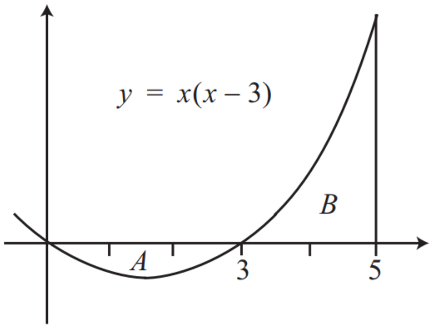
MAT 201 Answer

Solution

If we set $y = 0$ we see that $x(x-3) = 0$, and so $x = 0$ or $x = 3$. Thus the curve cuts the x-axis at $x = 0$ and at $x = 3$. The x^2 term is positive, and so we know that the curve forms a U-shape as shown below.

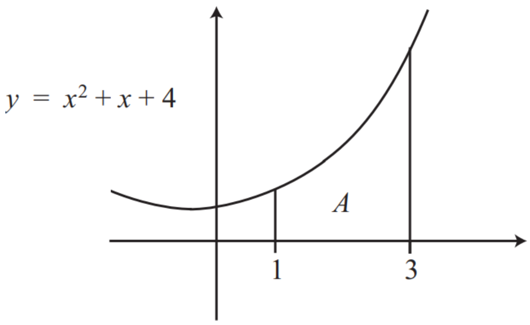
1.



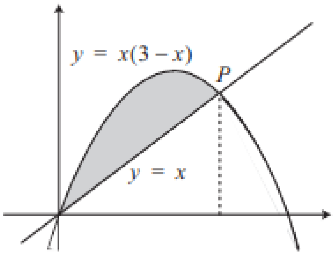
Solution

If we set $y = 0$ we obtain the quadratic equation $x^2 + x + 4 = 0$, and for this quadratic $b^2 - 4ac = 1 - 16 = -15$ so that there are no real roots. This means that the curve does not cross the x-axis. Furthermore, the coefficient of x^2 is positive and so the curve is U-shaped. When $x = 0$, $y = 4$ and so the curve looks like this.

2.



Sketching both curves on the same axes, we can see by setting $y = 0$ that the curve $y = x(3-x)$ cuts the x-axis at $x = 0$ and $x = 3$. Furthermore, the coefficient of x^2 is negative and so we have an inverted U-shape curve. The line $y = x$ goes through the origin and meets the curve $y = x(3-x)$ at the point P. It is this point that we need to find first of all.



At P the y co-ordinates of both curves are equal. Hence:

$$\begin{aligned} x(3-x) &= x \\ 3x - x^2 &= x \\ 2x - x^2 &= 0 \\ x(2-x) &= 0 \end{aligned}$$

so that either $x = 0$, the origin, or else $x = 2$, the x co-ordinate of the point P.

We now need to find the shaded area in the diagram. To do this we need the area under the upper curve, the graph of $y = x(3-x)$, between the x-axis and the ordinates $x = 0$ and $x = 2$. Then we need to subtract from this the area under the lower curve, the line $y = x$, and between the x-axis and the ordinates $x = 0$ and $x = 2$.

The area under the curve is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x(3-x)\mathrm{d}x \\ &= \int_0^2 (3x - x^2)\mathrm{d}x \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 \\ &= [6 - \tfrac{8}{3}] - [0] \\ &= 3\tfrac{1}{3}, \end{aligned}$$

and the area under the straight line is

$$\begin{aligned} \int_0^2 y\,\mathrm{d}x &= \int_0^2 x\,\mathrm{d}x \\ &= \left[\frac{x^2}{2}\right]_0^2 \\ &= [2] - [0] \\ &= 2\,. \end{aligned}$$

3. Thus the shaded area is $3\frac{1}{3} - 2 = 1\frac{1}{3}$ units of area.

1. $\dfrac{x^4+3x^2}{(x^2+1)^2}$ **2.** $\dfrac{16x}{(4-x^2)^2}$ **3.** $\dfrac{3e^{3x}+2e^{4x}}{(1+e^x)^2}$ **4.** $\dfrac{2+\sqrt{x}}{(1+\sqrt{x})^2}$ **5.** $2^x \ln(2) \cot x - 2^x \csc^2 x$ **6.** $10x \sec(5x^2) \tan(5x^2)$

