PS8 | Muath Hamidi

In [1]:

```python
#===================================================
# Course: PHYS 512
# Problem: PS8
#===================================================
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#===================================================
# Libraries
#===================================================
import numpy as np # For math
import matplotlib.pyplot as plt # For graphs

```

Problem 1

The solution is a complex exponential and has the form;

$$f(x,t) = \xi^t exp(ikx)$$

Substitute in the leapfrog scheme

$$\frac{exp(ikx)(\xi^{t+dt} - \xi^{t-dt})}{2dt} = \frac{exp(ikx)\xi^t(\xi^{+dt} - \xi^{-dt})}{2dt}$$

Take this as the LHS. Now, the RHS;

$$-v\frac{exp(ikx)\xi^t(exp(+ikdx) - exp(-ikdx)}{2dx}$$

Match the both sides;

$$\frac{exp(ikx)\xi^t(\xi^{+dt} - \xi^{-dt})}{2dt} = -v\frac{exp(ikx)\xi^t(exp(+ikdx) - exp(-ikdx)}{2dx}$$

So,

$$\rightarrow \xi^{+dt} - \xi^{-dt} = v\frac{dt}{dx}(2isin(kdx))$$

Multiply both sides by ξ^{+dt}

$$\rightarrow \xi^{2dt} - 2iv\frac{dt}{dx}\xi^{dt} sin(kdx) - 1 = 0$$

So, the solution is;

$$\rightarrow \xi^{dt} = \frac{2iv\frac{dt}{dx}sin(kdx) \pm \sqrt{-4v^2(\frac{dt}{dx})^2 sin^2(kdx) + 4}}{2}$$

$$\rightarrow \xi^{dt} = iv\frac{dt}{dx}sin(kdx) \pm \sqrt{-v^2(\frac{dt}{dx})^2 sin^2(kdx) + 1}$$

This is a complex solution. If $v\frac{dt}{dx} \leq 1$ then $|\xi^{dt}|^2 = 1$. If $v\frac{dt}{dx} > 1$ then the solution is not in our interest.

Problem 2

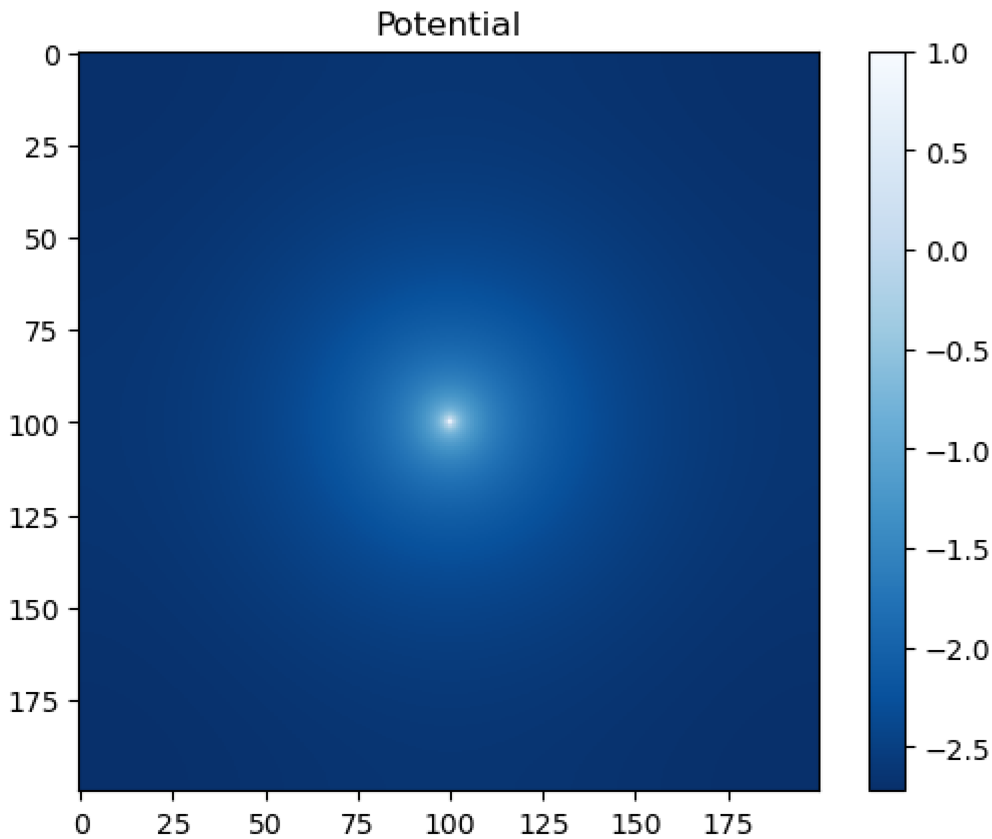
First, let's use the average neighbours method to find the potential;

