponding to the differential equation
$(D^6 + 12D^4 + 48^2 + 64) y = 0$

Solution: Replacing D by m in the linear differential operator of the given equation, the auxiliary equation becomes
$m^6 + 12m^4 + 48m^2 + 64 = 0$

Example 6: Write the characteristic equation corresponding to the differential equation
$(D^2 + 2aD + b^2) y = c \sin wx$.

Solution: On replacing D by m in the homogeneous part of the given equation and equating it to zero, we arrive at the following characteristic equation
$m^1 + 2am + b^2 = 0$

remember that while writing the auxiliary equation for non-homogeneous differential equation, he non-homogeneous part is neglected.
Auxiliary Eqn. (12) is a polynomial in m of degree n and, it can have at the most n roots.

Let $m_1, m_2, \ldots, m_n$ be the n roots. Then the following three possibilities arises;

I) Roots of auxiliary equation may be **all real** and distinct,
II) Roots of auxiliary equation may be **all real,** but **some** of the roots may be **repeated.**
III) Auxiliary equation may have **complex roots.**

We now proceed to find the solution of Eqn. (10) for these three cases one by one.

Case 1: Auxiliary equation has real and distinct roots:
Let the roots $m_1, m_2, \ldots, m_n$ of auxiliary Eqn. (12) be real and distinct.

Now suppose $m = m_1$. Since m_1 is a root of auxiliary Eqn. 912), clearly e^{m_1x} is an integral of Eqn. (10) and satisfies it on the interval $]-\infty, \infty[$.

Similarly, for $m = m_2$, e^{m_2x} is a solution of Eqn. (10) and e^{m_1x} and e^{m_2x} are also linearly independent on the interval since

$$W(e^{m_1x}, e^{m_2x}) = \begin{vmatrix} e^{m_1x} & e^{m_2x} \\ m_1e^{m_1x} & m_2e^{m_2x} \end{vmatrix}$$

$$= (m_2 - m_1)\ e^{(m_1+m_2)x} \neq 0 \text{ for } m_1 \neq m_2.$$

Now, the n roots of Eqn (12), namely $m_1, m_2, \ldots, m_n$ are real and distinct solutions, $\boxed{e^{m_1x}}$, $\boxed{e^{m_2x}}$, …, e^{m_nx} are all distinct and linearly independent solutions of Eqn. 910).

Sin Eqn. (10) is of nth ordr and we have n distinct and linearly independent solutions, therefore, we can express the complete solution of Eqn (10) as

$$y = c_1\boxed{e^{m_1x}} + c_2\boxed{e^{m_2x}} + \ldots + c_n\boxed{e^{m_nx}}, \qquad \ldots(13)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

We now illustrate this case with the help of a few examples.

Example 7: Solve $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 12y = 0$

Solution: The given equation can be written as
$(2D^2 + 5D – 12) y = 0$
the auxiliary equation is
$2m^2 + 5m – 12 = 0$
$\Rightarrow$ $(2m – 3) (m + 4) = 0$
$m = 3/2, - 4$

here the roots are real and distinct.

Hence complete solution of the given differential equation is $y = c_1e^{(3/2)x} + c_2e^{-4x}$, where c_1 and c_2 are arbitrary constants.

Let us look at another example.

Example 8: If $\frac{d^2y}{dx^2}$ $- a^2y = 0$, show that y = Acosh ax + Bsinh ax is the complete solution.

Solution: The auxiliary equation corresponding to the given differential equation is
$m^2 – a^2 = 0$
$(m – a) (m + a) = 0$
$m = a, - a.$

roots being real and distinct, the general solution of the given equation is
$y = c_1e^{ax} + c_2e^{-ax}$
From the definition of hyperbolic functions, we know that

$$\text{Cosh } ax = \frac{1}{2} (e^{ax} + e^{-ax} \qquad \ldots(14)$$

$$\text{and sinh } ax = \frac{1}{2} (e^{ax} – e^{-ax)} \qquad \ldots(15)$$

adding relations (14) and (15), we get
$e^{ax} = \cosh ax + \sinh ax$
Substrating relation (15) from (14), we get
$e^{-ax} = \text{coshax} – \text{sinhax}$
the general solution of given differential equation can thus be written as
$y = c_1 (\cosh ax + \sinh ax) c_2 (\cosh ax – \sinh ax)$
$y = (c_1 + c_2) \cosh ax +(c_1 – c_2) \sinh ax$
$A \cosh ax + B \sinh ax,$
where $A = c_1 + c_2$ and $B = c_1 – c_2$ are two arbitrary constants.

We now consider an initial value problem.

Example 9: Solve the equation

$$\frac{d^2x}{dx^2} - 4x = 0$$

with the conditions that when t = 0, x = 0 and $\frac{dx}{dt} = 3$.

Solution: The auxiliary equation corresponding to the given equation is
$m^2 - 4 = 0$
$\Rightarrow$ $(m - 2)(m + 2) = 0$
$m = 2, -2$

hence the general solution of the differential equation is
$x = c_1e^{-2t}$.
we now apply the given conditions at t = 0.we have
$= 2c_1e^{-2t}$
Condition that x = 0 when t = 0 gives
$0 = c_1 + c_2$,
and the condition that $= 3$ when t = 0 gives
$3 = 2c_1 - 2c_2$
From the two equations for c_1 and c_2, we conclude tht
$C_1 = \frac{3}{4}$ and $c_2 = -$. Therefore,

$$X = (e^{2t} - e^{-2t})$$

Which can also be put in the form
$x =$ sinh 2t.
Now you may try the following exercises.

We now take up the case when the roots of auxiliary equation are all real but some of them are repeated.

Case II: Auxiliary Equation has real and repeat roots:

Let two roots of auxiliary Eqn. (12) be equal, say $m_1 = m_2$. then solution (13) of Eqn. (10) becomes
$y = (c_1 + c_2)\ e^{m_1 x} + c_3\ 3^{m_3 x} + \dots + c_n\ e^{m_n x}$

Since $(c_1 + c_2)$ can be replaced by a single constant, this solution will have (n -1) arbitrary constants.

We know that the general or complete solution of an nth order linear differential equation must contain n arbitary constants; hence the above solution having (n -1) arbitrary constants is not the general solution.

To obtain general solution in this case let us reqrite Eqn. (10) in the form
$L_1 (y) = (D^n + a_1 D^{..} + {}^{a}{}_{n-1} D + a_n 0\ y = 0$ …(16)
Where $D = \frac{d}{dx}$ and $D^n = \frac{d^n}{dx^n}$ and L_1 is a linear differential operator.

If $m_1, m_2, \dots, m_n$ are the roots of auxiliary equation corresponding to, Eqn. (16), then Eqn.(16) can be written as

$D - m_1) (D - m_2) \dots (D - m_n)\ y = 0$ (17)

It is clear that when all the n roots $m_1, m_2, \dots, m_n$ are real and distinct the complete solution of Eqns. (16) or (17) is constituted by the solutions of the n equations.

$(D - m_1)\ y = 0, (D - m_2)\ y = 0 \dots. (D - m_n)\ y = 0$
in case the two roots are equal say $m_1 = m_2$, then Eqn. (17) takes the form
$(D =- m_1)^2 (D - m_3) \dots (D - m_n)\ y = 0$
and then solutions corresponding to two equatl roots are the solutions of
$(D - m_1)^2\ y = 0$
$(D - m_1) [(D - m_1)\ y] = 0$ …(18)
Let $(D - m_1)\ y = v$ …(19)
Then Eqn. (18) reduces to
$(D - m_1)\ V = 0$
$\frac{dV}{dx} - m_1 V = 0$

it is a linear differential equation of the first order and its solution (ref. Sec. 3.3 of unit 3) is
$V = c_1 \boxed{e^{m_1 x}}$

With this value of V, Eqn. (19) becomes
$(D - m_1)\, y = c_1 e^{m_1 x}$
which is again a linear differential equation of the first order and is solution is
$y = e^{m_1 x} (c + c_1 x)$,
c_1, c_2 being constants.

Similarly, the solution of Eqn. (17) corresponding to three equal roots say $m_1 = m_2 = m_3$, are the solutions of
$(D - m_1)^3 y = 0$
$(D - m_1)\, [(D = m_1)^2 y] = 0$

te $(D - m_1)^2\, y = z$ in the above equation. solving the equation for z and putting the value of z obtained in the above equation, we have
$(D - m_1)^2 y = c_1 e^{m_1 x}$

Substituting gain $(D - m_1)\, y = t$ and proceeding as before, we get
$(D - m_1)\, y = e^{m_1 x}\ (c_2 + c_1 x)$
The solution of above linear differential equation of first order is

$$y = e^{m_1 x} \left(\frac{c_1}{2} x^2 + c_2 x + c_3 \right)$$

thus, it is clear that if a root m_1 of Eqn. (16) is repeated r times, then solution corresponding to this root will be of the form

$y = e^{m_1 x}\ (A_1 + A_2 x + A_3 x^2 + \ldots + A_r x^{r-1})$
and the general solution of Eqn. (16) will then be
$$y = e^{m_1 x} (A_1 + A_2 x + A_3 x^2 + \ldots + A_r x^{r-1}) + A_{r+1}\, e^{m_{r+1} x} + \ldots + A_n e^{m_n x} \qquad \ldots(20)$$
We now illustrate the above discussion with the help of a few examples.

Example 10: Solve $\dfrac{d^4 y}{dx^4} - \dfrac{d^3 y}{dx^3} - 9\dfrac{d^2 y}{dx^2} - 11\dfrac{dy}{dx} - 4\, y = 0$

Solution: The given differential equation can be written as

$(D^4 - m^3 - 9D^2 - 11D - 4)\, y = 0$
Auxiliary equation of the given equation is
$M^4 - m^3 - 9m^2 - 11m - 4 = 0$
$(m + 1)^3\, (m - 4) = 0$
$m = -1, -1, -1, 4$

here the root – 1 is repeatred three times and root 4 is distinct. Hence, using Eqn. (20), the general solution of the given differential equation is
$y = (A + Bx + Cx^2)\, e^{-x} + De^{4x}$,
where A, B, C and D are arbitrary constants.

Let us consider another example.

Example 11: Find the complete solution of
$(D^4 – 8D^2 + 16)\, y = 0$

Solution: In this case the auxiliary equation is
$m^4 – 8m^2 + 16 = 0$
$\Rightarrow (m^2 – 4)^2 = 0$
$\Rightarrow (m- 2)^2 (m + 2)^2 = 0$
$\Rightarrow m = 2,2, - 2, - 2$

here 2 and -2 are both repeated. Therefore, the method of repeated real roots will be separately applied to each repeated root. Hence the complete solution of the given differential equation is

$y = (A + Bx)\, e^{2x} + (C + Dx)\, e^{-2x}$.

and now some exercises for you.

Now we shall discuss the case when the auxiliary equation may have complex roots.

Case III: Auxiliary Equation has complex roots:

If the roots of auxiliary Eqn. (12) are not all real, then some or, may be, all the roots are complex. We know from the theory of equations that if all the coefficients of a polynomial equation are real, then its complex roots occur in conjugate pairs. In Eqn. (12), all the coefficients are assumed to be real constants and hence complex roots, if any, must occur in conjugate pairs.

