Total Probability & Baye's Theorem

If there are two or more causes of an outcome, it is often desirable to determine the prob. that the outcome was due to a particular one of the possible causes. Even though this kind of problem can be solved by merely applying the addition and multiplication rule, much compact procedure has been developed called the B.T.

Baye's Theorem

Let a sample space S of an expt. be partitioned into n mutually exclusive (i.e $A_i \cap A_j = \phi$ for $i \neq j$) and exclusive events ($\sum A_i = S$) $A_1, A_2 \ldots A_n$. Let B be an arbitrary event that occurred when the expt. was performed. Such that $P(A_i) \neq 0$ $i = 1, 2 \ldots n$ then,

$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B/A_i) \quad \text{and}$$

$$P(A_i | B) = \frac{P(A_i)\, P(B/A_i)}{P(B)}$$

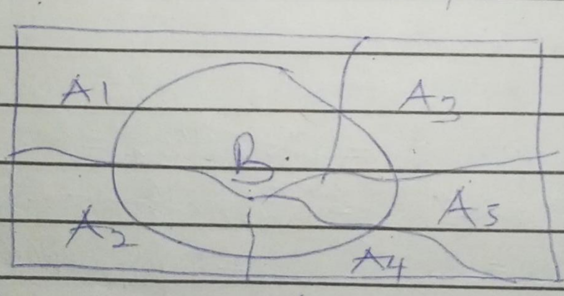
Proof:

Let the events A_i & B be depicted as in below fig.

20-29 — C

30-35 — B

35.5 above A



By defn of conditional prob., we have

$$P(B/A_i) = \frac{P(A_i \cap B)}{P(A_i)} \quad \text{—— (i)}$$

$$\Rightarrow P(A_i \cap B) = P(A_i) \cdot P(B/A_i)$$

We Know that,

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)} \quad \text{(ii)}$$

But, total prob is

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

Using eqn (i), we have

$$P(B) = P(A_1)\cdot P(B/A_1) + P(A_2)\cdot P(B/A_2) + \cdots + P(A_n)\cdot P(B/A_n) \quad \text{(iii)}$$

$$= \sum_{i=1}^{n} P(A_i)\cdot P(B/A_i)$$

Now, putting (iii) in (ii) we have

$$P(A_i/B) = \frac{P(A_i)\cdot P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\cdot P(B/A_i)}$$ Baye's Formula

Examples:

Suppose 15% of Apple and 10% of mangoes in a consignment were toxic. If the consignment consist of 60% apple & 40% Mango, what is the prob that a fruit selected at random is toxic.

Solution:

Let B be the event of toxic fruit and A_1, A_2 be the event of selecting fruit being an apple an a mango resp.

$\therefore P(A_1) = 60/100 = 0.6$

$P(A_2) = 40/100 = 0.4$

$P(B/A_1) = 15/100 = 0.15$

$P(B/A_2) = 10/100 = 0.10$

$P(B) = P(A_1)\cdot P(B/A_1) + P(A_2)\cdot P(B/A_2)$

$= (0.6 \times 0.15) + (0.4 \times 0.10) = 0.13$

Example 2:
A urn contains 2 white and 4 red balls, another urn contains 1 white and 1 red ball. A ball is chosen at random from the first urn and placed unseen in the second urn.

(a) What is the prob. that the ball selected from the second urn is white?

(b) What is the conditional prob. that the ball placed unseen in the second urn was white given that a white ball is selected from the second urn?

Soln:

(a) Let W_1 = white ball from 1st urn

R_1 = Red ✓ ✓ ✓ ✓

W_2 = White ✓ ✓ 2nd urn

R_2 = Red ✓ ✓ ✓ ✓

$$P(W_2) = P(W_1) \cdot P(W_2|W_1) + P(R_1) \cdot P(W_2|R_1)$$

where $P(W_1) = 2/6$, $P(R_1) = 4/6$

$P(W_2|W_1) = 2/3$, $P(W_2|R_1) = 1/3$

$$= \left(2/6 \times 2/3\right) + \left(4/6 \times 1/3\right) = 4/9$$

(b) $$P(W_1|W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{P(W_2|W_1) \cdot P(W_1)}{P(W_2)}$$

$$= \frac{2/3 \times 2/6}{4/9} = \frac{2/9}{4/9} = \frac{2}{4} = 1/2$$