Essence of Linear Algebra

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Disclaimer

All the credits for the content goes to the respective authors listed in the Appendix: Further Learning.

The Storyline

The Geometry of Linear Equations
- Vectors and Basis vectors
- Linear combinations and Span

The box game: Matrices
- Elimination and Multiplication, A=LU

Transforming your LIFE Leenearly
- Cool Video, The Determinant

Space Tour
- Column space, Null space, Inverses

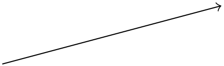
Celebrity: The Rank
- Solution concept

Some things of Eigen
- Eigen values, Eigen vectors, Change of Basis

The Storyline

Vectors

What even are they?

- A line with a arrowhead? 
  OR

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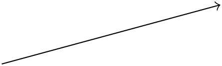
- A line with a arrowhead? 
  OR
- A set of numbers arranged vertically?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

OR

Vectors

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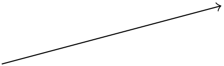
- A line with a arrowhead? 
  OR
- A set of numbers arranged vertically?

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

  OR
- An abstract $\vec{v}$

Vectors

Abstract view[1]

Associative property of vector addition

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

Commutative Property of vector addition

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

There is a zero vector $\vec{0}$ such that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v}$. For every vector $\vec{v}$ there is a vector $-\vec{v}$ so that $\vec{v} + (-\vec{v}) = \vec{0}$.

Associative of scalar multiplication

$$a(\overrightarrow{bv}) = (ab)\vec{v} \qquad 1\vec{v} = \vec{v}$$

Distributive property of scalar multiplication

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w}$$

$$(a + b)\vec{v} = a\vec{v} + b\vec{v}$$

[1] Abstract vector spaces — Essence of Linear Algebra, Chapter 11

Basis Vectors

These are the vectors which can define the entire coordinate space.

- Do you recognize this special vector?

  v = x i + y j + z k

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

Basis Vectors

These are the vectors which can define the entire coordinate space.

- Do you recognize this special vector?

$$\begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}$$

- Also, have you seen this?

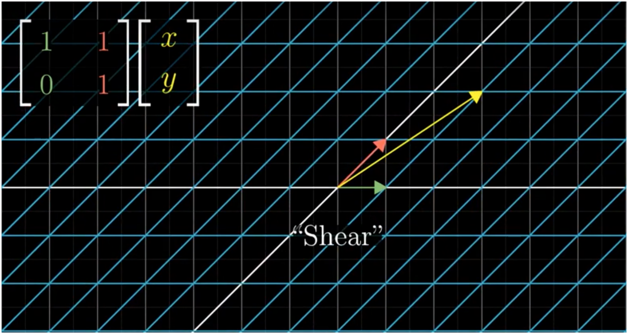
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is a special matrix called Shear matrix.

Basis vectors

Example

The image[2] below shows the basis vectors of a Shear matrix.



[2] Essence of Linear Algebra, Chapter 3

The Storyline

Linear combinations

Additivity

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad \vec{u} + \vec{v} = \vec{w}$$

Scaling

$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad 2\vec{v} = \overrightarrow{(2v)}$$

Hybrid

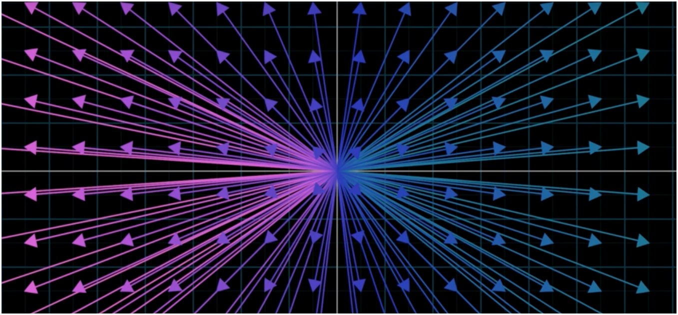
$$2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 20 \\ 13 \end{bmatrix} \qquad a\vec{u} + b\vec{v} = \vec{w}$$

Span

The **span** of a set of vectors is the collection of all possible linear combinations of those vectors.

