Lecture 3: Predicates and Quantifiers.

Agenda

- Predicates and Quantifiers
  - Existential Quantifier $\exists$
  - Universal Quantifier $\forall$

Motivating example

Consider the compound proposition:

If Zeph is an octopus then Zeph has 8 limbs.

Q1: What are the atomic propositions and how do they form this proposition.

Q2: Is the proposition true or false?

Q3: Why?

Motivating example

A1: Let p = “Zeph is an octopus” and q = “Zeph has 8 limbs”. The compound proposition is represented by $p \rightarrow q$.

A2: True!

A3: Conditional always true when p is false!

Q: Why is this not satisfying?

Motivating example

A: We wanted this to be true because of the fact that octopi have 8 limbs and not because of some (important) non-semantic technicality in the truth table of implication.

But recall that propositional calculus doesn't take semantics into account so there is no way that *p* could impact on *q* or affect the truth of $p \rightarrow q$.

Logical Quantifiers help to fix this problem. In our case the fix would look like:

For all *x*, if *x* is an octopus then *x* has 8 limbs.

Motivating example

Expressions such as the previous are built up from *propositional functions* –statements that have a variable or variables that represent various possible subjects. Then quantifiers are used to bind the variables and create a proposition with embedded semantics. For example:

"For all *x*, if *x* is an octopus then *x* has 8 limbs."

there are two atomic propositional functions

- $P(x)$ = "*x* is an octopus"
- $Q(x)$ = "*x* has 8 limbs"

whose conditional $P(x) \rightarrow Q(x)$ is formed and is bound by "For all *x* ".

Semantics

If logical propositions are to have meaning, they need to describe something. Up to now, propositions such as "Abdullah is short", "Yusuff is 19 years old", and "Habib is immature" had no intrinsic meaning. Either they are true or false, but no more.

In order to endow such propositions with meaning, we need to have a ***universe of discourse***, i.e. a collection of *subjects* (or *nouns*) about which the propositions relate.

Q: What is the universe of discourse for the three propositions above?

Semantics

A: There are many answers. Here are some:

- Abdullah, Habeeb and Yusuf (This is also the smallest correct answer)
- People in the world
- Animals

Java: The Java analog of a universe of discourse is a **type**. There are two categories of types in Java: *reference types* which consist of objects and arrays, and *primitive types* like **`int`**, **`boolean`**, **`char`**, etc. Examples of Java "universes" are:

`int`, **`char`**, **`int[][]`**, **`Object`**, **`String`**, **`java.util.LinkedList`**, **`Exception`**, etc.

Predicates

A ***predicate*** is a property or description of subjects in the universe of discourse. The following predicates are all *italicized* :

- Mariam is *short*.
- The bridge *is structurally sound*.
- 17 *is a prime number.*

Java: predicates are **`boolean`**-valued method calls-

- **`someLinkedList.`*`isEmpty`*`()`**
- ***`isPrime`*`(17)`**

L4

Propositional Functions

By taking a variable subject denoted by symbols such as *x*, *y*, *z*, and applying a predicate one obtains a ***propositional function*** (or ***formula***). When an object from the universe is plugged in for *x*, *y*, etc. a truth value results:

- *x* is tall. …e.g. plug in *x* = Johnny
- *y* is structurally sound. …e.g. plug in *y* = GWB
- *n* is a prime number. ...e.g. plug in *n* = 111

Java: propositional functions are **`boolean`** methods, rather than particular calls.

- **`boolean isEmpty(){…} //in LinkedList`**
- **`boolean isPrime(int n){…}`**

Multivariable Predicates

Multivariable predicates generalize predicates to allow descriptions of relationships between subjects. These subjects may or may not even be in the same universe of discourse. For example:

– Johnny *is taller than* Debbie.

– 17 *is greater than one of* 12, 45*.*

– Johnny *is at least* 5 *inches taller than* Debbie.

Q: What universes of discourse are involved?

Multivariable Predicates

A: Again, many correct answers. The most obvious answers are:

- For "Johnny *is taller than* Debbie" the universe of discourse of **both** variables is all people in the universe.
- For "17 *is greater than one of* 12, 45" the universe of discourse of all three variables is **Z** (the set of integers)
- For "Johnny *is at least* 5 *inches taller than* Debbie" the first and last variable have people as their universe of discourse, while the second variable has **R** (the set of real numbers).

Multivariable Propositional Functions

The multivariable predicates, together with their variables create ***multivariable propositional functions***. In the above examples, we have the following generalizations:

- x is taller than y
- a is greater than one of b, c
- x is at least n inches taller than y

In Java, a multivariable predicate is a **`boolean`** method with several arguments:

```
tallerByNumInches(Person x, double n, Person y)
   { … }
```

Quantifiers

There are two quantifiers

- Existential Quantifier

  "$\exists$" reads "there exists"

- Universal Quantifier

  "$\forall$" reads "for all"

Each is placed in front of a propositional function and ***binds*** it to obtain a proposition with semantic value.

