0.1 Homework 5

0.1.1 Partition Function

Show that the partition function of an Ising lattice can be written as

$$Q_N(B,T) = \sum_{N_+,N_{+-}} g_N(N_+,N_{+-})\mathbf{exp}\{-\beta H_N(N_+,N_{+-})\},$$

where

$$H_N(N_+,N_{+-}) = -J\left(\frac{1}{2}qN - 2N_{+-}\right) - \mu B(2N_+ - N), \tag{1}$$

while other symbols have their usual meanings; compare these results to equations

$$H_N(N_+,N_{++}) = -J(N_{++} + N_{--} - N_{+-}) - \mu B(N_+ - N_-) \tag{2}$$

$$= -J\left(\frac{1}{2}qN - 2qN_+ + 4N_{++}\right) - \mu B(2N_+ - N) \tag{3}$$

and

$$Q_N(B,T) = \sum_{N_+,N_{++}} g_N(N_+,N_{++})\mathbf{exp}\left\{-\beta H_N(N_+,N_{++})\right\}.$$

The Hamiltonian of the Ising model is given by

$$H = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j - \mu B\sum_i \sigma_i, \quad \sigma_i = \pm 1 \quad \forall i.$$

The total number of neighbor pairs is

$$N_{++} + N_{--} + N_{+-} = \frac{1}{2}qN$$

So the interaction energy component of the Hamiltonian becomes

$$-J\sum_{\langle i,j\rangle}\sigma_i\sigma_j = -J(N_{++} + N_{--} - N_{+-}),$$

where $\sigma_i\sigma_j = +1$ for N_{++} and N_{--}, and $\sigma_i\sigma_j = -1$ for N_{+-}.

The magnetic energy component is

$$-\mu B\sum_i \sigma_i = -\mu B(N_+ - N_-) = -\mu B(2N_+ - N), \quad N_- = N - N_+.$$

Combining these two components gives the total Hamiltonian

$$H_N = -J(N_{++} + N_{--} - N_{+-}) - \mu B(2N_+ - N)$$

Using the relation $N_{++} + N_{--} = \frac{1}{2}qN - N_{+-}$, we can rewrite the Hamiltonian as

$$\boxed{H_N = -J\left(\frac{1}{2}qN - 2N_{+-}\right) - \mu B(2N_+ - N)},$$

So the partition function can be expressed as

$$\begin{aligned} Q_N(B,T) &= \sum_{N_+,N_{+-}} g_N(N_+,N_{+-})\text{exp}\{-\beta H_N(N_+,N_{+-})\} \\ &= \sum_{N_+,N_{+-}} g_N(N_+,N_{+-})\text{exp}\left\{-\beta\left[-J\left(\frac{1}{2}qN - 2N_{+-}\right) - \mu B(2N_+ - N)\right]\right\} \end{aligned}$$

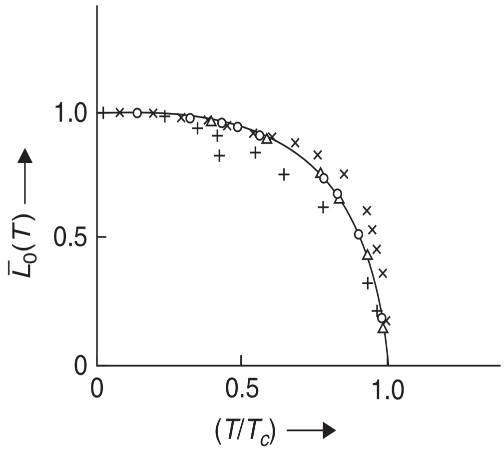

Figure 1: The spontaneous magnetization of a Weiss ferromagnet as a function of temperature. The experimental points (after Becker) are for iron (x), nickel (o), cobalt (Δ), and magnetite (+).

which matches the provided expression.

