Lecture4
Sets

Agenda

– Curly brace notation “{ … }”
– Cardinality “| … |”
– Element containment $\in$
– Subset containment $\subseteq$
– Empty set { } = $\varnothing$
– Power set P(S) = 2^S
– N-tuples “(…)” and Cartesian product $\times$

Agenda(2)

- Set Builder
- Set Operations
  - Union $\cup$ and Disjoint union
  - Intersection $\cap$
  - Difference “–”
  - Complement “ $\overline{}$ ”
  - Symmetric Difference $\oplus$

Sets

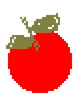
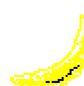
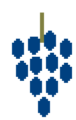
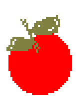
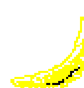
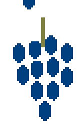
DEF: A ***set*** is a collection of elements.

Sets are the basic data structure out of which most mathematical theories are built.

Sets

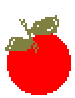
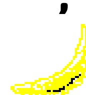
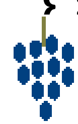
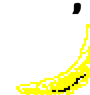
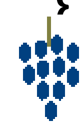
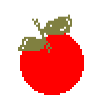
Curly braces "{" and "}" are used to denote sets.

Java note: In Java curly braces denote arrays, a data-structure with inherent ordering. Mathematical sets are *unordered* so different from Java arrays. Java arrays require that all elements be of the same type. Mathematical sets don't require this, however. EG:

- { 11, 12, 13 }
- {  ,  ,  }
- {  ,  ,  , 11, Leo }

Sets

A set is defined only by the elements which it contains. Thus repeating an element, or changing the ordering of elements in the description of the set, does not change the set itself:

- { 11, 11, 11, 12, 13 } = { 11, 12, 13 }
- {  ,  } = { , ,  }

Standard Numerical Sets

- The natural numbers:
  N = { 0, 1, 2, 3, 4, … }
- The integers:
  Z = { … -3, -2, -1, 0, 1, 2, 3, … }
- The positive integers:
  $\mathbf{Z}^+$ = {1, 2, 3, 4, 5, … }
- The real numbers: **R** --contains any decimal number of arbitrary precision
- The rational numbers: **Q** --these are decimal numbers whose decimal expansion repeats

Q: Give examples of numbers in **R** but not **Q**.

Standard Numerical Sets

A:

$\in$-Notation

The Greek letter "$\in$" (epsilon) is used to denote that an object is an *element* of a set. When crossed out "$\notin$" denotes that the object is *not an element*."

EG: $3 \in S$ reads:

"3 is an element of the set S".

Q: Which of the following are true:

1. $3 \in \mathbf{R}$
2. $-3 \in \mathbf{N}$
3. $-3 \in \mathbf{R}$
4. $0 \notin \mathbf{Z}^+$
5. $\exists x \quad x \in \mathbf{R} \land x^2 = -5$

$\in$-Notation

A: 1, 3 and 4

1. $3 \in \mathbf{R}$. True: 3 is a real number.
2. $-3 \in \mathbf{N}$. False: natural numbers don't contain negatives.
3. $-3 \in \mathbf{R}$. True: -3 is a real number.
4. $0 \notin \mathbf{Z}^+$. True: 0 isn't positive.
5. $\exists x \;\; x \in \mathbf{R} \wedge x^2 = -5$. False: square of a real number is non-neg., so can't be -5.

$\subseteq$-Notation

DEF: A set S is said to be a ***subset*** of the set T iff every element of S is also an element of T. This situation is denoted by

$$S \subseteq T$$

A synonym of "subset" is "contained by".

Definitions are often just a means of establishing a logical equivalence which aids in notation. The definition above says that:

$$S \subseteq T \qquad \Leftrightarrow \qquad \forall x \ (x \in S) \rightarrow (x \in T)$$

We already had all the necessary concepts, but the "$\subseteq$" notation saves work.

$\subset$-Notation

When "$\subset$" is used instead of "$\subseteq$", *proper* containment is meant. A subset S of T is said to be a ***proper subset*** if S is not equal to T. Notationally:

$$S \subset T \quad \Leftrightarrow \quad S \subseteq T \wedge \exists x \ (x \notin S \wedge x \in T)$$

Q: What algebraic symbol is $\subset$ reminiscent of?

$\subset$-Notation

A: $\subset$ is to $\subseteq$, as $<$ is to $\leq$.

The Empty Set

The ***empty set*** is the set containing no elements. This set is also called the ***null set*** and is denoted by:

- {}
- $\emptyset$

Subset Examples

Q: Which of the following are true:

1. $\mathbf{N} \subset \mathbf{R}$
2. $\mathbf{Z} \subseteq \mathbf{N}$
3. $-3 \subseteq \mathbf{R}$
4. $\{1,2\} \not\in \mathbf{Z}^+$
5. $\varnothing \subseteq \varnothing$
6. $\varnothing \subset \varnothing$

Subset Examples

A: 1, 4 and 5

1. $\mathbf{N} \subset \mathbf{R}$. All natural numbers are real.

2. $\mathbf{Z} \subseteq \mathbf{N}$. Negative numbers aren't natural.

3. $-3 \subseteq \mathbf{R}$. Nonsensical. -3 is not a subset but an element! (This could have made sense if we viewed -3 as a set –which in principle is the case– in this case the proposition is **false**).

4. $\{1,2\} \notin \mathbf{Z}^+$. This actually makes sense. The set {1,2} is an object in its own right, so could be an element of some set; however, {1,2} is not a number, therefore is not an element of **Z**.

5. $\emptyset \subseteq \emptyset$. Any set contains itself.

6. $\emptyset \subset \emptyset$. No set can contain itself properly.

Cardinality

The ***cardinality*** of a set is the number of distinct elements in the set. $|S|$ denotes the cardinality of S.

Q: Compute each cardinality.

1. $|\{1, -13, 4, -13, 1\}|$
2. $|\{3, \{1,2,3,4\}, \emptyset\}|$
3. $|\{\}|$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}|$

Cardinality

Hint: After eliminating the redundancies just look at the number of top level commas and add 1 (except for the empty set).

A:

1. $|\{1, -13, 4, -13, 1\}| = |\{1, -13, 4\}| = 3$
2. $|\{3, \{1,2,3,4\}, \varnothing\}| = 3$. To see this, set $S = \{1,2,3,4\}$. Compute the cardinality of $\{3, S, \varnothing\}$
3. $|\{\}| = |\varnothing| = 0$
4. $|\{ \{\}, \{\{\}\}, \{\{\{\}\}\} \}| = |\{ \varnothing, \{\varnothing\}, \{\{\varnothing\}\}\}| = 3$

Cardinality

DEF: The set *S* is said to be ***finite*** if its cardinality is a nonnegative integer. Otherwise, *S* is said to be *infinite*.