In [137]:

```python
#=====================================================
# Part a
#=====================================================
# Functions
#=====================================================
def average_neighbors(mat):
    out=0*mat
    out=out+np.roll(mat,1,0)
    out=out+np.roll(mat,-1,0)
    out=out+np.roll(mat,1,1)
    out=out+np.roll(mat,-1,1)
    return out/4

#=====================================================
# Average Neighbours Method
#=====================================================
size = 200
s2 = int(abs(size/2))
V = np.zeros([size,size]) # V array
V[s2, s2] = 1 # origion
iterations = 10000

for i in np.arange(iterations):
    V = average_neighbors(V)
    V[s2, s2] +=1 # origin


scale = 1 - V[s2, s2]
V = V + scale

#=====================================================
# Print Chosen Values
#=====================================================
r0 = V[s2, s2] + average_neighbors(V)[s2, s2]
v0 = V[s2, s2]
v1 = V[s2 + 1, s2]
v2 = V[s2 + 2, s2]
v5 = V[s2 + 5, s2]
print("V[0,0] = ", v0)
print("rho[0,0] = ",r0)
print("V[1,0] = ", v1)
print("V[2,0] = ", v2)
print("V[5,0] = ", v5)

#=====================================================
# Plot - Greens' Function
#=====================================================
plt.imshow(V, cmap="Blues_r")
plt.colorbar()
plt.title("Potential")
```

```
V[0,0] =  1.0
rho[0,0] =  1.0
V[1,0] =  0.0
V[2,0] =  -0.4533841134643257
V[5,0] =  -1.050788709610979
```

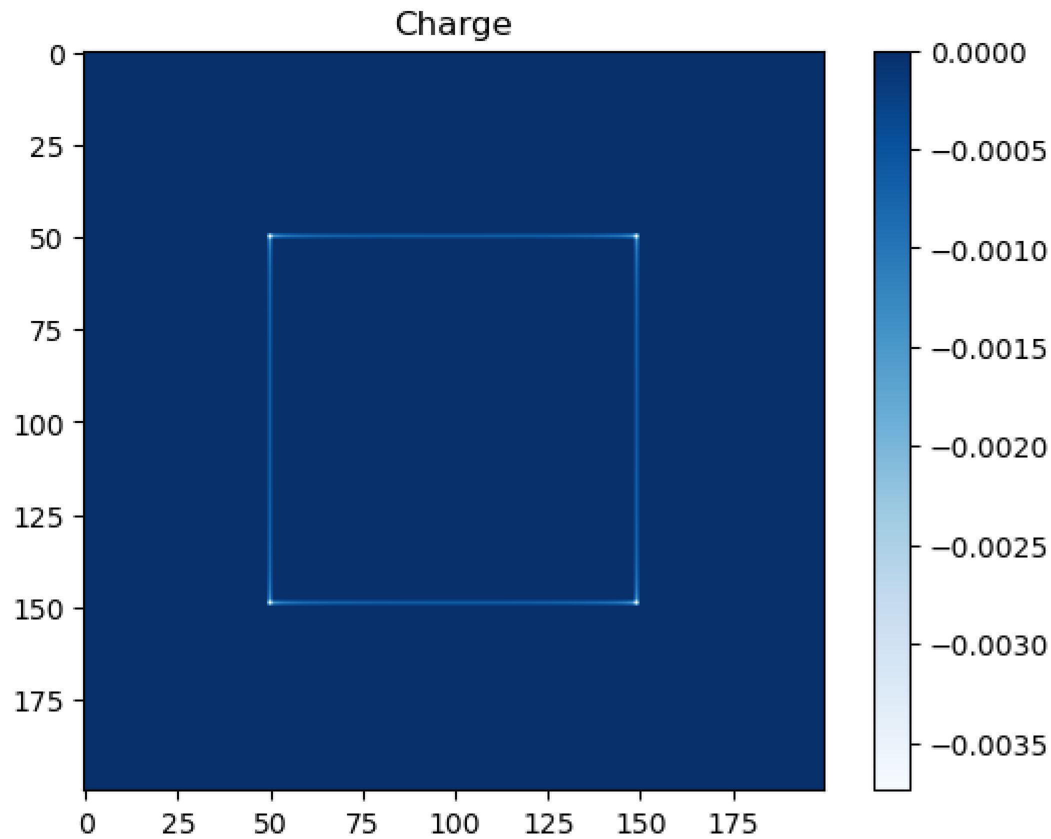
```
Out[137]: Text(0.5, 1.0, 'Potential')
```



