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MOMENTS

The rth raw moment of the random variable X is given by $E(X^r) = \mu_r' = \sum x^r f(x)$ --- dist d_n

and

$$E(X^r) = \int x^r f(x)\,dx \quad \text{--- cont. } d_n$$

The rth central moment is given by

$$\mu_r = E(X-\mu)^r = \sum (x-\mu)^r f(x) \quad \text{--- dist. } d_n$$

and

$$\mu_r = E(X-\mu)^r = \int (x-\mu)^r f(x) \quad \text{--- cont. } d_n$$

We know that when $r=2$, then using dist. X

$$E(X-\mu)^2 = \sum (x-\mu)^2 f(x)$$

$$= \sum (x^2 - 2\mu x + \mu^2) f(x)$$

$$= \sum x^2 f(x) - 2\mu \sum x f(x) + \mu^2 \sum f(x)$$

$$= \mu_2' - 2\mu_1'\mu_1' + (\mu_1')^2$$

$$= \mu_2' - 2(\mu_1')^2 + (\mu_1')^2$$

$$= \mu_2' - (\mu_1')^2$$

Thus; $E(X-\mu)^2 = \mu_2' - (\mu_1')^2$

Note that the variance of a random variable is the second central moment.

Example

Give $f(x) = \begin{cases} \frac{4x(9-x^2)}{81} & 0 \le x \le 3 \\ 0, & o/w \end{cases}$

find the first four raw moment / moment about origin

$$\mu_1' = E(X) = \frac{4}{81}\int_0^3 x \cdot x(9-x^2)\,dx$$

$$= \frac{4}{81}\int_0^3 9x^2 - x^4\,dx$$

$$= \frac{4}{81}\left(3x^3 - \frac{x^5}{5}\right)\Big|_0^3$$

$$= \frac{4}{81}\left(81 - \frac{243}{5}\right) \ldots 6$$

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$$\mu_2' = E(x^2) = 4/81 \int_0^3 x^2 (9x - x^3) dx$$

$$= 4/81 \left(9x^4/4 - x^6/6 \right) \Big|_0^3$$

$$E(x^2) = \frac{4}{81} (182.25 - 121.5) = \frac{4}{81} (60.75) = 3$$

$$\mu_3' = E(x^3) = \frac{4}{81} \int_0^3 x^3 \cdot x (9 - x^2) dx$$

$$= \frac{4}{81} \int_0^3 (9x^4 - x^6) dx$$

$$= \frac{4}{81} \left(\frac{9x^5}{5} - \frac{x^7}{7} \right) \Big|_0^3 = \frac{4}{81} \left(\frac{2187}{5} - \frac{2187}{7} \right)$$

$$\mu_3' = 6.17$$

Note:

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3 = E(x-\mu)^3$$

Also;

$$\mu_4 = E(x-\mu)^4$$

$$= {}^4C_0 x^4 \mu^0 - {}^4C_1 x^3 \mu^1 + {}^4C_2 x^2 \mu^2 - {}^4C_3 x \mu^3 + {}^4C_4 x^0 \mu^4$$

$$= x^4 - 4x^3\mu + 6x^2\mu^2 - 4x\mu^3 + \mu^4$$

Recall the Binomial Expansion

$$= x^4 - 4x^3\mu + 6x^2\mu^2 - 4x\mu^3 + \mu^4$$

$$= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 4\mu_1(\mu_1')^3 + (\mu_1')^4$$

$$= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

$$(x-a)^b = \cancel{{}^bC_0 x^b a^{b-0}} - \cancel{{}^bC_1 x^{b-1} a^{1}} + \cancel{{}^bC_2 x^{b-2} a^{2}}$$

$$\text{i.e. } (x-a)^b = {}^bC_0 x^b a^0 - {}^bC_1 x^{b-1} a^1 + {}^bC_2 x^{b-2} a^2$$

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CHEBYCHEV'S INEQUALITY

Definition: If the random var. X has a finite mean μ and a finite variance σ^2 then for every $K \geq 1$

(i) $P(|X-\mu| \geq K\sigma) \leq 1/K^2$ and

(ii) $P(|X-\mu| < K\sigma) \geq 1 - 1/K^2$

Simply put in words, Cheb ineq says that the prob that X differs from its mean by at least K standard deviation is at most $1/K^2$. And the prob that X deviate from its mean by less than K S.D is at least $1 - 1/K^2$.

