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1 Introduction to probability theory

Bayes' theorem

$$p(B|A)=\frac{p(A|B)\cdot p(B)}{p(A)}=\frac{p(A|B)\cdot p(B)}{\sum_{B'}p(A|B)\cdot p(B')}$$

Expectation and covariance

$$\langle f\rangle=\sum_i f(i)p_i \text{ or } \langle f\rangle=\int f(x)p(x)dx$$

$$\mu=\langle i\rangle=\sum_i ip_i \text{ or } \mu=\langle x\rangle=\int xp(x)dx$$

$$\sigma^2=\langle i^2\rangle-\langle i\rangle^2$$

$$\sigma_{ij}^2=\langle ij\rangle-\langle i\rangle^2$$

Binomial distribution

$$\frac{N!}{(N-i)!i!}=\binom{N}{i} \text{ binomial coefficient}$$

$$p_i=\binom{N}{i}\cdot p^i q^{N-i} \text{ distribution}$$

$$\mu=\langle i\rangle=N\cdot p$$

$$\langle i^2\rangle=p\cdot N+p^2\cdot N\cdot(N-1)$$

$$\sigma^2=N\cdot p\cdot q$$

$$\sum_{i=0}^N p_i=\sum_{i=0}^N\binom{N}{i}\cdot p^iq^{N-i}=(p+q)^N=1$$

Gauss distribution

$$p(x)=\frac{1}{\left(2\pi\sigma^2\right)^{\frac12}}\cdot e^{-\frac{x-\mu}{2\sigma^2}},\quad \langle x^2\rangle=\sigma^2$$

Poisson distribution

$$p(k;\mu)=\frac{\mu^k}{k!}e^{-\mu},\quad E[k]=\mu,\ V[k]=\mu$$

Information entropy

$$S=-\sum_i p_i\ln(p_i)$$

2 The microcanonical ensemble

$E\approx$ const , V = const , N = const.

The fundamental postulate

$$\Omega(E)=\sum_{n:E-\delta E\le E_n\le E}1$$

$$\Omega(E;\delta E)=\frac{1}{h^{3N}N!}\iint_{E-\delta E\le\mathcal{H}(\vec q,\vec p)\le E}d\vec q d\vec p$$

$$S=-k_B\sum_{i=1}^{\Omega}p_i\ln(p_i)=k_B\ln(\Omega)$$

n_0 different particles

$$\Omega=\frac{1}{h^{3N}\prod_{j=0}^{n_0}N_j!}\iint_{E-\delta E\le\mathcal H(\vec q,\vec p)\le E}d\vec q d\vec p$$

Equilibrium conditions

Entropy S must be maximal

Thermal contact

$$\left.\frac{\partial S(E,V,N)}{\partial E}\right|_{V,N}=\frac{1}{T(E,V,N)}$$

Contact with volume excahnge

$$\left.\frac{\partial S(E,V,N)}{\partial V}\right|_{E,N}=\frac{p(E,V,N)}{T(E,V,N)}$$

Contact with exchange of particle number

$$\left.\frac{\partial S(E,V,N)}{\partial N}\right|_{E,V}=-\frac{\mu(E,V,N)}{T(E,V,N)}$$

Equations of state

$$dE=TdS-pdV+\mu dN$$

Specific heat

$$c_v=\frac{dE}{dT}$$

solution concept

- Set up Hamiltonian
- Calculate phasevolume Ω
- Calculate entropy S
- determine T,p,μ
- Calculate $U=\langle E\rangle$
- thermodynamic potentials:
  $F(T,V,N)=U-TS$
  $\hat H(S,p,N)=U+pV$
  $G(T,p,N)=U+pV-TS$