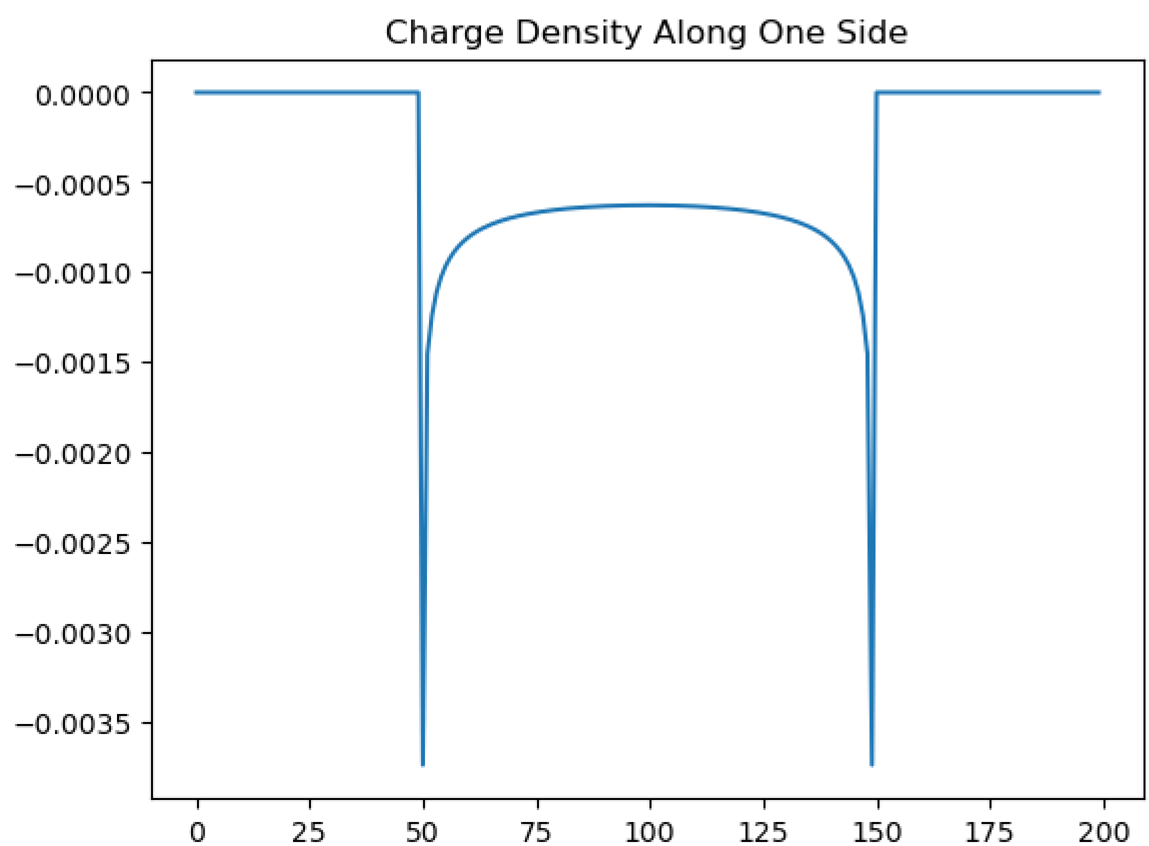
Nice and realistic, with true potential values as in the question.

