Derivatives of Composite Functions

Recall that the **composite function** or **composition** of two functions is the function obtained by applying them one after the other.

For example, If $f(x) = \dfrac{1}{x}$ and $g(x) = x^3 + 2$, then

$$f(g(x)) = \frac{1}{g(x)} = \frac{1}{x^3+2}$$

and $g(f(x)) = (f(x))^3 + 2 = \left(\dfrac{1}{x}\right)^3 + 2 = \dfrac{1}{x^3} + 2$

Try a **Java applet.**

The derivative of the composition of two non-constant functions is equal to the product of their derivatives, evaluated appropriately.

The Chain Rule

We have the Chain Rule: $$\boxed{(g(h(x)))' = g'(h(x))h'(x)}$$

Example 1: Using $g(x) = \dfrac{1}{x} = x^{-1}$ and $h(x) = x^3 + 2$,

we have $g'(x) = (-1)x^{-2}$ and $h'(x) = 3x^2$, $g'(h(x)) = (-1)(h(x))^{-2}$, so we get

$$\left(\frac{1}{x^3+2}\right)' = g'(h(x))h'(x) = (-1)(h(x))^{-2}(3x^2) =$$

$$(-1)(x^3+2)^{-2}(3x^2) = \boxed{\frac{-3x^2}{(x^3+2)^2}}$$

On the other hand, $\left(\dfrac{1}{x^3} + 2\right)' = (h(g(x)))' = h'(g(x))g'(x) =$

$3(g(x))^2(-x^{-2}) = 3(x^{-1})^2(-x^{-2}) = \boxed{-3x^{-4}}$, as expected.

Example 2: Let $g(x) = x^3$, and $h(x) = x^2$, so that

$g(h(x)) = h(x)^3 = (x^2)^3 = x^6$.

Then $g'(x) = 3x^2$, so $g'(h(x)) = 3(h(x))^2$, and $h'(x) = 2x$,

so the Chain Rule gives us

$(g'(h(x)))' = g'(h(x))h'(x) = \left(3(h(x))^2\right)(2x) =$

$\left(3(x^2)^2\right)(2x) = \left(3x^4\right)(2x) = \boxed{6x^5}$, as expected.

Example 3: Let $g(x) = x^3 + 3$, and $h(x) = x^2 + 2$, so that

$g(h(x)) = (h(x))^3 + 3 = (x^2 + 2)^3 + 3$.

Then $g'(x) = 3x^2$, so $g'(h(x)) = 3(h(x))^2$, and $h'(x) = 2x$,

so the Chain Rule gives us

$(g'(h(x)))' = g'(h(x))h'(x) = \left(3(h(x))^2\right)(2x) = \left(3(x^2 + 2)^2\right)(2x) =$

$\boxed{6x(x^2 + 2)^2}$

Example 4: Find $f'(x)$ if $f(x) = \sqrt[3]{x^4 + x^2 + 1}$.

We let $g(x) = x^{\frac{1}{3}}$ and $h(x) = x^4 + x^2 + 1$ so that $f(x) = g(h(x))$.

Then $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, $g'(h(x)) = \dfrac{1}{3}(h(x))^{-\frac{2}{3}}$, and $h'(x) = 4x^3 + 2x$,

so we have $f'(x) = g'(h(x))h'(x) = \dfrac{1}{3}(h(x))^{-\frac{2}{3}}(4x^3 + 2x) =$

$$\boxed{\frac{2x(2x^2 + 1)}{3\left(x^4 + x^2 + 1\right)^{\frac{2}{3}}}}$$

Example 5: Find $f'(x)$ if $f(x) = \sin(x^2)$.

We let $g(x) = \sin x$ and $h(x) = x^2$ so that $f(x) = g(h(x))$.

Then $g'(x) = \cos x$, $g'(h(x)) = \cos(x^2)$, and $h'(x) = 2x$,

so we have $f'(x) = g'(h(x))h'(x) = \left(\cos(x^2)\right)(2x) = \boxed{2x\cos(x^2)}$

Example 6: Find $f'(x)$ if $f(x) = (\sin x)^2$.

We let $g(x) = x^2$ and $h(x) = \sin x$ so that $f(x) = g(h(x))$.

Then $g'(x) = 2x$, $g'(h(x)) = 2\sin x$, and $h'(x) = \cos x$,

so we have $f'(x) = g'(h(x))h'(x) = (2\sin x)(\cos x) = \boxed{2\sin x\cos x = \sin 2x}$

The Power Rule $\left((g(x))^n\right)' = n\left(g(x)\right)^{n-1} g'(x)$

Example 4a: Find $f'(x)$ if $f(x) = \sqrt[3]{x^4 + x^2 + 1}$.

We write $f(x) = (g(x))^{\frac{1}{3}}$ where $g(x) = x^4 + x^2 + 1$.

Then $f'(x) = \frac{1}{3}(g(x))^{-\frac{2}{3}}g'(x) = \frac{1}{3}\left(x^4 + x^2 + 1\right)^{-\frac{2}{3}}(4x^3 + 2x) =$

$$\boxed{\frac{2x(2x^2+1)}{3\left(x^4 + x^2 + 1\right)^{\frac{2}{3}}}}$$

Example 7: Find $f'(x)$ if $f(x) = \left(\frac{4x-3}{2x+1}\right)^8$.

We have $f'(x) = 8\left(\frac{4x-3}{2x+1}\right)^7\left(\frac{4x-3}{2x+1}\right)' =$

$$8\left(\frac{4x-3}{2x+1}\right)^7\left(\frac{(2x+1)(4x-3)' - (4x-3)(2x+1)'}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7\left(\frac{(2x+1)4 - (4x-3)2}{(2x+1)^2}\right) =$$

$$8\left(\frac{4x-3}{2x+1}\right)^7\left(\frac{8x+4-8x+6}{(2x+1)^2}\right) = 8\left(\frac{4x-3}{2x+1}\right)^7\left(\frac{10}{(2x+1)^2}\right) = \boxed{80\frac{(4x-3)^7}{(2x+1)^9}}$$

Example 8: Find $f'(x)$ if $f(x) = e^{\cos x}$.

We have $f'(x) = e^{\cos x}(\cos x)' = e^{\cos x}(-\sin x) =$ $-\sin x e^{\cos x}$

Example 9: Find $f'(x)$ if $f(x) = \sin\left(e^{\tan x}\right)$.

We have $f'(x) = \cos\left(e^{\tan x}\right)\left(e^{\tan x}\right)' = \cos\left(e^{\tan x}\right)e^{\tan x}(\tan x)' =$

$\cos\left(e^{\tan x}\right)e^{\tan x}\sec^2 x$
