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(1)

$F(x) = \Sigma P(x)$

Question 1

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RANDOM VARIABLE & THEIR PROB. DN.

Random Variable

A random Variable is a numerical Valued function defined on every element in a sample space.

Example 1:

In a throw of two coins; Let X represent the number of heads. Then the random variable X (no of heads) can take values; 0 1 1 2

$S = \{TT, HT, TH, HH\}$

Example 2:

A die is thrown twice; Let X and Y rep. the no that turns up in the first and second throw respectively.

(i) Define the sample space

(ii) If $Z = X+Y$, find (a) $P(Z \leq 8)$ (b) $P(3 \leq Z \leq 8)$

Soln:

(i) The sample space can be rep. thus;

| X/Y | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | (1,1) | - | - | - | | (1,6) |
| 2 | | | | | | |
| ⋮ | | | | | | |
| 6 | | | | | | (6,6) |

(ii) The r.v $Z = X+Y$ can take values

2 3 - - - - - - 7

3

⋮

7 - - - - - - - 12

(a) $P(Z \leq 8) = 26/36$ (b) $P(3 \leq Z \leq 8) = 25/36$

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~~Example 2~~ Discrete Random Variables:

A discrete random variable is a r.v that can take on at most a countable number of possible values.

For a dist. r.v. X, the prob. mass fun (PMF) P(x), is defined as follows;

$$P(x) = P(X = x)$$

which satisfies the following conditions

(i) $P(x) \geq 0$ for all $x$

(ii) $\sum_x P(x) = 1$

(iii) $P\ a \leq x \leq b = \sum_{x=a}^{b} P(x)$ for $a<b$

Examples

(1) The prob. Mass fn is given by;

$$P(x) = \begin{cases} 1/3 & \text{for } x = -1 \\ 2/5 & \text{for } x = 0 \\ 4/15 & \text{for } x = 2 \\ 0 & \text{o/w} \end{cases}$$

Can the fn be said to be a pmf?

Soln: it is necessary to show $\sum P(x) = 1$

i.e. $1/3 + 2/5 + 4/15 = 1$

Hence the fn is a pmf.

(2) Let the r.v X denote the no. of heads in 3 tosses of a fair coin, what is the Pmf of X?

soln: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

The different events defined by the r.v X are as follows;

$[X = 0] = \{TTT\}$

$[X = 1] = \{HTT, THT, TTH\}$

$[X = 2] = \{HHT, THH, HTH\}$

$[X = 3] = \{HHH\}$

Hence, Pmf is

$$P(x) = \begin{cases} 1/8, & x=0 \\ 3/8, & x=1 \\ 3/8, & x=2 \\ 1/8, & x=3 \end{cases}$$

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(3) Let the r.v X denote the sum obtained in rolling a pair of fair dice. Determine the PMF of X.

Soln:

Let the pair (a,b) denote the outcomes of the roll, where a & b = Outcome of one & 8 other die resp.

$\therefore X = a+b$

The diff. events defined by the r.v X is as follows;

$[X=2] = \{(1,1)\}$

$[X=3] = \{(1,2)\ (2,1)\}$

$[X=4] = \{(1,3), (2,2)\ (3,1)\}$

$[X=5] = \{(1,4)\ (2,3), (3,2)\ (4,1)\}$

$\vdots$

$[X=12] = \{(6,6)\}$

$$P(X) = \begin{cases} 1/36, & x=2 \\ 2/36, & x=3 \\ 3/36, & x=4 \\ 4/36, & x=5 \\ 5/36, & x=6 \\ 6/36, & x=7 \\ 5/36, & x=8 \\ 4/36 & x=9 \\ 3/36 & x=10 \\ 2/36 & x=11 \\ 1/36 & x=12 \end{cases}$$

(4). A discrete r.v X has the PMF given by

$$P(X) = \begin{cases} K(x-1)^2 ; & x = 0,1,\ldots 4 \\ 0 & o/w \end{cases}$$

(a) Determine the value of K.