In [138]:

```
#==================================================
# Part b
#==================================================
# Functions
#==================================================
dim = s2 # the square (shape) mask dimension
dim2 = int(dim/2)
mask = np.zeros([size, size], dtype = bool)
mask[dim - dim2 : dim + dim2, dim - dim2 : dim + dim2] = 1

#==================================================
# Functions
#==================================================
def MaskConv(rho, mask):
    rho_mask = np.zeros(mask.shape)
    rho_mask[mask] = rho
    return np.fft.fftshift(np.fft.irfft2(np.fft.rfft2(V) * np.fft.rfft2(rho_

def conjgrad(A, b, mask, x): # Conjugate gradient
    iterarions = 5000
    r = b - A(x, mask)
    p = r.copy()
    rtr = np.sum(r**2)

    for i in range(iterarions):
        Ap=A(p, mask)
        pAp=np.sum(p*Ap)
        alpha=rtr/pAp
        x=x+alpha*p
        r=r-alpha*Ap
        rtr_new=np.sum(r**2)
        beta=rtr_new/rtr
        p=r+beta*p
        rtr=rtr_new
    return x

#==================================================
# Distribution
#==================================================
rho = conjgrad(MaskConv, mask[mask], mask, mask[mask])

mask_rho = np.zeros(mask.shape)
mask_rho[mask] = rho

#==================================================
# Plots
#==================================================
plt.imshow(mask_rho, cmap="Blues")
plt.title("Charge")
plt.colorbar()
plt.show()
plt.close()

plt.plot(mask_rho[:, s2 + dim2 - 1])
plt.title("Charge Density Along One Side")
```



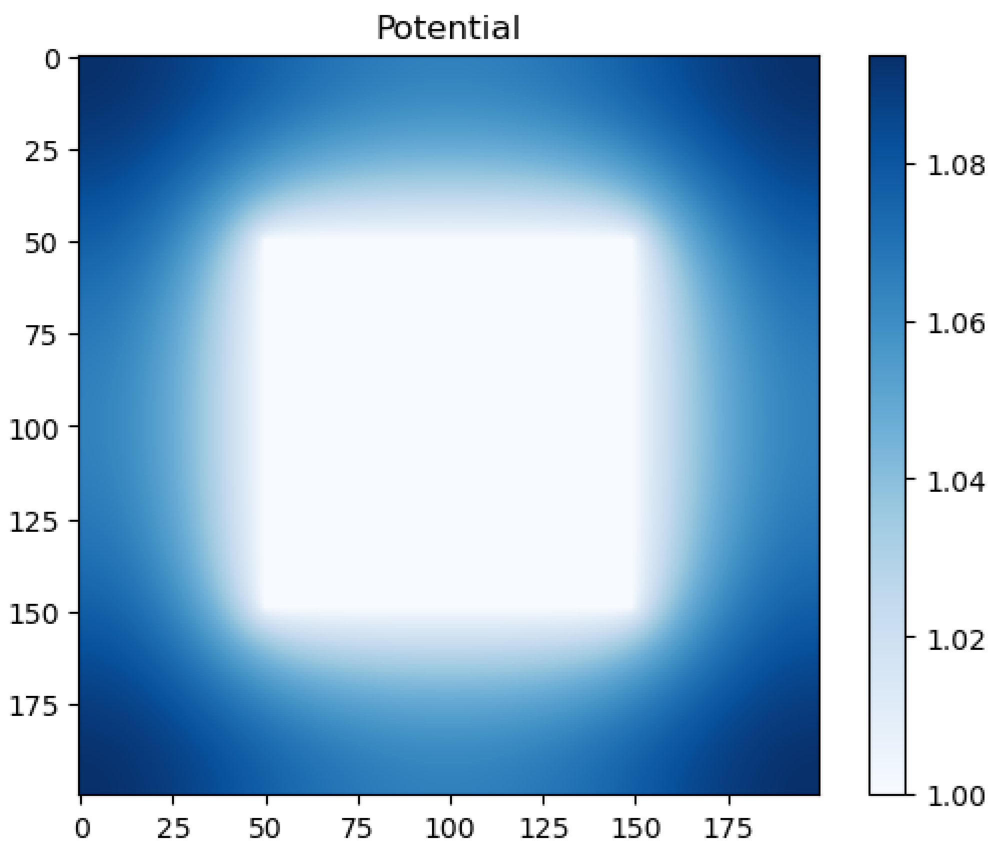

```
Out[138]: Text(0.5, 1.0, 'Charge Density Along One Side')
```