EG: **N**, **Z**, $\mathbf{Z}^+$, **R**, **Q** are each infinite.

Note: We'll see later that not all infinities are the same. In fact, **R** will end up having a bigger infinity-type than **N**, but surprisingly, **N** has same infinity-type as **Z**, $\mathbf{Z}^+$, and **Q**.

Power Set

DEF: The ***power set*** of S is the set of all subsets of S.

Denote the power set by $P(S)$ or by 2^S.

The latter weird notation comes from the following.

Lemma: $|2^S| = 2^{|S|}$

Power Set – Example

To understand the previous fact consider

$$S = \{1,2,3\}$$

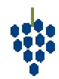
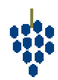
Enumerate all the subsets of S :

0-element sets: {}	1
1-element sets: {1}, {2}, {3}	+3
2-element sets: {1,2}, {1,3}, {2,3}	+3
3-element sets: {1,2,3}	+1

Therefore: $|\, 2^{s} \,| = 8 = 2^{3} = 2^{|s|}$

Ordered *n*-tuples

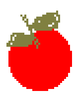
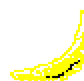
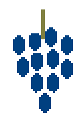
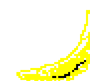
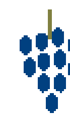
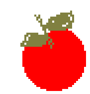
Notationally, *n*-tuples look like sets except that curly braces are replaced by parentheses:

- (11, 12) – a 2-tuple aka ***ordered pair***
- ( ) -  -tuple
- (  ,  Leo) – a 5-tuple

Java: *n* -tuples are similar to Java arrays "{…}", except that type-mixing isn't allowed in Java.

Ordered *n*-tuples

As opposed to sets, repetition and ordering do matter with *n*-tuples.

- (11, 11, 11, 12, 13) $\neq$ (11, 12, 13)
- ( ,  , ) $\neq$ ( ,  , )

Cartesian Product

The most famous example of 2-tuples are points in the Cartesian plane $\mathbf{R}^2$. Here ordered pairs (x,y) of elements of $\mathbf{R}$ describe the coordinates of each point. We can think of the first coordinate as the value on the x-axis and the second coordinate as the value on the y-axis.

DEF: The ***Cartesian product*** of two sets A and B – denoted by $A \times B$– is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Q: Describe $\mathbf{R}^2$ as the Cartesian product of two sets.

Cartesian Product (2)

A: $\mathbf{R}^2 = \mathbf{R}\times\mathbf{R}$. I.e., the Cartesian plane is formed by taking the Cartesian product of the x-axis with the y-axis.

One can generalize the Cartesian product to several sets simultaneously.

Q: If $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$ what is $A \times B \times C$?

Cartesian Product(3)

A: $A = \{1,2\}$, $B = \{3,4\}$, $C = \{5,6,7\}$

$A \times B \times C$ =

{ (1,3,5), (1,3,6), (1,3,7),
(1,4,5), (1,4,6), (1,4,7),
(2,3,5), (2,3,6), (2,3,7),
(2,4,5), (2,4,6), (2,4,7) }

Lemma: The cardinality of the Cartesian product is the product of the cardinalities:

$$| A_1 \times A_2 \times \ldots \times A_n | = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_n|$$

Q: What does $\varnothing \times S$ equal?

Cartesian Product(4)

A: From the lemma:

$$|\varnothing \times S| = |\varnothing| \cdot |S| = 0 \cdot |S| = 0$$

There is only one set with no elements –the empty set– therefore, $\varnothing \times S$ must be the empty set $\varnothing$.

One can also check this directly from the definition of the Cartesian product.

Blackboard Exercise

Prove the following:

If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

Set Builder Notation

Up to now sets have been defined using the curly brace notation "{ … }" or descriptively "the set of all natural numbers". The set builder notation allows for concise definition of new sets. For example

- $\{ x \mid x$ is an even integer $\}$
- $\{ 2x \mid x$ is an integer $\}$

are equivalent ways of specifying the set of all even integers.

Set Builder Notation

In general, one specifies a set by writing

$$\{ f(x) \mid P(x) \}$$

Where $f(x)$ is a function of x (okay we haven't really gotten to functions yet…) and $P(x)$ is a propositional function of x. The notation is read as

"the set of all elements $f(x)$ such that $P(x)$ holds"

- Stuff between "{" and "|"
  - specifies how elements look
- Stuff between the "|" and "}"
  - gives properties elements satisfy
- Pipe symbol "|" is
  - short-hand for "such that".

Set Builder Notation. Shortcuts.

- To specify a subset of a pre-defined set, $f(x)$ takes the form $x \in S$. For example

  $$\{x \in \mathbf{N} \mid \exists y\,(x = 2y)\}$$

  defines the set of all even natural numbers (assuming universe of reference **Z**).
- When universe of reference is understood,

  don't need to specify propositional function EG: $\{x^3 \mid \}$ or simply $\{x^3\}$ specifies the set of perfect cubes

  $$\{0,1,8,27,64,125, \ldots\}$$

  assuming U is the set of natural numbers.

Set Builder Notation. Examples.

Q1: $U = \mathbf{N}$. $\{ x \mid \forall y\,(y \geq x) \} = ?$

Q2: $U = \mathbf{Z}$. $\{ x \mid \forall y\,(y \geq x) \} = ?$

Q3: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^2 = x)\} = ?$

Q4: $U = \mathbf{Z}$. $\{ x \mid \exists y\ (y \in \mathbf{R} \wedge y^3 = x)\} = ?$

Q5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = ?$

Q6: $U = \mathbf{R}$. $\{ |x| \} = ?$

Set Builder Notation. Examples.

A1: $U = \mathbf{N}$. $\{ x \mid \forall y\, (y \geq x) \} = \{ 0 \}$

A2: $U = \mathbf{Z}$. $\{ x \mid \forall y\, (y \geq x) \} = \{ \}$

A3: $U = \mathbf{Z}$. $\{ x \mid \exists y\, (y \in \mathbf{R} \wedge y^2 = x) \}$

$= \{ 0, 1, 2, 3, 4, \ldots \} = \mathbf{N}$

A4: $U = \mathbf{Z}$. $\{ x \mid \exists y\, (y \in \mathbf{R} \wedge y^3 = x) \} = \mathbf{Z}$

A5: $U = \mathbf{R}$. $\{ |x| \mid x \in \mathbf{Z} \} = \mathbf{N}$

A6: $U = \mathbf{R}$. $\{ |x| \}$ = non-negative reals.