Existential Quantifier

- "$\exists x\ P(x)$" is true when *an* instance can be found which when plugged in for x makes $P(x)$ true
- Like disjunctioning over entire universe $\exists x\ P(x) \Leftrightarrow P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots$

Existential Quantifier. Example

Consider a universe consisting of

- Leo: a lion
- Jan: an octopus with all 8 tentacles
- Bill: an octopus with only 7 tentacles

And recall the propositional functions

- $P(x)$ = "x is an octopus"
- $Q(x)$ = "x has 8 limbs"

$$\exists x \ (P(x) \rightarrow Q(x))$$

Q: Is the proposition true or false?

Existential Quantifier. Example

A: True. Proposition is equivalent to

$(P\,(\text{Leo}) \rightarrow Q\,(\text{Leo})) \vee (P\,(\text{Jan}) \rightarrow Q\,(\text{Jan})) \vee (P(\text{Bill}) \rightarrow Q\,(\text{Bill}))$

P (Leo) is false because Leo is a Lion, not an octopus, therefore the conditional

P (Leo) $\rightarrow Q$ (Leo) is true, and the disjunction is true.

Leo is called a *positive example*.

The Universal Quantifier

- "$\forall x\ P(x)$" true when *every* instance of x makes $P(x)$ true when plugged in
- Like conjunctioning over entire universe

$$\forall x\, P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots$$

Universal Quantifier. Example

Consider the same universe and propositional functions as before.

$$\forall x \ (P(x) \rightarrow Q(x))$$

Q: Is the proposition true or false?

Universal Quantifier. Example

A: False. The proposition is equivalent to

$$(P\,(\text{Leo}) \rightarrow Q\,(\text{Leo})) \land (P\,(\text{Jan}) \rightarrow Q\,(\text{Jan})) \land (P\,(\text{Bill}) \rightarrow Q\,(\text{Bill}))$$

Bill is the ***counter-example***, i.e. a value making an instance –and therefore the whole universal quantification– false.

P (Bill) is true because Bill is an octopus, while *Q* (Bill) is false because Bill only has 7 tentacles, not 8. Thus the conditional *P* (Bill)→*Q* (Bill) is false since T→F gives F, and the conjunction is false.

Illegal Quantifications

Once a variable has been bound, we cannot bind it again. For example the expression

$$\forall x \,(\, \forall x \, P\,(x)\,)$$

is nonsensical. The interior expression $(\forall x\, P\,(x))$ bounded x already and therefore made it unobservable to the outside. Going back to our example, the English equivalent would be:

Everybody is an *everybody is an octopus*.

Multivariate Quantification

Quantification involving only one variable is fairly straightforward. Just a bunch of OR's or a bunch of AND's.

When two or more variables are involved each of which is bound by a quantifier, the order of the binding is important and the meaning often requires some thought.

Parsing Example

A: True.
For any "exists" we need to find a positive instance.
Since x is the first variable in the expression and is "existential", we need a number that works for all other y, z. Set $x = 0$ (want to ensure that $y - x$ is not too small).
Now for each y we need to find a positive instance z such that $y - x \geq z$ holds. Plugging in $x = 0$ we need to satisfy $y \geq z$ so set $z := y$.

Q: Did we have to set $z := y$?

Parsing Example

A: No. Could also have used the constant $z := 0$. Many other valid solutions.

Q: Isn't it simpler to satisfy

$$\exists x \, \forall y \, \exists z \, (y - x \geq z)$$

by setting $x := y$ and $z := 0$?

Parsing Example

A: No, this is illegal ! The existence of *x* comes before we know about *y*. I.e., the *scope* of *x* is higher than the scope of *y* so as far as *y* can tell, *x* is a *constant* and cannot affect *x*.

A Java example helps explain this point. To evaluate $\exists x \; \forall y \; Q(x,y)$ might do the following.

Parsing Example —Java

```
boolean exists_x_forAll_y(){
  for(int x=firstInt; x<=lastInt; x++){
      if (forAll_y(x)) return true;
  }
  return false;
}

boolean forAll_y(int x){
  for(int y=firstInt; y<=lastInt; y++){
      if ( !Q(x,y) ) return false;
  }
  return true;
}
```

Order matters

Set the universe of discourse to be all natural numbers {0, 1, 2, 3, … }.

Let $R(x,y)$ = “$x < y$”.

Q1: What does $\forall x\, \exists y\, R(x,y)$ mean?