Let one such pair of complex roots of Eqn 912) be $m_1 = \alpha - i\beta$, where α and β are real and $i^2 = - 1$. Formally, there is no difference between this case and case I, and hence the corresponding terms of solution are

$$y = c_1\, e^{(\alpha + i\beta)x} + c_2\, e^{(\alpha - i\beta)x}$$
$$= e^{\alpha x}\, [c_1[e^{i\beta x} + c_2 e^{- i\beta x}] \qquad \ldots(21)$$

however, in practice we would prefer to work with real functions instead pf complex exponentials. To achieve this, we make use of the **Euler's formula**, namely,

$e^{i\theta}$ = cos θ i sin θ and $e^{-i\theta}$ = cos$\square$ - i sin$\square$,

where $\square$ is any real number. Using these results, the expression (21), which is the part pf the solution corresponding to complex roots, becomes

$e^{\alpha x}$ [c_1 (cos$\square$x + i sin$\square$x) + c_2 (cos$\square$x – i sin$\square$x)]
= $e^{\alpha x}$ [(c_1 + c_2) cos$\square$x + (c_1 – c_2) i sin$\square$x]

Since c_1 + c_2 are arbitrary constants, we may write

A = c_1 + c2 and B = i(c_1 – c_2),

So that A and B are again arbitrary constants, though not real. Expression (21) now takes the form

$e^{\alpha x}$ [Acos$\square$x + Bsin$\square$x]
…(22)

Further, if the complex root is repeated, then the complex conjugate root will also be repeated and the corresponding terms in the solution can be written, using the form (20), as

$e^{x(\alpha + i\beta)}$ (c_1 + c_2x) + $e^{x(\alpha - i\beta)}$ (c_3 + c_4x)

Proceeding as above and writing

A = c_1 + c_3 B = i(c_1 – c_3) C = c_2 + c_4 D = i(c_2 – c_4),

The above expression can be written as

$e^{\alpha x}$ [(A + Cx) cos$\square$x + (B + Dx) sin$\square$x]
…(23)

in the case of multiple repetition of complex roots, the results are obtained analogous to those in the case of multiple repetition of real roots.

We now illustrate this case of complex roots with the help of a dew examples.

Example 12: For the differential equation

$\boxed{\frac{d^4y}{dx^4}}$ - m^4y = 0, show that its solution can be expressed in the form

y = c_1 cos mx + $c_2$sin mx + c_3 cosh mx + c_4 sinh mx.

Solution: The given differential equation can be expressed as
$(D^4 – m^4) y = 0$
In this case since m is used as a constant in the given differential equation, we can replace D by some other letter, λ say.
So, the auxiliary equation is
$(\ ^4 – m^4) = 0$
$(\ ^2 – m^2) (\ ^2 + m^2) = 0$
$= m, - m, \pm\ im$

Now the solution corresponding to roots + m and – m can be obtained as we have done in Example 8 and write it as

C_3 cosh mx + c_4 sinh mx
Solution corresponding to imaginary roots + im and –im will be
$Ae^{imx + Be}$-imx
Which can be written as

A(cos mx + i sin mx + B(cos mx – i sin mx)
= (A + B) cos mx + i (A – B) sin mx
= c_1 cosmx + c_2 sinmx

where c_1 = (A + B) and c_2 = i(A – B) are constants.
Hence the general solution of the given differential equation is

Y = c_1 cos mx + c_2 sin mx + c_3 cosh mx + c_4 sinh mx

Let us look at another example.

Example 13: Solve $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$

Solution: in this case is
$M^4 – 4m^3 + 8m^2 – 8m + 4 = 0$
$(m^2 – 2m + 2)^2 = 0$
$[m – (1 + i)]^2 [m – (1 – 9)]^2 = 0$
m = 1 + i, 1 + i, 1 – i.

Roots are complex and repeated in this case.
Hence the general solution can be written as
$Y= (c_1 + xc_2) e^{(1+i)x} + (c_3 + xc_4) e^{(1-i)x}$
$= e^x [(c_1 + xc_2) e^{ix} + (c_3 + xc_4) e^{-ix}]$
$= e^x [(c_1 + xc_2) (cosx + i sinx) + (c_3 + xc_4) (cosx – i cinx)]$

$= e^x \{[(c_1 + xc_2) + x(c_2 + c_4)] \cos x + i [(c_1 – c_3) + x(c_2 – c_4)] \sin x\}$
$= e^x [(A + Bx) \cos x + (C + Dx) \sin x]$
where $A = (c_1 + c_3)$, $B = (c_2 + c_4)$, $C = i (c_1 – c_3)$ and $D = i)c_2 – c_4)$ are all constants.
You may now try the following exercise.

4.0 CONCLUSION

We now end this unit by given a summary of what we have covered in it.

5.0 SUMMARY

in this unit we have covered the following.

1) The general linear differential equation with dependent variable y and independent variable x is termed as an equation.
   a) with variable coefficients if the coefficients of y and its derivatives are functions of x.
   b) with constant coefficients if the coefficients of y and its derivatives are all constants.
   c) homogeneous if the terms other than those of y and its derivatives are absent.
   d) non-homogeneous if the terms other than those of y and derivatives are present and are constants or functions of independent variable x.

2) A solution of general linear differential equation exists and is unique if conditions of Theorem 1 are satisfied.

3) A set of functions $y_1(x), y_2(x), \ldots, y_n(x)$ defined on an interval I is linearly dependent if for constants $c_1, c_2, .., c_n$ not all zero, we have for every x in I, $c_1y_1(x) + c_2y_2(x) + \ldots, + c_ny_n(x) = 0$.
4) A set of functions $y_1(x), y_2(x),\ldots, y_n(x)$ on I is linearly independent on I if it is not linearly dependent on I.
5) If $y = y_1$ is a solution of homogeneous linear differential equation on I, so is $y = c_1$ on I, where c is arbitrary constant.
6) If $y = y_1, y_2, \ldots, y_m$ are solutions of linear homogeneous differential equation on I, so is $y = c_1y_1 + c_2y_2 + \ldots + c_my_m$ on I, where $c_1, c_2, \ldots, c_m$ are arbitrary constants.
7) If $y_1, y_2,\ldots, y_n$ are linearly independent solutions of an nth order homogeneous linear differential equation on an interval I,
   then
   $y = c_1y_1 + c_2y_2 + \ldots + c_ny_n$
   (where $c_1,c_2,\ldots, c_n$ being arbitrary constants)

is defined as the complete primitive of the given equation on I.

8) For a non-homogeneous equation
  a) the complete primitive of the corresponding homogeneous part is called it complementary function.
  b) particular solution of the non-homogeneous part involving no arbitrary constant is called its particular integral.
  c) Complementary function and particular integral constitute its general solution.

9) Solution y, of an nth order linear differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \boxed{\frac{dy}{dx}} + a_n y = 0$$

with constant coefficients $a_1, \ldots, a_{n-1}, a_n$ having n roots $m_1, m_2, \ldots, m_n$, when
  a) roots are real and distinct, is
  $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$
  b) roots are real and repeated, say $m_1 = m_2 = \ldots, = m_r$, is
  $y = (c_1 + c_2 x + \ldots + c_r x^{r-1}) e^{m_1 r} + c_{r+1} e^{m_{r+1} x} + \ldots, + c_n e^{m_n x}$.
  c) roots are complex and one such pair is $\square \pm i\square$, is
  $y = e^{\square x} (c_1 \cos\square x + c_2 \sin\square x)$
  corresponding to that pair of roots.

6.0 TUTOR MARKED ASSIGNMENT

1. Verify if the function $y = \frac{1}{4} \sin 4x$ is a unique solution of the initial value problem

$y' + 16y = 0$
$y(0) = 0, y'(0) = 1.$

2. In the following problems verify that the given function y_1 and y_2 are the solutions of the corresponding equations. Decide whether the set $\{y_1, y_2\}$ of solutions is linearly dependent or independent.

  a) $y'' - y = 0$, $y_1 = e^x$ and $y_2 = e^{-x}$ over $-\square < x < \square$
  b) $y'' + 9y = 0$, $y_1 = \cos 3x$ and $y_2 = \cos\left(3x + \frac{\pi}{2}\right)$ over $-\square < x < \square$.
  c) $y'' - 2y' + y = 0$ $y_1 = e^x$ and $y_2 = xe^x$ over $-\square < x < \square$.

3. Construct an example to show that a set of functions could be linearly independent on some interval and yet have a vanishing Wronskian.

4. Verify that $y = 1/x$ is a solution of the non-linear differential equation $y'' = 2y^3$ on the interval $]0, \infty[$, but the constant multiple $y = c/x$ is not a solution of the equation when $c \neq 0$, and $c \neq \pm 1$.

5. Functions $y_1 = 1$ and $y_2 = \text{Inx}$ are solutions of the non-linear differential equation $y'' + (y')^2 = 0$ on the interval]0, [. Then

   a) is $y_1 + y_2$ a solution of the equation?
   b) is $c_1y_1 + c_2y_2$, a solution of the equation, where c_1 and c_2 are arbitrary constants?

6. Solve the following equations:

   a) $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$
   b) $9\frac{d^2y}{dx^2} + 18\frac{dy}{dx} - 16y = 0$
   c) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0$

7. In the following equations find the solution y for x = 1:

   a) $(d^2 - 2d - 3)Y = 0$; when $x = 0$, $y = 4$ and $y' = 0$
   b) $(D^3 - 4D)y = 0$, when $x = 0$, $y = 0$ $y' = 0$ and $y'' = 2$.

8. Find the complete primitive of the following equations:

   a) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = 0$
   b) $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4Y = 0$
   c) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0$
   d) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

9. Find the general solution of the following equations subject to the conditions mentioned alongside:

   a) $(D^2 + 4d + 4)y = 0$; when $x = 0$, $y = 1$ and $y' = -1$

b) $(D^3 - 3D - 2)\, y = 0$; when $x = 0$, $y' = 9$ and $y'' = 0$

c) $(D^4 + 3D^3 + 2D^2)\, y = 0$; when $x = 0$, $y = 0$, $y' = 4$, $y'' = -6$, $y'' = 14$.

10. Find the general solution of the following Equations:

a) $\frac{d^2y}{dx^2} - 2\,\square\,\frac{dy}{dx} + (\square^2 + \square^2)\, y = 0$

b) $\frac{d^4y}{dx^4} + a^4y = 0$

c) $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

7.0 REFERENCES/FURTHER READINGS

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UNIT 2 METHOD OF UNDETERMINED COEFFICIENTS

CONTENTS

1.0 INTRODUCTION

In Unit 5, we learnt that in order to find the complete integral of a general non-homogeneous linear differential equation, namely

$$L(y) = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = b(x) \qquad \ldots(1)$$

where ao, a,, ... , a, are constants, it is necessary to find a general solution of the corresponding homogeneous equation that is, the complementary function and then add to it any particular solution of Eqn. (1). In Sec. S.3 we discussed the methods of determining complementary function of linear differential equations with constant coefficients having auxiliary equations with different types of roots. But how do we find a particular solution of these equations? We shall now be considering this problem in this unit.

Variety of methods exist for finding particular integral of a non-homogeneous linear differential equations. The simplest of these methods is the method of undetermined coefficients. Basically, this method consists in making a guess as to the form of trial solution and then determine the coefficients involved in the trial solution so that it actually satisfies the given equation. You may recall that we had touched upon this method in Sec. 3.3 of Unit 3 for finding the particular integral of non-homogeneous linear differential equations of the first order having constant coefficients. In this unit we shall be discussing this method in general for finding the particular integral of second and higher order linear differential equations with constant coefficients.

2.0 OBJECTIVES

After studying this unit you should be able to

- identify the types of non-homogeneous terms for which method of undetermined coefficients can be successfully applied.
- write the form of trial solutions when non-homogeneous terms are polynomials, exponential functions or their combinations.
- describe the constraints of this method.