Example

The image[3] below shows how two vectors can span the 2D space.



[3] Essence of Linear Algebra, Chapter 2

The Storyline

Elimination

Gauss-Jordan Elimination

A method of solving a linear system of equations. This is done by transforming the system's augmented matrix into reduced row-echelon form (rref) by means of row operations.
Types of row Operations:
Type 1: Swap the positions of two rows.
Type 2: Multiply a row by a nonzero scalar.
Type 3: Add to one row a scalar multiple of another.

Example

$$\text{RREF} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Elimination using Multiplication

Example

$$x + 2y + z = 2$$

$$3x + 8y + z = 12$$

$$4y + z = 2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_{32}E_{31}E_{21}A = U$$

$A = LU$

Example

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$EA = U$$

$$A = (E_{32}E_{31}E_{21})^{-1}U$$

$$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

$$A = LU$$

$$So, A = E^{-1}U. Therefore, L = E^{-1}$$

The Storyline

Cool Video

Definition Cool

The phrase "cool" is very relaxed, never goes out of style, and people will never laugh at you for using it.

Definition Video

Make a video recording of (something broadcast on television). YouTube nowadays.

Theorem

Cool + Video = JUST START PLAYING THE VIDEO!

Example

Here You Go: HUGO

Cool Video

0000 0018 6674 7970 6d70 3432 0000 0000 6973 6f6d 6d70
3432 0003 34bf 6d6f 6f76 0000 006c 6d76 6864 0000 0000
d42c 59a6 d42c 59a6 0000 0258 0006 06bf 0001 0000 0100
0000 0000 0000 0000 0000 0001 0000 0000 0000 0000 0000
0000 0000 0001 0000 0000 0000 0000 0000 0000 0000 4000
0000 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000
0000 0000 0000 0003 0000 0015 696f 6473 0000 0000 1007
004f ffff 2915 ff00 0151 fd74 7261 6b00 0000 5c74 6b68
6400 0000 0f00 0000 00d4 2c59 ae00 0000 0100 0000 0000
0606 6600 0000 0000 0000 0000 0000 0000 0000 0000 0100
0000 0000 0000 0000 0000 0000 0000 0100 0000 0000 0000
0000 0000 0000 0040 0000 0002 8000 0001 6800 0000 0000
2465 6474 7300 0000 1c65 6c73 7400 0000 0000 0000 0100
0606 6600 0000 0000 0100 0000 0151 756d 6469 6100 0000
206d 6468 6400 0000 0000 0000 00d4 2c59 aa00 0075 3001
2d40 0355 c400 0000 0000 2d68 646c 7200 0000 0000 0000
0076 6964 6500 0000 0000 0000 0000 0000 0056 6964 656f
4861 6e64 6c65 7200 0001 5120 6d69 6e66 0000 0014 ......

The Determinant

Block Title

The determinant of a matrix A is denoted det(A) or det A. It can be viewed as the scaling factor of the transformation described by the matrix.[4]

The formula:

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \text{ad - bc}$$

[4] Source: Wikipedia

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- The determinant can tell us whether or not a given transformation associated with that matrix squishes everything into a smaller dimension.

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The formula:

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

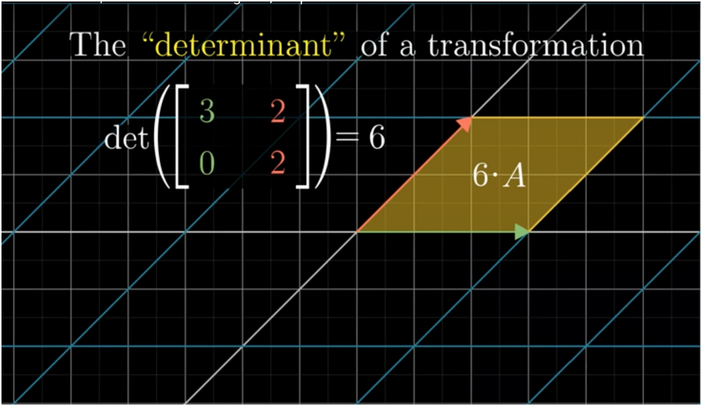
- The determinant can tell us whether or not a given transformation associated with that matrix squishes everything into a smaller dimension.
- Also, if the value of determinant is negative then the transformation is equivalent to inverting the orientation of space.