Ideal Gas

$$\mathcal H=\sum_{i=1}^{3N}\frac{p_i^2}{2m}+V(q_1,\dots,q_{3N})$$

microcanonical partition sum for an ideal gas

$$\Omega(E)=\frac{V^N\pi^{3N/2}(2mE)^{3N/2}}{h^{3N}N!\left(\frac{3N}{2}\right)!}$$

$$S=k_BN\left\{\ln\left[\left(\frac VN\right)\left(\frac{4\pi mE}{3h^2N}\right)^{3/2}\right]+\frac52\right\}$$

Equations of state for ideal gas

$$\frac1T=\left(\frac{\partial S}{\partial E}\right)_{N,V}=\frac32\frac{Nk_B}{E}\to U=\frac32Nk_BT$$

$$p=T\left(\frac{\partial S}{\partial V}\right)_{E,N}=TNk_B\frac1V\to pV=Nk_BT$$

$$\mu=k_BT\ln\left(\frac{N\lambda^3}{V}\right) \quad \text{chemical potential}$$

$$\lambda=\frac{h}{\sqrt{2\pi mk_BT}} \quad \text{Thermal de Broglie}$$

Einstein model for specific heat of a solid

$E=\hbar\omega\left(\frac N2+Q\right)\to Q=\left(\frac{E}{\hbar\omega}-\frac N2\right)$

$$\Omega(E,N)=\frac{(Q+N)!}{Q!N!}$$

$$S=k_B\left[Q\ln\left(\frac{Q+N}{Q}\right)+N\ln\left(\frac{Q+N}{N}\right)\right]$$

$$=k_BN\left[(e+\frac12)\ln(e+\frac12)-(e-\frac12)\ln(e-\frac12)\right]$$

$$e=E/E_0\,;E_0=N\hbar\omega\,;\beta=\hbar\omega/k_BT$$

$$\frac1T=\frac{\partial S}{\partial E}\Rightarrow E=N\hbar\omega\left(\frac12+\frac{1}{e^\beta-1}\right)$$

Entropic elasticity of polymers

$N_+-N_-=\frac La=m\to N_+=\frac12(N+m)$

$\Omega=\frac{N!}{N_+!N_-!}=\frac{N!}{\left(\frac12(N+m)\right)!\left(\frac12(N-m)\right)!}$

if both directions are possible $x2$

$$S=-k_B\left(N_+\ln\left(\frac{N_+}{N}\right)+N_-\ln\left(\frac{N_-}{N}\right)\right)$$

Statistical deviation from average

Two ideal gases in thermal conact $T_1=T_2$

$$S_i=\frac32k_BN_i\ln(E_i)+\text{independent of}E_i$$

$$S=S_1+S_2$$

$$dS=0\ \to\ \frac{\partial S_1}{\partial E_1}=\frac{\partial S_2}{\partial E_2}$$

$$\to\overline E_1=\frac{N_1}{N}E$$

consider small deviation:

$$E_1=\overline E_1+\Delta E,\quad E_2=\overline E_2-\Delta E$$

$$S(\overline E_1+\Delta E)\approx\frac32k_B\left[N_1\ln\overline E_1+N_2\ln\overline E_2-\frac{N_1}{2}\left(\frac{\Delta E}{\overline E_1}\right)^2-\frac{N_2}{2}\left(\frac{\Delta E}{\overline E_2}\right)^2\right]$$

$$\to\Omega=\overline\Omega e^{\left[-\frac34\left(\frac{\Delta E}{E}\right)^2N^2\left(\frac{1}{N_1}+\frac{1}{N_2}\right)\right]}$$

3 The canonical ensemble

T = const , V = const , N = const.