3.0 MAIN CONTENT

3.1 Types of Non-Homogeneous Terms for Which the Method is Applicable

The method of undetermined coefficients, as we have already mentioned in Sec. 6.1, is a procedure for finding particular integral y^ in a general solution y(x) = yc(x) °f equations of the form (1). The success of this method is based on our ability to guess the probable form of particular solutions.

We know that the result of differentiating functions such as x^r ($r > 0$, an integer) an exponential function $e^{\alpha x}$ (α constant) or sinmx or cosmx (m constant) is again a polynomial, an exponential or a linear combination of sine or cosine functions respectively. Hence, if the non-homogeneous term b(x) in Eqn. (1) is a polynomial an exponential function, or a sine or cosine function then we can choose the particular integral to be a suitable combination of polynomial, an exponential. I sinusoidal function with a number of undetermined constants. These constant! I then be determined so that the trial solution satisfies the given equation.

Note: A function which is a combination of a sine function (or cosine function) with an exponential function and/or a polynomial is a sinusoidal function.

Thus the types of non-homogeneous term for which the method of undetermined coefficients is successfully applicable are

i) polynomials
ii) exponential functions
iii) sine or cosine functions
iv) a combination of the terms of types (i), (ii) and (iii) above.

We shall now discuss the method of undetermined coefficients to find the particular integral for these various types of non-homogeneous terms one by one.

3.1.2 Non-homogeneous term is an Exponential Function:

Let us suppose that the non-homogeneous term b(x) in Eqn. (1) is an exponential function of the form $e^{\alpha x}$ (α a constant).

In other words, suppose we have to solve an equation.

$$L(y) = a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_{n-1}\frac{dy}{dx} + a_n y = e^{\alpha x} \qquad \ldots(6)$$

The appropriate form of the trial solution can be taken as
$y_p(x) = Ae^{\alpha x}$ …(7)
provided $e^{\alpha x}$ is not a solution of the homogeneous differential equation corresponding to Eqn. (1) (i.e., α is not a root of the auxiliary equation).

If α is a root of Eqn. (6), then the choice (7) would not give us any information for determining the value of A. in that case, we can take $y_p(x) = Axe^{\alpha x}$ as the trial solution. If α is r-times repeated root of the auxiliary equation, then the suitable form of the trial solution for determining particular integral will be

$$y_p(x) = Ax^r e^{\alpha x} \qquad \ldots(8)$$

substituting this value of y_p in Eqn. (6) and equating coefficients of $e^{\alpha x}$ on both sides, we can find the value of undermined coefficient A and thus find the particular integral (8).

For a better understanding of whatever we have discussed above let us take up a few examples.

Example 3: Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 3e^x.$$

solution: Auxiliary equation is
$(m + 1)(m + 2) = 0$
$\Rightarrow m = -1, -2,$
$\therefore$C.F. = $c_1e^{-x} + c_2e^{-2x}$
Since e^x is not a part of the complementary function, hence trial solution for finding a particular integral can be taken as
$y_p(x) = Ae^x$
Substituting this value of y_p in the given differential equation,
we get
$2Ae^x + 3Ae^x + Ae^x = 3e^x$
$\Rightarrow 6Ae^x = 3e^x$
Equating coefficient of e^x on both sides, we get
$6A = 3, \Rightarrow A = \frac{1}{2}$
hence
P. I = $\frac{1}{2}e^x$
$\therefore$The general solution fo the given differential equation is
$y = c_1e^{-x} + c_2e^{-2x} +$ ---- e^x
let us consider another example which illustrate the case of repeated roots of an auxiliary equation.

Example 4: solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 12e^x$

Solution: Auxiliary equation is
$(m - 1)^3 = 0$
$\Rightarrow m = 1, 1, 1.$
$\therefore$C.F. = $(c_1 + c_2x + c_3x^2)\,e^x$
since non-homogeneous term of the given differential equation is e^x which is present in the complementary function and moreover I is 3-times repeated root of the auxiliary equation, we take the form of trial solution to be
$y_p(x) = Ax^3e^x$.

Note that in the selection of the trial solution $y_p(x)$ no smaller power of x will give us the particular integral. Moreover, it is not similar to any term of complementary function of the given equation.
On substituting this value pf y_p in the given differential equation, we get
$- Ax^33^x + 3A\,[x^3e^x + 3x^2e^x] - 3A\,[x^3e^x + 6x^2e^x\ 6xe^x]$
$+ A\,[x^3e^x + 9x^2e^x + 18xe^x + 6e^x] = 12e^x$

Equating coefficients of e^x on both sides, we get
$6A = 12, \Rightarrow a = 2.$
Thus, P.I. = $2x^3e^x$
$\therefore$The general solution of the given differential equation is
$y = (c_1 + c_2x + c_3x^2)\, e^x + 2x^3e^x$
And now an exercise for you.

You may also come across the situation when b(x) in Eqn. (1) is a sum of two or more functions. Suppose $b(x) = b_1(x) + b_2(x)$; then from the superposition principle we have the P.I. $y_p(x)$ of $L(y) = b(x)$ to the equal to $y_p = y_{p1} + y_{p2}$,
where y_{p1} is a P.I. of $L(y) = b_1(x)$ and y_{p2} is a P.I. of $L(y) = b_2(x)$. This enables us to decompose the problem of solving linear equation $L(y) = b(x)$ into simpler problem an example.

Example 5: find a general solution of

$$\frac{d^2y}{dx^2} - 2\,\frac{dy}{dx} - + y = e^x + 4.$$

Solution: Auxiliary equation is
$M^2 - 2m + 1 = 0$
$\Rightarrow (m - 1)^2 = 0$
$\Rightarrow m = 1, 1.$
$\therefore$C.F. = $(c_1 + xc_2)\, e^x$
To find the particular solution we first consider equation
$+ y = e^x$ …(9)

1 is a repeated root of the auxiliary equation, we consider the trial solution
$y_{p1} = Ax^2e^x$

on substituting y_{p1} in Eqn. (9), we find that
$(2Ae^x + 4xAe^x + x^2Ae^x) - 2(2xAe^x + x^2Ae^x) + Ax^2e^x = e^x$

comparing the coefficient of e^x on both sides, we have
$2Ae^x = e^x$
$\Rightarrow A = \frac{1}{2}$
$\therefore y_{p1} \frac{x^2}{2}\, e^x$
Now consider the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} \quad y = 4 \qquad ...(10)$$

Since no-homogeneous term is a constant, we try y_{p2} = A and find that A = 4 satisfies (10). Hence a particular solution of the given equation is

$$y_p = y_{p1} + y_{p2} = \frac{x^2}{2}e^x = 4$$

A general solution will them be

$$y = c_1e^x + c_2xe^x + 4 + \frac{x^2}{2}e^x$$

The term b(x) can be a combination of many more terms like this. We may have $b(x) = x + e^x$, $b(x) = x + x^3$, $b(x) = 3 + x^2$ etc. In these cases, we can obtain particular integral using I and II discussed above and by finding y_{p1} and y_{p2} as we have done in Example 5.

We shall give you the general method of finding P.I. when we discuss cases IV and V.

You may try these exercises.

We can now take up the case when b(x) in Eqn. (1) is either a sine or a cosine function.

3.1.3 Non-homogeneous Term is a Sine or a Cosine Function

After going through I and II above and attempting the exercises given so far, you know how to handle b(x) when it is polynomial, an exponential function or a combination of both. Now can you say how this case is handled when b(x) is a sine or a cosine function?

We know that the linear differential operator when applied to $\sin\beta x$ or $\cos\beta x$ will yield a linear combination of $\sin\beta x$ and $\cos\beta x$. Therefore, if the non-homogeneous term b(x) of differential Eqn. (1) is of the form

$$B(x) = \alpha_1\sin\beta x \text{ or } \alpha_2\cos\beta x \text{ or } \alpha_1\sin\beta x + \alpha_2\cos\beta x$$

We can take the trial solution in the form

$$y_p(x) = A\cos\beta x + B\sin\beta x \qquad \ldots(11)$$

provide $\pm i\beta$ are not roots of the auxiliary equation corresponding to the given differential equation.

If $\pm i\beta$ are r-times repeated roots of the auxiliary equation, then we can take the form of trial solution to be
$y_p(x) = x^r(A\cos\beta x + B\sin\beta x)$ …(12)

We then substitute the value of $y_p(x)$ in the form (11) or (12), whichever is applicable in Eqn. (1) and equate the coefficients of $\sin\beta x$ and $\cos\beta x$ on both sides of the resulting equation. This gives us equations for obtaining the values of A and B in terms of known quantities. Knowing the values of A and B, particular integral of Eqn. (1) is obtained from relations (11) or (12).

We now illustrate this theory with the help of a few examples.

Example 6: Find the general solution of

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = \sin x$$

Solution: Auxiliary Equation is
$(m^4 - 2m^2 + 1) = 0$
$\Rightarrow (m^2 - 1)^2 = 0$
$\Rightarrow m = 1, 1, -1, -1$
$\therefore$ C.F. $= (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$

Since ------ i is not a root of the auxiliary equation, that is, term sinx does not appear in the complementary function, we can take the trial solution in the form
$y_p(x) = A\sin x + B\cos x$.
Substituting this value of y_p in the given differential equation, we get
$(A\sin x + B\cos x) - 2(-A\sin x - B\cos x) + (A\sin x + B\cos x) = \sin x$
$\Rightarrow 4A\sin x + 4B\cos x = \sin x$
Equating coefficients of sinx and cosx on both sides, we get

$4A = 1 \Rightarrow A = \frac{1}{4}$
and $4B = 0 \Rightarrow B = 0$
Thus, $y_p(x) = \frac{1}{4}\sin x$
and the complete solution of the differential equation is

$y = (c_1 + c_2x)\, e^x + (c_3 + c_4x)\, e^{-x} + \frac{1}{4}$ sinx.

Let us look at another example.

Example 7: Solve the initial value problem

$$\frac{d^2y}{dx^2} + y = 2\cos x,\ y(0) = 1,\ y'(0) = 0$$

Solution: The auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = \pm\, i$

$\Rightarrow$C.F. = $c_1\cos x + c_2\sin x$

Now since $\pm\, i$ is a root of the auxiliary equation i.e., cosx itself appears in the complementary function, we take the form of the trial solution as

$y_p(x) = x\,(A\sin x + B\cos x)$

Substituting the value of $y_p(x)$ in the equation, we get

$2(A\cos x - B\sin x) + x(-A\sin x - B\cos x) + x(A\sin x + B\cos x) = 2\cos x$

$\Rightarrow 2A\cos x - 2B\sin x = 2\cos x$

Comparing the coefficients of sinx and cosx on both sides, we get

$2A = 2 \Rightarrow A = 1$ and $B = 0$.

Therefore,

$y_p(x) = x\sin x$

and the general solution is

$y(x) = c_1\cos x + c_2\sin x + x\sin x$

we now use initial conditions to determine c_1 and c_2

Now $y(0) = 1$ gives $c_1 = 1$

And $y'(0) = 0$ gives $c_2 = 0$

Thus, $y(0) = \cos x + x\sin x$

You may now try the following exercises.

In the example considered so far, did you notice that the function b(x) itself suggested the form of the particular solution $y_p(x)$? in fact, we can expand the list of functions b(x) for which the method of undetermined coefficients can be applied to include products of these functions as well. We now discuss such cases.

