Rules of Inference
Detailed w/ Step-by-Step 7 Examples!

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Have you heard of the rules of inference?

They're especially important in logical arguments and proofs, let's find out why!

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While the word "argument" may mean a disagreement between two or more people, in mathematical logic, an argument is a sequence or list of statements called premises or assumptions and returns a conclusion.

An argument is only valid when the conclusion, which is the final statement of the opinion, follows the truth of the discussion's preceding assertions.

Consequently, it is our goal to determine the conclusion's truth values based on the rules of inference.

Definition

The **rules of inference** (also known as inference rules) are a logical form or guide consisting of premises (or hypotheses) and draws a conclusion.

A **valid argument** is when the conclusion is true whenever all the beliefs are true, and an **invalid argument** is called a fallacy as noted by Monroe Community College.

In other words, an argument is valid when the conclusion logically follows from the truth values of all the premises.

There are two ways to form logical arguments, as seen in the image below. We will be utilizing both formats in this lesson to become familiar and comfortable with their framework.

$$\begin{array}{l} \textit{Premise } 1 \\ \textit{Premise } 2 \\ \quad\vdots \\ \underline{\textit{Premise } \mathrm{n}} \\ \therefore \textit{Conclusion} \end{array}$$

OR

$$\textit{Premise } 1, \textit{Premise } 2, \ldots, \textit{Premise } 2 \mapsto \textit{Conclusion}$$

$$P_1, P_2, \ldots, P_n \Rightarrow C$$

Valid Argument — Definition

Basic Example

Now, before we jump into the inference rules, let's look at a basic example to help us understand the notion of assumptions and conclusions.

Is this argument valid?

If Marcus is a poet, then he is poor.
Marcus is a poet.

∴ Marcus is poor.

Logical Argument — Example

Without using our rules of logic, we can determine its truth value one of two ways.

1. Surmising the fallacy of each premise, knowing that the conclusion is valid only when all the beliefs are valid.
2. Construct a truth table and verify a tautology.

From the above example, if we know that both premises "If Marcus is a poet, then he is poor" and "Marcus is a poet" are both true, then the conclusion "Marcus is poor" must also be true.

And using a truth table validates our claim as well.

$$\begin{array}{l} Poet(P) \to Poor(Q) \\ Poet(P) \\ \hline \therefore Poor(Q) \end{array}$$

$(P \to Q) \wedge P \Rightarrow Q$

P	Q	$P \to Q$	$(P \to Q) \wedge P$	$((P \to Q) \wedge P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Valid Argument Truth Table

Rules Of Inference Examples

But what if there are multiple premises and constructing a truth table isn't feasible?

Thankfully, we can follow the Inference Rules for Propositional Logic!

Rule	Tautology	Name
$p \to q$ p $\therefore q$	$((p \to q) \wedge p) \Rightarrow q$	Modus Ponens (Law of Detachment)
$p \to q$ $\neg q$ $\therefore \neg p$	$((p \to q) \wedge \neg q) \Rightarrow \neg q$	Modus Tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \Rightarrow (p \to r)$	Hypothetical Syllogism (Transitivity)
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \Rightarrow q$	Disjunctive Syllogism
p $\therefore p \vee q$	$p \Rightarrow p \vee q$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \Rightarrow p$	Simplification
p q $\therefore p \wedge q$	$(p) \wedge (q) \Rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \Rightarrow (q \vee r)$	Resolution

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Rules Of Inference — Chart

Now, these rules may seem a little daunting at first, but the more we use them and see them in action, the easier it will become to remember and apply them.

Let's look at an example for each of these rules to help us make sense of things.
Let p be "It is raining," and q be "I will make tea," and r be "I will read a book."