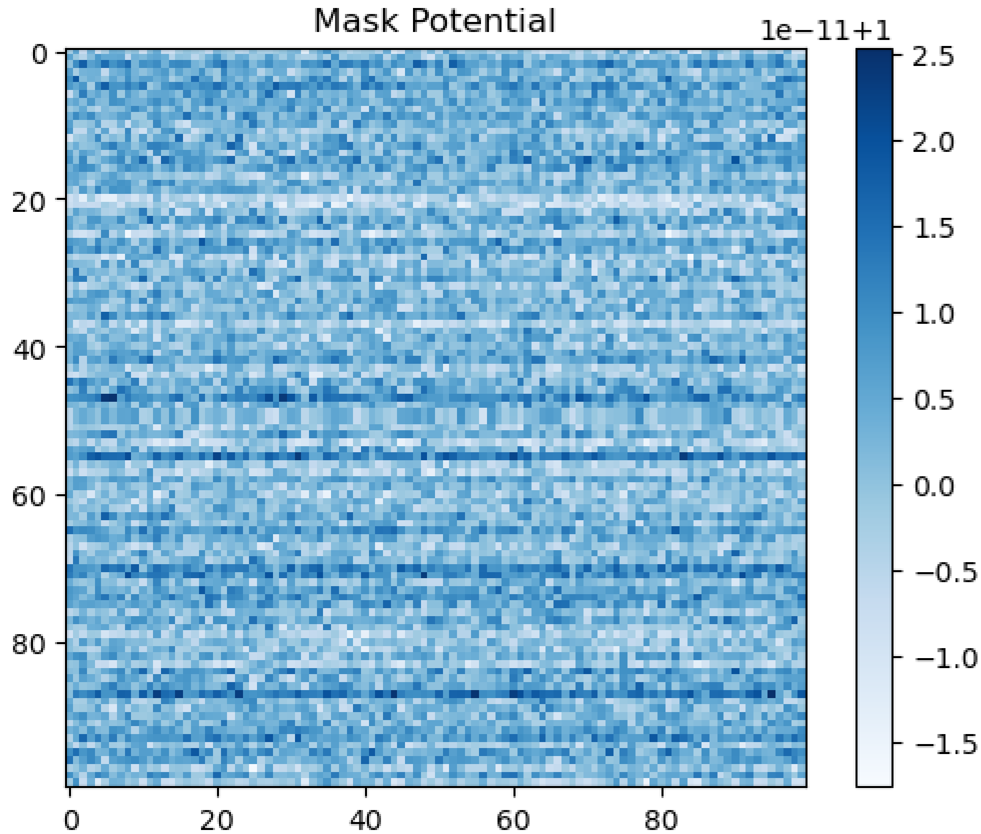
This is expected as the charge density will be maximum at the corners due to their repulsive force.

In [139]:

```
#======================================================
# Part c
#======================================================
# Plot Potential
#======================================================
Pot = np.fft.fftshift(np.fft.irfft2(np.fft.rfft2(V) * np.fft.rfft2(mask_rho)
plt.imshow(Pot, cmap="Blues")
plt.title("Potential")
plt.colorbar()
plt.show()
plt.close()

#======================================================
# Plot - Mask Potential
#======================================================
plt.imshow(Pot[50:150, 50:150], cmap="Blues")
plt.title("Mask Potential")
plt.colorbar()

#======================================================
# Calculations - Maximum potential difference on the mask
#======================================================
vmin = np.min(Pot[50:150, 50:150])
vmax = np.max(Pot[50:150, 50:150])
print("Potential mean = ", Pot[50:150, 50:150].mean())
print("Maximum potential difference on the mask = ", vmax - vmin)
```



