Relations

AGENDA

- Definitions
- Relations on Set
- Types of Relations
- Properties of relation.
- Combining relations
- Equivalence Relations
- Equivalence classes
- Databases and relations
- Representing Relations

Relations

If we want to describe a relationship between elements of two sets A and B, we can use **ordered pairs** with their first element taken from A and their second element taken from B.

Since this is a relation between **two sets**, it is called a **binary relation**.

Definition: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, for a binary relation R we have $R \subseteq A \times B$. We use the notation aRb to denote that $(a, b) \in R$ and $a\underline{R}b$ to denote that $(a, b) \notin R$.

Relations

When (a, b) belongs to R, a is said to be **related** to b by R.

Example: Let P be a set of people, C be a set of cars, and D be the relation describing which person drives which car(s).

P = {Hamzah, Biriri, Mariam, Sani},

C = {Mercedes, BMW, tricycle}

D = {(Hamzah, Mercedes), (Biriri, Mercedes),
(Biriri, BMW), (Sani, tricycle)}

This means that Hamzah drives a Mercedes, Biriri drives a Mercedes and a BMW, Sani drives a tricycle, and Mariam does not drive any of these vehicles.

Functions as Relations

You might remember that a **function** f from a set A to a set B assigns a unique element of B to each element of A.

The **graph** of f is the set of ordered pairs (a, b) such that b = f(a).

Since the graph of f is a subset of A×B, it is a **relation** from A to B.

Moreover, for each element **a** of A, there is exactly one ordered pair in the graph that has **a** as its first element.

Functions as Relations

Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph.

This is done by assigning to an element $a \in A$ the unique element $b \in B$ such that $(a, b) \in R$.

Relations on a Set

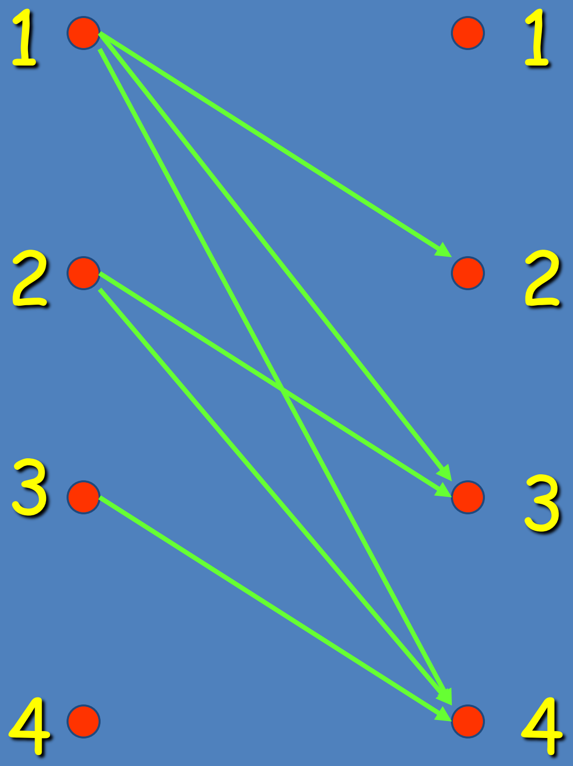
Definition: A relation on the set A is a relation from A to A.

In other words, a relation on the set A is a subset of A×A.

Example: Let $A = \{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a < b\}$?

Relations on a Set

Solution: R = { (1, 2),(1, 3),(1, 4),(2, 3),(2, 4),(3, 4)}



R	1	2	3	4
1		X	X	X
2			X	X
3				X
4				

Relations on a Set

How many different relations can we define on a set A with n elements?

A relation on a set A is a subset of A×A.

How many elements are in A×A ?

There are n^2 elements in A×A, so how many subsets (= relations on A) does A×A have?

The number of subsets that we can form out of a set with m elements is 2^m.

Therefore, 2^{n^2} subsets can be formed out of A×A.