Set Theoretic Operations

Set theoretic operations allow us to build new sets out of old, just as the logical connectives allowed us to create compound propositions from simpler propositions. Given sets *A* and *B*, the set theoretic operators are:

- Union ($\cup$)
- Intersection ($\cap$)
- Difference ($-$)
- Complement ("—")
- Symmetric Difference ($\oplus$)

give us new sets $A \cup B$, $A \cap B$, $A - B$, $A \oplus B$, and $\overline{A}$.

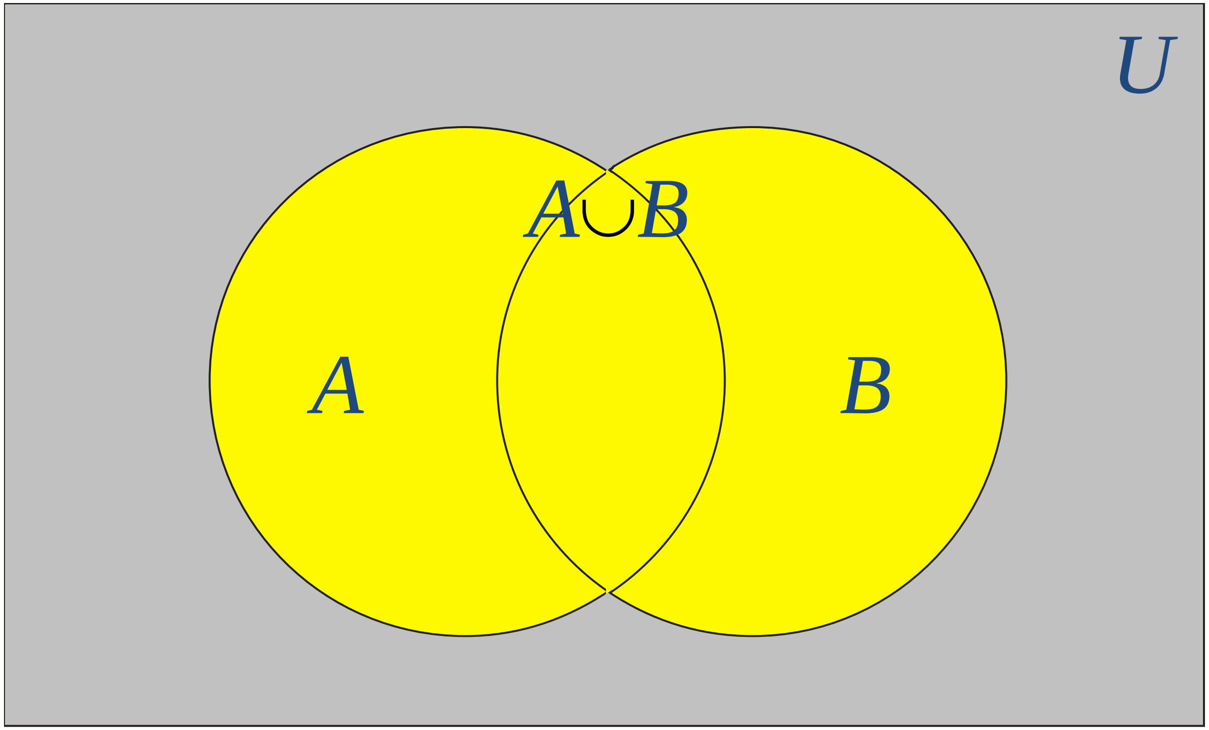
Venn Diagrams

Venn diagrams are useful in representing sets and set operations. Various sets are represented by circles inside a big rectangle representing the universe of reference.

Union

Elements in at least one of the two sets:

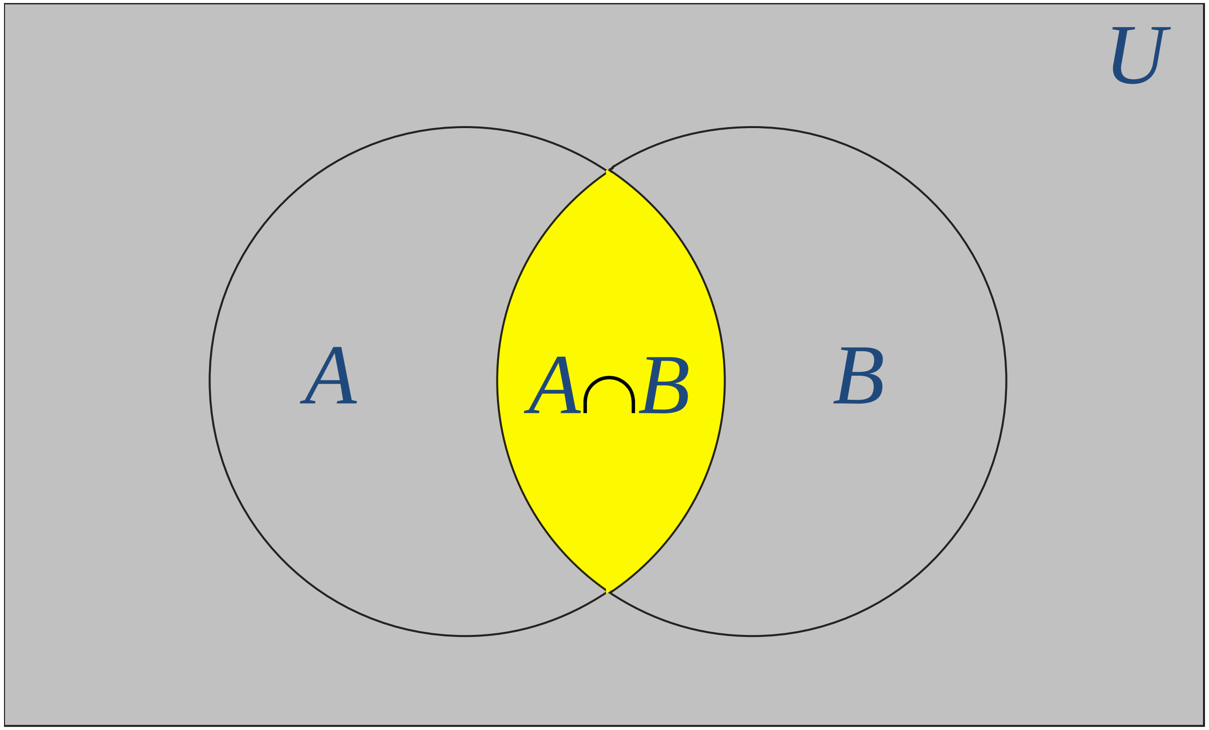
$$A \cup B = \{ x \mid x \in A \vee x \in B \}$$