Boltzmann distribution

$$p_i=\frac1Ze^{-\beta E_i}\quad\text{Boltzmann distribution}$$

$$Z=\sum_i e^{-\beta E_i}\quad\text{partition sum}$$

For classical Hamiltonian systems:

$$p(\vec q,\vec p)=\frac{1}{ZN!h^{3N}}e^{-\beta\mathcal H(\vec q,\vec p)}$$

$$Z_N(T,V)=\frac{1}{N!h^{3N}}\iint d\vec q d\vec p e^{-\beta\mathcal H(\vec q,\vec p)}$$

For common Hamiltonian:

$$Z_N(T,V)=\frac{1}{\lambda^{3N}N!}\int_V d\vec q e^{-\beta\hat V(\vec q)}$$

Free energy

$$F(T,V,N)=-k_BT\ln Z_N(T,V)$$

$$\langle E\rangle=U=-\partial_\beta\ln Z_N$$

total differential:

$$dF=dE+d(TS)=-SdT-pdV+\mu N$$

equations of state

$$S=-\frac{\partial F}{\partial T},\quad p=-\frac{\partial F}{\partial V},\quad \mu=\frac{\partial F}{\partial N}$$

Non-interacting systems

ϵ_{ij} is the j^{th} state of the i^{th} element

$$Z=\sum_{j1}\sum_{j2}\dots\sum_{jN}e^{-\beta\sum_{i=1}^N\epsilon_{ij_i}}$$

$$=\left(\sum_{j1}e^{-\beta\epsilon_{1j_1}}\right)\dots\left(\sum_{jN}e^{-\beta\epsilon_{Nj_1N}}\right)$$

$$=z_1\cdot\dots\cdot z_N=\prod_{i=1}^N z_i$$

$$\to F=-k_BT\sum_{i=1}^N\ln(z_i)=-k_BT\ln(Z)$$

$$Z=z^N,\quad F=-k_BTNln(z)$$

Ideal Gas

$\mathcal H=\sum_{i=1}^N\frac{\vec p_i^2}{2m}$

$$Z_N(T,V)=\frac{V^N}{N!}\left(\int_{-\infty}^{+\infty}\frac{dp}{h}e^{-\beta\frac{p^2}{2m}}\right)^{3N}$$

$$=\frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N$$

Equipartition theorem

f_{dof} are the degrees of freedom.

harmonic Hamiltonian with $f_{dof}=2$

$$\mathcal H=Aq^2+Bp^2$$

$$z\propto\int dqdpe^{-\beta\mathcal H}$$

$$=\left(\frac{\pi}{A\beta}\right)^{\frac12}\cdot\left(\frac{\pi}{B\beta}\right)^{\frac12}\propto\left(T^{\frac12}\right)^{f_{dof}}$$

For sufficiently high temperture (classical limit), each quadratic term in the Hamiltonian contributes a factor $T^{\frac12}$ to the partition sum ('equipartition theorem')

$$F=-k_BT\ln(z)=-\frac{f_{dof}}{2}k_BTln(T)$$

$$S=-\frac{\partial F}{\partial T}=\frac{f_{dof}}{2}k_B(\ln(T)+1)$$

$$U=-\partial_\beta\ln(z)=\frac{f_{dof}}{2}k_BT$$

$$c_v=\frac{dU}{dT}=\frac{f_{dof}}{2}k_B$$

$$c_p=\frac{f_{dof}+2}{2}k_B$$

Molecular gases

N molecules; x different mode types:

$Z=Z_{trans}\cdot Z_{vib}\cdot Z_{rot}\cdot Z_{elec}\cdot Z_{nuc}$

$Z_x=z_x^N$

Vibrational modes

often described by the Morse potential:

$V(r)=E_0\left(1-e^{-\alpha(r-r_0)}\right)^2$

An exact solution of the Schrödinger equation gives:

$E_n=\hbar\omega_0\left(n+\frac12\right)-\frac{\hbar^2\omega_0^2}{eE_0}\left(n+\frac12\right)^2$

$\omega_0=\frac{\alpha}{2\pi}\sqrt{\frac{2E_0}{\mu}},\quad\mu=\frac m2$

For $\hbar\omega_0\ll E_0$ we can use the harmonic approximation:

$z_{vib}=\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega_0}}$

$T_{vib}\approx\frac{\hbar\omega_0}{k_B}\approx6.140K$ for H_2

Rotational modes

standart approximation is the one of a rigid rotator. The moment of inertia is given as:

$I=\mu r_0^2\quad T_{rot}=\frac{\hbar^2}{Ik_B}\quad\omega_n=(2J+1)$

$\to E_l=\frac{\hbar^2}{2I}l(l+1)$

Nuclear contributions: ortho- and parahydrogen

$S=1\,,z_{ortho}=\sum_{l=1,3,5,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$

$S=0\,,z_{para}=\sum_{l=0,2,4,\dots}(2l+1)e^{-\frac{l(l+1)T_{rot}}{T}}$

Specific heat of a solid

Debye model

$$\rightarrow \omega(k) = \left(\frac{4\kappa}{m}\right)^{\frac{1}{2}} \left|\sin\left(\frac{ka}{2}\right)\right|$$

$$\omega = \frac{2\pi}{T}, \quad k = \frac{2\pi}{\lambda}$$

Debye frequency:

$$\omega_D = c_s \left(\frac{6\pi^2 N}{V}\right)^{\frac{1}{3}}$$

$$c_s = \left.\frac{d\omega}{dk}\right|_{k=0} = \sqrt{\frac{\kappa}{m}}a$$

density of states in ω-space:

$$D(\omega) = 3\frac{\omega^2}{\omega_D^3} \quad \text{for } \omega \le \omega_D$$

count modes in frequency-space:

$$\sum_{modes}(\dots) = 3\sum_k (\dots) = 3N\int_0^{\omega_D} d\omega D(\omega)(\dots)$$

partition sum:

$$z(\omega) = \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}$$

$$\rightarrow Z = \prod_{modes} z(\omega)$$

$$\rightarrow E = -\partial_\beta \ln(Z) = \sum_{modes}\hbar\omega\left(\frac{1}{e^{\beta\hbar\omega}-1}+\frac{1}{2}\right)$$

$$= E_0 + 3N\int_0^{\omega_D} d\omega \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\frac{3\omega^2}{\omega_D^3}$$

$$c_v(T) = \frac{\partial E}{\partial T}$$

$$= \frac{3\hbar^2 N}{k_B T^2}\int_0^{\omega_D} d\omega \frac{3\omega^2}{\omega_D^3}\frac{e^{\beta\hbar\omega}\omega^2}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

$$u = \beta\hbar\omega$$

$$c_v(T) = \frac{9Nk_B}{u_m^3}\int_0^{u_m}\frac{e^u u^4}{(e^u-1)^2}du$$

the limit for $\hbar\omega_D \ll k_B T$:

$$c_v(T) = 3Nk_B$$

the limit for $k_B T \ll \hbar\omega_D$: ($T_D = \frac{\hbar\omega_D}{k_B}$)

$$c_v(T) = \frac{12\pi^4}{5}Nk_B\left(\frac{T}{T_D}\right)^3$$

Black body radiation

$$E = \frac{4\sigma}{c}VT^4, \quad \sigma = \frac{\pi^2 k_B^4}{60\hbar^3c^2}$$

$$c_v = \frac{16\sigma}{c}VT^3$$

$$J = \frac{P}{A} = \sigma T^4 \quad \text{Stefan- Boltzmann law}$$

Plank's law for black body radiation

$$u(\omega) := \frac{\hbar}{\pi^2c^3}\frac{\omega^3}{e^{\hbar\omega/(k_BT)}-1}$$

The Plank distribution has a maximum at:

$\hbar\omega_{max} = 2.82k_BT$ Wien's displacement law

4 The grandcanonical ensemble

$T, \mu = const.$

$$p_N(q,p) = \frac{1}{\Xi_\mu(T,V)}e^{-\beta(H_N(q,p)-\mu N)}$$

$$\Xi_\mu(T,V) = \sum_{N=0}^{\infty}\frac{1}{h^{3N}N!}\iint d^{3N}qd^{3N}pe^{-\beta(H_N-\mu N)}$$

$$\rightarrow \Xi_z = \sum_{N=0}^{\infty} z^N Z_N(T,V)$$

$z = e^{\beta\mu} \rightarrow$ Fugacity

Mean phase space observable

$$\langle F\rangle = \frac{1}{\Xi_\mu(T,V)}\sum_{N=0}^{\infty}\frac{1}{h^{3N}N!}\iint d^{3N}qd^{3N}p\dots$$

$$\dots e^{-\beta(H_N-\mu N)}F_N(q,p)$$

mean particle number:

$$\langle N\rangle = \frac{1}{\beta}\left(\frac{\partial}{\partial\mu}\ln\left(\Xi_\mu(T,V)\right)\right)_{T,V}$$

$$= z\left(\frac{\partial}{\partial z}\ln\left(\Xi_z(T,V)\right)\right)_{T,V}$$

pressure:

$$p = -\left\langle\frac{\partial H}{\partial V}\right\rangle = \frac{1}{\beta}\left(\frac{\partial}{\partial V}\ln\left(\Xi_\mu(T,V)\right)\right)$$

energy U:

$$U = \langle H\rangle = -\left(\frac{\partial}{\partial\beta}\ln\left(\Xi_\mu(T,V)\right)\right)_{\mu,V} + \mu\langle N\rangle$$

$$= -\left(\frac{\partial}{\partial\beta}\ln\left(\Xi_z(T,V)\right)\right)_{z,V}$$

Grandcanonical potential

grandcanonical potential:

$$\Psi(T,V,\mu) = -k_BT\ln\left(\Xi_\mu(T,V)\right)$$

p is maximal, if Ψ is minimal.

Total differential:

$$d\Psi = -SdT - pdV - \langle N\rangle d\mu$$

Equations of state:

$$S = -\frac{\partial\Psi}{\partial T}, p = -\frac{\partial\Psi}{\partial V}, N = -\frac{\partial\Psi}{\partial\mu}$$

Fluctuations

$$\sigma_N^2 = \langle N^2\rangle - \langle N\rangle^2 = \frac{1}{\beta^2}\left(\partial_\mu^2\ln(\Xi_\mu)\right)$$

$$\frac{\sigma_N}{\langle N\rangle} \propto \frac{1}{\sqrt{N}}$$

Ideal gas

$$Z_N(T,V) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\right)^N, \lambda = \frac{h}{(2\pi mk_BT)^{\frac{1}{2}}}$$

$$\Xi = \sum_{N=0}^{\infty} Z_N(T,V)z^N$$

$$= \sum_{N=0}^{\infty}\frac{1}{N!}\left(e^{\beta\mu}\frac{V}{\lambda^3}\right)^N$$

$$= e^{z\frac{V}{\lambda^3}} \quad \text{fugacity: } z := e^{\beta\mu}$$

$$\langle N\rangle = \frac{1}{\beta}\partial_\mu\ln(Z_G) = \frac{V}{\lambda^3}d^{\beta\mu}$$

$$\mu = k_BT\ln\left(\frac{N\lambda^3}{V}\right)$$

Molecular adsorption onto a surface

$$Z_G = z_G^N; z_G = 1 + e^{-\beta(\epsilon-\mu)}$$

$$\langle n\rangle = \frac{1}{e^{-\beta(\mu-\epsilon)}+1} \quad \text{per site}$$

$$\langle\epsilon\rangle = \epsilon\langle n\rangle$$

5 Quantum fluids

Fermion vs. bosons

1. Fermions: Pauli-principle + not distinguishable
2. Bosons: symetric wave function + not distinguishable
3. Boltzmann: particles are distinguishable

Canonical ensemble

$\omega_n \rightarrow$ degeneracy of state n

$$z = \sum_n \omega_n\exp(-\beta E_n)$$

Grand canonical ensemble

only two states $0, \epsilon$

Fermions:

$$z_F = 1 + e^{-\beta(\epsilon-\mu)}$$

average occupation number n_F:

$$n_F = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \quad \text{Fermi function}$$