Answer: We can define 2^{n^2} different relations on A.

Properties of Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if (a, a)$\in$R for every element a$\in$A.

Are the following relations on {1, 2, 3, 4} reflexive?

R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} No.

R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} Yes.

R = {(1, 1), (2, 2), (3, 3)} No.

Definition: A relation on a set A is called **irreflexive** if (a, a)$\notin$R for every element a$\in$A.

Properties of Relations

Definitions:

A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

A relation R on a set A is called **antisymmetric** if $a = b$ whenever $(a, b) \in R$ and $(b, a) \in R$.

A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Relations

Are the following relations on {1, 2, 3, 4} symmetric, antisymmetric, or asymmetric?

R = {(1, 1), (1, 2), (2, 1), (3, 3), (4, 4)}	symmetric
R = {(1, 1)}	sym. and antisym.
R = {(1, 3), (3, 2), (2, 1)}	antisym. and asym.
R = {(4, 4), (3, 3), (1, 4)}	antisym.

Properties of Relations

Definition: A relation R on a set A is called **transitive** if whenever (a, b)$\in$R and (b, c)$\in$R, then (a, c)$\in$R for a, b, c$\in$A.

Are the following relations on {1, 2, 3, 4} transitive?

R = {(1, 1), (1, 2), (2, 2), (2, 1), (3, 3)}

Yes.

R = {(1, 3), (3, 2), (2, 1)}

No.

R = {(2, 4), (4, 3), (2, 3), (4, 1)}

No.

Counting Relations

Example: How many different reflexive relations can be defined on a set A containing n elements?

Solution: Relations on R are subsets of A×A, which contains n^2 elements.

Therefore, different relations on A can be generated by choosing different subsets out of these n^2 elements, so there are 2^{n^2} relations.

A **reflexive** relation, however, **must** contain the n elements (a, a) for every $a \in A$.

Consequently, we can only choose among $n^2 – n = n(n – 1)$ elements to generate reflexive relations, so there are $2^{n(n – 1)}$ of them.

Combining Relations

Relations are sets, and therefore, we can apply the usual **set operations** to them.

If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

In each case, the result will be **another relation from A to B**.

Combining Relations

… and there is another important way to combine relations.

Definition: Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and
$(b, c) \in S$. We denote the composite of R and S by
$S \circ R$.

In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

Example: Let D and S be relations on A = {1, 2, 3, 4}.

$D = \{(a, b) \mid b = 5 - a\}$ “b equals (5 – a)”

$S = \{(a, b) \mid a < b\}$ “a is smaller than b”

$D = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

$S \circ D = \{ (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in $S \circ D = \{(a,b) \mid b > 5 - a\}$ or $S \circ D = \{(a,b) \mid a + b > 5\}$.

Combining Relations

We already know that **functions** are just **special cases** of **relations** (namely those that map each element in the domain onto exactly one element in the codomain).

If we formally convert two functions into relations, that is, write them down as sets of ordered pairs, the composite of these relations will be exactly the same as the composite of the functions (as defined earlier).

Combining Relations

Definition: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by

$R^1 = R$

$R^{n+1} = R^n \circ R$

In other words:

$R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Combining Relations

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Remember the definition of transitivity:

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for a, b, $c \in A$.

The composite of R with itself contains exactly these pairs (a, c).

Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.

Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

n-ary Relations

In order to study an interesting application of relations, namely **databases**, we first need to generalize the concept of binary relations to **n-ary relations**.

Definition: Let A_1, A_2, …, A_n be sets. An **n-ary relation** on these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$.

The sets A_1, A_2, …, A_n are called the **domains** of the relation, and n is called its **degree**.

n-ary Relations

Example:

Let R = {(a, b, c) | $a = 2b \wedge b = 2c$ with a, b, c$\in$**N**}

What is the degree of R?

The degree of R is 3, so its elements are triples.

What are its domains?

Its domains are all equal to the set of integers.

Is (2, 4, 8) in R?

