0.1 Homework 5

0.1.1 Landau's Theory

Derive the critical exponents based on Landau's theory for second-order phase transition.

$$\psi_0(t, m_0) = q(t) + r(t)m_0^2 + s(t)m_0^4 + \cdots \quad \left(t = \frac{T - T_c}{T_c}, |t| \ll 1\right);$$

Assuming that

- Symmetry: The free energy is even in m_0;
- Analticity: ψ_0 is analytic in m_0 and t, which allows a Taylor expansion;
- Critical behavior: Near T_c, the coefficients behave as $r(r) \approx r_0 t, \quad s(t) \approx s_0 > 0$.

The exponents are given by:

$$m_0 \sim (-t)^\beta, \quad \chi \sim |-t|^{-1}, \quad m_0 \sim h^{1/\delta}, \quad \xi \sim |t|^{-\nu}$$

The equilibrium order parameter m_0 minimizes the free energy:

$$\begin{aligned} \frac{\partial \psi_0}{\partial m_0} = 0 &\Rightarrow 2r(t)m_0 + 4s(t)m_0^3 = 0 \\ &\Rightarrow m_0[r(t) + 2s(t)m_0^2] = 0 \end{aligned}$$

So

- Disordered phase($T > T_c$): $m_0 = 0$, since $r(t) > 0$;
- Ordered phase($T < T_c$): $m_0^2 = -\dfrac{r(t)}{2s(t)} \approx -\dfrac{r_0 t}{2s_0}$, since $r(t) \approx r_0 t$ and $s(t) \approx s_0$.

1. For $T < T_c, t < 0$, $m_0 \sim \sqrt{-t} \Rightarrow m_0 \sim (-t)^{1/2} \Rightarrow \boxed{\beta = \frac{1}{2}}$
2. Susceptibility χ, which is defined as $\chi^{-1} = \left.\dfrac{\partial^2 \psi_0}{\partial m_0^2}\right|_{m_0 = m_{eq}}$.
   - For $T > T_c$, $m_0 = 0$. $\chi^{-1} = 2r(t) \approx 2r_0 t \Rightarrow \chi \sim t^{-1}$
   - For $T < T_c$, $m_0^2 = -\dfrac{r(t)}{2s(t)}$:

$$\frac{\partial^2 \psi_0}{\partial m_0^2} = 2r(t) + 12s(t)m_0^2 = 2r(t) + 12s(t)\left[-\frac{r(t)}{2s(t)}\right] = -4r(t)$$
$$\chi^{-1} = -4r(t) \approx -4r_0 t \Rightarrow \chi \sim (-t)^{-1} \Rightarrow \boxed{\gamma = 1}$$

3. Specific heat.
   - For $T > T_c$, $\psi_0 = q(t)$;
   - For $T < T_c$, $\psi_0 = q(t) + r(t)m_0^2 + s(t)m_0^4 = q(t) - \dfrac{r(t)^2}{4s(t)}$. And the specific heat is defined as $C = -T\dfrac{\partial^2 \psi_0}{\partial T^2}$. Since $r(t) \sim t$, the singular part is C, which jumps at $t = 0$. So $\boxed{\alpha = 0}$.
4. Critical isotherm. At $T = T_c$, the free energy is $\psi_0 = q(0) + s(0)m_0^4 + \cdots$. Applying an external field h, the equilibrium condition is

$$h = \frac{\partial \psi_0}{\partial m_0} = 4s(0)m_0^3 \Rightarrow m_0 \sim h^{1/3} \Rightarrow \boxed{\delta = 3}.$$

5. Correlation length, which is defined as $\xi \sim \sqrt{\dfrac{c}{r(t)}} \sim t^{-1/2} \Rightarrow \boxed{\nu = \frac{1}{2}}$