For $T \rightarrow 0$, the fermi function approaches a step function:

$$n_F = \Theta(\mu-\epsilon)$$

Bosons:

$$z_B = \frac{1}{1-e^{-\beta(\epsilon-\mu)}}$$

average occupation number n_B:

$$n_B = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$$

- Fermions tend to fill up energy states one after the other
- Bosons tend to condense all into the same low energy state

The ideal Fermi fluid

density of states:

$$D(\epsilon) = \frac{V}{2\pi N}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{\epsilon}$$

Fermi energy

$$N = \sum_{\vec{k},m_s} n_{\vec{k},m_s} = N\int_0^\infty d\epsilon D(\epsilon)n_F(\epsilon)$$

Limit $T \rightarrow 0$. $\mu(T=0)$ is called Fermi energy:

$$\epsilon_F = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2\rho^{\frac{2}{3}}}{2m}$$

specific heat

$$\mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] \text{ for } T \ll \frac{\epsilon_F}{k_B}$$

$$c_V = \left.\frac{\partial E}{\partial T}\right|_V = N\frac{\pi^2}{3}k_B^2D(\epsilon_F)T$$

$$c_V = N\frac{\pi^2}{2}\frac{k_BT}{\epsilon_F}k_B$$

Fermi pressure

$$p \overset{T\rightarrow 0}{\rightarrow} \frac{2}{5}\frac{N}{V}\epsilon_F = \frac{(2\pi^2)^{\frac{2}{3}}}{5}\frac{\hbar^2}{mv^{\frac{5}{3}}}$$

The ideal Bose fluid

$\epsilon = \frac{\hbar^2k^2}{2m}$ and conserved particle number N.

$$N = \frac{N}{\lambda^3}g_{\frac{3}{2}}(z)$$

$$z = e^{\beta\mu}, \quad \lambda = \frac{h}{(2\pi mk_BT)^{\frac{1}{2}}}$$

$$T_c = \frac{2\pi}{\left(\zeta\left(\frac{3}{2}\right)\right)^{\frac{3}{2}}}\frac{\hbar^2\rho^{\frac{2}{3}}}{k_Bm}$$

$$E = \frac{3}{2}k_BT\frac{V}{\lambda^3}g_{\frac{5}{2}}(z) = \frac{3}{2}k_BTN_e\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}$$

$$c_V = \frac{15}{4}k_BN\left(\frac{T}{T_c}\right)^{\frac{3}{2}}\frac{\zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} \text{ (for } T \le T_c)$$

$$c_V = \frac{15}{4}k_BN\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)} - \frac{9}{4}k_BN\frac{g_{\frac{3}{2}}(z)}{g_{\frac{1}{2}}(z)} \quad (T > T_c)$$

Classical limit

$\mu \rightarrow -\infty$ the two grandcanonical distr. become the Maxwell-Boltzmann distr.

$$n_{F/B} = \frac{1}{e^{\beta(\epsilon-\mu)}\pm 1} \rightarrow e^{\beta\mu}e^{-\beta\epsilon}$$

$$N = g\frac{V}{\lambda^3}e^{\beta\mu}$$

$$E = \frac{3}{2}k_BTN$$

6 Phase transitions

Ising model

Hamiltonian

$$\mathcal{H} = -\sum_{i,j}J_{ij}S_iS_j - \mu B_0\sum_i S_i$$

special cases:

Ferromagnetic systems:

$\mathcal{H} = -J\sum_{\langle i,j\rangle}\vec{J}_i\vec{J}_j - \mu\vec{B}\sum_i\vec{J}_i$

lattice gases:

$\mathcal{H} = -\sum_{\langle i,j\rangle}J_{ij}S_iS_j$

1. Dimensional

Only Next Neighbor and $B_0 = 0$

$J_{i,i+1} \rightarrow J_i, \quad \mathcal{H} = -\sum_{i=1}^{N-1}J_iS_iS_{i+1}, \quad j_i = \beta J_i$

$$Z_N = \sum_{S_1}\dots\sum_{S_N}\exp\left(\sum_{i=1}^{N-1}j_iS_iS_{i+1}\right)$$

$$= 2^N\prod_{i=1}^{N-1}\cosh(\beta J_i)$$

Spin correlation function:

$$\langle S_iS_{i+1}\rangle = \tanh(\beta J)$$

spontanious magnetisation:

$$M_S(T) = \mu\langle S\rangle$$

$$M_S^2(T) = \mu^2\lim_{j\rightarrow\infty}\langle S_iS_{i+1}\rangle$$

No phase transition for $T > 0$. But for $T = 0$ $M_S(T=0) = \mu$

Transfer matrix

$j = \beta J, \quad b = \beta\mu B_0, \quad S_i = \pm 1$

$$T_{i,i+1} = e^{jS_iS_{i+1}+\frac{1}{2}b(S_i+S_{i+1})}$$

$$\rightarrow e^{-\beta\mathcal{H}} = T_{1,2}\cdot T_{2,3}\dots T_{N,1}$$

$$T = \begin{pmatrix} T(+1,+1) & T(+1,-1) \\ T(-1,+1) & T(-1,-1)\end{pmatrix}$$

$$Z_N = \lambda_1^N + \lambda_2^N = E_+^N + E_-^N$$

for $N \gg 1 \rightarrow E_+ \gg E_-$

Renormalization of the Ising chain

$$K' = \frac{1}{2}\ln(\cosh(2K))$$

Renormalization of the 2d Ising model

$$\overline{K}' = K' + K_1 = \frac{3}{8}\ln(\cosh(4K))$$

The 2d Ising model

$$\beta\mathcal{H} = -K\sum_{r,c} S_{r,c}S_{r+1,c} - K\sum_{r,c} S_{r.c}S_{r,c+1}$$

$$1 = \sinh(2K_c)$$

$$K_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.4407$$

$$T_c = 2J/\ln\left(1+\sqrt{2}\right) \approx 2.269J/k_B$$

Perturbation theory

$F \leq F_u = F_0 + \langle\mathcal{H}_1\rangle_0$ Bogoliubov inequality

Mean field theory for the Ising model

$$\mathcal{H} = -J\sum_{\langle i,j\rangle} S_i S_j$$

$$\mathcal{H}_0 = -B\sum_i S_i$$

$$\begin{aligned}
F_0 &= -Nk_BT\ln\left(e^{\beta B} + e^{-\beta B}\right)\\
&= -Nk_BT\ln(2\cosh(\beta B))\\
F &\leq F_0 + \langle\mathcal{H} - \mathcal{H}_0\rangle_0\\
&= -Nk_BT\ln(2\cosh(\beta B)) - N\frac{z}{2}\langle S\rangle_0^2\\
&+ N\langle S\rangle_0 = F_u\\
&\rightarrow z = 2\cdot\text{dimension}\\
B &= Jz\langle S\rangle_0 = Jz\tanh(\beta B)
\end{aligned}$$

$$K_c = \frac{1}{z} \rightarrow T_c = \frac{zJ}{k_B}$$

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7 Classical fluids

Virial expansion

$$F = Nk_BT\left[\ln(\rho\lambda^3) - 1 + B_2\rho\right]$$

$$p = \rho k_BT\left[1 + B_2\rho\right]$$

Second virial coefficient

$$B_2(T) = -2\pi\int r^2 dr\left(e^{-\beta U(r)} - 1\right)$$

8 Others

Stirling's formula

$$\ln(n!) = n\ln(n) - n + \frac{1}{2}\ln(2\pi n)$$

de Broglie relation

$$\epsilon = \frac{p^2}{2m} = \frac{\hbar^2k^2}{2m}$$

Energies

$$E_{kin} = \frac{1}{2}M\overline{v^2}$$

$$E_{rot} = \frac{1}{2}I\overline{\omega^2}$$