Q2: What does $\exists y\, \forall x\, R(x,y)$ mean?

Order matters

$R(x,y)$ = “$x < y$”

A1: $\forall x\, \exists y\, R(x,y)$:

“All numbers x admit a bigger number y”

A2: $\exists y\, \forall x\, R(x,y)$:

“Some number y is bigger than all x”

Q: What’s the true value of each expression?

Order matters

A: 1 is true and 2 is false.

$\forall x\, \exists y\, R(x,y)$: "All numbers *x* admit a bigger number *y*" --just set $y = x + 1$

$\exists y\, \forall x\, R(x,y)$: "Some number *y* is bigger than all numbers *x*" --*y* is never bigger than itself, so setting $x = y$ is a counterexample

Q: What if we have two quantifiers of the same kind? Does order still matter?

Order matters – but not always

A: No! If we have two quantifiers of the same kind order is irrelevent.

$\forall x \, \forall y$ is the same as $\forall y \, \forall x$ because these are both interpreted as "for every combination of x and y…"

$\exists x \, \exists y$ is the same as $\exists y \, \exists x$ because these are both interpreted as "there is a pair x , y…"

Logical Equivalence with Formulas

DEF: Two logical expressions possibly involving propositional formulas and quantifiers are said to be ***logically equivalent*** if no-matter what universe and what particular propositional formulas are plugged in, the expressions always have the same truth value.

EG: $\forall x\, \exists y\, Q(x,y)$ and $\forall y\, \exists x\, Q(y,x)$ are equivalent –names of variables don't matter.

EG: $\forall x\, \exists y\, Q(x,y)$ and $\exists y\, \forall x\, Q(x,y)$ are not!

DeMorgan Revisited

Recall DeMorgan's identities:

- Conjunctional negation:

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \Leftrightarrow (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n)$$

- Disjunctional negation:

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \Leftrightarrow (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$$

Since the quantifiers are the same as taking a bunch of AND's ($\forall$) or OR's ($\exists$) we have:

- Universal negation:

$$\neg\ \forall x\ \mathrm{P}(x) \Leftrightarrow \exists x\ \neg \mathrm{P}(x)$$

- Existential negation:

$$\neg\ \exists x\ \mathrm{P}(x) \Leftrightarrow \forall x\ \neg \mathrm{P}(x)$$

Negation Example

Compute: $\neg \forall x \, \exists y \;\; x^2 \leq y$

In English, we are trying to find the opposite of "every x admits a y greater or equal to x's square". The opposite is that "some x does not admit a y greater or equal to x's square"

Algebraically, one just flips all quantifiers from $\forall$ to $\exists$ and vice versa, and negates the interior propositional function. In our case we get:

$$\exists x \, \forall y \, \neg(x^2 \leq y) \;\; \Leftrightarrow \;\; \exists x \, \forall y \;\; x^2 > y$$

Blackboard Exercises

1.3.41) Show that the following are logically equivalent:

- $(\forall x\ A(x\ )) \vee (\forall x\ B(x\ ))$
- $\forall x,y\ A(x\ ) \vee B(y\ )$

Blackboard Exercises

Need to show that

$(\forall x\, A(x)) \vee (\forall x\, B(x)) \leftrightarrow \forall x,y\, A(x) \vee B(y)$

is a tautology, so LHS and RHS must always have same truth values.

Blackboard Exercises

$(\forall x\, A(x)) \vee (\forall x\, B(x)) \leftarrow\rightarrow \forall x,y\, A(x) \vee B(y)$

CASE I) Assuming LHS true show RHS true.

Either $\forall x\, A(x)$ true OR $\forall x\, B(x)$ true

Case I.A) For all x, $A(x)$ true.

As $(T \vee \textit{anything}) = T$, we can set

$\textit{anything} = B(y)$ and obtain

For all x and y, $A(x) \vee B(y)$ – the RHS!

Case I.B) For all x, $B(x)$ true… similar to case (I.A)

Blackboard Exercises

$(\forall x\, A(x)) \vee (\forall x\, B(x)) \leftarrow\rightarrow \forall x,y\, A(x) \vee B(y)$

CASE II) Assume LHS false, show RHS false.

Both $\forall x\, A(x)$ false AND $\forall x\, B(x)$ false.

Thus $\exists x\, \neg A(x)$ true AND $\exists x\, \neg B(x)$ true.

Thus there is an example x_1 for which $A(x_1)$ is false and an example x_2 for which $B(x_2)$ is false.

As $F \vee F = F$, we have $A(x_1) \vee B(x_2)$ false.

Setting $x = x_1$ and $y = x_2$ gives a counterexample to $\forall x,y\, A(x) \vee B(y)$ showing the RHS false.