(b) Find the $P(1 \le X \le 3)$

soln: (a) Since $\sum P(X) = 1$

then $\sum_{i=0}^{4} K(x-1)^2 = 1$

i.e $K(1+0+1+4+9) = 1$

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$P(x) = P(X=x)$

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Question.....

$\therefore K = 1/15$

$$P \begin{cases} 1/15, & x=0 \\ 0, & x=1 \\ 1/15, & x=2 \\ 4/15, & x=3 \\ 2/5, & x=4 \\ 0 & o/w \end{cases}$$

(b) $P(1 \le X \le 3) = P(X=1) + P(X=2) + P(X=3)$

$= 0 + 1/15 + 4/15 = 1/3$

(5) The PMF of the no. N of customers that arrive at local library within one hour interval is defined by

$$P(N) = \begin{cases} 5^n/n!\, e^{-5} & ; n = 0, 1, \ldots \\ 0 & o/w \end{cases}$$

What is the prob that at most two customers arrive at the library within an given hour?

Soln: The prob that at most two customers arrive at a library with 1 hr is a prob that zero or 1 or 2 customers

i.e. $P(N \le 2) = P\{(N=0) \cup (N=1) \cup (N=2)\} = P\{N=0\} + P(N=1) + P(N=2)$

$= e^{-5}\left(1 + 5 + 25/2\right)$

$= 18.5e^{-5} = 0.1246$

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# Continous Random Variable

Discrete random variables have a set of possible value that are either finite or countably infinite. However, there exists another group of random variables that can assume an uncountable set of possible values. Such random variables are called continous random variables. Thus, we define a r.v X to be a cont- r.v. if $\exists$ a non-ve function $f(x)$ defined for all real $x \in (-\infty, \infty)$, having the following properties;

(i) $f(x) \geq 0$ for $-\infty \leq x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $\int_a^b f(x)\,dx = P(a < x < b)$ for $a < b$

The funct $f(x)$ is called the prob density fn (PDF) of r.v X

Examples

(i) Assume that X is a continous r.v with the following Pdf

$$f(x) = \begin{cases} A(2x - x^2) & 0 < x < 2 \\ 0 & o/w. \end{cases}$$

(a) what is the value of A? (b) Find $P(x > 1)$

soln: (a) Since $f(x)$ is a Pdf, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 A(2x - x^2)\,dx = 1$$

$$\therefore A\left[x^2 - x^3/3\right]_0^2 = 1$$

$$A\left(4 - 8/3\right) = 4A/3 = 1$$

$$A = 3/4$$

(b) $\therefore P(X > 1) = \int_1^{\infty} f(x)\,dx$

$$= \frac{3}{4}\int_1^{\infty}(2x - x^2)\,dx = 3/4\left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= 3/4\left(4/3 - 7/3\right)$$

$$= 1/2$$

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Question.....

(3) Is the following fn a legitimate Pdf?

$$f(x) = \begin{cases} x^2/9 & 0 \le x \le 3 \\ 0 & o/w \end{cases}$$

soln: for $f(x)$ to be a legitimate Pdf, we need to check if $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 \frac{x^2}{9}\,dx = \frac{x^3}{27}\Big|_0^3 = 1$$

$\therefore$ $f(x)$ is a Pdf.

(3) Consider the funtn

$$g(x) = \begin{cases} c & a \le x \le b \\ 0 & o/w \end{cases}$$

(a) For what value of $c$ is $g(x)$ a legitimate Pdf?