Intersection

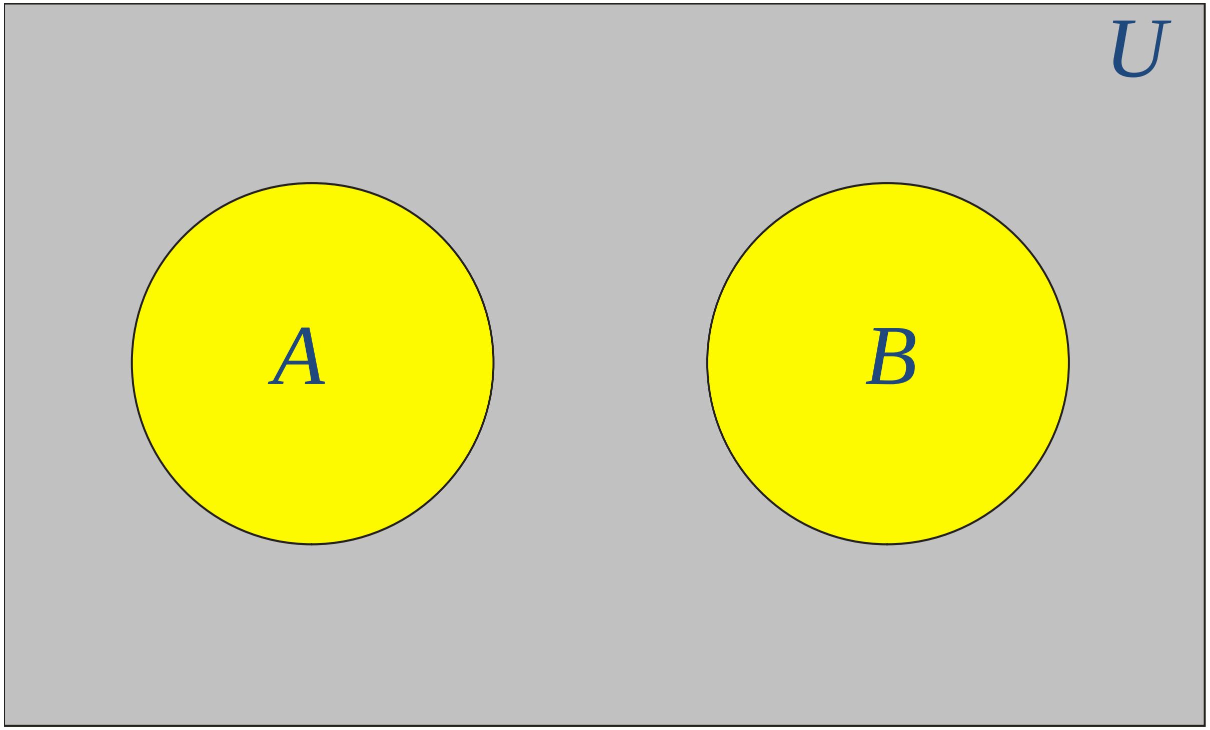
Elements in exactly one of the two sets:

$$A \cap B = \{\, x \mid x \in A \wedge x \in B \,\}$$



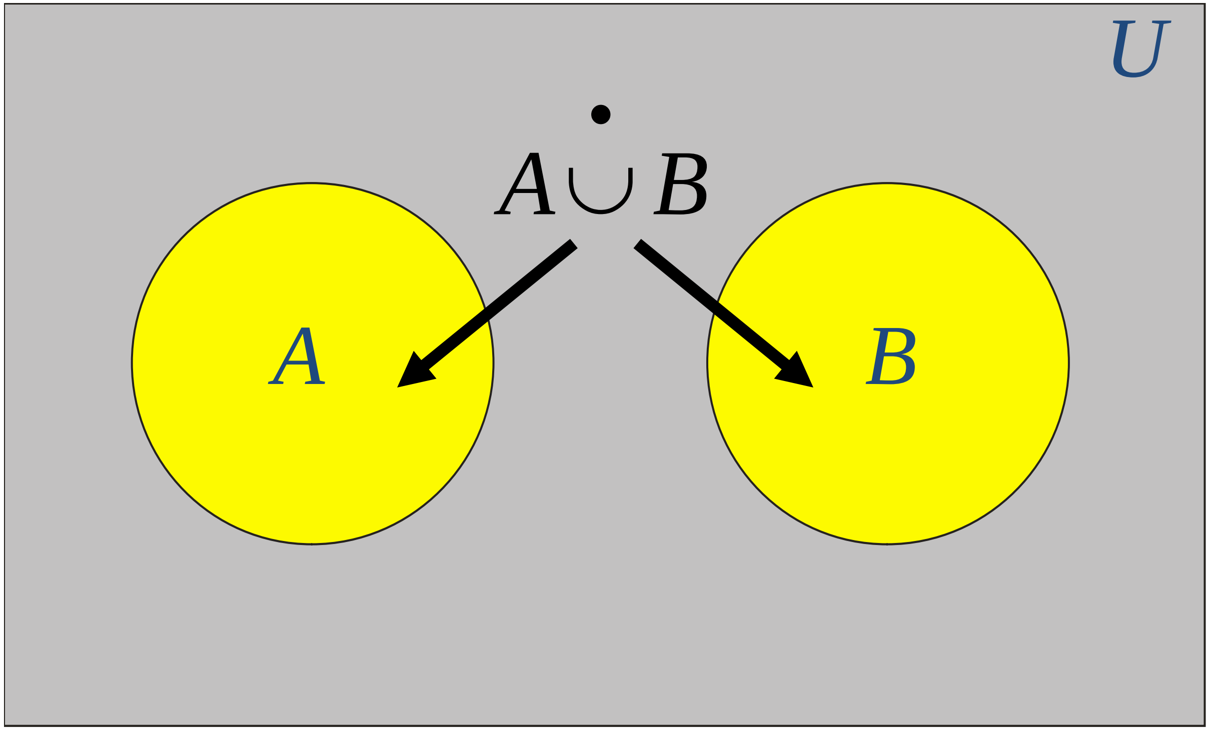
Disjoint Sets

DEF: If A and B have no common elements, they are said to be ***disjoint***, i.e. $A \cap B = \varnothing$.



Disjoint Union

When A and B are disjoint, the ***disjoint union*** operation is well defined. The circle above the union symbol indicates disjointedness.