3.1.4 Non-homogeneous Term is a Product of an Exponential and a Polynomial

Let us suppose that b(x) is of the form

$b(x) = e^{\alpha x}[b_0x^k + b_1x^{k-1} + \ldots. + b_{k-1}x + b_k] = e^{\alpha x}P_k(x)$

with this form of b(x), Eqn. (1) reduces to

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots. + a_{n-1}\frac{dy}{dx} + a_n = e^{\alpha x}\ [b_0x^k + b_1x^{k-1} + \ldots + b_{k-1}x + b_k] \qquad \ldots(13)$$

We now take the trial solution in the form

$$y_p(x) = e^{\alpha x}\ [A_0x^k + A_1x^{k-1} + \ldots. + A_{k-1}x + A_k] \qquad \ldots(14)$$

provide α is not a root of the auxiliary equation corresponding to Eqn. (13). If α is a root of the auxiliary equation, say, it is r-times repeated root of the auxiliary equation then we modify the trial solution as

$$y_p(x) = x^r e^{\alpha x}\ [A_0x^k + A_1x^{k-1} + \ldots + A_{k-1}x + A_k] \qquad \ldots(15)$$

Remember that in Eqn. (15) no smaller power of x will yield a particular integral. Here r is the smallest positive integer for which every term in the trial solution (15) will differ from every term occurring in the complementary function corresponding to Equ (13).

In order to determine the constants $A_0, A_1, \ldots., A_k$ we substitute $y_p(x)$ in the form (14) or (15) as the case may be in eqn. (13) and then compare the coefficients of $e^{\alpha x}$ on both sides. For a better understanding of whatever we have discussed above, let us consider an example.

Example 8: Solve $\frac{d^3y}{dx^3} - \frac{dy}{dx} = xe^{-x}$

Solution: Auxiliary equation is

$m^3 – m = 0$

$\Rightarrow m(m^2 –1) = 0$

$\Rightarrow m = 0, - 1, 1$

$\therefore$ C.F. $= c_1 + c_2e^{-x} + c_3e^x$

Here the non-homogeneous term is xe^{-x} appears in the complementary function. Further, (-1) is a non-repeated root of the auxiliary equation. Thus, we take the form of trial solution as

$y_p(x) = x [B + Ax] e^{-x} = Ax^2e^{-x} = Ax^2e^{-x} + Bxe^{-x}$

substituting this value of y_p in the given differential equation, we get

$$- A [-x^2e^x + 2xe^{-x}] + A [-x^2e^{-x} + 6xe^{-x} – 6e^{-x}] – B (-xe^{-x} + e^{-x}) + B(-xe^{-x} + 3e^{-x}) = xe^{-x}$$

Comparing the coefficients of xe^{-x} and e^{-x} on both sides, we get

$4A = 1 \Rightarrow A = \frac{1}{4}$

and $-6A + 2B = 0 \Rightarrow B = \frac{3}{4}$

Hence $y_p(x) = \frac{1}{4} x^2e^{-x} + \frac{3}{4} xe^{-x} = \frac{e^{-x}}{4} (x^2 + ex)$

And the general solution is

$$y = c_1 + c_2e^{-x} + \frac{e^{-x}}{4}(x^2 + 3x).$$

You may now try the following exercise.

Lastly, we take up the case when b(x) is a product of a polynomial, an exponential function and a sinusoidal function.

3.1.5 Non-homogeneous Term is a Product of a Polynomial, an Exponential and a Sinusoidal function

Let us suppose that the non-homogeneous term b9x) in Eqn. (1) has one of the following two forms:

$$b(x) = e^{\alpha x} P_k(x) \sin\beta x \text{ or } b(x)\ e^{\alpha x}\ P_k(x) \cos\beta x, \qquad \ldots(16)$$

where $P_k(x)$, as given by Eqn. (2), is apolynomial of degree k or less and α and β are any real numbers. You may recall Euler's formula and write

$e^{(\alpha + i\beta)x} = e^{\alpha x} \cos\beta x + i\, e^{\alpha x} \sin\beta x$

or, equivalently, we have

$$e^{\alpha x} \cos\beta x = \text{Real}\,(e^{(\alpha + i\beta)x}) = \frac{e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x}}{2}$$

and $$e^{\alpha x} \sin\beta x = \text{Imaginary}\,(e^{(\alpha + i\beta)x}) = \frac{e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}}{2i}$$

Hence b(x) in Eqn. (16) reduces to

$$b(x) = (b_0x^k + b_1x^{k-i} + \ldots + b_k) \left\{\frac{e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}}{2i}\right\}$$

or $b(x) = (b_0x^k + b_1x^{k-1} + \ldots + b_k)\left\{\dfrac{e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x}}{2}\right\}$

In either of the above two cases, we take the trial solution in the form

$y_p(x) = (A_0x^k + A_1x^{k-1} + \ldots + A_k)\,(e^{(\alpha+i\beta)x})$

$\qquad (B_0x^k + B_1x^{k-1} + \ldots + B_k)\,(e^{(\alpha-i\beta)x})$

or, equivalently,

$y_p(x) = (A_0x^k + A_1x^{k-1} + \ldots + A_k)\, e^{\alpha x}\cos\beta x +$

$\qquad (B_0x^k + B_1x^{k-1} + \ldots + B_k)\, e^{\alpha x}\sin\beta x,$

provided $\alpha \pm i\beta$ is not a root of the auxiliary equation.

If $(\alpha \pm i\beta)$ is r-times repeated root of the auxiliary equation, we can then modify the trial solution by multiplying it by x^r. We then substitute the trial solution in Eqn. (1) and equate the coefficients of like terms on both sides to determine A_0, A_1, …., A_k and B_0, B_1, …., B_k. Substituting these values of undetermined coefficients in the trial solution, we get the particular integral.

Let us now illustrate the above case with the help of a few examples.

Example 9: find the appropriate form of trial solution for the differential equation

$$\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 3e^x + 2xe^{-x} + e^{-x}\sin x$$

Solution: Auxiliary equation is

$m^4 + 2m^3 + 2m^2 = 0$

$\Rightarrow m^2(m^2 + 2m + 2) = 0$

$\Rightarrow m = 0, 0, -1 \pm i$

$\therefore$C.F. $= c_1 + c_2x + e^{-x}(c_3\sin x + c_4\cos x)$

Here the non-homogeneous term is $3e^x + 2xe^{-x} + e^{-x}\sin x$

Since the term $e^{-x}\sin x$ also appear in C.F., the appropriate form of the trial solution is

$y_p = Ae^x + (Bx + C)\,e^{-x} + xe^{-x}(D\cos x + E\sin x).$

We now take up an example in which b(x) is a product of a polynomial an exponential and a sinusoidal function.

Example 10: Write down the form of the trial solution for the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} \; 5y = x^2e^{-x} \sin x$$

Solution: the auxiliary equation is
$m^2 + 2m + 5 = 0$
$\Rightarrow m = -1 \pm 2i$
The roots are not equal to $-1 \pm i$. Hence the for of trial solution is
$y_p = (A_0x^2 + A_1x + A_2)\, e^{-x}\cos x + (B_0x^2 + B_1x + B_2)\, e^{-x}\sin x$

Note that the form of trial solution taken in Case v above is the most general form. This is because the trial solutions taken in Cases I –IV are particular forms of Case V.

And now some exercise for you.

After going through the Cases I – V above and attempting the exercises given, you must have understood the method of undetermined coefficient quite well. Did you make certain observations about the method? Let us now summarize the observations and constraints of this method.

3.2 Observations and Constraints of the Method

1) Method is straight forward in application.
2) It can be used by any learner who is not familiar with more elegant techniques of finding the solutions f the differential equations such as inverse operators and variation of parameters, which involve integrations and which we shall be discussing in the subsequent units.
3) Success of this method depends to a certain extent on the ability to guess an appropriate form of the trial solution.
4) If the non-homogeneous term is complicated and the trial solution involves a large number of terms, then determination of coefficients in the trial solution becomes laborious.
5) This method is not a general method of finding the particular solution of differential equations. It is applicable to linear non-homogeneous equations with **constant coefficients and with restricted** forms of the no-homogeneous terms.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit, we have covered the following:

1) Method of undetermined coefficients is applicable if
   a) The equation is a linear equation with constant coefficients.
   b) The non-homogeneous term is either a polynomial, an exponential function, a sinusoidal function or a product of these functions.
2) The results giving trial solutions corresponding to different non-homogeneous terms in the equation L(y) = b(x), where the equation L(y) = 0 has r-times repeated roots are summarized in the following table.

Non-homogeneous term b(x)	Trial solution, $y_p(x)$
$P_p(x) = b_0x^k + b_1x^{k-1} + \ldots + b_{k-1}{}^{x+b}{}_k$	$x^r(A_0x^{k-1} + \ldots + A_k)$
$e^{\alpha x}$	$x^r(Ae^{\alpha x})$
$\begin{cases}\sin\beta x \\ \cos\beta x\end{cases}$	$x^r(A\sin\beta x + B\cos\beta x)$
$e^{\alpha x}\ P_k(x)$	$x^r e^{\alpha x}(A_0x^k + \ldots + A_k)$
$e^{\alpha x} P_k(x) \begin{cases}\sin\beta x \\ \cos\beta x\end{cases}$	$x^r[(A_0x^k + \ldots + A_k)e^{\alpha x}\sin\beta x$ $+ (B_0x^k + \ldots) + B_x)e^{\alpha x}\cos\beta x]$

3) Observations and constrains of the method.

6.0 TUTOR MARKED ASSIGNMENT

1. Find a form of particular integral of the following equations

   a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2 + 1$

   b) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2$

2. Determine the general solution of the following equations.

   a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$

   b) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = x$

3. Find a particular integral of the following differential equations.

a) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = e^{-2x}$

b) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} + 1 = e^{-x}$

4. Find a general solution of the following differential equations:

a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2 (e^x + e^{-x})$

b) $2\frac{d^2y}{dx^2} + 8y = x^3 + e^{2x}$

5. Solve the following initial value problems:

a) $\frac{d^2y}{dx^2} - y = e^{2x}$. $y(0) = -1$, $y'(0) = 1$.

b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y + e^{2x} = 0$, $y(0) = y'(0) = 0$.

6. Solve the following equations:

a) $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = \sin 2x$

b) $\frac{d^3y}{dx^3} - \frac{dy}{dx} = 2\cos x$

7. Solve the following initial value problems:

a) $\frac{d^2y}{dx^2} + 4y = \sin x$, $y(0) = 2$, $y'(0) = -1$

b) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \cos x - \sin 2x$, $y(0) = \frac{-7}{20}$, $y'(0) = \frac{1}{5}$

8. Solve the following equations:

a) $\frac{d^2y}{dx^2} + 9y = x^2 e^{3x}$

b) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 4x\,e^{2x}$

9. Write the form of the trial solution for each of the following:

a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = x\cos 3x - \sin 3x$

b) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = xe^{-x}\cos 2x$

c) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = xe^{x}\cos 2x$

d) $\frac{d^2y}{dx^2} + y = x^2 \sin x$

e) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = xe^{2x}\sin x$

10. Find the general solution of the following equations.

a) $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = x + 3\cos x + e^{-2x}$

b) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} = x^2 + 4 + x\sin x$

c) $\frac{d^3y}{dx^3} + \frac{dy}{dx} = x^3 + \cos x$

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UNIT 3 METHOD OF VARIATION OF PARAMETERS

CONTENTS

1.0 INTRODUCTION

In unit 6, we discussed the method of undetermined coefficients for determining particular solution of the differential equation with constant coefficients when its non-homogeneous term is of a particular form (viz, a polynomial, an exponential, a sinusoidal function etc).