No.

Is (4, 2, 1) in R?

Yes.

Databases and Relations

Let us take a look at a type of database representation that is based on relations, namely the **relational data model.**

A database consists of n-tuples called **records**, which are made up of **fields**.

These fields are the **entries** of the n-tuples.

The relational data model represents a database as an n-ary relation, that is, a set of records.

Databases and Relations

Example: Consider a database of students, whose records are represented as 4-tuples with the fields **Student Name**, **ID Number**, **Major**, and **GPA**:

R = {(Ackermann, 231455, CS, 3.88),
(Adams, 888323, Physics, 3.45),
(Chou, 102147, CS, 3.79),
(Goodfriend, 453876, Math, 3.45),
(Rao, 678543, Math, 3.90),
(Stevens, 786576, Psych, 2.99)}

Relations that represent databases are also called **tables**, since they are often displayed as tables.

Databases and Relations

A domain of an n-ary relation is called a **primary key** if the n-tuples are uniquely determined by their values from this domain.

This means that no two records have the same value from the same primary key.

In our example, which of the fields **Student Name**, **ID Number**, **Major**, and **GPA** are primary keys?

Student Name and **ID Number** are primary keys, because no two students have identical values in these fields.

In a real student database, only **ID Number** would be a primary key.

Databases and Relations

In a database, a primary key should remain one even if new records are added.

Therefore, we should use a primary key of the **intension** of the database, containing all the n-tuples that can ever be included in our database.

Combinations of domains can also uniquely identify n-tuples in an n-ary relation.

When the values of a **set of domains** determine an n-tuple in a relation, the **Cartesian product** of these domains is called a **composite key**.

Databases and Relations

We can apply a variety of **operations** on n-ary relations to form new relations.

Definition: The **projection** $P_{i_1, i_2, \ldots, i_m}$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

In other words, a projection $P_{i_1, i_2, \ldots, i_m}$ keeps the m components $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ of an n-tuple and deletes its (n – m) other components.

Example: What is the result when we apply the projection $P_{2,4}$ to the student record (Stevens, 786576, Psych, 2.99) ?

Solution: It is the pair (786576, 2.99).

Databases and Relations

In some cases, applying a projection to an entire table may not only result in fewer columns, but also in **fewer rows**.

Why is that?

Some records may only have differed in those fields that were deleted, so they become **identical**, and there is no need to list identical records more than once.

Databases and Relations

We can use the **join** operation to combine two tables into one if they share some identical fields.

Definition: Let R be a relation of degree m and S a relation of degree n. The **join** $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree m + n – p that consists of all (m + n – p)-tuples

$(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$,

where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

Databases and Relations

In other words, to generate Jp(R, S), we have to find all the elements in R whose p last components match the p first components of an element in S.

The new relation contains exactly these matches, which are combined to tuples that contain each matching field only once.

Databases and Relations

Example: What is $J_1(Y, R)$, where Y contains the fields **Student Name** and **Year of Birth**,

Y = {(1978, Ackermann),
(1972, Adams),
(1917, Chou),
(1984, Goodfriend),
(1982, Rao),
(1970, Stevens)},

and R contains the student records as defined before ?

Databases and Relations

Solution: The resulting relation is:

{(1978, Ackermann, 231455, CS, 3.88),
(1972, Adams, 888323, Physics, 3.45),
(1917, Chou, 102147, CS, 3.79),
(1984, Goodfriend, 453876, Math, 3.45),
(1982, Rao, 678543, Math, 3.90),
(1970, Stevens, 786576, Psych, 2.99)}

Since Y has two fields and R has four, the relation $J_1(Y, R)$ has 2 + 4 – 1 = 5 fields.

Equivalence Relations(1)

Equivalence relations are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relations(2)

Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

Since R is **reflexive**, every element is equivalent to itself.

Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relations (3)

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?

Solution:

- R is reflexive, because l(a) = l(a) and therefore aRa for any string a.
- R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
- R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.