(b) soln

$$\int_a^b c\,dx = cx\Big|_a^b = c(b-a) = 1$$

$$\therefore c = 1/b-a$$

[4]

A continuous r.v $X$ has Pdf

$$f(x) = \begin{cases} Kx, & 0 \le x \le 1 \\ K(2-x); & 1 \le x \le 2 \\ 0 & o/w \end{cases}$$

(a) Find $K$ (b) Find $P(0.8 \le X \le 1.8)$

Soln:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$ we have

$$\int_{-\infty}^{\infty} Kx\,dx + \int_1^2 K(2-x)\,dx = 1$$

$$K\left(\frac{x^2}{2}\right)\Big|_0^1 + K\left(2x - \frac{x^2}{2}\right)\Big|_1^2 = 1$$

$$K\left(\frac{1}{2} - 0\right) + K(4-2) - \left(2 - \frac{1}{2}\right) = 1$$

$$K\left(\frac{1}{2} + 2 - 2 + \frac{1}{2}\right) = 1$$

$$\therefore K = 1$$

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$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & o/w \end{cases}$$

$$P(0.8 \le x \le 1.8) = \int_{0.8}^{1.0} x\,dx + \int_{1.0}^{1.8} (2-x)\,dx$$

$$= x^2/2 \Big|_{0.8}^{1.0} + 2x - x^2/2 \Big|_{1.0}^{1.8}$$

$$= (0.5) - 0.32) + (3.6 - 1.52 - 2 + 0.5) = 0.18 + 0.48$$
$$= 0.66$$

## Cummulative Distribution Function F(x)

The cummulative distn function (cdf) of a random variable X, denoted by F(x) is a function which gives the prob. that the observed value of X does not exceed a given value x. i.e. F(x) denotes the prob. that the random var. X takes on a value that is less than or equal to x. Some properties of F(x) include;

(i) $0 \le F(x) \le 1$

(ii) $F(\infty) = 1$ (iii) $F(-\infty) = 0$

(iv) $P(a < X \le b) = F(b) - F(a)$

(v) $P(X > a) = 1 - P(X \le a) = 1 - F(a)$

(vi) If X is a continous varible; $F(x) = \int f(x)\,dx$

or $f(x) = d/dx\, F(x)$

Examples

(1) The cdf of the random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x + 1/2, & 0 \le x \le \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$$

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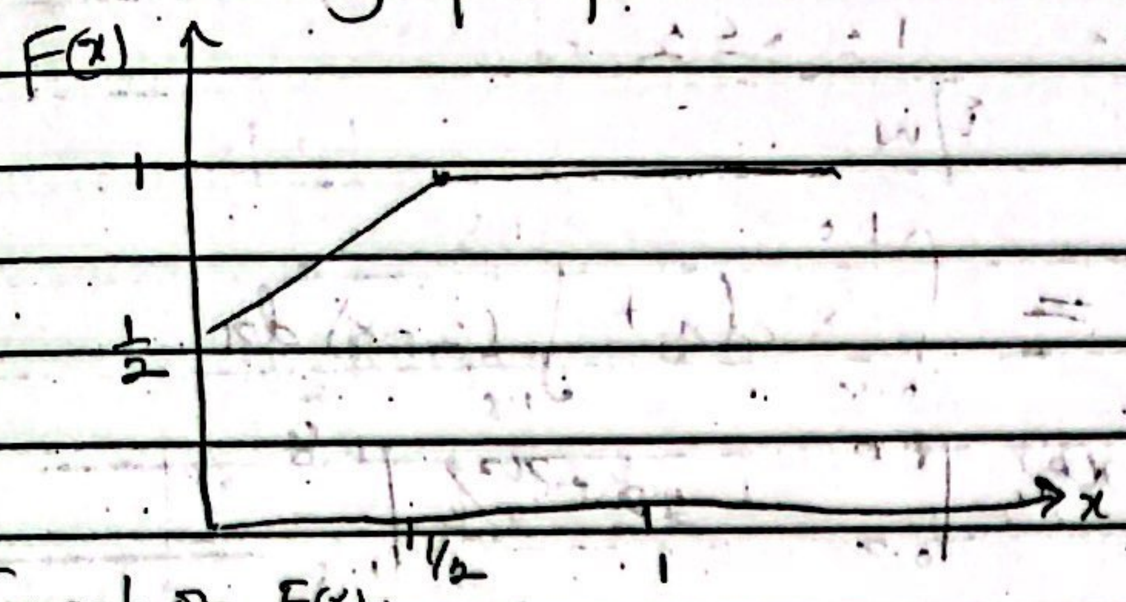
Draw the graph of the CDF




Graph of F(x).