Example:

(1) A r.v X has mean $\mu = 40$ and variance $\sigma^2 = 4$

Find: (i) $P(|X-40|) \geq 5$ (ii) $P(|X-40|) < 5$

Soln:

$$P(|X-40| \geq 5) = P(|X-40| \geq 2K)$$

Since $\sigma^2 = 4$ then $\sigma = 2$

$$\therefore 2K = 5$$

$$K = 5/2$$

$$\therefore P(|X-40| \geq 5) \leq 1/K^2 = 1/(5/2)^2$$

$$\therefore P(|X-40| \geq 5) \leq 4/25$$

(ii) $P(|X-40| < 5) \geq 1 - 4/25$

$$P(|X-40| < 5) \geq 21/25$$

$$P(|X-40| < 5) \geq 21/25$$

Ex2:

Let

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & o/w \end{cases}$$

Let $a = 2$ & $b = 8$

find (i) $P(|X-\mu| \geq 3)$ (ii) $P(|X-\mu| < 3)$

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Sol^n :

Now, from the above, $\mu = E(x) = a+b/2 = 3+8/2 = 5$

$$\sigma^2 = Var(x) = \frac{1}{12}(b-a)^2 = \frac{1}{12}(8-2)^2 = 36/12 = 3$$

$$\therefore \sigma = \sqrt{3}$$

$$\therefore P(|x-\mu| \geq 3) = P(|x-\mu| \geq K(\sqrt{3}))$$

$$\therefore K\sqrt{3} = 3$$

$$K = \frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$K = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$\therefore P(|x-\mu| \geq 3) \leq \frac{1}{(\sqrt{3})^2} \leq 1/3$$

and $P(|x-\mu| < 3) = 1 - 1/3 = 2/3$.

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MOMENT GENERATING FUNCTION

The moment generating function $M_x(t)$ of the random Variable x is defined for all real values of t as;

$$M_x(t) = E(e^{tx})$$

$$= \sum e^{tx} f(x) \text{ if X is discrete}$$

$$= \int e^{tx} f(x)\,dx \text{ if X is continous}$$

Now, by Taylor series expansion of e^{tx}, we have

$$e^{tx} = 1 + tx + (tx)^2/2! + (tx)^3/3! + \cdots + (tx)^r/r!$$

$$E(e^{tx}) = 1 + t\,E(x) + t^2 \frac{E(x^2)}{2!} + t^3 \frac{E(x^3)}{3!} + \cdots + t^r \frac{E(x^r)}{r!}$$

$$M_x(t) = E(e^{tx}) = 1 + t\mu_1' + t^2 \frac{\mu_2'}{2!} + \cdots + \frac{t^r \mu_r'}{r!}$$

$$\frac{d\,M_x(t)}{dt} = 0 + \mu_1' + \frac{2t\mu_2'}{2!} + \cdots + rt^{r-1}\frac{\mu_r'}{r!}$$

Set $t = 0$, to obtain

$$\mu_1' = E(x) = \text{Mean}$$

Also;

$$M_x''(t) = 0 + \mu_2' + \cdots + \frac{t^{r-2}\mu_r'}{(r-2)!}$$

$$\therefore M_x''(0) = \mu_2'$$

$\therefore$

$$Var(x) = M_x''(0) - (M_x'(0))^2$$

$$= \mu_2' - (\mu_1')^2$$