In this unit we familiarize you with an alternative approach fir determining a particular solution that can be applied even when the coefficients of the differential Equation are functions of the independent variable and the non-homogeneous term may not be of a particular form. Such an approach is due to Joseph Louis Lagrange (1736 – 1813) and is termed as variation of parameters. Even though the approach is quite general bit is limited in its scope in the sense that it can be utilized in situations where the fundamental solution set for the reduced equation is known. Also, it can be used for first and higher order equations alike though its appreciation can be well understood for the later set of equations. The method requires for its applicability the complete knowledge of fundamental solution set of the reduced equation and for equations with variable coefficients the determination of this set may be extremely difficult. In the case of linear differential equations with variable coefficients, at times, it may not be possible to find all linearly independent solutions of the reduced but at least one or more may be obtainable. For such situations Jean le Rond d'Alembert (1717 – 1783), a French mathematician and a physicist, developed a method that is often called the method of **reduction of order.** When one or more solutions of reduced equation are known that D'Alembert's method can be used to derive an equation of order lower than that of a given equation and obtain the rest of the solutions of a reduced equation as well as the particular integral of the non-homogeneous term. We shall be discussing the method of reduction of order in Sec. 7.3 of the unit. For some particular forms of the second order linear differential equations with variable coefficients, we have also listed some rule by which one integral of the homogeneous equation an be guessed.

However, there exist linear differential equation with variable coefficients if second and higher order for which we may not be able to guess any integral of its complementary function. But, among such equations is a class of equations known as Euler's equation or homogeneous linear differential equations, where, by certain substitution, it is possible to find all the integrals of its complementary function. In Sec. 7.4, we shall be discussing the method of solving Euler's equations and those equations which are reducible to Euler's form.

2.0 OBJECTIVES

After reading this unit you should be able to

- Use the method of variation of parameters to find particular integral of non-homogeneous linear differential equations with constant or variable coefficients.
- Use the method of reduction of order to find the complete integral of the linear non-homogeneous equation of second order when one integral of the corresponding homogeneous equation is known.
- Write down one integral for second order linear homogeneous differential equation with variable coefficients in certain cases merely through inspection.
- Solve Euler's equations.

3.0 MAIN CONTENT

3.1 Variation of Parameters

Let us not discuss the details of the method by considering the non-homogeneous second order linear equation.

$L[y] = y'' + a_1(x)\, y' + a_2(x)y = b(x),$ …(1)

Where we have taken the coefficients of y'' to be 1 and $a_1(x)$, $a_2(x)$, and $b(x)$ are defined and continuous on some interval J. Let $[y_1(x), y_2(x)]$ be a fundamental solution set for the corresponding homogeneous equation

$L[y] = 0$ …(2)

Then we know that the general solution of (2) is given by

$Y_c(x) = c_1y_1(x) + c_2y_2(x),$ …(3)

Where c_1 and c_2 are constants. To find a particular solution of the non-homogeneous equation, the idea associated with the method of variation of parameters is to replace the constants in Eqn. (3) by function of x. That is, we seek a solution of Eqn. (1) of the form

$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$ …(4)

where $u_1(x)$ and $u_2(x)$ are unknown functions to be determined. Since we have introduced two unknowns, we need two equation involving these functions for their determination.

In other words, we impose two conditions which the functions u_1 and u_2 must satisy in order that relation (4) is a solution of Eqn. (1). We call these conditions the **auxiliary conditions.** These conditions are imposed in such a way that the calculations are simplified. Let us see how this is done.

Now if relation (4) is a solution of Eqn. (1), then it must satisfy it. Thus, first we compute $y'_p(x)$ and $y''_p(x)$ from Eqn. (4).

$y_p' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$ …(5)

To simplified the computation and to avoid second order derivatives for the unknown u_1, u_2 in the expression for y_p'', let us choose the first **auxiliary condition** as

$u'_1 y_1 + u\square y_2 = 0$ …(6)

Thus relation (5) becomes

$y\square = u_1 y\square + u_2 y\square$

…(7)

and

$y\square = u\square y\square + u\square y_1'' + u\square y\square + u_2 y_2''$

…(8)

Substituting in Eqn (1), the expressions for y_p, $y\square$ and $y\square$ as given b Eqn. (4), (7) and (8), respectively, we get

$b(x) = L[y_p]$

$= (u\square y\square + u_1 y_1'' + u\square y\square + u_2 y\square) + a_1(u_1 y_1' + u_2 y_2') + a_2(\square y_1 + \square y_2)$

$= (u_1'\square + u_2' y_2') + \square(y_1'' + a_1\square + a_2 y_1) + u_2(y_2'' + a_1 y_2' + a_2 y_2)$

$= (\square\square + u_2' y_2') + \square L[y_1] + \square L[y_2]$ …(9)

since y_1 and y_2 are the solution of the homogeneous equation, we have

$L[y_1] = L[y_2] = 0$

Thus Eqn. (9) becomes

$\square\square + u_2' y_2' = b(x)$

…(10)

whicg is the **seond auxiliary condition.**
Now if we can find u_1 and y_2 satisfying Eqns. (16) and (10), viz.,

$$\left.\begin{array}{l} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = b(x) \end{array}\right] \qquad \ldots(11)$$

then y_p given by Eqn. (4) will be a particular solution of Eqn. (1). In order to determined u_1, u_2 we first solve the linear system of Eqns (11) for $\square$ and u_2'.
Algebraic manipulations yield

$$\square(x) = \frac{-b(x)y_2(x)}{W(y_1, y_2)},\ u_2'(x) = \frac{b(x)\ y_1(x)}{W(y_1, y_2)}, \qquad \ldots(12)$$

where

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1\ y_2' - y_2 \square$$

is the **Wronskian** of $y_1(x)$ and y_2 (x).

Note that this Wronskian is never zero on J, because $\{y_1, y_2\}$ is a fundamental solution set.
On integrating $\square$(x) and $\square$given by Eqn. (12), we obtain

$$y_p(x) = \int \frac{-b(x)\ y_2(x)}{W(y_1, y_2)}\, dx,\ u_2(x) = \int \frac{b(x)\ y_1(x)}{W(y_1, y_2)}\, dx \qquad \ldots(13)$$

Hence

$$y_p(x) = y_1(x) \int \frac{-b(x)\ y_2(x)}{W(y_1, y_2)}\, dx + y_2(x) \int \frac{b(x)\ y_1(x)}{W(y_1, y_2)}\, dx \qquad \ldots(14)$$

is a particular integral of Eqn. (1).

We now sum up the various steps involved in determining a particular solution of Eqn. (1).

Step 1: Find a fundamental solution set $\{y_1(x), y_2(x)\}$ for the corresponding homogeneous equation.

Step II: Assume the particular integral of Eqn (1) in the form
$y_p(x) = u_1(x)\ y_1(x) + u_2(x)\ y_2(x)$
and determine $u_1(x)$ and $u_2(x)$ by using the formula (13) directly or by first solving the system of Eqns. (11) for ------- and then integrating.

Setp III: Substitute $u_1(x)$ and $u_2(x)$ into the expression for $y_p(x)$ to obtain a particular solution.

We now illustrate these steps with the help of the following examples.

Example 1: Determine the general solution of the differential equation

$$\frac{d^2y}{dx^2} + y = \sec x,\ 0 < x < \frac{\pi}{2}$$

Solution: Step I: The auxiliary equation corresponding to the given equation is
$m^2 + 1 = 0$
$\Rightarrow m = \pm\ i$
and the two solutions of the reduced equation are
$y_1(x) = \cos x$
and
$y_2(x) = \sin x.$
Hence the complementary function is given by
$Y_c(x) = c_1\cos x + c_2\sin x.$

Step II: To find particular integral, we write

$$y_p(x) = u_1(x) \cos x + u_2(x)\sin x \qquad \ldots(15)$$

$$\therefore \frac{dy_p}{dx} = [-u_1(x)\sin x + u_2(x)\cos x] + \frac{du_1}{dx} \cos x + \frac{du_2}{dx} \sin x$$

Let us take the first auxiliary condition as

$$\frac{du_1}{dx} \quad \cos x + \frac{du_2}{dx} \sin x = 0 \qquad \ldots(16)$$

So that

$$\frac{dy_p}{dx} = - u_1(x)\sin x + u_2(x)\cos x$$

Differentiating the above equation once again, we get

$$\frac{dy_p}{dx} = - u_1(x)\cos x = u_2(x)\sin x - \sin \frac{du_1}{dx} + \cos x \frac{du_2}{dx} \qquad \ldots(17)$$

Since $y_p(x)$ must satisfy the given equation, we substitute in the given equatin the expression for y_p and ----- fom Eqns. (15) and (17), respectively, and obtain

$$- \sin x \frac{du_1}{dx} + x\mathrm{os}x \frac{du_2}{dx} = \sec x \qquad \ldots(18)$$

On solving Eqns. (16) and (18) for $\frac{du_1}{dx}$ and $\frac{du_2}{dx}$, we get

$\frac{du_1}{dx} = -\tan x$, $\frac{du_2}{dx} = 1$,

which on integration yields

$u_1(x) = \text{In}(\cos x)$ and $u_2(x) = x$

Step III: Substituting the values of $u_1(x)$ and $u_2(x)$ in Eqn. (15) we obtain a particular solution of the given equation in the form

$y_p(x) = \cos x \, \text{In}(\cos x) + x\sin x$

and the general solution is

$y = c_1\cos x + c_2\sin x + x\sin x + \cos x \, \text{In}(\cos x)$

Note that in Eqn. (1) we have taken the coefficients of y'' to be 1. if the given equation is of the form $a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)$, then before applying the method it must be put in the form $y'' + p(x)y' + q(x)y = g(x)y = g(x)$ as we have done in the following example.

Example 2: Find the general solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} = f(x) \qquad \ldots(19)$$

Solution: Step 1: We first rewrite the given equation in the form

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\frac{dy}{dx} = \frac{f(x)}{(1-x^2)}$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} - \frac{1}{x(1-x^2)}\frac{dy}{dx} = 0 \qquad \ldots(20)$$

This is a first order equation in $\frac{dy}{dx}$. To solve this we put $\frac{dy}{dx} = p$. Then Eqn. (20) reduces to

$$\frac{dp}{dx} - \frac{1}{x(1-x^2)}p = 0$$

$$\text{or } \frac{1}{p}dp = \frac{dx}{x(1-x^2)} \qquad \ldots(21)$$

Now Eq. (21) is in variable separable form and can be expressed as

$$\frac{dp}{p} = \left[\frac{1}{x} + \frac{x}{1-x^2}\right]dx$$

Integrating we get

$$P = \frac{c_1 x}{\sqrt{1-x^2}}$$

or $$\frac{dy}{dx} = \frac{c_1 x}{\sqrt{1-x^2}} \qquad \ldots(22)$$

Integrating Eqn. (22), once again, we get the solution of Eqn. (20) in the form

$$Y_c(x) = -\sqrt{1-x^2} + c_2 \qquad \ldots(23)$$

where c_1 and c_2 are arbitrary constants.