7. $\dfrac{12x^3}{1+9x^8}$ **8.** $\dfrac{e^x}{\sqrt{1-e^{2x}}}$ **9.** $\dfrac{1}{2\sqrt{x}(1+x)}$ **10.** $-3^{\csc x} \ln 3 \csc x \cot x$ **11.** $4e^{\sin(4x)}\cos(4x)$ **12.** $\dfrac{\cot \sqrt[3]{x}}{3x^{2/3}}$

13. $-2e^{-x}\sin(e^{-x})\cos(e^{-x})$ **14.** $-\dfrac{\cos(\cos(\ln x))\sin(\ln x)}{x}$ **15.** $-\dfrac{2\cos 4x}{(\sin 4x)^{3/2}}$ **16.** $-\dfrac{6e^{2x}}{(1+e^{2x})^4}$

17. $(x^2+2x)e^x\cos(x^2e^x)$ **18.** $\dfrac{1-x^2}{(1+x^2)^2}\cos\left(\dfrac{x}{1+x^2}\right)$ **19.** $\dfrac{(x^3+3x^2)e^x}{1+x^3e^x}$ **20.** $e^{x\sin x}(\sin x + x\cos x)$

21. $e^x + e^x \ln x + xe^x \ln x$ **22.** $(3x^2+2x^3)e^{2x}\cos x - x^3e^{2x}\sin x$

4.

22. Hint. $2^{x+\cos x}$. There are at least three different ways of finding the derivative. One is apply the rule

$$(a^x)' = a^x \ln a$$

which applies whenever the base a is constant. That requires memorizing that rule.

A second method uses the identity

$$a^x = e^{x \ln a}$$

which is useful to convert exponentiation with different bases to exponentiation with the natural base e.

$y = x^2 \sin^{-1} x$

This must be differentiated using the **product rule**.

$$u = x^2 \qquad\qquad v = \sin^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = 2x \qquad\qquad \frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= x^2 \cdot \frac{1}{\sqrt{1-x^2}} + 2x\sin^{-1} x\\ &= \frac{x^2}{\sqrt{1-x^2}} + 2x\sin^{-1} x\end{aligned}$$

5.

$$y = \sqrt{1-x^2}\cos^{-1} x$$

This must be differentiated using the **product rule.**

$$u = \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} \qquad v = \cos^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \qquad \frac{dv}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= \sqrt{1-x^2} \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) - \frac{x}{\sqrt{1-x^2}} \cdot \cos^{-1} x$$

$$= -1 - \frac{x\cos^{-1} x}{\sqrt{1-x^2}}$$

6.

Problem 1 $f(x)$ and $g(x)$ are continuous and differentiable everywhere. Since $f(-1) = f(3) = 11$
$g(-2) = g(2) = -6$ they satisfy hypotheses of Rolle's Thm.

For $f(x)$: $f'(x) = 4x - 4$ so $4x - 4 = 0 \Rightarrow x = 1$.
Therefore, the only choice for c is $\boxed{c = 1}$

For $g(x)$: $g'(x) = 3x^2 - 4x - 4$
Set $3x^2 - 4x - 4 = 0 \Rightarrow (3x+2)(x-2) = 0$
$\Rightarrow x = 2, -\frac{2}{3}$

Inside the interval $(-2,2)$ the only choice for c is $\boxed{-\frac{2}{3} = c}$

Note: Rolle's Thm says $c \in (a,b)$ so c is not an endpoint.

7.

Solutions

1. First,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[(x^2-1)(y+2)\right] \\
&= (y+2)\frac{\partial}{\partial x}\left[(x^2-1)\right] \\
&= (y+2)(2x) \\
&= 2x(y+2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[(x^2-1)(y+2)\right] \\
&= (x^2-1)\frac{\partial}{\partial x}\left[(y+2)\right] \\
&= (x^2-1)\cdot 1 \\
&= (x^2-1).
\end{aligned}$$

8.