Example — Modus Ponens

"If it is raining, then I will make tea."	$p \to q$	**Modus Ponens (Law of Detachment)**
"It is raining."	p	$((p \to q) \wedge p) \Rightarrow q$
"Therefore, I will make tea."	$\therefore q$	

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Modus Ponens — Example

Example — Modus Tollens

"If it is raining, then I will make tea." $p \to q$
"I do not make tea." $\neg q$
"Therefore, it is not raining." $\therefore \neg p$

Modus Tollens
$$\left(\left(p \to q\right) \wedge \neg q\right) \Rightarrow \neg q$$

Modus Tollens — Example

Example — Hypothetical Syllogism

"If it is raining, then I will make tea." $p \to q$
"If i make tea, then I will read a book." $q \to r$
"Therefore, if it rains, then I will read a book." $\therefore p \to r$

Hypothetical Syllogism
$$\left(\left(p \to q\right) \wedge \left(q \to r\right)\right) \Rightarrow \left(p \to r\right)$$

Hypothetical Syllogism — Example

Example — Disjunctive Syllogism

"I will make tea, or I will read a book." $q \vee r$
"I will not make tea." $\neg q$
"Therefore, I will read a book." $\therefore r$

Disjunctive Syllogism
$$\left(\left(q \vee r\right) \wedge \neg q\right) \Rightarrow r$$

Disjunctive Syllogism — Example

Disjunctive Syllogism — Example

Example — Addition

"I will make tea."	q	**Addition**
"Therefore, I will make tea, or I will read a book."	$\therefore q \vee r$	$q \Rightarrow q \vee r$

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Rules Of Inference Addition — Example

Example — Simplification

"I will make tea and I will read a book."	$q \wedge r$	**Simplification**
"Therefore, if it rains, then i will read a book."	$\therefore q$	$(p \wedge q) \Rightarrow p$

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Simplification Discrete Math — Example

"It is raining, or I will make tea."	$p \vee q$	**Resolution**
"It is not raining, or I will read a book."	$\neg p \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \Rightarrow (q \vee r)$
"Therefore, I will make tea, or I will read a book."	$\therefore q \vee r$	

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Discrete Math Resolution — Example

Valid Vs Invalid Argument

Alright, so now let's see if we can determine if an argument is valid or invalid using our logic rules.

Test the validity of the argument:

- If it snows, Paul will miss class.
- Paul did not miss class.
- Therefore, it did not snow.

First, we will translate the argument into symbolic form and then determine if it matches one of our rules.

If it snows, Paul will miss class.	$p \to q$	
Paul did not miss class.	$\neg q$	
It did not snow.	$\therefore \neg p$	

VALID

Modus Tollens: $\left(\left(p \to q\right) \wedge \neg q\right) \Rightarrow \neg q$

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Example Of Modus Tollens Argument

Because the argument matches one of our known logic rules, we can confidently state that the conclusion is valid.

Let's look at another example.

Test the validity of the argument:

- If it snows, Paul will miss class.
- It did not snow.
- Therefore, Paul did not miss class.

So, now we will translate the argument into symbolic form and then determine if it matches one of our rules for inference.

Because the argument does not match one of our known rules, we determine that the conclusion is invalid.

Here's a big hint…

…translating arguments into symbols is a great way to decipher whether or not we have a valid rule of inference or not.

Discrete Math Quantifiers

But what about the quantified statement? How do we apply rules of inference to universal or existential quantifiers?

A quantified statement helps us to determine the truth of elements for a given predicate. And if we recall, a predicate is a statement that contains a specific number of variables (terms).

There are types of quantifiers:

1. Universal Quantification (all, any, each, every)
2. Existential Quantification (there exists, some, at least one)

And what you will find is that the inference rules become incredibly beneficial when applied to quantified statements because they allow us to prove more complex arguments.

Let's look at the logic rules for quantified statements and a few examples to help us make sense of things.

Name	Rule	Example
Universal Instantiation	$\forall x P(x)$ $\therefore P(c)$	"All women are brave." "Therefore, Lily is brave."
Universal Generalization	$P(c)$ *for an arbitrary c* $\therefore \forall x P(x)$	"Lily is brave." "Therefore, all women are brave."
Existential Instantiation	$\exists x P(x)$ $\therefore P(c)$ *for some element c*	"There is someone who ran a mile in 4 minutes." "Let's call him Sparky and say that Sparky ran a mile in 4 minutes."
Existential Generalization	$P(c)$ *for some element c* $\therefore \exists x P(x)$	"Sparky ran a mile in 4 minutes." "Therefore, someone ran a mile in 4 minutes."