Step II: For the given differential equation, assume a particular solution in the form

$$y_p(x) = u_1(x)\sqrt{1-x^2} + u_2(x)$$

$$\therefore \frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}}\, u_1 + \left[\sqrt{1-x^2}\,\frac{du_1}{dx} + \frac{du_2}{dx}\right]$$

We choose the first auxiliary condition as

$$\sqrt{1-x^2}\,\frac{du_1}{dx} + \frac{du_2}{dx} = 0 \qquad \ldots(24)$$

Then

$$\frac{dy_p}{dx} = \frac{-x}{\sqrt{1-x^2}}\, u_1$$

and $$\frac{d^2y_p}{dx^2} = -\frac{1}{(1-x^2)^{3/2}}\, u_1 - \frac{1}{\sqrt{1-x^2}}\,\frac{du_1}{dx}$$

Substituting, from above, the expressions for y_p' and y_p'' in Eqn. (19), we get

$$-x\sqrt{1-x^2}\,\frac{du_1}{dx} = f(x) \qquad \ldots(25)$$

as our second auxiliary condition.

Solving Eqns. 924) and (25) for [illegible] and [illegible] and integrating, we get

$$U_1(x) = -\int \frac{f(x)}{x\sqrt{1-x^2}}\,dx \text{ and } u_2(x) = \int \frac{f(x)}{x}\,dx$$

Step III: The expressions for $u_1(x)$ and $u_2(x)$ when substituted in $y_p(x)$ gives a particular integral in the form

$$Y_p(x) = -\sqrt{1-x^2}\int \frac{f(x)}{x\sqrt{1-x^2}}\,dx + \int \frac{f(x)}{x}\,dx$$

Hence a general integral o fthe given differential equation is

$$Y = - c_1 \boxed{\sqrt{1-x^2}} + c_2 - \boxed{\sqrt{1-x^2}} \int \frac{f(x)}{x\sqrt{1-x^2}}\, dx + \int \frac{f(x)}{x}\, dx$$

You may now try the following exercises.

If you have carefully gone through Example 1 and 2 above, and also attempted 1. and 2., you will find that the results of second order non-homogeneous linear differential equations can be put in the form of the following theorem.

Theorem 1: If the functions $a_0(x)$, $a_1(x)$, $a_2(x)$ and $b(x)$ are continuous on some interval J and if y_1 and y_2 are the linearly independent solutions of the homogeneous equations associated with the differential equation

$$A_0(x) \boxed{\frac{d^2y}{dx^2}} + a_1(x) \boxed{\frac{dy}{dx}} + a_2(x)y = b(x), \qquad \ldots(26)$$

then a particular solution of Eqn. (26) is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x)\ b(x)}{a_0(x)\ W(y_1, y_2)}\, dx + y_2(x) \int \frac{y_1(x)\ b(x)}{a_0(x)\ W(y_1, y_2)}\, dx \qquad \ldots(27)$$

where $w(y_1, y_2)$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

Remark: In using the method of variation of parameters for finding a particular integral of a given equation, it is advisable to choose a particular integral $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, and then proceed to find $u_1(x)$ and $u_2(x)$ as we have done in Examples 1 and 2 above. It is usually avoided to memorise formulas given by Eqns. (13) or (27). But since the procedure involved is somewhat long and complicated and moreover, it may not always be easy or even possible to evaluate the integrals involved, these formulas turn out to be useful. In such cases, the formulas for $y_p(x)$ provide a starting point for the numerical evaluation of $y_p(x)$.

The method of variation of parameters which we have discussed for non-homogeneous second order equations can be easily generalized to nth order equations of the form

$$A_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n(x)y = b(x)$$

where $a_0(x)$, $a_1(x)$,……, $a_n(x)$, $b(x)$ are continuous in some interval J. The learner interested into the details of the method for a higher order equation may refer to the Appendix at the end of the Unit. We shall not be giving the details at this stage but, however, illustrate it through an example.

Example 3: Find the general solution of

$$\frac{d^3y}{dx^3} - 6 \frac{d^2y}{dx^2} + 11 \boxed{\frac{dy}{dx}} - 6y = e^{2x}$$

Solution: Step I: The auxiliary equation corresponding to the given equation is
$m^3 - 6m^2 + 11m - 6 = 0$
$\Rightarrow (m-1)(m^2 - 5m + 6) = 0$
$\Rightarrow (m-1)(m-2)(m-1) = 0$
thus the linearly independent solutions are
$y_1(x) = e^x,\ y_2(x) = e^{2x},\ y_3(x) = e^{3x}$,
and the complementary function is given by
$y_c(x) = c_1e^x + c_2e^{2x} + c_3e^{3x}$ …(28)

Step II: To find particular integral, we write
$y_p(x) = u_1(x)e^x + u_2(x)e^{2x} + u_3(x)e^{3x}$ …(29)
$\square\ \frac{dy_p}{dx} + (\square e^x + \square e^{2x} + u_3' e^{3x}) + (u_1 e^x + 2u_2e^{2x} + 3u_3e^{3x})$

Let the **first auxiliary condition** be
$\square ex + \square e^{2x} + u_3' e^{3x}) = 0$ …(30)
Thus
$y_p' = u_1e^x + 2u_2e^{2x} + 3u_3e^{3x}$
and
$y_p'' = \square e^x + 2\square e^{2x} + 3u_3' e^{3x}) + (u_1e^x + 4u_2e^{2x} + 9u_3e^{3x)}$
Let us choose the **second condition** as

$\square e^x + 2\square e^{2x} + 3u_3' e^{3x} = 0$ …(31)
Then
$y_p'' = u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}$
$\square\ y_p'' = (u\square e^x + 4\square e^{2x} + 9\square e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$
Substituting the values of y_p, y_p', y_p'' and y_p''' in the given equation, we get

$(\square e^x + 4\square e^{2x} + 9\square e^{3x}) + (u_1e^x + 8u_2e^{2x} + 27u_3e^{3x})$
$- 6(u_1e^x + 4u_2e^{2x} + 9u_3e^{3x}) + 11(u_1e^x + 2u_2e^x + 2u_2e^{2x} + 3u_3e^{3x})$
$- 6(u_1e^x + u_2e^{2x} + u_3e^{3x}) = e^{2x}$ …(32)
$\Rightarrow \square e^x + 4\square e^{2x} + 9\square e^{3x} = e^{2x}$,
which is our **third auxiliary condition**
Thus, we get the system of equations

$$\left.\begin{aligned} u_1'e^x + u_2'e^{2x} + u_3'e^{3x} &= 0 \\ u_1'e^x + 2u_2'e^{2x} + 3u_3'e^{3x} &= 0 \\ u_1'e^x + 4u_2'e^{2x} + 9u_3'e^{3x} &= e^{2x} \end{aligned}\right] \qquad \ldots(33)$$

Solving Eqns. (33) for u_1', u_2' and u_3', we get

$u_1' = \frac{1}{2}e^x$, $u_2' = -1$ and $u_3' = \frac{1}{2}e^{-x}$

Integrating, we get

$u_1 = \frac{1}{2}e^x$, $u_2 = -x$ and $u_3 = \frac{1}{3}e^{-x}$

Step III: We get a particular integral in the form

$y_p(x) = \frac{1}{2}e^{2x} - xe^{2x} - \frac{1}{2}e^{2x} = -xe^{2x}$,

and the general solution is

$y = c_1e^x + c_2e^{2x} + c_3e^{2x} - c_3e^{3x} - xe^{2x,}$

You may now try this exercise.

Clearly the method of variation of parameters has an advantage over the method of undermined coefficients in the sense that it always yields a particular solution y_p provided all the solutions of the corresponding homogeneous equation are known. Moreover, its application is not restricted to particular forms of the non-homogeneous term. In the next section we will discuss a technique which is very similar to the method of variation of parameter.

3.2 Reduction of Order

For a given nth order linear homogeneous differential equation, if one nontrivial solution is known, then the method of reduction of order, as the name suggests, reduces the equation to an (n- 1)th order equation. Thus, if we can find in some way, one or more linearly independent solutions of the reduced equation, we can accordingly reduce the order of the given differential equation. In other words, if p independent solutions of a homogeneous linear corresponding to an nth order equation are known, where $p < n$, then the technique can be used to obtain a linear equation of order (n- p). This fact is particularly interesting when $n = 2$, since the resulting first order equation ca always be solved by the methods we have done in Block 1. That is, if we know one solution of the homogeneous linear differential equation of the second order, we can solve the non-homogeneous equation by the method of reduction of order and obtain both a particular solution and a second linearly independent solution of the homogeneous equation. Let us now see how method works for a second order linear equation.

Consider a second order non-homogeneous equation of the form (1), viz.,

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = b(x),$$

where $a_1(x)$, $a_2(x)$ and $b(x)$ are continuous on some interval J. Suppose that $y = y_1(x)$ is a nontrivial solution of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \qquad \ldots(34)$$

Then $y = cy_1(x)$ is also a solution of Eqn. (34) for some constant c. We now replace the constant c by an unknown function v(x) and take a second trial solution in the form

$y = v(x)y_1(x)$

Now,

$y' = v'y_1 + vy_1'$

$y'' = v''y_1 + 2v'\,y_1' + vy_1''$

substituting from above the expression for y, y' and y" in the given equation, we get

$(v''y_1 + 2v'y_1' + vy_1'') + a_1(v'y_1 + vy_1') + a_2vy_1 = b(x)$

$$\Rightarrow\ v''y_1 + v'\,(2y_1' + a_1y_1) + v(y_1'' + a_1y_1' + a_2y_1) = b(x) \qquad \ldots(35)$$

Since y_1 is a solution of Eqn. 934), the last term on the l.h.s. of Eqn. (35) is zero. Therefore Eqn. (35) reduces to

$$v''y_1 + v'(2y_1' + a_1y_1) = b(x) \qquad \ldots(36)$$

Let $\frac{dv}{dx} = p(x)$, so that Eqn. (36) becomes

$$\frac{dp}{dx} + \frac{2y_1' + a_1y_1}{y_1}\, p = \frac{b(x)}{y_1} \qquad \ldots(37)$$

This is a first order linear differential equation with integrating factor

$$\text{I.F.} = \text{EXP}\left[\int \frac{2y_1' + a_1y_1}{y_1}\,dx\right]$$

Now $\int \frac{2y_1' + a_1y_1}{y_1}\,dx = 2\text{Iny}_1 + \int a_1(x)\,dx$

$\therefore$ I.F. $= y_1^2\ e^{\int a_1(x)dx} = y_1^2\ h(x)$, where $h(x) = e^{\int a\text{--}(x)dx}$

Thus Eqn. (37) reduces to

$y_1^2\, h(x)p(x) = c_1 + \int b(x)\, y_1h(x)\, dx$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{y_1^2 h(x)}\left[c_1 + \int b(x)\ y_1h(x)\ dx\right]$$

Integrating the above equation once again, we obtain

$$V(x) = c_2 + c_1 \int \frac{1}{y_1^2 h(x)}\, dx + \int \frac{1}{y_1^2 h(x)}\left[\int b(x)\ y_1h(x)\ dx\right] dx$$

Thus the general solution of the given equation can be expressed as

$$y = v(x)y_1(x) = c_2y_1(x) + c_1y_1(x) \int \frac{1}{y_1^2 h(x)}$$

$$+ y_1(x) \int \frac{1}{y_1^2 h(x)}\left[\int b(x)\ y_1h(x)\ dx\right] dx \qquad \ldots(38)$$

Note that the function $y_1(x) \int \frac{1}{y_1^2 h(x)}$, in the second term on the r.h.s. of Eqn. (38), is the 2nd linearly independent solution of Eqn. (34) and the last term on the r.h.s. is a particular integral of the given non-homogeneous equation.

We now take up an example to illustrate the theory.