(2) Assume that X has the Pmf given by

$$P(X) = \begin{cases} 1/4, & x=0 \\ 1/2, & x=1 \\ 1/4, & x=2 \end{cases}$$

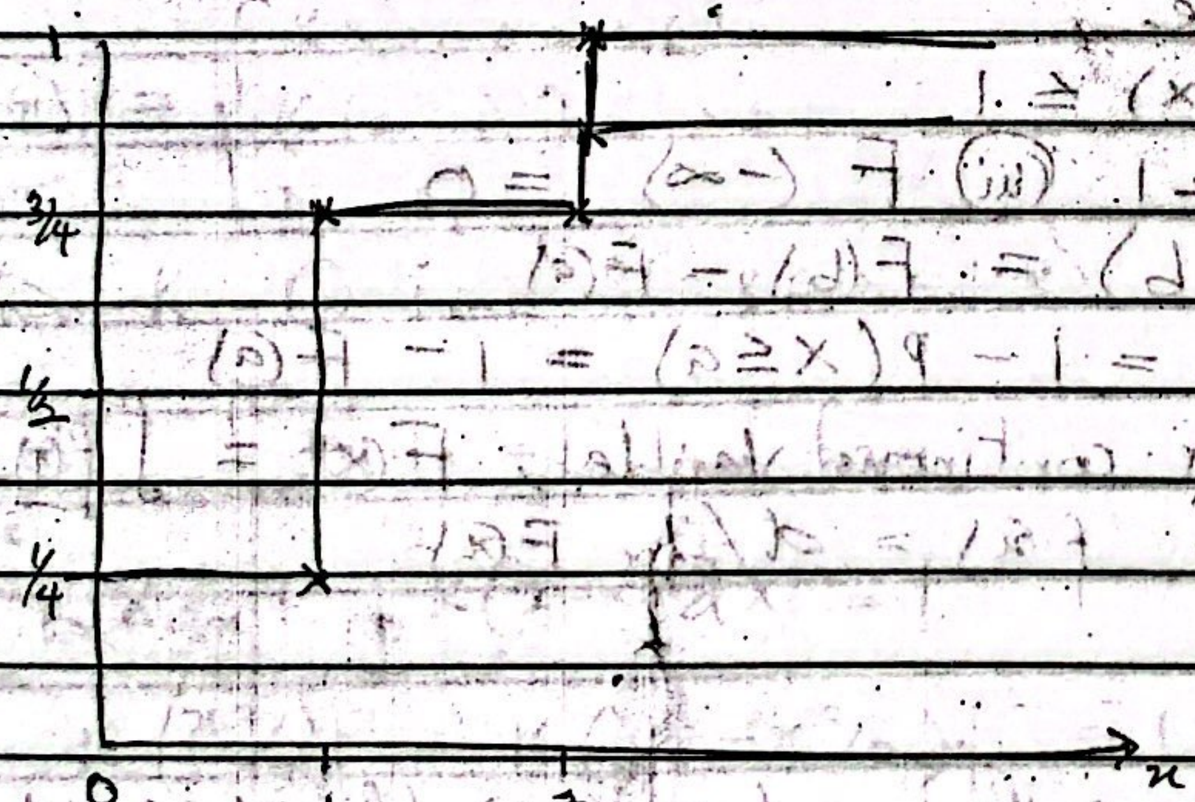
(a) Obtain its cdf (b) Graph of F(x)

Soln:

(a)

$$F(x) = \begin{cases} 0 & x<0 \\ 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

(b)

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## Distribution function for Discrete Random Variable

The dn fn or Cummulative dn fn for a discrete random var. X is defined by:

$$F(x) = P(X = x) = \sum_{x}^{\infty} f(x)$$

Where $f(x)$ is the prob fn of X. If X takes on only a finite no. of values $x_1, x_2 \ldots x_n$, then the dn fn is given by

$F(x) = 0, \quad -\infty < x < x_1$

$f(x_1), \quad x_1 \le x < x_2$

$f(x_2) + f(x_1), \quad x_2 \le x < x_3$

$\vdots$

$f(x_1) + f(x_2) + \ldots f(x_n), \quad x_n \le x < \infty$

It should be noted that dn fn here is a step fn.