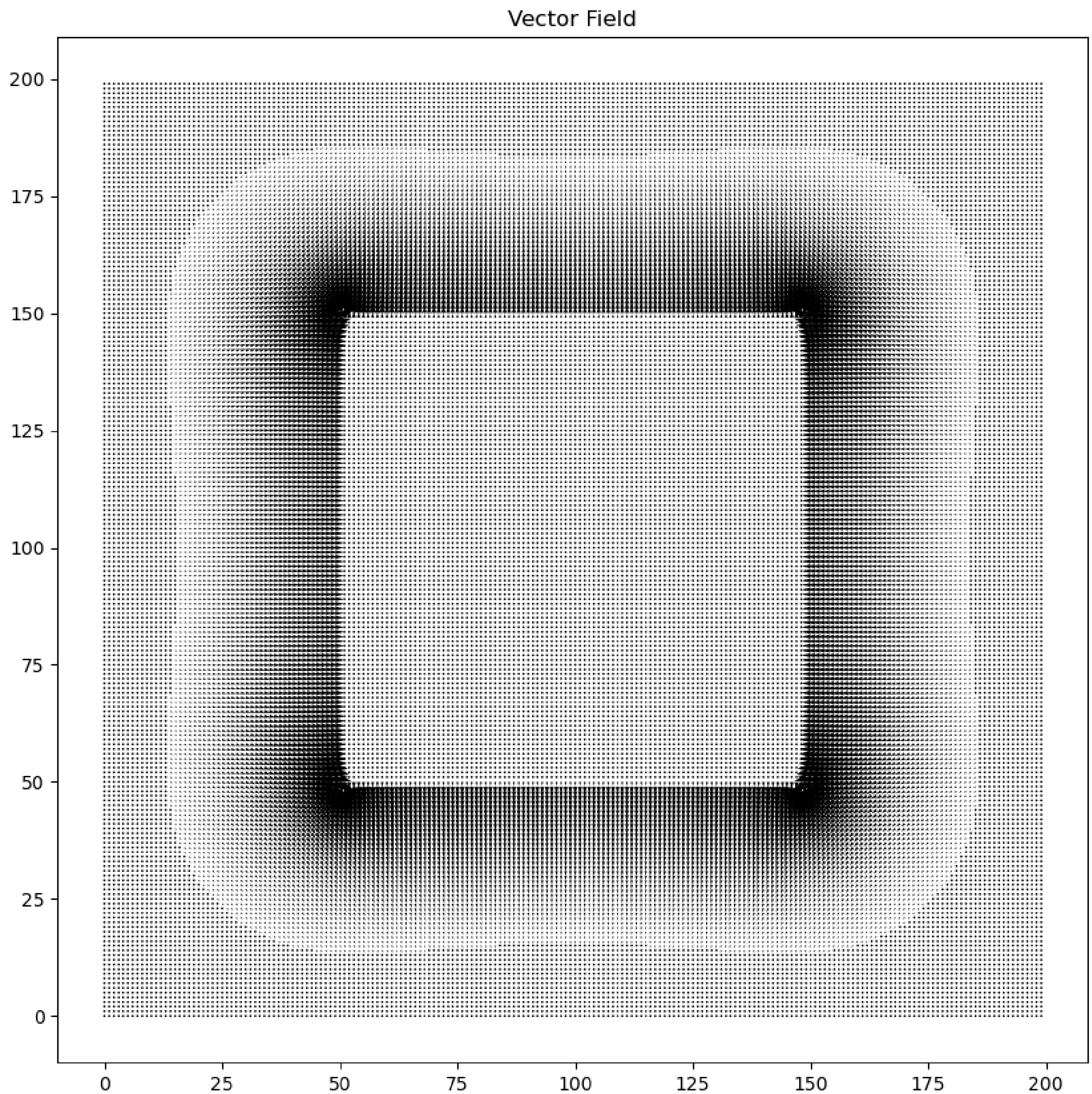

```
Potential mean =  1.0000000000024702
Maximum potential difference on the mask =  4.2953640644327606e-11
```



As you can see, the maximum potential difference on the mask is in order of 10^{-11}, so the potential in the interior is really close to constant (=1).