2. We can start by observing that

$$e^{x+y+1} = e^x e^y e.$$

So

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(e\, e^x e^y\right) \\
&= e\, e^y \frac{\partial}{\partial x}\left(e^x\right) \\
&= e\, e^y (e^x) \\
&= e^{x+y+1}.
\end{aligned}$$

1.

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(e\, e^x e^y\right) \\
&= e\, e^x \frac{\partial}{\partial y}\left(e^y\right) \\
&= e\, e^x (e^y) \\
&= e^{x+y+1}.
\end{aligned}$$

3. Using the Product Rule,

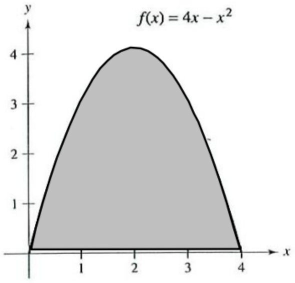
$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(e^{-x}\right)\sin(x+y) + e^{-x}\frac{\partial}{\partial x}\left(\sin(x+y)\right) \\
&= -e^{-x}\sin(x+y) + e^{-x}(1 \cdot \cos(x+y)) \\
&= e^{-x}\left(\cos(x+y) - \sin(x+y)\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(e^{-x}\right)\sin(x+y) + e^{-x}\frac{\partial}{\partial y}\left(\sin(x+y)\right) \\
&= 0 + e^{-x}(1 \cdot \cos(x+y)) \\
&= e^{-x}\cos(x+y).
\end{aligned}$$

2.

Graph:



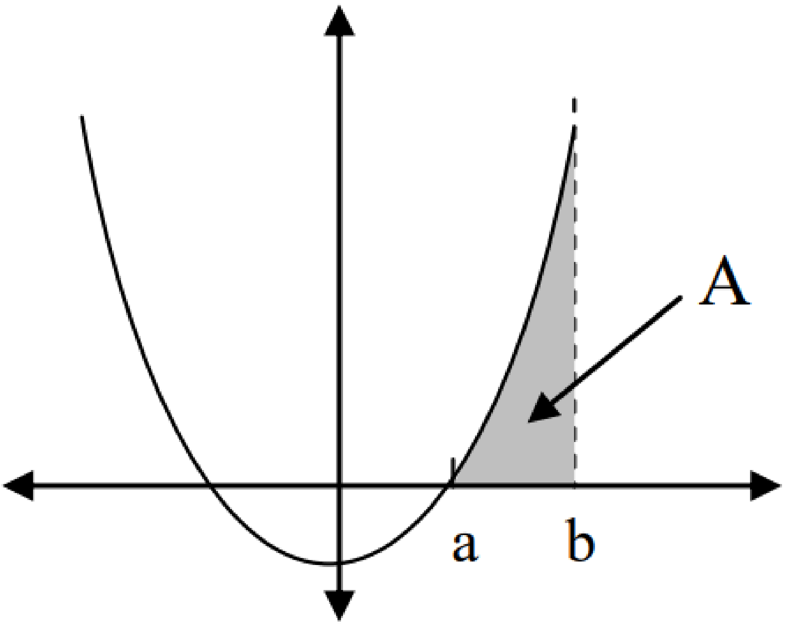
To find the boundaries, determine the x-intercepts: $f(x)=0 \to 4x-x^2=0$

$x(4-x)=0$

$x=0$ or $(4-x)=0$ so $x=0$ and $x=4$

Therefore the boundaries are $a=0$ and $b=4$

9.



10.

Now we need to find our *g(y)*. That is easily done by solving $y=x^2+2$ for x.

$$y=x^2+2$$
$$x^2=y-2$$
$$x=\pm\sqrt{y-2}$$

We will ignore the negative radical, since our area is in the first quadrant.