Ex: Let X be a discrete r.v that takes on values $-1, 0, \& 1$ with prob $P(x=-1) = \frac{1}{4}$, $P(x=0) = \frac{1}{2}$ & $P(x=1) = \frac{1}{4}$. Obtain the cummulative dn fn (cdf) of X and the graph.

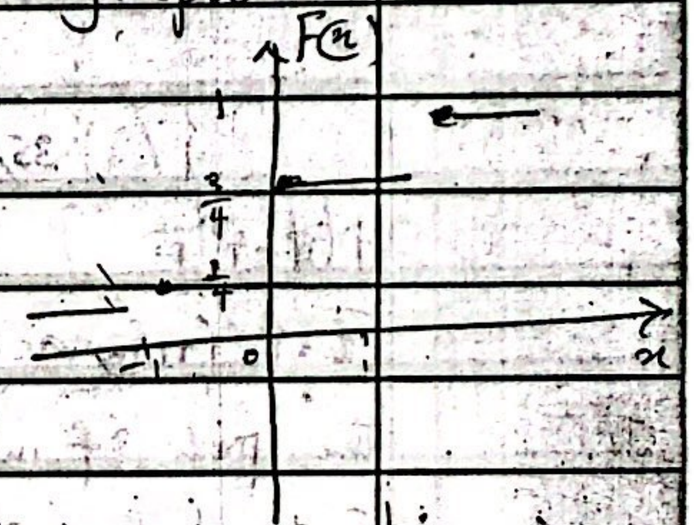
Soln

$F(x) = 0, \quad x < -1$ or $-\infty < x < -1$

$= \frac{1}{4}, \quad -1 \le x < 0$

$= \frac{3}{4}, \quad 0 \le x < 1$

$= 1, \quad x > 1$ or $1 \le x < \infty$




It should be noted that $F(x)$ is a monotonically increasing function i.e for any two particular values of $x_1, x_2$ & $x_3$ where $x_1 \le x_2$, then, $F(x_1) \le F(x_2)$

Properties of C.D.F

If $F(x)$ is a cummulative dn fn (cdf) for a r.v X then

1) $0 \le F(x) \le 1$ for all $x \in (-\infty, \infty)$
2) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$
3) For any $x_1 < x_2$, $F(x_1) < F(x_2)$ where $x_1, x_2 \in (-\infty, \infty)$
4) $\lim_{h \to 0} F(x+h) = F(x)$ for all $h > 0$

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Ex 21 A pair of dice is rolled once. Let X be a random var. denoting the sum of two nos that appears
(a) Obtain the prob. dn of X (b) Obtain the cdf & d graph

Soln:

(a) $P(X=x)$; $x = 2, 3, 4 \ldots 12$

thus, $P(X=2) = 1/36$, $P(X=3) = 2/36$, ..... $P(X=12) = 1/36$

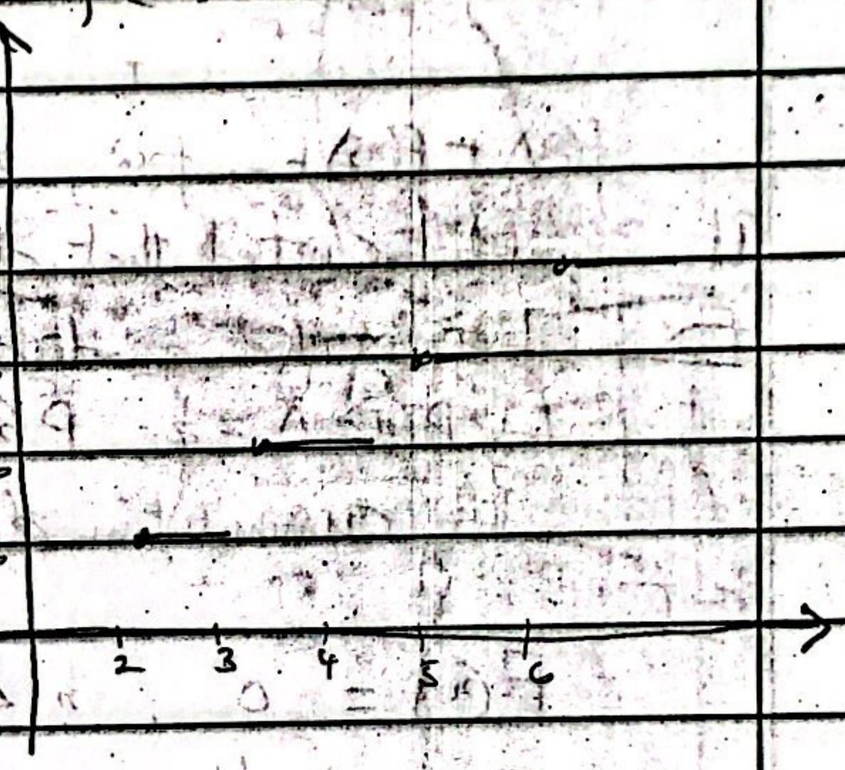
Hence we can represent this in a table as:

| X | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| P(X) | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

(b) $f(2) = 1/36$, $f(3) = 2/36$ ..... $f(12) = 1/36$

Now, for the cdf; we have

$$F(x) = \begin{cases} 0, & x < 2 \\ 1/36, & 2 \le x < 3 \\ 3/36, & 3 \le x < 4 \\ 6/36, & 4 \le x < 5 \\ 10/36, & 5 \le x < 6 \\ \vdots & \\ 35/36, & x \ge 12 \end{cases}$$




Note:

That from a given cdf we can specify the pdf & use it to compute the prob. of interest.

Example: consider the fn below:

$$F(x) = \begin{cases} 0, & x < 3 \\ 1/3, & 3 \le x < 4 \\ 1/2, & 4 \le x < 5 \\ 2/3, & 5 \le x < 6 \\ 1, & x \ge 6 \end{cases}$$

Obtain the pdf and then compute the prob. that;

(i) $3 < x \le 5$ (ii) $x \le 5$ (iii) $x > 4$

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Soln;

$$f(x) = \begin{cases} 1/3, & x = 3 \\ 1/2 - \frac{1}{3} = 1/6, & x = 4 \\ 2/3 - \frac{1}{2} = \frac{1}{6}, & x = 5 \\ 1 - 2/3 = \frac{1}{3}, & x = 6 \\ 0 & o/w \end{cases}$$

(i) $P(3 < x \leq 5) = \sum_{x=4}^{5} f(x) = \frac{1}{6} + \frac{1}{6} = 1/3$

(ii) $P(x \geq 5) = 1 - f(6) = 1 - 1/3 = 2/3$

(iii) $(x > 4) = \frac{1}{6} + \frac{1}{3} = 1/2$

WEEK 6

## Distribution fn for a Continous R.V.

Defn: The dn fn $F(x)$ for a continous random var. $X$ is defined as;

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

Example

Suppose $X$ is a cont. r.v with pdf

$$f(x) = \begin{cases} \frac{1}{9}x^2, & 0 \leq x \leq 3 \\ 0 & o/w \end{cases}$$

(a) Find the dn fn for $X$ (i.e Determine cdf)

soln: when $x < 0$, $F(x) = \int_{-\infty}^{x} f(u)\,du = 0 = \int_{-\infty}^{0} \frac{1}{9}x^2\,dx = 0$

when $0 \leq x \leq 3$, $F(x) = \int_0^3 \frac{1}{9}x^2\,dx = \frac{1}{9}\int_0^3 x^2\,dx = \frac{x^3}{27}$

for $x > 3 \Rightarrow \int_3^{\infty} f(x)\,dx = 1$

thus, $F(x) = \begin{cases} 0, & x < 0 \\ x^3/27, & 0 \leq x \leq 3 \\ 1, & x > 3 \end{cases}$