[4] Source: Wikipedia

The Determinant

Example

The image[5] below shows the significance of calculating the Determinant of a matrix.

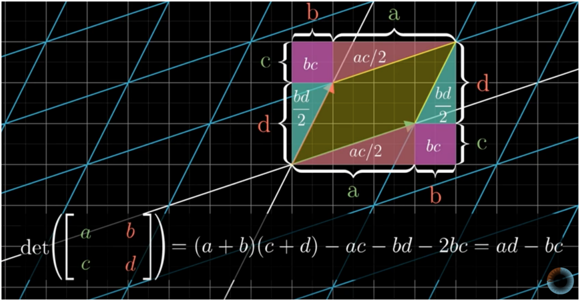


[5] The Determinant — Essence of Linear Algebra, Chapter 5

The Determinant

Example

The image[6] below shows how to calculate the Determinant of a 2D matrix.



[6] The Determinant — Essence of Linear Algebra, Chapter 5

The Storyline

Column space and Null space

Column space: C(A)

The column space of a matrix A is the vector space generated by all the linear combinations of the column vectors.

Null space: N(A)

The set of all vectors $\vec{v}$ such that

$$A\vec{v} = \vec{0}$$

Example

$$\text{Ax} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow N(A) = c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Inverse of a matrix A

Inverse of $A = A^{-1}$

In terms of transformation, the matrix that undoes all the transformations made by matrix A is called as inverse of matrix A (A^{-1}).

$$A^{-1}A = AA^{-1} = I$$

$$Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$$

Example

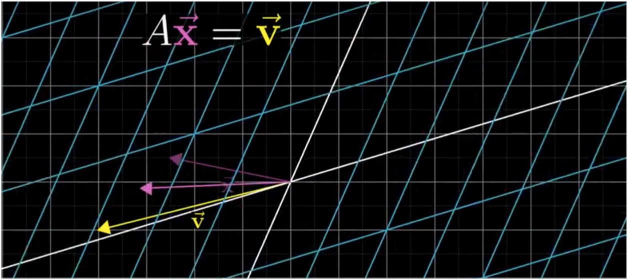
$$\mathrm{A} = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & 1 \\ -3 & 4 & 2 \end{bmatrix} \ \mathrm{b} = \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix} \ \mathrm{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathrm{x} = \mathrm{A}^{-1}b \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix} \begin{bmatrix} 21 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Inverse of A

Example

The image[7] below shows how two vectors are linked to each other via Inverse transformation.



[7] Essence of Linear Algebra, Chapter 6

The Storyline

Find Mr. Rank

What is the rank of the following matrix?

Be Quick![8]

$$\begin{bmatrix} 12 & 15 & 14 & 19 & 13 & 21 & 05 & 07 & 41 & 51 \\ 22 & 26 & 26 & 32 & 27 & 36 & 21 & 24 & 59 & 70 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 24 & 30 & 28 & 38 & 26 & 42 & 10 & 14 & 82 & 102 \\ 30 & 33 & 36 & 39 & 42 & 45 & 48 & 51 & 54 & 57 \\ 15 & 16.5 & 18 & 19.5 & 21 & 22.5 & 24 & 25.5 & 27 & 28.5 \end{bmatrix}$$

[8] You'll be given choc!