Example 4: Find the general solution of
$x^2y'' - xy' + y = x^{1/2}$, $0 < x < \infty$,
Given that $y_1 = x$ is a solution of the corresponding homogeneous equation.

Solution: The given equation is
$x^2y'' - xy' + y = x^{1/2}$ …(39)
Let us take $y = xv(x)$ as a trial solution for Eqn. (39). So that
$y' = v + xv'$
$v'' = 2v' + xv''$
Substituting for y, y' and y'' from above in Eqn. (39), we obtain
$x^2(2v' + xv'') - x(v + xv') + xv = x^{1/2}$
$\Rightarrow x^3v'' + x^2v' = x^{1/2}$
$\Rightarrow v'' + \text{-----}\ v' = x^{-5/2}$ …(40)

Eqn. (40) is a linear differential equation in v'. ts integrating factor is

I.F. = $e^{\int \frac{1}{x} dx} = e^{\text{Inx}} = x$

Therefore, Eqn. (40) yields

$V' x = \int x.x^{-5/2} dx + c_1$

$\Rightarrow v' = c_1 x^{-1} - 2x^{-3/2}$

Integrating once again, we have

$v = c_1 \text{ Inx} + 4x^{-1/2} + c_2$

Thus,

$y = xv = c_1 x \text{ Inx} + c_2 x + 4x^{1/2}$

is the general solution of Eqn. 939).

And now some exercise for you.

From that above it is seen that if one solution of the second order linear homogeneous Eqn. (34) is known, then the second linearly independent solution and a particular integral of the associated non-homogeneous equation can be determined.

We now give some rules, which will help you to find one integral included in the complementary function merely by inspection.

For a homogeneous equation of the form (34) if

Rule I: $1 + a_1(x) + a_2(x) = 0$, then $y = e^x$ is an integral of the Eqn. (34).

For instance, consider an equation

$xy'' - y' + (1 - x) y = x^2e^{-x}$ …(41)

To bring it to the form (34), we write it a

$$y'' - \frac{1}{x} y' + \left(\frac{1-x}{x}\right) y = xe^{-x}$$

Thus, $a_1(x) = -\frac{1}{x}$ and $a_2(x) = \frac{1}{x} - 1$

Now $1 + a_1(x) + a_2(x) = 1 - \frac{1}{x} + \frac{1}{x} - 1 = 0$

Thus, according to Rule 1, $y = e^x$ is an integral of the equation. You can verify your result by substituting $y = e^x$ in the given equation and check if it satisfies the given equation.

Rule II: $a_1(x) + xa_2(x) = 0$ then $y = x$ is an integral of the Eqn. (34)

Consider the equation,

$91 - x^2) y'' + xy' - y = x((1 - x^2)^{3/2}$

This equation can be written as

$$y'' + \frac{x}{1-x^2}\,y' - \frac{1}{1-x^2}\,y = x\sqrt{1-x^2}$$

Comparing the above equation with Eqn. (34), we have

$a_1(x) = \dfrac{x}{1-x^2}$ and $a_2(x) = -\dfrac{1}{1-x^2}$.

Here $a_1(x) + xa_2(x) = 0$, hence by the above rule $y = x$ is an integral of the homogeneous equation corresponding to the equation.

Rule III: $1 - a_1(x) + a_2(x) = 0$, then $y = e^{-x}$ is an integral of the Eqn. (34)

Rule IV: $2 + 2xa_1(x) + x^2 a_2(x) = 0$, then $y = x^2$ is an integral of the Eqn. (34).

Rule V: $1 + \dfrac{a_1(x)}{\alpha} + \dfrac{a_2(x)}{\alpha^2} = 0$, $\alpha > 0$, then $y = e^{\alpha x}$ is an integral of the Eqn. (34).

Note that in applying Rules I – V the given equation should be first put in the form of Eqn. (34).

You may now try the following exercise.

So far you have seen that the method of variation of parameters can be used only for those differential equation for which we know all the linearly independent solutions of the corresponding homogeneous equation. Method of reduction of order is helpful for finding complete solution of the second order non-homogeneous linear equations even if **one** solution of the corresponding homogeneous equation is known. There exists certain rules which, at once, give **one** solution merely through an inspection, included in the complementary function of the second order linear equations with constant coefficients. But, no rules exist which may help to guess **one or more** integrals included in the complementary function when the equation is or order higher than two and is having variable coefficients. However, there exists a class of linear differential equations with variable coefficients known as Euler's **equations** for which it is possible to find all the linearly independent integrals of the complementary function. In the next section we take up the method of solving Euler's Equations.

3.3 Euler's Equations

Consider the following differential equations

$$x^3 \frac{d^3y}{dx^3} + x^2 \frac{d^2y}{dx^2} + 4x + 2y = e^x \qquad \ldots(42)$$

$$x \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + 2y = e^x \qquad \ldots(43)$$

$$(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = \sin x \qquad \ldots(44)$$

All the three equations given above are linear as the dependent variable y and its derivative appear in their first degree and moreover there is no term involving the product of the two. Out of the three equations, only Eqn. (42) is such that the **powers of x in the coefficients are equal to the orders of the derivatives associated with them.** This type of equation known as **homogeneous linear differential equation or Euler's Equation.** Eqn (43) is linear but not homogeneous. Eqn. (44) is not of Euler's from form but can be reduced to Euler's form by the substitution $X = 2x - 1$. Here we shall consider only equations of the form (42) and (44).

The general form of Euler's equation of nth order is

$$x^n \frac{d^ny}{dx^n} + P_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_ny = f(x), \qquad \ldots(45)$$

where $P_1, P_2, \ldots, P_n$ are constants and right hand side is a constant or a function of x alone.

Eqn. (45) can be transformed to an equation with constant coefficients by changing the independent variable through the transformation

$z = \text{In}x$ or $x = e^z$

with this substitution, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = D_1y, \text{ where } D_1 = \frac{d}{dz}$$

$$\text{Also } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dx^2}\frac{dz}{dx}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = (D_1^2 - D_1)\,y = D_1(D_1 - 1)\,y$$

Proceeding as above, we shall, in general, get

$$x^n\frac{d^ny}{dx^n} = D_1(D_1-1)(D_1-2)\ldots(D_1 - \overline{n-1})\,y$$

Thus, Eqn (45) is transformed to the equation

$$[D_1 (D_1 - 1) - (D_1 - \overline{n - 1}) + P_1 D_1(D_1 - 1) \ldots.(D_1 \overline{n - 2}) + \ldots$$
$$\ldots + P_{n-2} D_1 (D_1 - 1) + P_{n-1} D_1 + P_n] y = f(e^z) \qquad \ldots(46)$$

Eqn. (46) is an equation with constant coefficients and its complementary function can be determined by the methods given in Unit 5. for obtaining its particular integral either the method of undetermined coefficients (as given in Unit 6 subject to the form of $f(e^z)$), or the method of variation of parameters can be utilized if the solution of Eqn. (46) is

$y = g(z)$,

then the solution of Eqn. (45) will be

$y = g(\text{In}x)$

we illustrate this method by the following examples

Example 5: Solve $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + y = \text{In}x$

Solution: It is Euler's equation of order 2. To solve it, let

$x = e^z$ or $z = \text{In}x$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} = D_1 y, \text{ where } D_1 = \frac{d}{dz}$$

$$\therefore \frac{d^2y}{dx^2} \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dz} \right) = - \frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2y}{dz^2} \frac{dz}{dx} = \frac{1}{x^2} \left(\frac{d^2y}{dz^2} - \frac{dy}{dz} \right)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = D_1 (D_1 - 1) y$$

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we get

$[D_1 (D_1 - 1) - D_1 + 1] y = z$

$\Rightarrow (\text{---} - 2 D_1 + 1] y = z \qquad \ldots(47)$

A.E. is

$m^2 - 2m + 1 \ \ 0$

$\Rightarrow m = 1, 1$

$\therefore$C.F. $= (c_1 + cz) e^z$

To find P.I. of Eqn. (47), let us assume that

$y_p = u_1(z) e^z + u_2(z) ze^z \qquad \ldots(48)$

$$\therefore \frac{dy_p}{dz} = u_1' e^z + u_2' ze^z + u_1 e^z + u_2 (ze^z + e^z)$$

As first auxiliary condition, assume that

$u_1^{'} \; e^z + u_2^{'} \; ze^z = 0,$ …(49)

so that

$\frac{dy_p}{dz} = u_1e^z + u_2(z + 1)\, e^z$ …(50)

Differentiating once again, we have

$\frac{dy_p}{dz} = u_1^{'} e^z + u_2^{'} (z + 1)\, e^z + u_1e^z + u_2e^z\,(z + 1) + u_2e^z$ …(51)

If $y_p(z)$ is a solution of Eqn. (47), it must satisfy it. Hence substituting the expression for y_p, and --- and --- from Eqn. (48), (50) and (51), respectively, in Eqn. (47), we obtain the second auxiliary condition as

$u_1^{'} e^z + (z + 1)\, e^z = z$ …(52)

Solving Eqn. (49) and (52) for-------, we get

$u_2^{'} e^z = z$ and $e^z \; u_1^{'} \; = - z^2$

$\Rightarrow u_1^{'} = - z^2 \, e^{-z}$ and --- $= ze^{-z}$

Integrating the above equation, we get

$u_1 = - \int \; z^2 \, e^{-z} \, dz$

$= - \left[z^2 \frac{e^{-z}}{-1} + 2 \int z \, e^{-z} dz \right]$

$= + z^2 \, e^{-z} - 2 \left[z^2 \frac{e^{-z}}{-1} + \int e^{-z} dz \right]$

$= z^2 \, e^{-2} + 2z \, e^{-z} + 2e^{-z}$

and $u_2 = \int \; z \, e^{-z} \, dz = - ze^{-z} + \int \; e^{-z} \, dz = - z \, e^{-z} \; e^{-z}$

Substituting the values of $u_1(z)$ and $u_2(z)$ in Eqn. (48), a particular integral of Eqn. (47) can be expressed in the form

$y_p(z) = (z^2 + 2z + 2)\, e^{-z}.\, e^z + (- z - 1)\, e^{-z} \, ze^z$

$= (z^2 + 2z + 2) - z(z + 1)$

$= z^2 + 2z + 2 - z^2 - z$

$= z + 2$

and the general solution of Eqn. (47) is

$y = (c_1 + c_2z)\, e^z + z \; z + 2$

Replacing z by In x, the general solution f the given equation is
$y = (c_1 + c_2 \ln x). X + \ln x + 2$

The complementary function of Euler's Eqn. (45) can also be found by assuming $y = x^m$ in the homogeneous part of the equation and then finding the values of m. we illustrate it through the following example.

Example 6: Solve $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} - 6x \frac{dy}{dx} + 18y = 0$

Solution: Let $y = x^m$

$\therefore \frac{dy}{dx} = m\, x^{m-1}$

$\frac{d^2y}{dx^2} = m\,(m-1)\,(m-2)\,x^{m-3}$

Substituting the above values in the given equation, we get

$[m\,(m-1)\,(m-2) \ldots m(m-1) - 6m + 18]\, x^m = 0$

$\Rightarrow\ (m^3 - 4m^2 - 3m + 18)\, x^m = 0$

thus, if

$m^3 - 4m^2 = 3m + 18 = 0,$ …(53)

then $y = x^m$ satisfies the given equation.

Equ. (53) is an algebraic equation of 3rd degree in m and its root are

M = - 2, 3, 3.