To prove that (1) and (3) are equivalent, we can use the relation between N_{+-} and N_{++}:

$$qN_+ = 2N_{++} + N_{+-} \Rightarrow N_{+-} = qN_+ - 2N_{++}$$

Substituting this into (1) gives:

$$\begin{aligned} H_N(N_+, N_{+-}) &= -J\left[\frac{1}{2}qN - 2(qN_+ - 2N_{++})\right] - \mu B(2N_+ - N) \\ &= \boxed{-J\left(\frac{1}{2}qN - 2qN_+ + 4N_{++}\right) - \mu B(2N_+ - N)} \end{aligned}$$

0.1.2 Equation of State

Show that the curve in 1 hits the horizontal and vertical axes at right angle according to the equation of state

$$\bar{L}_0 = \tanh\left(\frac{qJ\bar{L}_0}{kT}\right).$$

To show that the curve given by the equation of state $\bar{L}_0 = \tanh\left(\frac{qJ\bar{L}_0}{kT}\right)$ hits the horizontal and vertical axes at right angles, we need to analyze the slope of the curve at the boundaries($T = 0$ and $T = T_c = \frac{qJ}{k}$).

Differentiate both sides of the equation with respect to T, with chain rule:

$$\begin{aligned} \frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} &= \operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right)\left(\frac{qJ}{kT}\frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} - \frac{qJ\bar{L}_0}{kT^2}\right) \\ \left[1 - \operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right)\frac{qJ}{kT}\right]\frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} &= -\operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right)\frac{qJ\bar{L}_0}{kT^2} \\ \frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} &= \frac{\operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right)\frac{qJ\bar{L}_0}{kT^2}}{\operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right)\frac{qJ}{kT} - 1} \end{aligned}$$

1. At $T = 0$. Define $x = \dfrac{qJ\bar{L}_0}{kT}$, we have:

$$\begin{aligned}
\lim_{T\to 0} \tanh\left(\frac{qJ\bar{L}_0}{kT}\right) &= \lim_{x\to\infty} \tanh x = 1, \quad \forall \bar{L}_0 \neq 0 \\
&\Rightarrow \lim_{T\to 0} \bar{L}_0 = 1 \\
\lim_{T\to 0} \operatorname{sech}^2\left(\frac{qJ\bar{L}_0}{kT}\right) &= \lim_{x\to\infty} \operatorname{sech}^2 x = 0, \quad \forall \bar{L}_0 \neq 0 \\
&\Rightarrow \lim_{T\to 0} \frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} = \boxed{0}
\end{aligned}$$

Thus the curve hits the horizontal axis horizontally at $T = 0$.

2. At $T = T_c$. We have $\bar{L}_0 = 0$, and $\lim\limits_{x\to 0} \tanh x = x - \dfrac{x^3}{3} + o(x^3)$.

$$\begin{aligned}
\lim_{\bar{L}_0\to 0} \tanh\left(\frac{qJ\bar{L}_0}{kT}\right) &= \frac{qJ\bar{L}_0}{kT} - \frac{1}{3}\left(\frac{qJ\bar{L}_0}{kT}\right)^3 \\
\Rightarrow \bar{L}_0\left(1 - \frac{qJ}{kT}\right) &= -\frac{1}{3}\left(\frac{qJ}{kT}\right)^3 \bar{L}_0^3
\end{aligned}$$

Define $T_c = \dfrac{qJ}{k}$, so that $t = \dfrac{T}{T_c} = \dfrac{kT}{qJ}$ to substitute into the equation:

$$\bar{L}_0\left(1 - \frac{1}{t}\right) = -\frac{\bar{L}_0^3}{3t^3}$$

Let $t = 1 + \epsilon$ while $\epsilon \to 0$, we have $1 - \dfrac{1}{t} \approx \epsilon$. Then rewrite the equation as:

$$\begin{aligned}
\bar{L}_0\epsilon = -\frac{1}{3}\bar{L}_0^3 &\Rightarrow \bar{L}_0 \approx \sqrt{3}\sqrt{1 - \frac{T}{T_c}} \\
\Rightarrow \lim_{T\to T_c^-} \frac{\mathrm{d}\bar{L}_0}{\mathrm{d}T} &\approx -\frac{\sqrt{3}}{2}\frac{1}{\sqrt{1 - \frac{T}{T_c}}}\frac{1}{T_c} = \boxed{\infty}
\end{aligned}$$

Therefore the curve hits the vertical axis vertically at $T = T_c$.