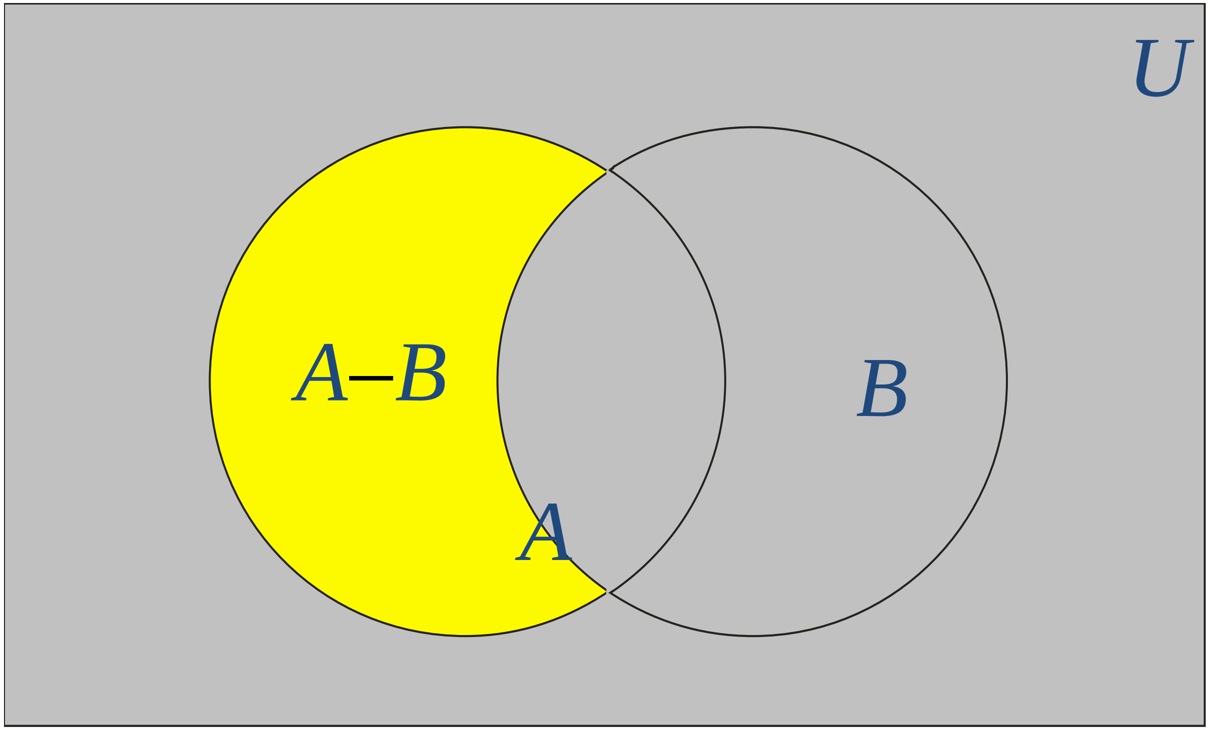
Disjoint Union

FACT: In a disjoint union of finite sets, cardinality of the union is the sum of the cardinalities. I.e.

Set Difference

Elements in first set but not second:

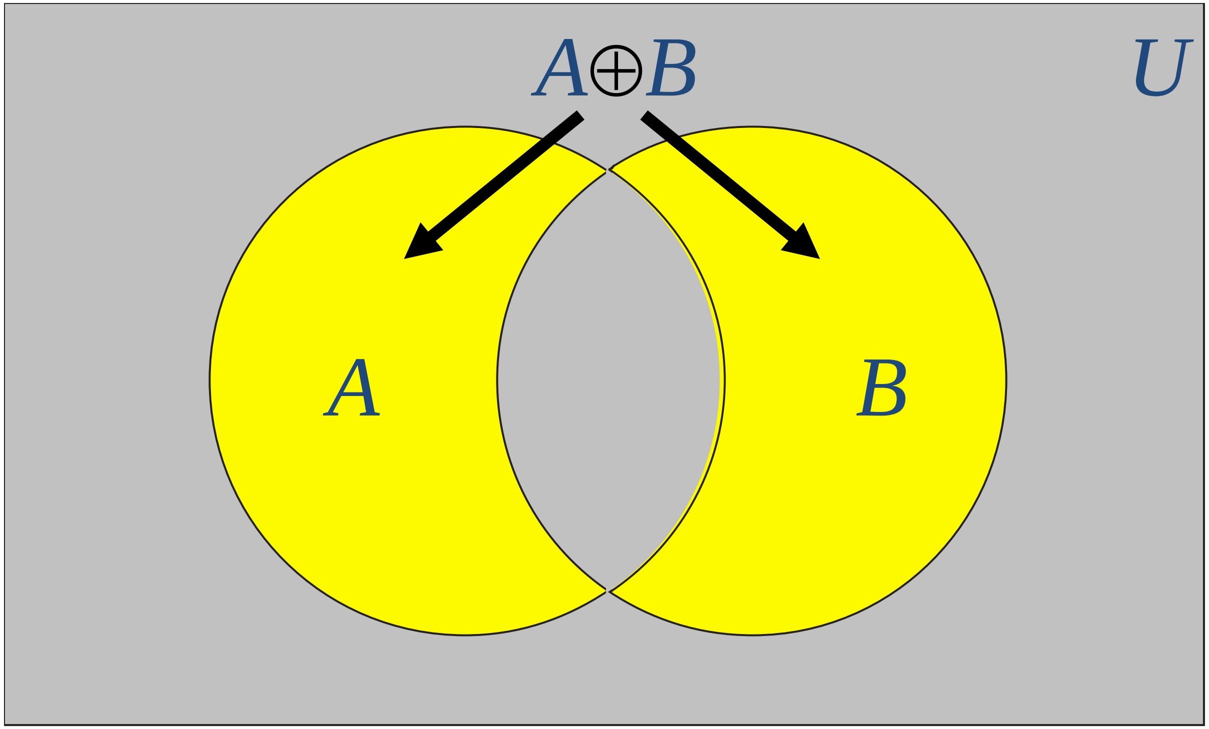
$$A - B = \{\, x \mid x \in A \wedge x \notin B \,\}$$



Symmetric Difference

Elements in exactly one of the two sets:

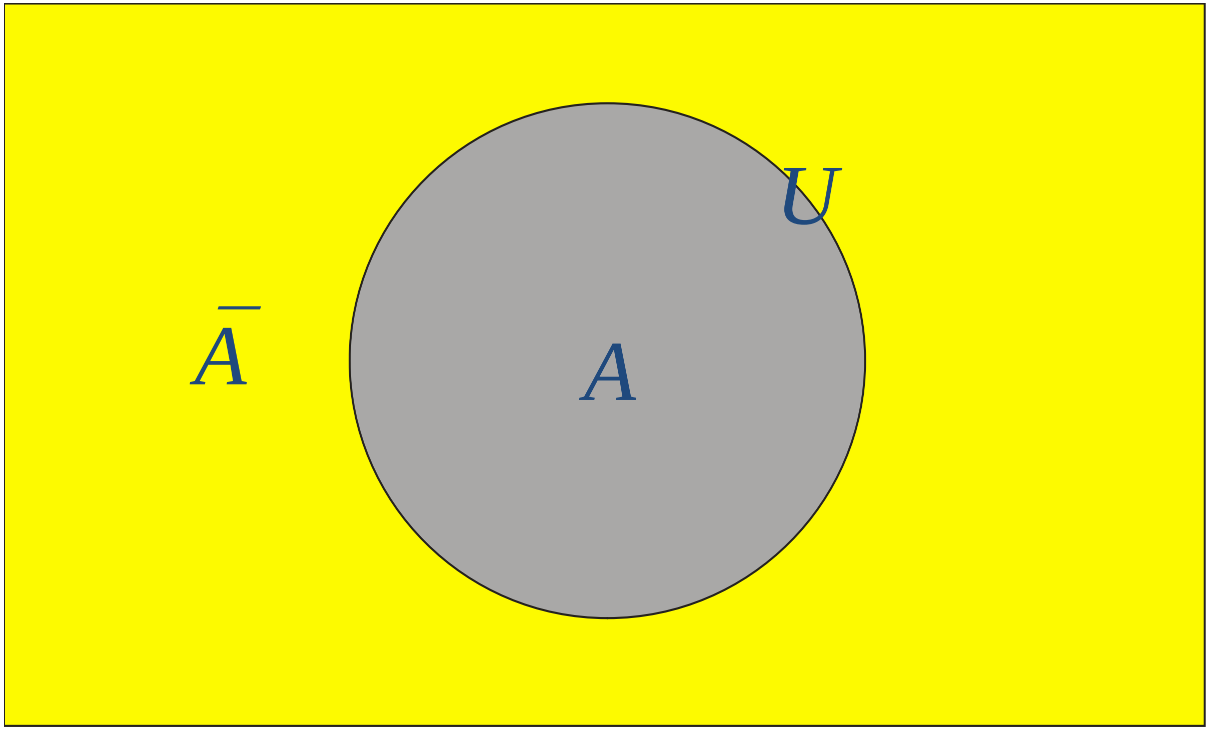
$$A \oplus B = \{\ x \mid x \in A \ \oplus\ x \in B\ \}$$



Complement

Elements not in the set (unary operator):

$$\bar{A} = \{\ x \mid x \notin A\ \}$$



Set Identities

Table 1, Rosen p. 49

- Identity laws
- Domination laws
- Idempotent laws
- Double complementation
- Commutativity
- Associativity
- Distribuitivity
- DeMorgan

This table is gotten from the previous table of logical identities (Table 5, p. 17) by rewriting as follows:

- *disjunction* "$\vee$" becomes *union* "$\cup$"
- *conjunction* "$\wedge$" becomes *intersection* "$\cap$"
- *negation* "$\neg$" becomes *complementation* "$-$"
- *false* "F" becomes *the empty set* $\varnothing$
- *true* "T" becomes *the universe of reference U*

Set Identities(2)

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof : $(A \cup B) \cup C = \{x \mid x \in A \cup B \lor x \in C\}$ (by def.)

Set Identities(3)

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof : $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)

$= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)

Set Identities(4)

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof : $(A \cup B) \cup C = \{x \mid x \in A \cup B \lor x \in C\}$ (by def.)

$= \{x \mid (x \in A \lor x \in B) \lor x \in C\}$ (by def.)

$= \{x \mid x \in A \lor (x \in B \lor x \in C)\}$ (logical assoc.)

Set Identities(5)

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof : $(A \cup B) \cup C = \{x \mid x \in A \cup B \vee x \in C\}$ (by def.)

$= \{x \mid (x \in A \vee x \in B) \vee x \in C\}$ (by def.)

$= \{x \mid x \in A \vee (x \in B \vee x \in C)\}$ (logical assoc.)

$= \{x \mid x \in A \vee x \in B \cup C)\}$ (by def.)

Set Identities(6)

In fact, the logical identities *create* the set identities by applying the definitions of the various set operations. For example:

LEMMA: (Associativity of Unions)

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof : $(A \cup B) \cup C = \{x \mid x \in A \cup B \lor x \in C\}$ (by def.)

$= \{x \mid (x \in A \lor x \in B) \lor x \in C\}$ (by def.)

$= \{x \mid x \in A \lor (x \in B \lor x \in C)\}$ (logical assoc.)

$= \{x \mid x \in A \lor (x \in B \cup C)\}$ (by def.)

$= A \cup (B \cup C)$ (by def.)

•

Other identities are derived similarly.

Set Identities via Venn

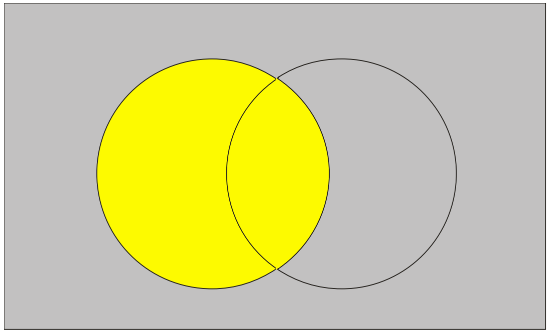
It's often simpler to understand an identity by drawing a Venn Diagram.

For example DeMorgan's first law

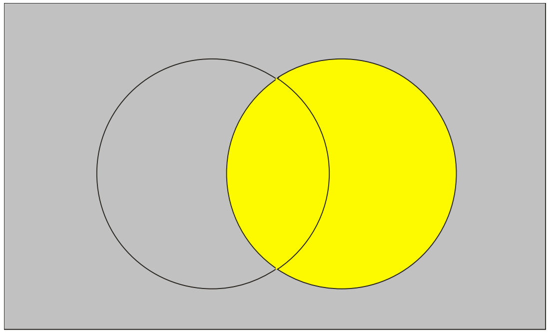
can be visualized as follows.