R is an equivalence relation.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class**
of a.

The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we will delete the subscript R and write **[a]** for this equivalence class.

If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes(2)

Example: In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse] ?

Solution: [mouse] is the set of all English words containing five letters.

For example, ‘horse’ would be a representative of this equivalence class.

Equivalence Classes(3)

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

- aRb
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i,

$i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

- $A_i \cap A_j = \emptyset$, if $i \neq j$
- $\cup_{i \in I} A_i = S$

Equivalence Classes(4)

Examples: Let S be the set {u, m, b, r, o, c, k, s}.
Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}}	yes.
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).
{{u, m, b, r, o, c, k, s}}	yes.
{{b, o, o, k}, {r, u, m}, {c, s}}	yes ({b,o,o,k} = {b,o,k}).
{{u, m, b}, {r, o, c, k, s}, ∅}	no (∅ not allowed).

Equivalence Classes(5)

Theorem: Let R be an equivalence relation on a set S. Then the **equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Equivalence Classes(6)

Example: Let us assume that Mubarak, Qudus and Aisha live in Abuja, Gimba and Omotosho live in Lagos, and Taiwo lives in Ibadan.

Let R be the **equivalence relation** {(a, b) | a and b live in the same city} on the set P = {Mubarak, Qudus, Aisha, Gimba, Omotosho, Taiwo}.

Then R = {(Mubarak,Mubarak), (Mubarak, Qudus), (Mubarak, Aisha), (Qudus, Mubarak), (Qudus, Qudus), (Qudus, Aisha), (Aisha, Mubarak),
(George, Qudus), (Aisha, Aisha), (Gimba,
Gimba), (Gimba, Omotosho), (Omotosho, Gimba), (Omotosho, Omotosho), (Taiwo, Taiwo)}.

Equivalence Classes(8)

Then the **equivalence classes** of R are:

{{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.

This is a **partition** of P.

The equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Equivalence Classes(9)

Another example: Let R be the relation $\{(a, b) \mid a \equiv b \pmod 3\}$ on the set of integers.

Is R an equivalence relation?

Yes, R is reflexive, symmetric, and transitive.

What are the equivalence classes of R ?

{{…, -6, -3, 0, 3, 6, …},
{…, -5, -2, 1, 4, 7, …},
{…, -4, -1, 2, 5, 8, …}}

Representing Relations

We already know different ways of representing relations. We will now take a closer look at two ways of representation: **Zero-one matrices** and **directed graphs**.

If R is a relation from A = $\{a_1, a_2, \ldots, a_m\}$ to B = $\{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

$m_{ij} = 1$, if $(a_i, b_j) \in R$, and

$m_{ij} = 0$, if $(a_i, b_j) \notin R$.

Note that for creating this matrix we first need to list the elements in A and B in a **particular, but arbitrary order**.

Representing Relations

Example: How can we represent the relation R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations Using Matrices

What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?

They are **square** matrices.

What do we know about matrices representing **reflexive** relations?

All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Representing Relations Using Matrices(2)

What do we know about the matrices representing **symmetric relations**?

These matrices are symmetric, that is, $M_R = (M_R)^t$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

symmetric matrix,
symmetric relation.

non-symmetric matrix,
non-symmetric relation.

Representing Relations Using Matrices(3)

The Boolean operations **join** and **meet** (you remember?) can be used to determine the matrices representing the **union** and the **intersection** of two relations, respectively.

To obtain the **join** of two zero-one matrices, we apply the Boolean “or” function to all corresponding elements in the matrices.

To obtain the **meet** of two zero-one matrices, we apply the Boolean “and” function to all corresponding elements in the matrices.

Representing Relations Using Matrices

Example: Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing R∪S and R∩S?

Solution: These matrices are given by

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Representing Relations Using Matrices

Example: How can we represent the relation $R = \{(2, 1), (3, 1), (3, 2)\}$ as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Representing Relations Using Matrices

Do you remember the **Boolean product** of two zero-one matrices?

