0.1 多项选择

1. **与总角动量算符的平方 $\vec{J}^2$ 对易的算符在 $(J_x, J_y, J_z, J_+, J_-)$ 中有?**

   已知角动量的基本对易关系 $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$, 那么

$$
\begin{aligned}
[J^2, J_l] = [\sum_i^3 J_i^2, J_l] &= \sum_i^3 [J_i^2, J_l] = \sum_i^3 \left(J_i[J_i, J_l] + [J_i, J_l]J_i\right) \\
&= \sum_i^3 \left(J_i i\hbar\epsilon_{ilk}J_k + i\hbar\epsilon_{ilk}J_kJ_i\right) \\
&= i\hbar\sum_i^3 (\epsilon_{ilk}J_iJ_k - \epsilon_{kli}J_kJ_i) = 0.
\end{aligned}
$$

   其中利用了 ϵ_{ijk} 的反对称性质以及 $k \iff i$ 的地位等价. 而 $J_\pm = J_x \pm iJ_y$ 是 $\{J_l\}$ 的线性组合, 根据对易关系的线性性质可知 $[J^2, J_\pm] = 0$, 所以待选项均为正确答案.

2. **在原子单位制下 $\hbar = c = 1$, 和能量同单位的量在 (距离, 动量, 时间, 质量, 角动量) 中有?**

   能量单位为 J=kg·m^2/s^2, 距离单位为 m, 动量单位为 kg·m/s, 时间单位为 s, 质量单位为 kg, 角动量单位为 kg·m^2/s. 现在要求 kg·m^2/s=m/s=1, 即寻找如何通过除以 $\hbar(\mathrm{kg\cdot m^2/s}), c(\mathrm{m/s})$ 来进行量纲变换

   (a) 距离. $\dfrac{E}{\hbar c} = \dfrac{\mathrm{kg\cdot m^2/s^2}}{\mathrm{kg\cdot m^2/s\cdot m/s}} = \dfrac{1}{\mathrm{m}}$, 说明距离和能量在单位上互为倒数.

   (b) $\boxed{\text{动量}}$. $E = pc$

   (c) 时间. $E = \hbar\omega = \hbar\dfrac{1}{\tau}$, 所以时间和能量单位互为倒数.

   (d) $\boxed{\text{质量}}$. $E = mc^2$.

   (e) 角动量. 角动量的量纲正好是 kg·m^2/s, 即无量纲数, 而能量无法通过除以 $\hbar$ 或 c 来变成角动量的量纲, 所以角动量和能量不同单位.

3. **宇称算符 $\mathbb{P}$ 连续作用两次为恒等变换, 这说明宇称算符 $\mathbb{P}$ 的本征值在 $(0, 1, -1, i, -i)$ 中有?**

   不妨设 $\mathbb{P}\psi = \lambda\psi$, 那么 $\mathbb{P}^2\psi = \lambda^2\psi = \psi$, 所以 $\lambda^2 = 1$, 即 $\lambda = \pm 1$. 所以宇称算符的本征值为 $\boxed{1, -1}$.

4. **如果算符 A 满足 $A^2 = A$, 那么算符 A 的本征值有 $(0, 1, -1, i, -i)$ 中有?**

   不妨设 $A\psi = \lambda\psi$, 那么 $A^2\psi = A(\lambda\psi) = \lambda^2\psi$, $\lambda^2 = \lambda$, 即 $\lambda = 0, 1$. 所以算符 A 的本征值为 $\boxed{0, 1}$.