The Rank

Solution concept provided by The Rank

The Rank tells you everything about
the number of solutions to a given system of linear equations.[9]

Matrix A with dimensions m x n, rank r and rref(A) = R			
$r = m = n$	$r = n < m$	$r = m < n$	$r < m, r < n$
$\text{R} = \text{I}$	$\text{R} = \begin{bmatrix} I \\ 0 \end{bmatrix}$	$\text{R} = \begin{bmatrix} I & F \end{bmatrix}$	$\text{R} = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
1	0 or 1	1 or ∞	0 or ∞

[9] Gilbert Strang, Linear Algebra lecture 8

The Storyline

Some things of Eigen

Eigen values and vectors

An Eigen vector of a linear transformation is a non-zero vector whose direction does not change when that linear transformation is applied to it. More formally, if T is a linear transformation from a vector space **V** and $\vec{v}$ is a vector in **V** that is not the zero vector, then $\vec{v}$ is an eigenvector of T if $T(\vec{v})$ is a scalar multiple of $\vec{v}$. This condition can be written as the equation

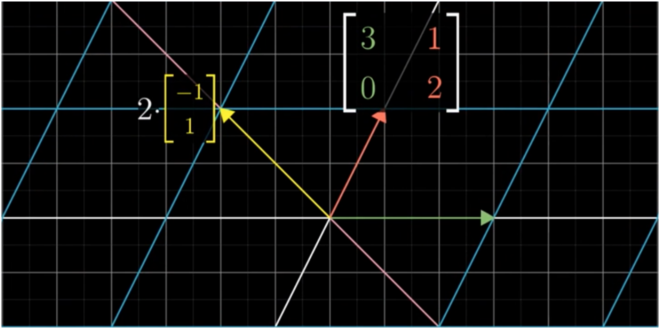
$$T(\vec{v}) = \lambda \vec{v}$$

where λ is a scalar known as the eigenvalue, characteristic value or root associated with the eigenvector $\vec{v}$.

Eigen values and vectors

Example

The image[10] below shows how the eigenvector does not change it's direction after applying a linear transformation.

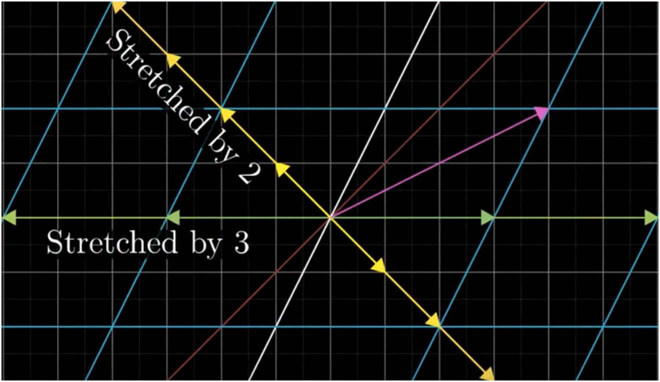


[10] Eigenvectors and eigenvalues, Essence of linear algebra, chapter 10

Eigen values and vectors

Example

The image[11] below shows how the eigenvector gets scaled after applying a linear transformation.



[11] Eigenvectors and eigenvalues, Essence of linear algebra, chapter 10

Change of Basis

New coordinates to Old coordinates

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-1) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (2) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Old coordinates to New coordinates

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}$$

Change of Basis

Transformation

Let $\vec{v}$ be a vector in the New coordinates i.e. change of Basis vectors, A be the matrix representing the transformation: Change of Basis, and M be the transformation in Old coordinate system and T be the final transformation in the New coordinate system.

$$A^{-1}MA\vec{v} = T\vec{v}$$

Example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix} \vec{v}$$

Summary

How do you possibly hope to *summarize the whole talk?*

Like the OLD MAN said: TOGETHER!

For Further Learning I

- Gilbert Strang
  https://tinyurl.com/gt7dy36
  MIT OCW
- Grant Sanderson
  https://goo.gl/R1kBdb
  Essence of linear algebra, YouTube channel
  3Blue1Brown
- Think Different
  You can always find more to learn... If you want to ;-)