Now let's set up the integral: $A=\int_2^4 \sqrt{y-2}\,dy$

Solve the integral using a simple *u* substitution:

$A=\int_2^4 \sqrt{y-2}\,dy=\frac{2}{3}\left(\sqrt{y-2}\right)^3\Big|_2^4=\frac{2}{3}\left(\sqrt{4-2}\right)^3-\frac{2}{3}\left(\sqrt{2-2}\right)^3=\frac{2}{3}\left(\sqrt{2}\right)^3-0=\frac{4\sqrt{2}}{3}$ square units.

The first quadrant area bounded by the following curves: $y=x^2+2$, $y=4$ and $x=0$ is equal to $\frac{4\sqrt{2}}{3}$ square units.

11.

things. So let's suppose that $x = 0$ and we'll see what happens. In the first equation, if $x = 0$, then $ye^0 = 0$, and so $y = 0$ as well. Let's check to see if $y = x = 0$ works in the second equation: if x and y are both 0, then we get $0 = 0$, and so the point $(0, 0)$ is definitely a possibility. But wait! I just realized that it's actually *not* a possibility.[2] Why? There is a third equation in our system of equations that I forgot to write down - our original constraint equation:

$$x^3 + y^3 = 16$$

If $x = y = 0$, then we get $0 = 16$, which is not true. So $(0, 0)$ does not work.

Now that we took care of the trivial case (or rather proved that there is no trivial case), we can assume that both x and y are nonzero. The first thing I think of trying is solving for λ in both equations and setting them equal to each other:

$$\begin{aligned} ye^{xy} = \lambda 3x^2 \quad &\Rightarrow \quad \lambda = \frac{ye^{xy}}{3x^2} \\ xe^{xy} = \lambda 3y^2 \quad &\Rightarrow \quad \lambda = \frac{xe^{xy}}{3y^2} \end{aligned}$$

This gives us:

$$\frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2}$$

After cross multiplying, we can divide by 3 on both sides, and - since e^{xy} is never 0 - we can divide by that as well. Our equation turns into:

$$y^3 = x^3$$

Now, we can plug this into our constraint equation:

$$x^3 + y^3 = 16 \quad \Rightarrow \quad x^3 + x^3 = 16 \quad \Rightarrow \quad x^3 = 8$$

and so $x = 2$. This implies that $y = 2$. And that's it! The only critical point we have is $(2, 2)$. Therefore, the maximum value of the function f on $x^3 + y^3 = 16$ is:

$$f(2, 2) = e^4$$

Lesson learned: use the constraint equation to your advantage when solving Lagrange multiplier problems. It's every bit as important as your λ equations. □

12.

We need all the first and second derivatives so lets work them out. we have

$$
\begin{aligned}
f_x &= 2x \\
f_y &= 2y \\
f_{xx} &= 2 \\
f_{yy} &= 2 \\
f_{xy} &= 0
\end{aligned}
$$

For stationary points we need $f_x = f_y = 0$. This gives $2x = 0$ and $2y = 0$ so that there is just one stationary point, namely $(x, y) = (0, 0)$. We now need to classify it. Now

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(2) - 0^2 = 4 > 0$$

so it is either a max or a min. But $f_{xx} = 2 > 0$ and $f_{yy} = 2 > 0$. Hence it is a minimum. Our conclusion is that this function has just one stationary point $(0, 0)$ and that it is a minimum.
The example we have just done is very straightforward. It is untypical in that most functions have more than one stationary point. The next example again has just one stationary point but the analysis is slightly more involved.

13.

The first and second order partial derivatives of this function are:

$$
\begin{aligned}
f_x &= -2xe^{-(x^2+y^2)} \\
f_y &= -2ye^{-(x^2+y^2)} \\
f_{xx} &= -2e^{-(x^2+y^2)}(1 - 2x^2) \quad \text{by the product rule} \\
f_{yy} &= -2e^{-(x^2+y^2)}(1 - 2y^2) \\
f_{xy} &= 4xye^{-(x^2+y^2)}
\end{aligned}
$$

Stationary points are when $f_x = 0$ and $f_y = 0$ and so there is only one stationary point, at $(x, y) = (0, 0)$. Substituting $(x, y) = (0, 0)$ into the expressions for f_{xx}, f_{yy} and f_{xy} gives

$$f_{xx} = -2, \quad f_{yy} = -2, \quad f_{xy} = 0$$

Therefore

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - 0^2 = 4 > 0$$

so that $(0, 0)$ is either a min or a max. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

14.