Visual DeMorgan

A:

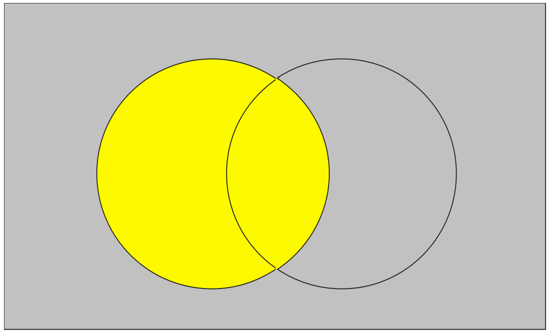


B:



Visual DeMorgan

A:



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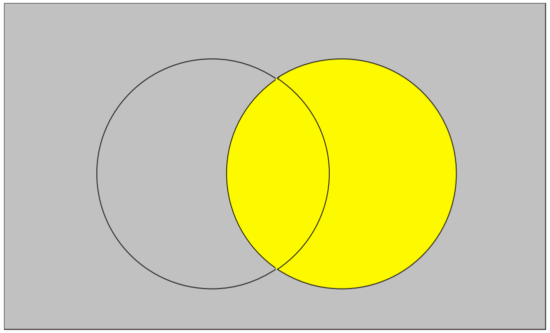
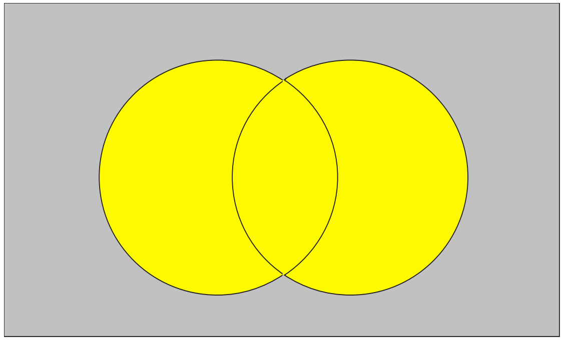
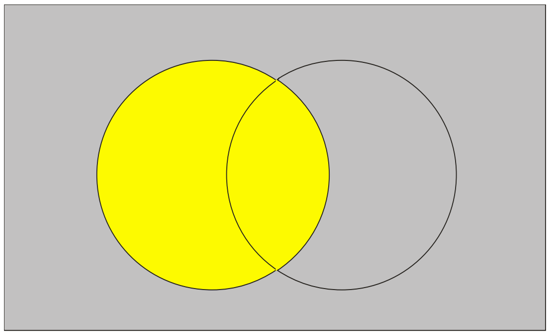


$A \cup B$:



Visual DeMorgan

A:



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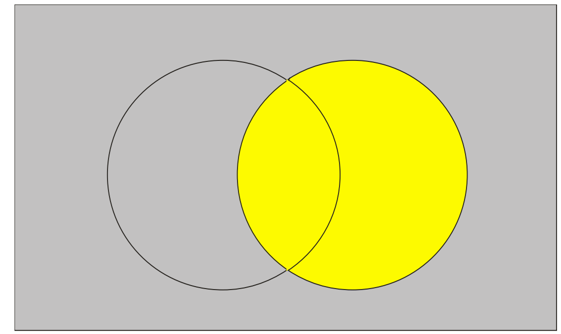

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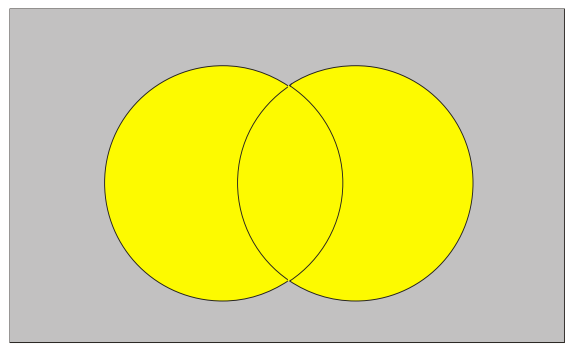

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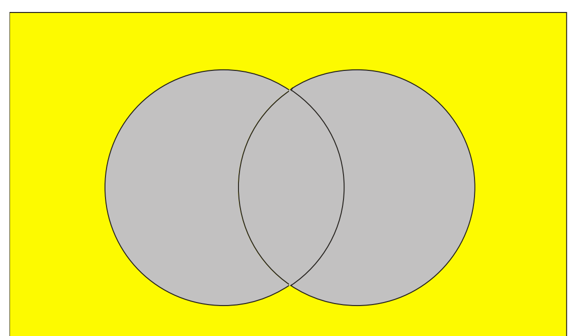
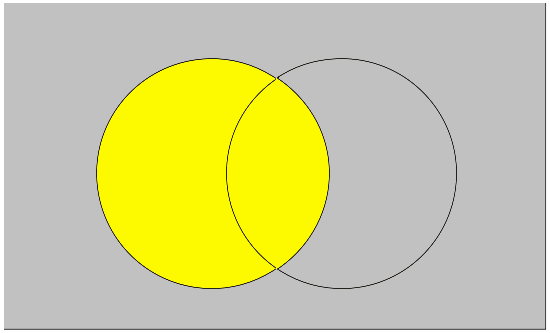
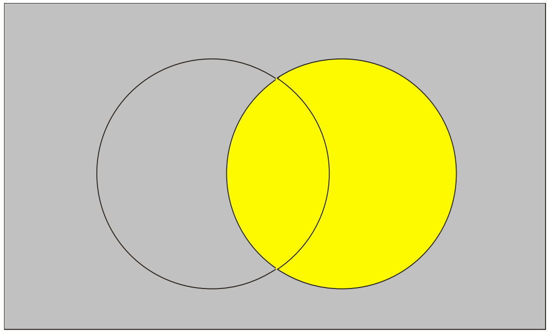


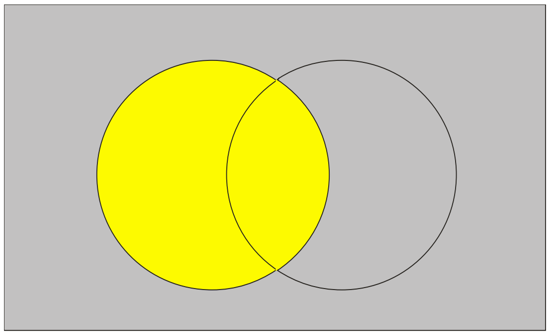
Visual DeMorgan

A: 

B: 

Visual DeMorgan

A:



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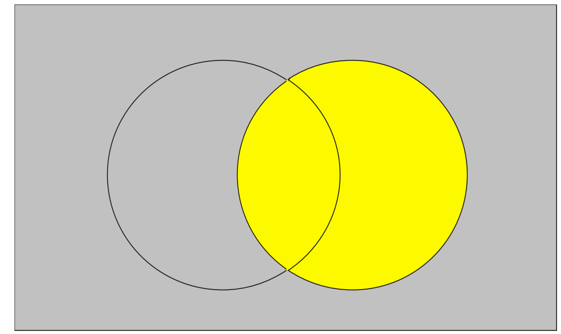