Quantified Statement

It is essential to point out that it is possible to infer invalid statements from true ones when dealing with Universal Generalization and Existential Generalization. So, we have to be careful about how we formulate our reasoning.

For example, suppose we said:

- "Lily is a gymnast."
- "Therefore, all women are gymnasts."

This line of reasoning is over-generalized, as we inferred the wrong conclusion, seeing that not all women are a gymnast.

Lewis Carroll – Example

Okay, so let's see how we can use our inference rules for a classic example, complements of Lewis Carroll, the famed author Alice in Wonderland.

- "All lions are fierce."
- "Some lions do not drink coffee."
- "Some fierce creatures do not drink coffee."

So, this means we are given to premises, and we want to know whether we can conclude "some fierce creatures do not drink coffee."

Let's let L(x) be "x is a lion," F(x) be "x is fierce," and C(x) be "x drinks coffee."

All lions are fierce. $\forall x\left(L(x) \rightarrow F(x)\right)$

Some lions do not drink coffee. $\exists x\left(L(x) \wedge \neg C(x)\right)$

Some fierce creatures do not drink coffee. $\exists x\left(F(x) \wedge \neg C(x)\right)$

Lewis Carroll Logic — Example

But the problem is, how do we conclude the last line of the argument from the two given assertions?

If we can prove this argument is true for one element, then we have shown that it is true for others.

Let's let Lambert be our element. This means that Lambert is a lion who is fierce and doesn't drink coffee.

1.	Some lions do not drink coffee.	$\exists x\left(F(x) \wedge \neg C(x)\right)$	Premise (given)
2.	Let's call him Lambert and say that Lambert is a lion that doesn't drink coffee	$L(Lambert) \wedge \neg C(Lambert)$	Existential Instantiation from (1)
3.	If Lambert is both a lion and a non-coffee drinker, then we can simplify and say Lambert is a lion	$L(Lambert)$	Simplification from (2)
4.	If Lambert it both a lion and a non-coffee drinker, then we can simplify and say Lambert does not drink coffee	$\neg C(Lambert)$	Simplification from (2)
5.	All lions are fierce.	$\forall x\left(L(x) \rightarrow F(x)\right)$	Premise (given)
6.	If all lions are fierce and Lambert is a lion, then Lambert is fierce.	$L(Lambert) \rightarrow F(Lambert)$	Universal Instantiation from (5)
7.	If Lambert is a lion, then Lambert is fierce. And we know that Lambert is a lion is true, then we can say that Lambert is fierce.	$F(Lambert)$	Modus Ponens (Law of Detachment) from (3) and (6)
8.	Lambert is fierce and doesn't drink coffee.	$F(Lambert) \wedge \neg C(Lambert)$	Conjunction from (4) and (7)
9.	Therefore, we have shown that if Lambert is fierce and does not drink coffee, then there is some fierce creature who does not drink coffee.	$\exists x\left(L(x) \wedge \neg C(x)\right)$	Existential Generalization form (8)

Proof Quantified Statements

We did it! By using a particular element (Lambert) and proving that Lambert is a fierce creature that does not drink coffee, then we were able to generalize this to say, "some creature(s) do not drink coffee."

Together we will use our inference rules along with quantification to draw conclusions and determine truth or falsehood for arguments.

Let's jump right in!

Video Tutorial w/ Full Lesson & Detailed Examples

1 hr 33 min

- Introduction to Video: Rules of Inference
- **00:00:57** Understanding logical arguments
- **Exclusive Content for Members Only**
  - **00:14:41** Inference Rules with tautologies and examples
  - **00:22:28** What rule of inference is used in each argument? (Example #1a-e)

- 00:22:28 What rule of inference is used in each argument? (Example #1a-e)
- **00:26:44** Determine the logical conclusion to make the argument valid (Example #2a-e)
- **00:30:07** Write the argument form and determine its validity (Example #3a-f)
- **00:33:01** Rules of Inference for Quantified Statement
- **00:35:59** Determine if the quantified argument is valid (Example #4a-d)
- **00:41:03** Given the predicates and domain
- **00:51:04** Construct a valid argument using the inference rules (Example #7)
- **Practice Problems** with Step-by-Step Solutions
- **Chapter Tests** with Video Solutions