5. **玻色子产生和湮灭算符满足对易关系 $\left[b_\alpha^\dagger, b_\beta^\dagger\right] = [b_\alpha, b_\beta] = 0$, $\left[b_\alpha, b_\beta^\dagger\right] = \delta_{\alpha\beta}$, 那么和总粒子数算符 $N = \sum\limits_\alpha b_\alpha^\dagger b_\alpha$ 对易的算符在 $(b_\alpha, b_\alpha^\dagger b_\alpha, b_\alpha^\dagger b_\beta, b_\alpha^\dagger b_\beta b_\mu, b_\alpha^\dagger b_\beta b_\mu^\dagger b_\nu)$ 中有?**

   已知 $[N, A] = \sum\limits_i \left[b_i^\dagger b_i, A\right] = \sum\limits_i \left\{b_i^\dagger [b_i, A] + \left[b_i^\dagger, A\right] b_i\right\}$, 代入以上各算符 A 判断是否对易.

   (a) $[N, b_\alpha] = \sum\limits_i \left\{b_i^\dagger [b_i, b_\alpha] + \left[b_i^\dagger, b_\alpha\right] b_i\right\} = \sum\limits_i \{0 + (-\delta_{i\alpha})b_\alpha\} = -b_\alpha$

   (b)

$$
\begin{aligned}
\boxed{[N, b_\alpha^\dagger b_\alpha]} &= \sum_i \left[b_i^\dagger b_i, b_\alpha^\dagger b_\alpha\right] = \sum_i \left\{b_i^\dagger [b_i, b_\alpha^\dagger b_\alpha] + \left[b_i^\dagger, b_\alpha^\dagger b_\alpha\right] b_i\right\} \\
&= \sum_i \left\{b_i^\dagger \left(b_\alpha^\dagger [b_i, b_\alpha] + [b_i, b_\alpha^\dagger]\, b_\alpha\right) + \left(b_\alpha^\dagger [b_i^\dagger, b_\alpha] + [b_i^\dagger, b_\alpha^\dagger] b_\alpha\right) b_i\right\} \\
&= \sum_i \left\{b_i^\dagger (b_\alpha^\dagger \cdot 0 + \delta_{i\alpha} b_\alpha) + (b_\alpha^\dagger(-\delta_{i\alpha}) + 0 \cdot b_\alpha) b_i\right\} \\
&= \sum_i \delta_{i\alpha}(b_i^\dagger b_\alpha - b_\alpha^\dagger b_i) = 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
\boxed{[N, b_\alpha^\dagger b_\beta]} &= \sum_i \left[b_i^\dagger b_i, b_\alpha^\dagger b_\beta\right] = \sum_i \left\{b_i^\dagger [b_i, b_\alpha^\dagger b_\beta] + \left[b_i^\dagger, b_\alpha^\dagger b_\beta\right] b_i\right\} \\
&= \sum_i \left\{b_i^\dagger (b_\alpha^\dagger [b_i, b_\beta] + [b_i, b_\alpha^\dagger] b_\beta) + (b_\alpha^\dagger [b_i^\dagger, b_\beta] + [b_i^\dagger, b_\alpha^\dagger] b_\beta) b_i\right\} \\
&= \sum_i \left\{b_i^\dagger (b_\alpha^\dagger \cdot 0 + \delta_{i\alpha} b_\beta) + (b_\alpha^\dagger (-\delta_{i\beta}) + 0 \cdot b_\beta) b_i\right\} \\
&= \sum_i \left(b_i^\dagger b_\beta \delta_{i\alpha} - b_\alpha^\dagger b_i \delta_{i\beta}\right) = 0.
\end{aligned}
$$

(d)

$$
[N, b_\alpha^\dagger b_\beta b_\mu] = b_\alpha^\dagger b_\beta [N, b_\mu] + [N, b_\alpha^\dagger b_\beta] b_\mu = -b_\alpha^\dagger b_\beta b_\mu
$$

(e)

$$
\boxed{[N, b_\alpha^\dagger b_\beta b_\mu^\dagger b_\nu]} = b_\alpha^\dagger b_\beta [N, b_\mu^\dagger b_\nu] + [N, b_\alpha^\dagger b_\beta] b_\mu^\dagger b_\nu = 0 + 0 = 0
$$

可以不严谨地总结出一条规律: 粒子数算符 $\hat{N}$ 只会与另一个粒子数算符对易, 而与单独的产生湮灭算符均不对易.