For this function

$$f_x = -2x - y$$
$$f_y = -x - 2y$$
$$f_{xx} = -2$$
$$f_{yy} = -2$$
$$f_{xy} = -1$$

For stationary points, $-2x - y = 0$ and $-x - 2y = 0$ so again the only possibility is $(x, y) = (0, 0)$. We have

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (-1)^2 = 3 > 0$$

so that $(0, 0)$ is either a max or a min. Since $f_{xx} < 0$ and $f_{yy} < 0$ it is a maximum.

15.

The first and second order partial derivatives of this function are

$$f_x = 6x^2 + 6y^2 - 150$$
$$f_y = 12xy - 9y^2$$
$$f_{xx} = 12x$$
$$f_{yy} = 12x - 18y$$
$$f_{xy} = 12y$$

For stationary points we need

$$6x^2 + 6y^2 - 150 = 0 \quad \text{and} \quad 12xy - 9y^2 = 0$$

i.e.

$$x^2 + y^2 = 25 \quad \text{and} \quad y(4x - 3y) = 0$$

The second of these equation s implies **either** that $y = 0$ **or** that $4x = 3y$ and both of these possibilities now need to be considered. If $y = 0$ then the first equation implies that $x^2 = 25$ so that $x = \pm 5$ giving $(5, 0)$ and $(-5, 0)$ as stationary points.
If $4x = 3y$ then $x = \frac{3}{4}y$ and so the first equation becomes

$$\frac{9}{16}y^2 + y^2 = 25$$

so that $y = \pm 4$. $y = 4$ gives $x = 3$ and $y = -4$ gives $x = -3$, so we have two further stationary points $(3, 4)$ and $(-3, -4)$.
Thus in total there are **four** stationary points $(5, 0)$, $(-5, 0)$, $(3, 4)$ and $(-3, -4)$. Each of these must now be classified into max, min or saddle.

- Lets start with $(5, 0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$ so it is either a max or a min. But $f_{xx} = 60 > 0$ and $f_{yy} = 60 > 0$. Hence $(5, 0)$ is a minimum.
- Now deal with $(-5, 0)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = (-60)^2 > 0$ so it is either a max or a min. But $f_{xx} = -60 < 0$ and $f_{yy} = -60 < 0$. Hence $(-5, 0)$ is a maximum.
- Now deal with $(3, 4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(3, 4)$ is a saddle.
- Now deal with $(-3, -4)$. For this stationary point, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(-3, -4)$ is a saddle.

16.

Solution. The partial derivatives of f are

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \quad \text{and} \quad \frac{\partial f}{\partial y} = 4y^3 - 4x.$$

Setting $f_x = 0$ and $f_y = 0$, we obtain $y = x^3$ and $x = y^3$. It follows that $x = (x^3)^3 = x^9$. We observe that

$$\begin{aligned} x^9 - x = x(x^8 - 1) &= x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1). \end{aligned}$$

Hence, the equation $x^9 - x = 0$ has three real roots: $x = 0, 1, -1$. Thus, the critical points of f are $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

17.

Next, we find the second-order partial derivatives of f:

$$f_{xx} = 12x^2, \quad f_{xy} = -4, \quad f_{yy} = 12y^2.$$

At the critical point $(1, 1)$, we have

$$D(1, 1) = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128 > 0.$$

Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, $f(1, 1) = -2$ is a local minimum of f. Similarly, $f(-1, -1) = -2$ is also a local minimum of f.

At the point $(0,0)$ we have

$$D(0,0) = \begin{vmatrix} 0 & -4 \\ -4 & 0 \end{vmatrix} = -16 < 0.$$

Therefore, $(0,0)$ is a saddle point of the graph of f.

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