Also, the pdf $f(x)$ of a r.v $X$ of continous type can be obtained from the dn fn $F(x)$ by finding the first derivative thus; $\frac{d}{dx}F(x) = f(x)$

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Example: The dis fn of a r.v. X is given by,

$$F(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0 \\ 0 & o/w \end{cases}$$

find (a) the pdf (ii) $P(x \geq 2)$ (iii) $P(-3 \leq x \leq 4)$

soln: for the Pdf, we have find $d/dx\, F(x)$

$$f(x) = \frac{d}{dx}\left(1 - e^{-2x}\right) = 2e^{-2x}$$

$$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0 & o/w \end{cases}$$

(b) $P(x \geq 2) = \int_0^\infty 2e^{-2x}\,dx = -e^{-2x}\Big|_2^\infty$

$= -\left(e^{-\infty} - e^{-4}\right) = e^{-4}$

(c) $P(-3 \leq x < 4) = \int_{-3}^{4} 2e^{-2x}\,dx = -e^{-2x}\Big|_{-3}^{4}$

$= -\left(e^{-8} - e^{6}\right) = -\left(0.0003355 - 0.00248\right)$

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MATHEMATICAL EXPECTATION $E(X)$

Defn: The math exp. of a discrete r.v $X$ with prob. function $f(x)$ is the quantity:

$$E(X) = \sum_x x f(x)$$

The notation $\sum_x$ denotes summation over all values $x$ that are possible values of the random variable $X$.

Note: $E(X)$ is often denoted by Greek letter $\mu$.

Ex:

Let $X$ be the numerical outcome of one roll of a die. Find $E(X)$:

soln

The possible values of $X$ are the numbers $1, 2, \ldots 6$

The prob fn of $X$ is $f(x) = \frac{1}{6}$ for $x = 1, 2, \ldots 6$

Therefore;

$$E(X) = \sum_x x f(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6}$$
$$= 3.5$$

Ex 2:

Let $X$ be a discrete r.v. with prob fn

$$f(x) = \begin{cases} 1/3 & \text{for } x = 3 \\ 2/3 & x = 9 \end{cases}$$

fnd $E(X)$

soln: $E(X) = \sum x f(x) = 3 \cdot \frac{1}{3} + 9 \cdot \frac{2}{3} = 7$

Defn 2: Math exp. of a continuous r.v. $X$ with prob density fn $f(x)$ is the quantity:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Ex:

Let $X$ be a continuous r.v. with prob density fn



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$$f(x) = \begin{cases} 81x^2 & \text{for } 0 < x < 1/3 \\ 0 & \text{o/w} \end{cases}$$

Soln:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^{1/3} x \cdot 81x^2\, dx = \frac{1}{4}$$

Ex 2:

Let X be the numerical outcome of selecting a no at random from the interval (0,1), find E(X)

Soln:

The prob. density fn of X is the uniform probability density function $f(x) = 1$ for $0 < x < 1$ & $f(x) = 0$ for all other values of X.

Thus; $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 x\, 1\, dx = 1/2$

Exercises

(1) X is a discrete r.v with prob fn

$$f(x) = \begin{cases} 1/9 & \text{for } x = -3 \\ 2/9 & \text{" } x = 0 \\ 6/9 & \text{" } x = 6 \end{cases}$$

find E(X).

(2) Find E(X) if X is;

(a) the sum of the numerical outcomes of two rolls of a die.

(b) the maximum of the numerical outcomes of two rolls of a die.

(3) Find E(X) if X is a cont. r.v with pdf given by

(a) $$f(x) = \begin{cases} 2(1-x) & ; 0 < x < 1 \\ 0 & \text{o/w} \end{cases}$$

(b) $$f(x) = \begin{cases} 1/2 & \text{for } 0 < x < 1 \\ 1/2 & \text{" } 2 < x < 3 \\ 0 & \text{o/w} \end{cases}$$

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