Thus, $y = x^{-2}$, $y = x^3$ and $y = x^3$ are the solutions of the given equation. hence the general solution of the given equation is

$Y = c_1x^{-2} + x^3(c_2 + c_3 (\ln x))$

Note: Had all the roots of Eqn. (53) been real and different, the solutions corresponding to these roots would have been independent solutions and the general solution would have been pf the form

$y = c_1 x^{m_1} + c_2 x^{m_2} + c_3 x^{m_3}$

In the case of repeadted real roots of Eqn. (53), if a root m_1 is repeated r times, the integral corresponding to root m_1 is

$[c_1 + c2\ln x + c3 (\ln x)2 + \ldots + cr (\ln x)^{r-}1]\ x^{m_1}$

Further, if Eqn. (53) had a pair of complex roots, say $\alpha \pm i\beta$, then the corresponding part of the complementary function would have been

$x^{\alpha}\ [c_1 \cos (\beta \ln x) + c_2 \sin (\beta \ln x)]$

we illustrate the case of complex roots by the following example.

Example 7: Solve $x^2y'' + xy' + 4y = 0$

Solution: Substituting $y = x^m$ in the given equation, we get
$[m(m-1) + m + 4]\, x^m = 0$
Thus $y = x^m$ satisfies the given equation if
$m(m-1) + m + 4 = 0$
$\Rightarrow\; m^2 + 4 = 0$
$\Rightarrow\; m = \pm\, 2i$

Hence, the general solution of the given equation is

$y = c_1 \cos(2\ln x) + c_2 \sin(2\ln x)$

You may now try the following exercises.
Earlier we mentioned that Eqn. (44) is not Euler's equation, but can be reduced to Euler's form by the substitution $X = 2x - 1$. We now consider such equations which are reducible to Euler's form.

Equations Reducible to Euler's form

Consider the general nth order equation

$$(ax+b)^n \frac{d^n y}{dx^n} + (ax+b)^{n-1} P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + (ax+b) P_{n-1} \frac{dy}{dx} + P_n y = f(x), \qquad \ldots(54)$$

where $a, b, P_1, \ldots, P_n$ are all constants.
Equations of the form (54) can be reduced to Euler's equations by substituting $X = ax + b$.
With this substitution

$$\frac{dy}{dx} = a\frac{dy}{dX},\; \frac{d^2y}{dx^2} = a^2\frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n} = a^n\frac{d^ny}{dX^n}$$

and Eqn. (54) reduces to the equation,

$$a^n X^n \frac{d^n y}{dx^n} + a^{n-1} X^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots + aX P_{n-1} \frac{dy}{dx} + P_n y = g(X), \qquad \ldots(55)$$

where g is transformed form of the function f.

Eqn. (55) is now in Euler's form and can solved by the methods given earlier.

However, Eqn. (54) can be directly reduced to an equation with constant coefficients by substituting $ax + b = e^z$, instead of first substituting $ax + b = X$ and then $X = e^z$.

We illustrate the above theory with the help of following example.

Example 8: Solve $(3x + 2)^2 \frac{d^2y}{dx^2} + 3\,(3x + 2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solution: The given equation is an equation reducible to Euler's equation. We can, however, reduce it to an equation with constant coefficients by a single substitution.

$3x + 2 = e^z$ or $z = \text{In}\,(3x + 2)$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{3x+2}.3\frac{dy}{dz} \Rightarrow (3x+2)\frac{dy}{dx} = 3\frac{dy}{dz}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{3}{3x+2}\frac{dy}{dz}\right] = \frac{-3^2}{(3x+2)^2}\frac{dy}{dx} + \frac{3}{3x+2}\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= \frac{3^2}{(3x+2)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

Substituting y' and y'' from above in the given equation, we get

$$9\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right] + 3.3\frac{dy}{dz} - 36y = \frac{1}{3}\,[e^{2z} - 1]$$

$$\Rightarrow \frac{d^2y}{dz^2} - 4y = \frac{1}{27}\,(e^{2z} - 1) \qquad \ldots(56)$$

A.E is

$m^2 - 4 = 0 \Rightarrow m = \pm\ 2$

Hence C.F. $= y_c = c_1e^{2z} + c_2e^{-2z}$

To find a particular integral, we write

$$y_p(z) = u_1(z)\,e^{2z} + u_2(z)\,e^{-2z} \qquad \ldots(57)$$

$$\therefore\ \frac{dy_p}{dz} = u_1'\,e^{2z} + u_2'\,e^{-2z} + 2\,(u_1e^{2z} - u_2e^{-2z})$$

As the first auxiliary condition, let

$$u_1'\,e^{2z} + u_2'\,e^{-2z} = 0 \qquad \ldots(58)$$

so that

$$\frac{dy_p}{dz} = 2\,(u_1e^{2z} - u_2e^{-2z}) \qquad \ldots(59)$$

Differentiating Eqn. (59) once again, we get

$$\frac{dy_p}{dz} = 2(u_1' e^{2z} - u_2' e^{-2z}) + 4u_1e^{-2z} + 4u_2e^{-2z} \qquad \ldots(60)$$

Since $y_p(z)$ must satisfy Eqn. (56), hence on combining Eqn. (57), (59) and (60) we get the second auxiliary condition as

$$2(u_1' e^{2z} - u_2' e^{-2z}) = \frac{1}{27} (e^{2z} - 1)$$

$$\Rightarrow u_1' e^{2z} - u_2' e^{-2z} = \frac{1}{54} (e^{2z} - 1) \qquad \ldots(61)$$

Solving Eqns. (58) and (61) for u_1' and u_2', we get

$$u_1' = \frac{1}{108} (1 - e^{-2z}) \text{ and } u_2' = -\frac{1}{108} (1 - e^{2z})$$

Integrating u_1' and u_2', we get

$$u_1 = \frac{1}{108} \left(z + \frac{e^{-2z}}{2} \right) e^{2z} - \frac{1}{108} \left(z - \frac{e^{-2z}}{2} \right)$$

on substituting the values of $u_1(z)$ and $u_2(z)$ in relation (57), a particular solution of Eqn. (56) is obtained in the form.

$$y_p = \frac{1}{108} \left(z + \frac{e^{-2z}}{2} \right) e^{2z} - \frac{1}{108} \left(z - \frac{e^{-2z}}{2} \right) e^{-2z}$$

$$= \frac{1}{108} z (e^{2z} - e^{-2z}) + \frac{1}{108}$$

$\therefore$ The general solution of Eqn. (56) is

$$y = c_1e^{2z} + c_2e^{-2z} + \frac{1}{108} z [e^{2z} - e^{-2z}] + \frac{1}{108}$$

and the required solution of the given equation is

$$y = c_1 (3x+2)^2 + \frac{c_2}{(3x+2)^2} + \frac{1}{108} \text{In}(3x + 2)\left[(3x+2)^2 - \frac{1}{(3x+2)^2} \right] + \frac{1}{108}$$

You may not try the following exercises.

4.0 CONCLUSION

We now end this unit by giving a summary of what we have covered in it.

5.0 SUMMARY

In this unit we have studied the details concerning the following results:

1) Let y_1 and y_2 be the linearly independent solutions of the reduced equation of a non-homogeneous second order linear differential equation with constant or variable coefficients. Then on substituting $y = y_1u_1(x) + y_2u_2(x)$ and imposing the conditions (11), the particular integral of the given equation can be found.

2) If $y = y_1(x)$ is one solution of the reduced equation, then on substituting
$y = y_1(x)\, v(x)$
the second solution of the reduced equation and a particular integral of the corresponding non-homogeneous equation can be determined.

3) Rules for finding one integral included ion the complementary function of equations of the form (34) by mere inspection are given by the following table:

Condition satisfied	One integral
$1 + a_1(x) + a_2(x) = 0$	$y = e^x$
$1 - a_1(x) + a_2(x) = 0$	$y = e^{-x}$
$a_1(x) + xa_2(x) = 0$	$y = x$
$2 + 2x\, a_1(x) + x^2\, a_2(x) = 0$	$y = x^2$
$1 + \frac{a_1(x)}{\alpha} + \frac{a_2(x)}{\alpha^2} = 0,\ \alpha > 0$	$y = e^{\alpha x}$

4) Differential equation with variable coefficient of the form

$$x^n \frac{d^n y}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + P_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}x\frac{dy}{dx} + P_ny = f(x),$$

where $P_1, P_2, \ldots, P_n$ are constants and in which the powers of x in the coefficients are equal to the orders of the derivatives associated with them, is known as Euler's equation. This equation can be reduced to an equation with constant coefficients by using the substitution $x = e^z$.

6.0 TUTOR MARKED ASSIGNMENT

1. Determine a particular integral, using the method of variation of parameter for the following differential equations:

a) $y'' + y = \text{cosec } x,\ 0 < x < \frac{\pi}{2}$

b) $y'' - 2y' + y\ xe^x\ \text{In}x,\ x > 0$

c) $y'' + y = \tan x,\ 0 < x < \frac{\pi}{x}$

2. Find a general solution of the following differential equations, given that the functions $y_1(x)$ and $y_2(x)$ for $x > 0$ are linearly independent solutions of the corresponding homogeneous equations.

a) $x^2y'' - 2xy' + 2y = x + 1;\ y_1(x) = x,\ y_2(x) = x^2$

b) $x^2y'' + xy' - y = x^2e^x;\ y_1(x) = x,\ y_2(x) = \frac{1}{x}$

c) $xy'' - (x + 1)y' + y = x^2;\ y_1(x) = e^x,\ y_2(x) = x + 1$

3. Using the method of variation of parameters, find the general solution of the following equations:

a) $y'' - y' = x^2$

b) $y'' - 2y'' - y' + 2y = e^{3x}$

4. Solve the following differential equations:

a) $x^2y'' - 2xy' + 2y = 4x^2,\ x > 0;\ y_1(x) = x$

b) $x^2y'' + 5xy' - 5y = x^{-1/2},\ x > 0;\ y_1(x) = x$

5. A solution of the differential equati0n

$$X^2(1 - x^2)\frac{d^2y}{dx^2} - x^3\frac{dy}{dx} - 2y = 0$$

Is $y_1 = \frac{\sqrt{1-x^2}}{x}$. Use the method of reduction of order to find a general solution.

6. Solve equation

$$X(x\cos x - 2\sin x)\frac{d^2y}{dx^2} + (x^2 + 2)\sin x\frac{dy}{dx} - 2(x\sin x + \cos x)y = 0$$

Given that $y = x^2$ is a solution.

7. Verify that $y_1(x) = e^x$ is a solution of the homogeneous equation corresponding to Eqn. (41).

8. Find an integral included in the complementary function of the following equations, merely by inspection:

a) $y'' - \cot x\, y' - (1 - \cot x)\, y = e^x \sin x$
b) $(x \sin x + \cos x)\, y'' + x (\cos x)\, y' - y \cos x = x$
c) $(3 - x)\, y'' - (9 - 4x)\, y' + (6 - 3x)\, y = 0$

9. Solve the following equations:

a) $(x^2D^2 + 3xD)\, y = \dfrac{1}{x}$
b) $(x^2D^2 + xD - 1)\, y = x^m$
c) $\left(D^3 - \dfrac{4}{x}D^2 + \dfrac{5}{x^2}D - \dfrac{2}{x^3}\right) y = 1$

10. Solve the following equations.

a) $[(x + a)^2 D^2 - 4 (x + a) D + 6]\, y = x$
b) $[(1 + x)^2 D^2 + (1 + x) D + 1]\, y = 4 \cos [\ln (x + 1)]$.

7.0 REFERENCES/FURTHER READINGS

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