0.1 Homework 2

1. **Show that the volume element**

$$\mathrm{d}\omega = \prod_{i=1}^{3N}(\mathrm{d}q_i \mathrm{d}p_i)$$

of the phase space remains invariant under a canonical transformation of the (generalized) coordinates (q, p) to any other set of (generalized) coordinates (Q, P).

[Hint: Before considering the most general transformation of this kind, which is referred to as a contact transformation, it may be helpful ti consider a point transformation - one in which the new coordinates Q_i and the old coordinates q_i transform only among themselves.]

$$(Q, P) = (Q(q, p), P(q, p))$$

So the volume element is

$$\mathrm{d}\omega' = \prod_{i=1}^{3N}\mathrm{d}Q_i\mathrm{d}P_i = \left|\frac{\partial(Q, P)}{\partial(q, p)}\right|\prod_{i=1}^{3N}\mathrm{d}q_i\mathrm{d}p_i$$

$$J = \frac{\partial(Q, P)}{\partial(q, p)} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$$

Since canonical transformations preserve the Poisson brackets

$$\{Q_i, Q_j\} = 0, \quad \{P_i, P_j\} = 0, \quad \{Q_i, P_j\} = \delta_{ij},$$

which gives the Jacobian matrix J

$$J^T \Omega J = \Omega, \quad \Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

So det$\Omega = 1$, which means det$J = 1$.

Therefore we have $\mathrm{d}\omega' = \mathrm{d}\omega$, or

$$\prod_{i=1}^{3N}\mathrm{d}Q_i\mathrm{d}P_i = \prod_{i=1}^{3N}\mathrm{d}q_i\mathrm{d}p_i$$

2. **The generalized coordinates of a simple pendulum are the angular displacement θ and the angular momentum $ml^2\dot{\theta}$. Study, both mathematically and graphically, the nature of the corresponding trajectories in the phase space of the system, and show that the area A enclosed by a trajectory is equal to the product of the total energy E and the time period τ of the pendulum.** With θ and $L = m\dot{\theta}l^2$, the Hamiltonian of the simple pendulum is

$$H = \frac{L^2}{2ml^2} + mgl(1 - \cos\theta)$$

So the area A enclosed by a trajectory is computed using the integral of $L\mathrm{d}\theta$:

$$A = \oint L\mathrm{d}\theta.$$

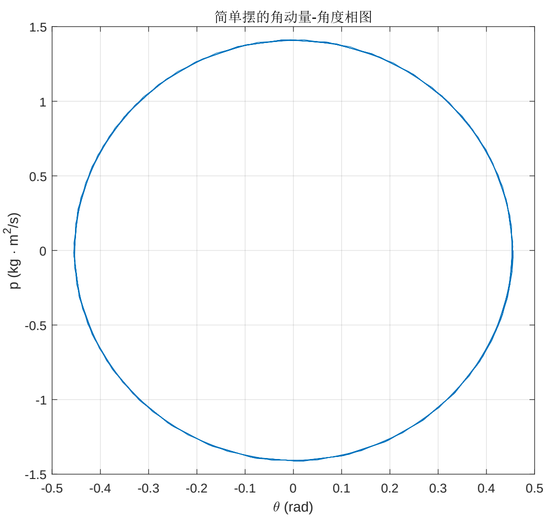
Deriative of A with respect to E gives the time period τ:

$$\frac{\mathrm{d}A}{\mathrm{d}E} = \frac{\mathrm{d}}{\mathrm{d}E}\oint L\mathrm{d}\theta = \oint \frac{\partial L}{\partial E}\mathrm{d}\theta$$

$$\frac{\partial H}{\partial L} = \frac{L}{ml^2} = \dot{\theta}$$

$$\Rightarrow \frac{\mathrm{d}A}{\mathrm{d}E} = \oint \frac{1}{\dot{\theta}}\mathrm{d}\theta = \tau$$

$$\Rightarrow A = E\tau.\square$$



简单摆的角动量-角度相图
1.5
1
0.5
0
-0.5
-1
-1.5
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
θ (rad)
p (kg · m²/s)