Let $A = [a_{ij}]$ be an $m \times k$ zero-one matrix and
$B = [b_{ij}]$ be a $k \times n$ zero-one matrix.

Then the **Boolean product** of A and B, denoted by $A \circ B$, is the $m \times n$ matrix with (i, j)th entry $[c_{ij}]$, where

$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2i}) \vee \ldots \vee (a_{ik} \wedge b_{kj})$.

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

Representing Relations Using Matrices

Let us now assume that the zero-one matrices $M_A = [a_{ij}]$, $M_B = [b_{ij}]$ and $M_C = [c_{ij}]$ represent relations A, B, and C, respectively.

Remember: For $M_C = M_A \circ M_B$ we have:

$c_{ij} = 1$ if and only if at least one of the terms $(a_{in} \wedge b_{nj}) = 1$ for some n; otherwise $c_{ij} = 0$.

In terms of the **relations**, this means that C contains a pair (x_i, z_j) if and only if there is an element y_n such that (x_i, y_n) is in relation A and (y_n, z_j) is in relation B.

Therefore, $C = B \circ A$ (**composite** of A and B).

Representing Relations Using Matrices

This gives us the following rule:

$M_{B\circ A} = M_A \odot M_B$

In other words, the matrix representing the **composite** of relations A and B is the **Boolean product** of the matrices representing A and B.

Analogously, we can find matrices representing the **powers of relations**:

$M_{R^n} = M_R^{[n]}$ (n-th **Boolean power**).

Representing Relations Using Matrices

Example: Find the matrix representing R^2, where the matrix representing R is given by

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution: The matrix for R^2 is given by

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Representing Relations Using Digraphs

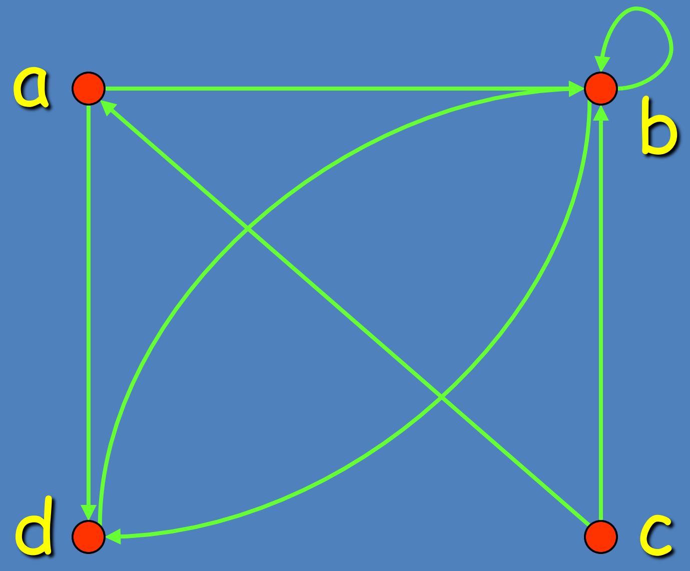
Definition: A **directed graph**, or **digraph**, consists of a set V of **vertices** (or **nodes**) together with a set E of ordered pairs of elements of V called **edges** (or **arcs**).

The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.

We can use arrows to display graphs.

Representing Relations Using Digraphs

Example: Display the digraph with V = {a, b, c, d}, E = {(a, b), (a, d), (b, b), (b, d), (c, a), (c, b), (d, b)}.



An edge of the form (b, b) is called a **loop**.

Representing Relations Using Digraphs

Obviously, we can represent any relation R on a set A by the digraph with A as its vertices and all pairs (a, b)$\in$R as its edges.

Vice versa, any digraph with vertices V and edges E can be represented by a relation on V containing all the pairs in E.

This **one-to-one correspondence** between relations and digraphs means that any statement about relations also applies to digraphs, and vice versa.