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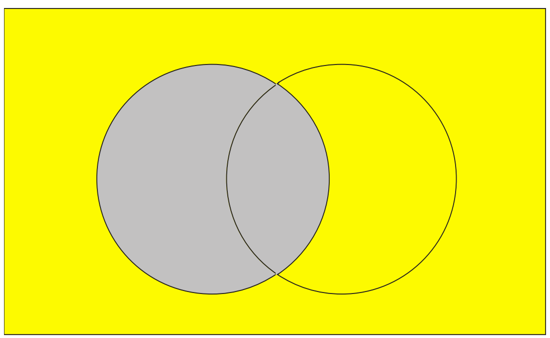

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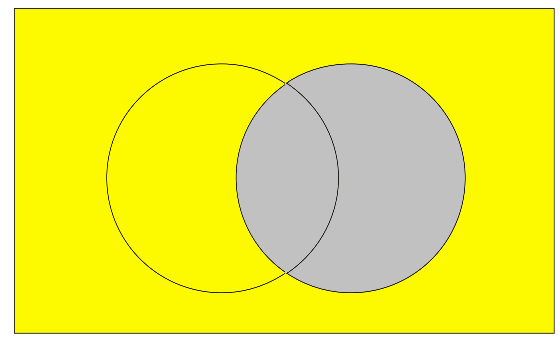
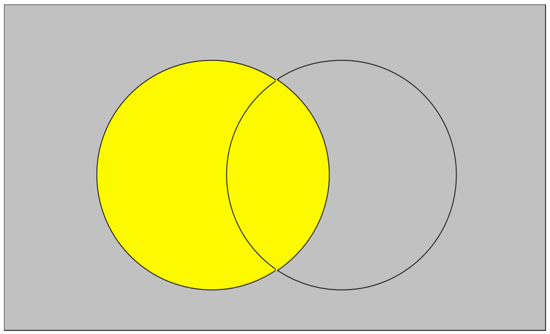


Visual DeMorgan

A:



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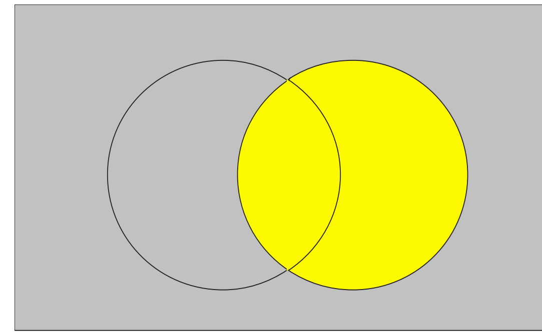

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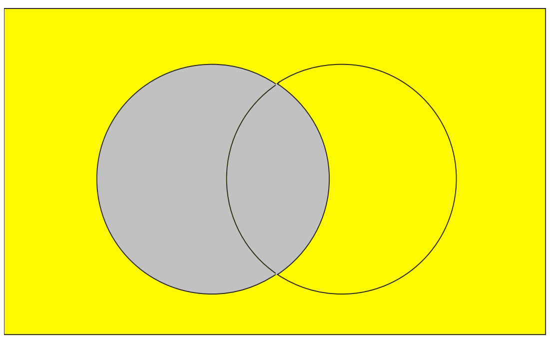

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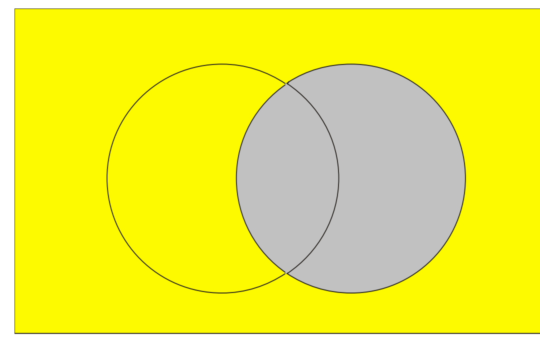
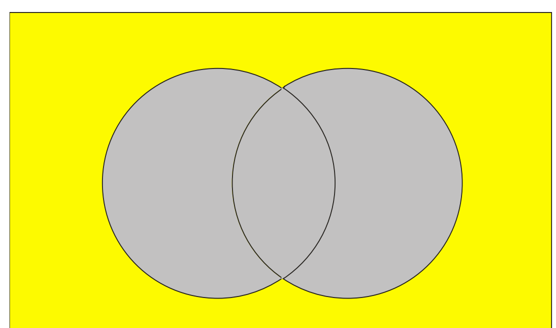
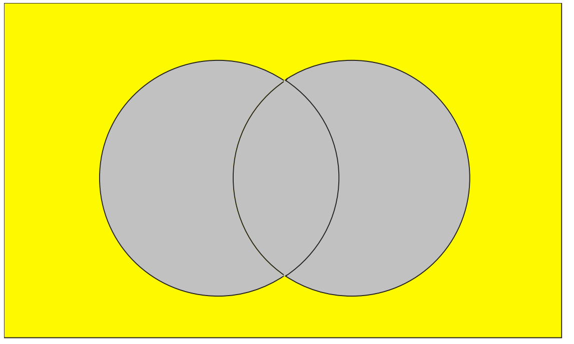


$\overline{A} \cap \overline{B}$:



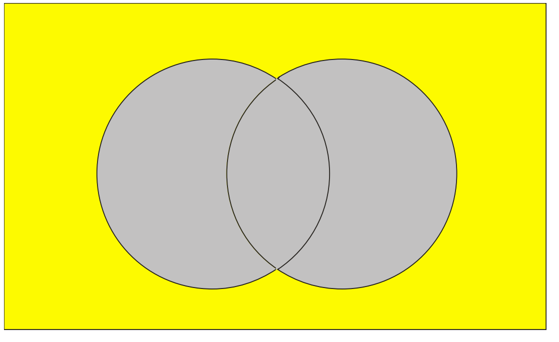
Visual DeMorgan

$$\overline{A \cup B} =$$



=

$$\overline{A} \cap \overline{B} =$$



Sets as Bit-Strings

If we order the elements of our universe, we can represent sets by bit-strings. For example, consider the universe

$$U = \{\text{ant, beetle, cicada, dragonfly}\}$$

Order the elements alphabetically. Subsets of U are represented by bit-strings of length 4. Each bit in turn, tells us whether the corresponding element is contained in the set. EG: {ant, dragonfly} is represented by the bit-string 1001.

Q: What set is represented by 0111 ?

Sets as Bit-Strings

A: 0111 represents

{beetle, cicada, dragonfly}

Conveniently, under this representation the various set theoretic operations become the logical bit-string operators that we saw before. For example, the symmetric difference of {beetle} with {ant, beetle, dragonfly} is represented by:

$$\begin{array}{r} 0100 \\ \oplus \quad \underline{1101} \\ 1001 \end{array} = \{\text{ant, dragonfly}\}$$