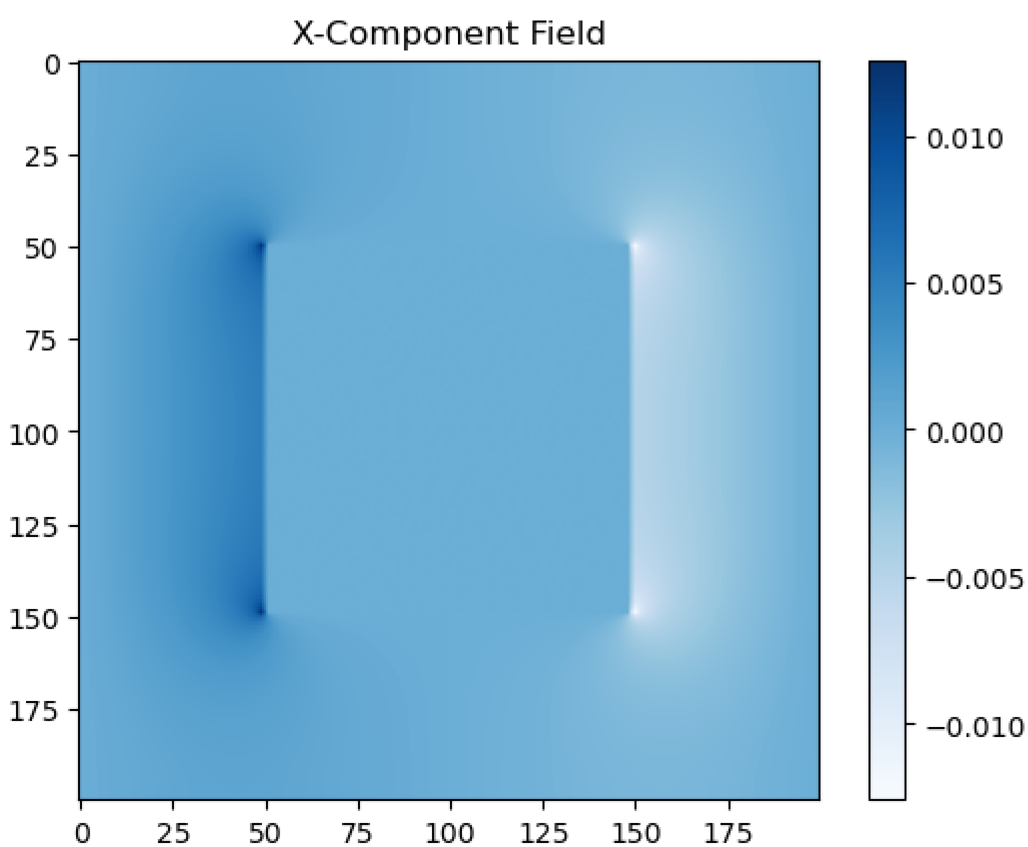
In [174]:

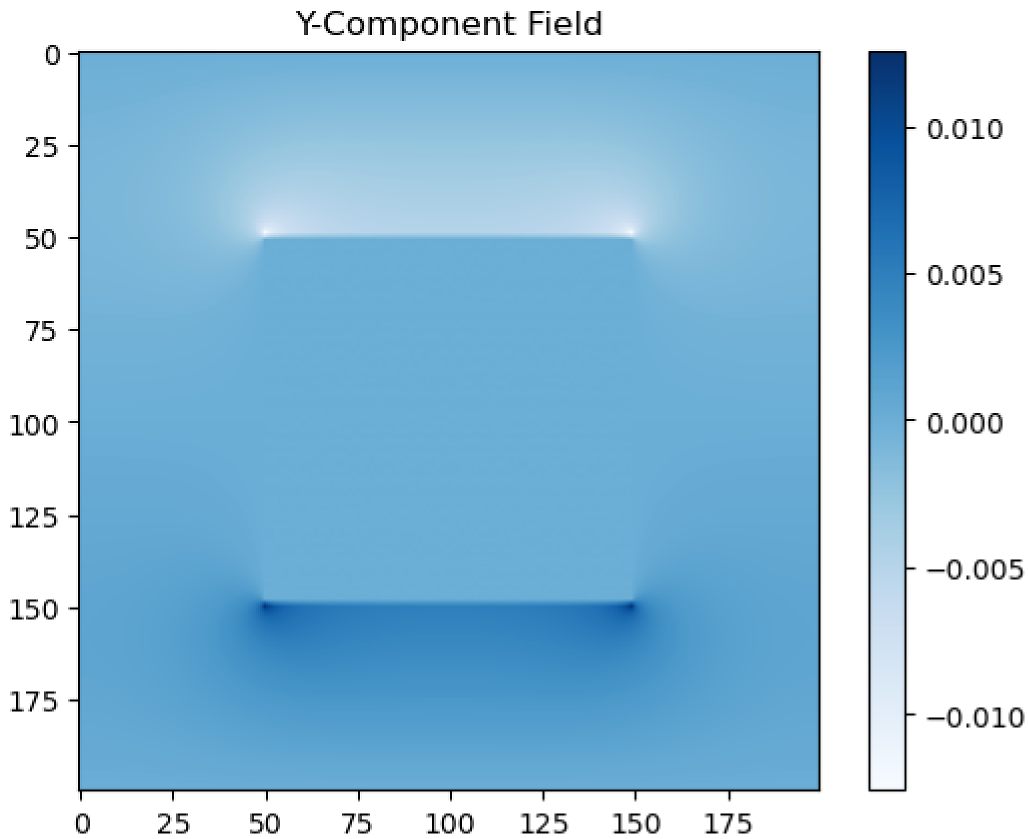
```
#===================================================
# Plot - Field Strength
#===================================================
Ex = (np.roll(Pot,1,axis=1) - np.roll(Pot,-1,axis=1))
Ey = (np.roll(Pot,-1,axis=0) - np.roll(Pot,1,axis=0))

# Vector Field
Vectors = np.zeros(Pot.shape)
X = np.arange(size)
Y = np.arange(size)
plt.figure(figsize=(10,10))
plt.quiver(X,Y,Ex,Ey, linewidth=0.5)
plt.title("Vector Field")
plt.show()
plt.close()

