0.1 Homework 3

0.1.1 Schwinger boson representation

A two-dimensional quantum harmonic oscillator contains two decoupled free bosons, whose annihilation operators can be represented as a and b respectively. $a = \frac{1}{\sqrt{2}}(x + ip_x)$, $b = \frac{1}{\sqrt{2}}(y + ip_y)$. They satisfy the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ and $[a, b] = [a, b^\dagger] = 0$. This system has U(2) symmetry, which includes an SU(2) subgroup. Let's explore how to construct the SU(2) representation using bosonic operators. Define $S^x = \frac{1}{2}(a^\dagger b + b^\dagger a)$, $S^z = \frac{1}{2}(a^\dagger a - b^\dagger b)$.

1. **Express S^y in terms of a and b. [Hint: Make $\vec{S} \times \vec{S} = i\vec{S}$]**

   To satisfy the commutation relation $\vec{S} \times \vec{S} = i\vec{S}$, we have

   $$[S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y$$

   So we have

   $$\begin{aligned} S^y &= \frac{1}{i}[S^z, S^x] = \frac{1}{i}\left[\frac{1}{2}\left(a^\dagger a - b^\dagger b\right), \frac{1}{2}\left(a^\dagger b + b^\dagger a\right)\right] \\ &= \frac{1}{4i}[a^\dagger a - b^\dagger b, a^\dagger b + b^\dagger a] \end{aligned}$$

   We have commutation formula that

   $$\begin{aligned} [\hat{A}, \hat{B} + \hat{C}] &= [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \\ [\hat{A} + \hat{B}, \hat{C}] &= [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \\ [\hat{A}, \hat{B}\hat{C}] &= \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C} \\ [\hat{A}\hat{B}, \hat{C}] &= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} \\ \Rightarrow [\hat{A}\hat{B}, \hat{C}\hat{D}] &= \hat{A}\hat{C}[\hat{B}, \hat{D}] + \hat{A}[\hat{B}, \hat{C}]\hat{D} + \hat{C}[\hat{A}, \hat{D}]\hat{B} + [\hat{A}, \hat{C}]\hat{D}\hat{B} \end{aligned}$$

   So we have

   $$\begin{aligned} S^y &= \frac{1}{4i}\left[a^\dagger a, a^\dagger b\right] + \frac{1}{4i}\left[a^\dagger a, b^\dagger a\right] - \frac{1}{4i}\left[b^\dagger b, a^\dagger b\right] - \frac{1}{4i}\left[b^\dagger b, b^\dagger a\right] \\ \left[a^\dagger a, a^\dagger b\right] &= \cancel{a^\dagger a^\dagger [a, b]} + a^\dagger [a, a^\dagger] b + \cancel{a^\dagger [a^\dagger, b] a} + \cancel{[a^\dagger, a^\dagger] b a} = a^\dagger b \\ \left[a^\dagger a, b^\dagger a\right] &= \cancel{a^\dagger b^\dagger [a, a]} + \cancel{a^\dagger [a, b^\dagger] a} + b^\dagger [a^\dagger, a] a + \cancel{[a^\dagger, b^\dagger] a a} = -b^\dagger a \\ \left[b^\dagger b, a^\dagger b\right] &= \cancel{b^\dagger a^\dagger [b, b]} + \cancel{b^\dagger [b, a^\dagger] b} + a^\dagger [b^\dagger, b] b + \cancel{[b^\dagger, a^\dagger] b b} = -a^\dagger b \\ \left[b^\dagger b, b^\dagger a\right] &= \cancel{b^\dagger b^\dagger [b, a]} + b^\dagger [b, b^\dagger] a + \cancel{b^\dagger [b^\dagger, a] b} + \cancel{[b^\dagger, b^\dagger] a b} = b^\dagger a \\ \Rightarrow S^y &= \frac{1}{4i}\left(a^\dagger b - b^\dagger a + a^\dagger b - b^\dagger a\right) = \boxed{\frac{1}{2i}\left(a^\dagger b - b^\dagger a\right)} \end{aligned}$$

2. **Prove that S^y is actually related to the angular momentum operator of the harmonic oscillator $L = xp_y - yp_x$, namely $S^y = \frac{L}{2}$.**

   Define

   $$\begin{aligned} x &= \frac{a + a^\dagger}{\sqrt{2}}, \quad p_x = \frac{i(a^\dagger - a)}{\sqrt{2}} \\ y &= \frac{b + b^\dagger}{\sqrt{2}}, \quad p_y = \frac{i(b^\dagger - b)}{\sqrt{2}} \end{aligned}$$

   So the angular momentum operator is

   $$\begin{aligned} L &= \left(\frac{a + a^\dagger}{\sqrt{2}}\right)\left(\frac{i(b^\dagger - b)}{\sqrt{2}}\right) - \left(\frac{b + b^\dagger}{\sqrt{2}}\right)\left(\frac{i(a^\dagger - a)}{\sqrt{2}}\right) \\ &= \frac{i}{2}\left[\left(a + a^\dagger\right)\left(b^\dagger - b\right) - \left(b + b^\dagger\right)\left(a^\dagger - a\right)\right] \\ &= \frac{i}{2}\left(ab^\dagger - \cancel{ab} + \cancel{a^\dagger b^\dagger} - a^\dagger b - ba^\dagger + \cancel{ba} - \cancel{b^\dagger a^\dagger} + b^\dagger a\right) \end{aligned}$$

Because $[a,b]=[a,b^\dagger]=0$, we have $ab^\dagger=b^\dagger a$ and $a^\dagger b=ba^\dagger$, so

$$L=\frac{i}{2}\left(ab^\dagger-a^\dagger b-a^\dagger b+ab^\dagger\right)=i(ab^\dagger-a^\dagger b)$$

While $S^y=\dfrac{1}{2i}(a^\dagger b-ab^\dagger)=\dfrac{i}{2}(ab^\dagger-a^\dagger b)$, so $S^y=\dfrac{L}{2}$. $\square$

3. **Define the following set of states, where $s=0,1/2,1,\cdots$, and $m=-s,-s+1,\cdots,s-1,s$ (they are called the Schwinger boson representation),**

$$|s,m\rangle=\frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}}\frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}}|\Omega\rangle$$

where $|\Omega\rangle$ is the state annihilated by a and b, i.e., $a|\Omega\rangle=b|\Omega\rangle=0$. Prove that the state $|s,m\rangle$ is indeed a simultaneous eigenstate of $\vec{S}^2=(S^x)^2+(S^y)^2+(S^z)^2$ and S^z , with eigenvalues $s(s+1)$ and m respectively. [Hint: Use the particle number basis.]

We have known that

$$S^z=\frac{1}{2}\left(a^\dagger a-b^\dagger b\right)$$
$$\vec{S}^2=(S^x)^2+(S^y)^2+(S^z)^2$$

where $a^\dagger a$ counts the number of particles in the a mode, and $b^\dagger b$ counts the number of particles in the b mode. So we have

$$a^\dagger a|s,m\rangle=(s+m)|s,m\rangle,\quad b^\dagger b|s,m\rangle=(s-m)|s,m\rangle$$
$$\Rightarrow S^z|s,m\rangle=\frac{1}{2}\left((s+m)-(s-m)\right)|s,m\rangle=\boxed{m|s,m\rangle}$$

So $|s,m\rangle$ is an eigenstate of S^z with eigenvalue m.

Define ladder operators $S^\pm=S^x\pm iS^y$:

$$S^+=a^\dagger b,\quad S^-=b^\dagger a$$
$$\Rightarrow S^2=S^zS^z+\frac{1}{2}\left(S^+S^-+S^-S^+\right)$$

接下来证明 Schwinger boson 表象下定义的态 $|s,m\rangle$ 以及对应的升降算符 $S^\pm$ 仍然满足传统的波函数关系. 以 $S^+=a^\dagger b$ 为例:

$$\begin{aligned}
S^+|s,m\rangle&=a^\dagger b\frac{(a^\dagger)^{s+m}}{\sqrt{(s+m)!}}\frac{(b^\dagger)^{s-m}}{\sqrt{(s-m)!}}|\Omega\rangle\\
&=\frac{\sqrt{s+m+1}}{\sqrt{s-m}}\frac{(a^\dagger)^{s+m+1}}{\sqrt{(s+m+1)!}}bb^\dagger\frac{(b^\dagger)^{s-m-1}}{\sqrt{(s-m-1)!}}|\Omega\rangle\\
&=\frac{\sqrt{s+m+1}}{\sqrt{s-m}}(b^\dagger b+1)|s,m+1\rangle\\
&=\frac{\sqrt{s+m+1}}{\sqrt{s-m}}(s-m-1+1)|s,m+1\rangle\\
&=\sqrt{s(s+1)-m(m+1)}|s,m+1\rangle
\end{aligned}$$

说明该定义下的算符仍然满足传统的数值关系, S^- 证明略. 则我们有

$$\begin{aligned}
S^+|s,m\rangle&=a^\dagger b|s,m\rangle=\sqrt{s(s+1)-m(m+1)}|s,m+1\rangle\\
S^-|s,m\rangle&=b^\dagger a|s,m\rangle=\sqrt{s(s+1)-m(m-1)}|s,m-1\rangle\\
\Rightarrow S^+S^-|s,m\rangle&=S^+\sqrt{s(s+1)-m(m-1)}|s,m-1\rangle=\Big[s(s+1)-m(m-1)\Big]|s,m\rangle\\
S^-S^+|s,m\rangle&=S^-\sqrt{s(s+1)-m(m+1)}|s,m+1\rangle=\Big[s(s+1)-m(m+1)\Big]|s,m\rangle\\
S^zS^z|s,m\rangle&=m^2|s,m\rangle
\end{aligned}$$

Combine the above results, and we have

$$
\begin{aligned}
S^2|s,m\rangle &= S^zS^z|s,m\rangle + \frac{1}{2}\left(S^+S^- + S^-S^+\right)|s,m\rangle \\
&= m^2|s,m\rangle + \frac{1}{2}\Big[s(s+1)-m(m-1)+s(s+1)-m(m+1)\Big]|s,m\rangle \\
&= \boxed{s(s+1)|s,m\rangle}
\end{aligned}
$$

□

0.1.2 1D tight-binding model

The Hamiltonian of a periodic tight-binding chain of length L is given by the following expression:

$$
H_{\text{chain}} = -t\sum_{n=1}^{L}\left(\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n\right)
$$

where t is the hopping matrix element between adjacent sites n and $n+1$, $\hat{a}_n^\dagger$ creates a fermion at site n, and the set of operators $\{a_n^\dagger, a_n; n=1,\cdots,L\}$ satisfies the standard anticommutation relations:

$$
\{a_n, a_{n'}^\dagger\} = \delta_{nn'}, \quad \{a_n, a_{n'}\} = 0, \quad \{a_n^\dagger, a_{n'}^\dagger\} = 0
$$

We assume periodic boundary conditions, i.e., we consider $a_{L+n}^\dagger = a_n^\dagger$.The purpose of this problem is to prove that this Hamiltonian can be diagonalized by a linear transformation of the discrete Fourier transform form:

$$
b_k^\dagger = \frac{1}{\sqrt{L}}\sum_{n=1}^{L} e^{ikn} a_n^\dagger
$$

1. **Let's require that $b_k^\dagger$ remains invariant under any shift of the summation index $n \to n+n'$ ("translation invariance"). Prove that this implies that the index k is quantized and determine the set of allowed k values. How many independent $b_k^\dagger$ operators are there?**

   不妨令 $n \to n+1$, 有

$$
\begin{aligned}
b_k^\dagger &= \frac{1}{\sqrt{L}}\sum_{n=1}^{L} e^{ik(n+1)} a_{n+1}^\dagger = \frac{1}{\sqrt{L}}\sum_{n'=2}^{L+1} e^{ikn'} a_{n'}^\dagger \\
&= \frac{1}{\sqrt{L}}\left[\sum_{n'=2}^{L} e^{ikn'} a_{n'}^\dagger + e^{ik(L+1)} a_{L+1}^\dagger\right] \\
&= \frac{1}{\sqrt{L}}\left[\sum_{n'=1}^{L} e^{ikn'} a_{n'}^\dagger - e^{ik} a_1^\dagger + e^{ik(L+1)} a_{L+1}^\dagger\right] \\
\Rightarrow e^{ik} a_1^\dagger &= e^{ik(L+1)} a_{L+1}^\dagger = e^{ik(L+1)} a_1^\dagger \\
\Rightarrow e^{ikL} &= 1 = e^{i2\pi m}, \quad m \in \mathbb{Z} \\
\Rightarrow k &= \frac{2\pi}{L} m, \quad m \in \{0,1,2,\cdots,L-1\}
\end{aligned}
$$

   So there are $\boxed{L}$ independent $b_k^\dagger$ operators.

2. **Verify that the set of b_k and $b_k^\dagger$ operators also satisfies the above standard anticommutation relations. That is:**

$$
\{b_k, b_{k'}^\dagger\} = \delta_{kk'}, \quad \{b_k, b_{k'}\} = 0, \quad \{b_k^\dagger, b_{k'}^\dagger\} = 0
$$

   Hint: Use the identity $\sum_{m=1}^{L} e^{i\frac{2\pi}{L}m} = 0$.

We have

$$b_k^\dagger = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} a_n^\dagger, \quad b_k = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{-ikn} a_n$$

So

$$\begin{aligned}
\{b_k, b_{k'}^\dagger\} &= \frac{1}{L} \sum_{n,n'} e^{-ikn} e^{ik'n'} \{a_n, a_{n'}^\dagger\} = \frac{1}{L} \sum_{n,n'} e^{-ikn} e^{ik'n'} \delta_{nn'} \\
&= \frac{1}{L} \sum_{n=1}^{L} e^{-ikn} e^{ik'n} = \frac{1}{L} \sum_{n=1}^{L} e^{i(k'-k)n} = \boxed{\delta_{kk'}} \\
\{b_k, b_{k'}\} &= \frac{1}{L} \sum_{n,n'} e^{-ikn} e^{-ik'n'} \{a_n, a_{n'}\} = \boxed{0} \\
\{b_k^\dagger, b_{k'}^\dagger\} &= \frac{1}{L} \sum_{n,n'} e^{ikn} e^{ik'n'} \{a_n^\dagger, a_{n'}^\dagger\} = \boxed{0}
\end{aligned}$$

3. **Prove that the inverse transformation of the above has the form:**

$$a_n^\dagger = \frac{1}{\sqrt{L}} \sum_k e^{-ikn} b_k^\dagger$$

where the sum is over the set of allowed k values determined in (a).

We have the definition

$$b_k^\dagger = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} a_n^\dagger$$

So

$$\begin{aligned}
\frac{1}{\sqrt{L}} \sum_k e^{-ikn} b_k^\dagger &= \frac{1}{\sqrt{L}} \sum_k e^{-ikn} \left(\frac{1}{\sqrt{L}} \sum_{n'} e^{ikn'} a_{n'}^\dagger \right) \\
&= \frac{1}{L} \sum_{n'} \sum_k e^{ik(n'-n)} a_{n'}^\dagger = \sum_{n'} \left(\frac{1}{L} \sum_k e^{ik(n'-n)} \right) a_{n'}^\dagger \\
&= \sum_{n'} (\delta_{nn'}) a_{n'}^\dagger = a_n^\dagger. \quad \square
\end{aligned}$$

4. **Show that $b_k^\dagger$ is indeed a creation operator of a single-particle eigenstate of H_{chain} by proving that its commutator with the Hamiltonian has the form $[H_{\text{chain}}, b_k^\dagger] = \varepsilon_k b_k^\dagger$. Give the explicit expression for the corresponding eigenvalue ε_k.**

We have known that

$$\begin{aligned}
H_{\text{chain}} &= -t \sum_{n=1}^{L} \left(\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a} \right), \quad \hat{a}_{L+1} = \hat{a}_1 \\
b_k^\dagger &= \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{ikn} a_n^\dagger
\end{aligned}$$

So the commutator

$$\begin{aligned}
[H_{\text{chain}}, b_k^\dagger] &= -t \sum_{n=1}^{L} \left(\left[a_n^\dagger a_{n+1}, b_k^\dagger \right] + \left[a_{n+1}^\dagger a_n, b_k^\dagger \right] \right) \\
&= -\frac{t}{L} \sum_{n=1}^{L} \sum_{n'}^{L} \left(\left[a_n^\dagger a_{n+1}, e^{ikn'} a_{n'}^\dagger \right] + \left[a_{n+1}^\dagger a_n, e^{ikn'} a_{n'}^\dagger \right] \right) \\
&= -\frac{t}{L} \sum_{n=1}^{L} \sum_{n'}^{L} e^{ikn'} \left(a_n^\dagger a_{n+1} a_{n'}^\dagger - \underbrace{a_{n'}^\dagger a_n^\dagger a_{n+1}}_{*} + a_{n+1}^\dagger a_n a_{n'}^\dagger - \underbrace{a_{n'}^\dagger a_{n+1}^\dagger a_n}_{*} \right)
\end{aligned}$$

根据 $a, a^\dagger$ 的反对易关系, 交换相邻的升算符和降算符满足关系 $\begin{cases} a^\dagger_{n'} a^\dagger_n = -a^\dagger_n a^\dagger_{n'} \\ a_{n'} a_n = -a_n a_{n'} \end{cases}$ 交换 * 项中的升算符, 从而使其变号:

$$
\begin{aligned}
[H_{\text{chain}}, b_k^\dagger] &= -\frac{t}{\sqrt{L}} \sum_{n=1}^{L} \sum_{n'}^{L} e^{ikn'} \left(a_n^\dagger a_{n+1} a_{n'}^\dagger + a_n^\dagger a_{n'}^\dagger a_{n+1} + a_{n+1}^\dagger a_n a_{n'}^\dagger + a_{n+1}^\dagger a_{n'}^\dagger a_n \right) \\
&= -\frac{t}{\sqrt{L}} \sum_{n=1}^{L} \sum_{n'}^{L} e^{ikn'} \left[a_n^\dagger \underbrace{\left(a_{n+1} a_{n'}^\dagger + a_{n'}^\dagger a_{n+1} \right)}_{\{a_{n+1}, a_{n'}^\dagger\}} + a_{n+1}^\dagger \underbrace{\left(a_n a_{n'}^\dagger + a_{n'}^\dagger a_n \right)}_{\{a_n, a_{n'}^\dagger\}} \right] \\
&= -\frac{t}{\sqrt{L}} \sum_{n=1}^{L} \sum_{n'}^{L} \left[e^{ikn'} a_n^\dagger \delta_{n+1,n'} + e^{ikn'} a_{n+1}^\dagger \delta_{n,n'} \right] \\
&= -\frac{t}{\sqrt{L}} \sum_{n=1}^{L} \left[e^{ik} e^{ikn} a_n^\dagger + e^{-ik} e^{ik(n+1)} a_{n+1}^\dagger \right] \\
&= -t \left[e^{ik} b_k^\dagger + e^{-ik} b_k^\dagger \right] \\
\varepsilon_k b_k^\dagger &= -2t \cos k b_k^\dagger
\end{aligned}
$$

So the corresponding eigenvalue $\boxed{\varepsilon_k = -2t \cos k}$.