```



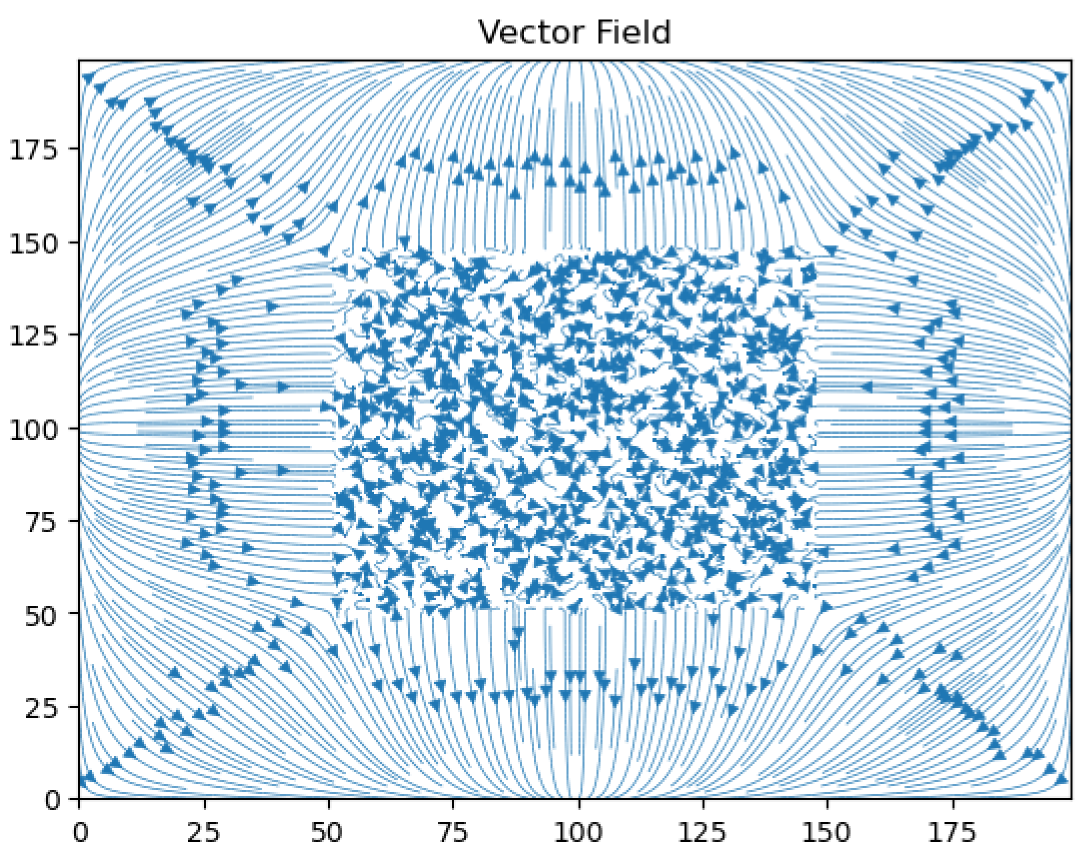
In [175]:

```python
# x - component
plt.imshow(Ex, cmap="Blues")
plt.colorbar()
plt.title("X-Component Field")
plt.show()
plt.close()

# y - component
plt.imshow(Ey, cmap="Blues")
plt.colorbar()
plt.title("Y-Component Field")
plt.show()
plt.close()

# Vector Field - Direction
Vectors = np.zeros(Pot.shape)
X = np.arange(size)
Y = np.arange(size)
plt.streamplot(X,Y,Ex,Ey, density=4, linewidth=0.5)
plt.title("Vector Field")
```






```
Out[175]: Text(0.5, 1.0, 'Vector Field')
```



This is what we expect. Strong field near the corners.

In [ ]: