

INSA Lyon ACM-ICPC Notebook 2018 (Python)

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1 Utilities

1.1 Bisect

```
from bisect import bisect_left, bisect_right, insort

data = [10, 20, 30, 40] # Must be sorted
insort(data, 25) # Insert 25 into the correct position
print(data) # Output: [10, 20, 25, 30, 40]

# Find the position to insert 35 while maintaining order
pos = bisect_left(data, 35)
print(pos) # Output: 4 (index where 35 should be inserted)

# Find the range of elements between 20 and 35
start = bisect_left(data, 20)
```

```
end = bisect_right(data, 35)
print(data[start:end]) # Output: [20, 25, 30]
```

1.2 Default Dictionary

```
from collections import defaultdict

# Using defaultdict with list
d = defaultdict(list)
d['a'].append(1)
d['b'].append(2)
print(d) # Output: defaultdict(<class 'list'>, {'a': [1], 'b': [2]})

# Using defaultdict with int
d_int = defaultdict(int)
d_int['a'] += 1
d_int['b'] += 2
print(d_int) # Output: defaultdict(<class 'int'>, {'a': 1, 'b': 2})
```

1.3 Array Input

```
arr = list(map(int, input().split()))
```

1.4 Heap Queue

```
import heapq

# Create an empty priority queue (heap)
pq = []

# Insert elements into the priority queue
heapq.heappush(pq, 15)
heapq.heappush(pq, 31)
heapq.heappush(pq, 3)
heapq.heappush(pq, 482)

# Pop the smallest element
print(heapq.heappop(pq)) # Output: 3

# The next smallest element
print(heapq.heappop(pq)) # Output: 15
```

1.5 Binary Tree

```
class TreeNode:
    def __init__(self, key, value):
        self.key = key
        self.value = value
        self.left = None
        self.right = None
        self.height = 1 # Height of this node

class BTreeMap:
    def __init__(self):
        self.root = None

    def _height(self, node):
        return node.height if node else 0

    def _balance_factor(self, node):
        return self._height(node.left) - self._height(node.right) if node else 0

    def _rotate_left(self, z):
        y = z.right
        T2 = y.left
        y.left = z
        z.right = T2
        z.height = 1 + max(self._height(z.left), self._height(z.right))
        y.height = 1 + max(self._height(y.left), self._height(y.right))
        return y

    def _rotate_right(self, z):
        y = z.left
```

```

T3 = y.right
y.right = z
z.left = T3
z.height = 1 + max(self._height(z.left), self._height(z.right))
y.height = 1 + max(self._height(y.left), self._height(y.right))
return y

def _balance(self, node):
    balance = self._balance_factor(node)
    if balance > 1: # Left heavy
        if self._balance_factor(node.left) < 0:
            node.left = self._rotate_left(node.left)
            return self._rotate_right(node)
        if balance < -1: # Right heavy
            if self._balance_factor(node.right) > 0:
                node.right = self._rotate_right(node.right)
                return self._rotate_left(node)
            return node
    def insert(self, key, value):
        def _insert(node, key, value):
            if node is None:
                return TreeNode(key, value)
            if key < node.key:
                node.left = _insert(node.left, key, value)
            elif key > node.key:
                node.right = _insert(node.right, key, value)
            else:
                node.value = value # Update existing key
                node.height = 1 + max(self._height(node.left), self._height(node.right))
                return self._balance(node) # return node if not balancing
            self.root = _insert(self.root, key, value)

        def get(self, key):
            def _get(node, key):
                if node is None:
                    return None
                if key < node.key:
                    return _get(node.left, key)
                elif key > node.key:
                    return _get(node.right, key)
                else:
                    return node.value
            return _get(self.root, key)

        def remove(self, key):
            def _remove(node, key):
                if node is None:
                    return None
                if key < node.key:
                    node.left = _remove(node.left, key)
                elif key > node.key:
                    node.right = _remove(node.right, key)
                else:
                    # Node with one or no children
                    if node.left is None:
                        return node.right
                    if node.right is None:
                        return node.left
                    # Node with two children: find inorder successor
                    succ = node.right
                    while succ.left:
                        succ = succ.left
                    node.key, node.value = succ.key, succ.value
                    node.right = _remove(node.right, succ.key)
                    node.height = 1 + max(self._height(node.left), self._height(node.right))
                    return self._balance(node) # return node if not balancing
                self.root = _remove(self.root, key)

        def range(self, lower, upper):
            def _range(node, lower, upper):
                if node is None:
                    return
                if lower is None or node.key >= lower:
                    yield from _range(node.left, lower, upper)
                if (lower is None or node.key >= lower) and (upper is None or node.key < upper):
                    yield (node.key, node.value)
                if upper is None or node.key < upper:
                    yield from _range(node.right, lower, upper)
            yield from _range(self.root, lower, upper)

```

1.6 Binary Tree Set

```

self.right = None
self.height = 1 # Height of this node

class BTreeSet:
    def __init__(self):
        self.root = None

    def _height(self, node):
        return node.height if node else 0

    def _balance_factor(self, node):
        return self._height(node.left) - self._height(node.right) if node else 0

    def _rotate_left(self, z):
        y = z.right
        T2 = y.left
        y.left = z
        z.right = T2
        z.height = 1 + max(self._height(z.left), self._height(z.right))
        y.height = 1 + max(self._height(y.left), self._height(y.right))
        return y

    def _rotate_right(self, z):
        y = z.left
        T3 = y.right
        y.right = z
        z.left = T3
        z.height = 1 + max(self._height(z.left), self._height(z.right))
        y.height = 1 + max(self._height(y.left), self._height(y.right))
        return y

    def _balance(self, node):
        balance = self._balance_factor(node)
        if balance > 1: # Left heavy
            if self._balance_factor(node.left) < 0:
                node.left = self._rotate_left(node.left)
                return self._rotate_right(node)
            if balance < -1: # Right heavy
                if self._balance_factor(node.right) > 0:
                    node.right = self._rotate_right(node.right)
                    return self._rotate_left(node)
                return node
        def insert(self, key):
            def _insert(node, key):
                if node is None:
                    return TreeNode(key)
                if key < node.key:
                    node.left = _insert(node.left, key)
                elif key > node.key:
                    node.right = _insert(node.right, key)
                return self._balance(node) # return node if not balancing
            self.root = _insert(self.root, key)

        def remove(self, key):
            def _remove(node, key):
                if node is None:
                    return None
                if key < node.key:
                    node.left = _remove(node.left, key)
                elif key > node.key:
                    node.right = _remove(node.right, key)
                else:
                    # Node with one or no children
                    if node.left is None:
                        return node.right
                    if node.right is None:
                        return node.left
                    # Node with two children: find inorder successor
                    succ = node.right
                    while succ.left:
                        succ = succ.left
                    node.key = succ.key
                    node.right = _remove(node.right, succ.key)
                    return self._balance(node) # return node if not balancing
                self.root = _remove(self.root, key)

        def contains(self, key):
            def _contains(node, key):
                if node is None:
                    return False
                if key < node.key:
                    return _contains(node.left, key)
                elif key > node.key:
                    return _contains(node.right, key)
                else:
                    return True
            return _contains(self.root, key)

        def range(self, lower, upper):
            def _range(node, lower, upper):
                if node is None:

```

```

class TreeNode:
    def __init__(self, key):
        self.key = key
        self.left = None

```

```

        return
    if lower is None or node.key >= lower:
        yield from _range(node.left, lower, upper)
    if (lower is None or node.key >= lower) and (upper is None or node.key < upper):
        yield node.key
    if upper is None or node.key < upper:
        yield from _range(node.right, lower, upper)
yield from _range(self.root, lower, upper)

```

1.7 Trie

```

class TrieNode:
    def __init__(self):
        self.children = {}
        self.is_end_of_word = False

class Trie:
    def __init__(self):
        self.root = TrieNode()

    def insert(self, word):
        node = self.root
        for char in word:
            if char not in node.children:
                node.children[char] = TrieNode()
            node = node.children[char]
        node.is_end_of_word = True

    def search(self, word):
        node = self.root
        for char in word:
            if char not in node.children:
                return False
            node = node.children[char]
        return node.is_end_of_word

    def starts_with(self, prefix):
        node = self.root
        for char in prefix:
            if char not in node.children:
                return False
            node = node.children[char]
        return True

    def _find_words_with_prefix(self, node, prefix, result):
        if node.is_end_of_word:
            result.append(prefix)
        for char, child in node.children.items():
            self._find_words_with_prefix(child, prefix + char, result)

    def list_words_with_prefix(self, prefix):
        node = self.root
        result = []
        for char in prefix:
            if char not in node.children:
                return result # No words start with this prefix
            node = node.children[char]
        self._find_words_with_prefix(node, prefix, result)
        return result

    def remove(self, word):
    def _remove(node, word, index):
        if index == len(word):
            if not node.is_end_of_word:
                return False # Word not found
            node.is_end_of_word = False # Unmark the end of word
            return len(node.children) == 0 # If no children, remove node
        char = word[index]
        if char not in node.children:
            return False # Word not found
        child_node = node.children[char]
        should_delete_child = _remove(child_node, word, index + 1)
        if should_delete_child:
            del node.children[char] # Delete the child node if no other word depends on it
            return len(node.children) == 0 and not node.is_end_of_word
        return False
    _remove(self.root, word, 0)

# Example usage
trie = Trie()
trie.insert("apple")
trie.insert("app")
trie.insert("bat")
trie.insert("banana")
trie.insert("batman")
print(trie.search("apple")) # Output: True
print(trie.search("app")) # Output: True

```

```

print(trie.search("batman")) # Output: True

trie.remove("app")
print(trie.search("app")) # Output: False
print(trie.search("apple")) # Output: True

print(trie.list_words_with_prefix("app")) # Output: ['apple']
print(trie.list_words_with_prefix("ba")) # Output: ['bat', 'banana', 'batman']

```

2 Dynamic Programming

2.1 Max Sum Subarray (Kadane's Algorithm)

```

def maxSubArraySum(a, size):
    max_so_far = 0
    max_ending_here = 0
    for i in range(0, size):
        max_ending_here = max_ending_here + a[i]
        if max_ending_here < 0:
            max_ending_here = 0
        elif (max_so_far < max_ending_here):
            max_so_far = max_ending_here
    return max_so_far

```

2.2 Longest Common Subsequence

```

def lcs(X , Y):
    # find the length of the strings
    m = len(X)
    n = len(Y)

    # declaring the array for storing the dp values
    L = [[None]*(n+1) for i in xrange(m+1)]

    """Following steps build L[m+1][n+1] in bottom up fashion
    Note: L[i][j] contains length of LCS of X[0..i-1]
    and Y[0..j-1]"""
    for i in range(m+1):
        for j in range(n+1):
            if i == 0 or j == 0 :
                L[i][j] = 0
            elif X[i-1] == Y[j-1]:
                L[i][j] = L[i-1][j-1]+1
            else:
                L[i][j] = max(L[i-1][j] , L[i][j-1])

    # L[m][n] contains the length of LCS of X[0..n-1] & Y[0..m-1]
    return L[m][n]

```

2.3 Levenshtein Distance

```

def levenshtein(s1, s2):
    if len(s1) < len(s2):
        return levenshtein(s2, s1)

    # len(s1) >= len(s2)
    if len(s2) == 0:
        return len(s1)

    previous_row = range(len(s2) + 1)
    for i, c1 in enumerate(s1):
        current_row = [i + 1]
        for j, c2 in enumerate(s2):
            insertions = previous_row[j + 1] + 1 # j+1 instead of j since previous_row and current_row
            # are one character longer
            deletions = current_row[j] + 1 # than s2
            substitutions = previous_row[j] + (c1 != c2)
            current_row.append(min(insertions, deletions, substitutions))
        previous_row = current_row
    return previous_row[-1]

```

2.4 Longest Increasing Subsequence

```
def lis(arr):
    n = len(arr)

    # Declare the list (array) for LIS and initialize LIS
    # values for all indexes
    lis = [1]*n

    # Compute optimized LIS values in bottom up manner
    for i in range(1, n):
        for j in range(0, i):
            if arr[i] > arr[j] and lis[i] < lis[j] + 1 :
                lis[i] = lis[j]+1

    # Initialize maximum to 0 to get the maximum of all
    # LIS
    maximum = 0

    # Pick maximum of all LIS values
    for i in range(n):
        maximum = max(maximum, lis[i])
    return maximum
```

3 Geometry

3.1 Convex Hull

```
def convex_hull(points):
    """Computes the convex hull of a set of 2D points.

    Input: an iterable sequence of (x, y) pairs representing the points.
    Output: a list of vertices of the convex hull in counter-clockwise order,
            starting from the vertex with the lexicographically smallest coordinates.
    Implements Andrew's monotone chain algorithm. O(n log n) complexity.
    """

    # Sort the points lexicographically (tuples are compared lexicographically).
    # Remove duplicates to detect the case we have just one unique point.
    points = sorted(set(points))

    # Boring case: no points or a single point, possibly repeated multiple times.
    if len(points) <= 1:
        return points

    # 2D cross product of OA and OB vectors, i.e. z-component of their 3D cross product.
    # Returns a positive value, if OAB makes a counter-clockwise turn,
    # negative for clockwise turn, and zero if the points are collinear.
    def cross(o, a, b):
        return (a[0] - o[0]) * (b[1] - o[1]) - (a[1] - o[1]) * (b[0] - o[0])

    # Build lower hull
    lower = []
    for p in points:
        while len(lower) >= 2 and cross(lower[-2], lower[-1], p) <= 0:
            lower.pop()
        lower.append(p)

    # Build upper hull
    upper = []
    for p in reversed(points):
        while len(upper) >= 2 and cross(upper[-2], upper[-1], p) <= 0:
            upper.pop()
        upper.append(p)

    # Concatenation of the lower and upper hulls gives the convex hull.
    # Last point of each list is omitted because it is repeated at the beginning of the other list.
    return lower[:-1] + upper[:-1]

# Example: convex hull of a 10-by-10 grid.
assert convex_hull([(i//10, i%10) for i in range(100)]) == [(0, 0), (9, 0), (9, 9), (0, 9)]
```

3.2 Misc Geometry Functions (C++)

```
// C++ routines for computational geometry.

#include <iostream>
#include <vector>
#include <cmath>
#include <cassert>

using namespace std;
```

```
double INF = 1e100;
double EPS = 1e-12;

struct PT {
    double x, y;
    PT() {}
    PT(double x, double y) : x(x), y(y) {}
    PT(const PT &p) : x(p.x), y(p.y) {}
    PT operator + (const PT &p) const { return PT(x+p.x, y+p.y); }
    PT operator - (const PT &p) const { return PT(x-p.x, y-p.y); }
    PT operator * (double c) const { return PT(x*c, y*c); }
    PT operator / (double c) const { return PT(x/c, y/c); }
};

double dot(PT p, PT q) { return p.x*q.x+p.y*q.y; }
double dist2(PT p, PT q) { return dot(p-q, p-q); }
double cross(PT p, PT q) { return p.x*q.y-p.y*q.x; }
ostream &operator<<(ostream &os, const PT &p) {
    os << "(" << p.x << ", " << p.y << ")";
}

// rotate a point CCW or CW around the origin
PT RotateCCW90(PT p) { return PT(-p.y, p.x); }
PT RotateCW90(PT p) { return PT(p.y, -p.x); }
PT RotateCCW(PT p, double t) {
    return PT(p.x*cos(t)-p.y*sin(t), p.x*sin(t)+p.y*cos(t));
}

// project point c onto line through a and b
// assuming a != b
PT ProjectPointLine(PT a, PT b, PT c) {
    return a + (b-a)*dot(c-a, b-a)/dot(b-a, b-a);
}

// project point c onto line segment through a and b
PT ProjectPointSegment(PT a, PT b, PT c) {
    double r = dot(b-a, b-a);
    if (fabs(r) < EPS) return a;
    r = dot(c-a, b-a)/r;
    if (r < 0) return a;
    if (r > 1) return b;
    return a + (b-a)*r;
}

// compute distance from c to segment between a and b
double DistancePointSegment(PT a, PT b, PT c) {
    return sqrt(dist2(c, ProjectPointSegment(a, b, c)));
}

// compute distance between point (x,y,z) and plane ax+by+cz=d
double DistancePointPlane(double x, double y, double z,
                           double a, double b, double c, double d)
{
    return fabs(a*x+b*y+c*z-d)/sqrt(a*a+b*b+c*c);
}

// determine if lines from a to b and c to d are parallel or collinear
bool LinesParallel(PT a, PT b, PT c, PT d) {
    return fabs(cross(b-a, c-d)) < EPS;
}

bool LinesCollinear(PT a, PT b, PT c, PT d) {
    return LinesParallel(a, b, c, d)
        && fabs(cross(a-b, a-c)) < EPS
        && fabs(cross(c-d, c-a)) < EPS;
}

// determine if line segment from a to b intersects with
// line segment from c to d
bool SegmentsIntersect(PT a, PT b, PT c, PT d) {
    if (LinesCollinear(a, b, c, d)) {
        if (dist2(a, c) < EPS || dist2(a, d) < EPS ||
            dist2(b, c) < EPS || dist2(b, d) < EPS) return true;
        if (dot(c-a, c-b) > 0 && dot(d-a, d-b) > 0 && dot(c-b, d-b) > 0)
            return false;
        return true;
    }
    if (cross(d-a, b-a) * cross(c-a, b-a) > 0) return false;
    if (cross(a-c, d-c) * cross(b-c, d-c) > 0) return false;
    return true;
}

// compute intersection of line passing through a and b
// with line passing through c and d, assuming that unique
// intersection exists; for segment intersection, check if
// segments intersect first
PT ComputeLineIntersection(PT a, PT b, PT c, PT d) {
    b=b-a; d=d-c; c=c-a;
    assert(dot(b, b) > EPS && dot(d, d) > EPS);
    return a + b*cross(c, d)/cross(b, d);
}
```

```

// compute center of circle given three points
PT ComputeCircleCenter(PT a, PT b, PT c) {
    b=(a+b)/2;
    c=(a+c)/2;
    return ComputeLineIntersection(b, b+RotateCW90(a-b), c, c+RotateCW90(a-c));
}

// determine if point is in a possibly non-convex polygon (by William
// Randolph Franklin); returns 1 for strictly interior points, 0 for
// strictly exterior points, and 0 or 1 for the remaining points.
// Note that it is possible to convert this into an *exact* test using
// integer arithmetic by taking care of the division appropriately
// (making sure to deal with signs properly) and then by writing exact
// tests for checking point on polygon boundary
bool PointInPolygon(const vector<PT> &p, PT q) {
    bool c = 0;
    for (int i = 0; i < p.size(); i++){
        int j = (i+1)%p.size();
        if ((p[i].y <= q.y && q.y < p[j].y ||
            p[j].y <= q.y && q.y < p[i].y) &&
            q.x < p[i].x + (p[j].x - p[i].x) * (q.y - p[i].y) / (p[j].y - p[i].y))
            c = !c;
    }
    return c;
}

// determine if point is on the boundary of a polygon
bool PointOnPolygon(const vector<PT> &p, PT q) {
    for (int i = 0; i < p.size(); i++)
        if (dist2(ProjectPointSegment(p[i], p[(i+1)%p.size()], q), q) < EPS)
            return true;
    return false;
}

// compute intersection of line through points a and b with
// circle centered at c with radius r > 0
vector<PT> CircleLineIntersection(PT a, PT b, PT c, double r) {
    vector<PT> ret;
    b = b-a;
    a = a-c;
    double A = dot(b, b);
    double B = dot(a, b);
    double C = dot(a, a) - r*r;
    double D = B*B - A*C;
    if (D < -EPS) return ret;
    ret.push_back(c+a+b*(-B+sqrt(D+EPS))/A);
    if (D > EPS)
        ret.push_back(c+a+b*(-B-sqrt(D))/A);
    return ret;
}

// compute intersection of circle centered at a with radius r
// with circle centered at b with radius R
vector<PT> CircleCircleIntersection(PT a, PT b, double r, double R) {
    vector<PT> ret;
    double d = sqrt(dist2(a, b));
    if (d > r+R || d+min(r, R) < max(r, R)) return ret;
    double x = (d+d-R*R+r*r)/(2*d);
    double y = sqrt(r*r-x*x);
    PT v = (b-a)/d;
    ret.push_back(a+v*x + RotateCCW90(v)*y);
    if (y > 0)
        ret.push_back(a+v*x - RotateCCW90(v)*y);
    return ret;
}

// This code computes the area or centroid of a (possibly nonconvex)
// polygon, assuming that the coordinates are listed in a clockwise or
// counterclockwise fashion. Note that the centroid is often known as
// the "center of gravity" or "center of mass".
double ComputeSignedArea(const vector<PT> &p) {
    double area = 0;
    for(int i = 0; i < p.size(); i++) {
        int j = (i+1) % p.size();
        area += p[i].x*p[j].y - p[j].x*p[i].y;
    }
    return area / 2.0;
}

double ComputeArea(const vector<PT> &p) {
    return fabs(ComputeSignedArea(p));
}

PT ComputeCentroid(const vector<PT> &p) {
    PT c(0,0);
    double scale = 6.0 * ComputeSignedArea(p);
    for (int i = 0; i < p.size(); i++){
        int j = (i+1) % p.size();
        c = c + (p[i]+p[j])*(p[i].x*p[j].y - p[j].x*p[i].y);
    }
    return c / scale;
}

```

```

}

// tests whether or not a given polygon (in CW or CCW order) is simple
bool IsSimple(const vector<PT> &p) {
    for (int i = 0; i < p.size(); i++) {
        for (int k = i+1; k < p.size(); k++) {
            int j = (i+1) % p.size();
            int l = (k+1) % p.size();
            if (i == 1 || j == k) continue;
            if (SegmentsIntersect(p[i], p[j], p[k], p[l]))
                return false;
        }
    }
    return true;
}

int main() {
    // expected: (-5,2)
    cerr << RotateCCW90(PT(2,5)) << endl;

    // expected: (5,-2)
    cerr << RotateCW90(PT(2,5)) << endl;

    // expected: (-5,2)
    cerr << RotateCCW(PT(2,5), M_PI/2) << endl;

    // expected: (5,2)
    cerr << ProjectPointLine(PT(-5,-2), PT(10,4), PT(3,7)) << endl;

    // expected: (5,2) (7.5,3) (2.5,1)
    cerr << ProjectPointSegment(PT(-5,-2), PT(10,4), PT(3,7)) << " "
        << ProjectPointSegment(PT(7.5,3), PT(10,4), PT(3,7)) << " "
        << ProjectPointSegment(PT(-5,-2), PT(2.5,1), PT(3,7)) << endl;

    // expected: 6.78903
    cerr << DistancePointPlane(4,-4,3,2,-2,5,-8) << endl;

    // expected: 1 0 1
    cerr << LinesParallel(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
        << LinesParallel(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
        << LinesParallel(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

    // expected: 0 0 1
    cerr << LinesCollinear(PT(1,1), PT(3,5), PT(2,1), PT(4,5)) << " "
        << LinesCollinear(PT(1,1), PT(3,5), PT(2,0), PT(4,5)) << " "
        << LinesCollinear(PT(1,1), PT(3,5), PT(5,9), PT(7,13)) << endl;

    // expected: 1 1 1 0
    cerr << SegmentsIntersect(PT(0,0), PT(2,4), PT(3,1), PT(-1,3)) << " "
        << SegmentsIntersect(PT(0,0), PT(2,4), PT(4,3), PT(0,5)) << " "
        << SegmentsIntersect(PT(0,0), PT(2,4), PT(2,-1), PT(-2,1)) << " "
        << SegmentsIntersect(PT(0,0), PT(2,4), PT(5,5), PT(1,7)) << endl;

    // expected: (1,2)
    cerr << ComputeLineIntersection(PT(0,0), PT(2,4), PT(3,1), PT(-1,3)) << endl;

    // expected: (1,1)
    cerr << ComputeCircleCenter(PT(-3,4), PT(6,1), PT(4,5)) << endl;

    vector<PT> v;
    v.push_back(PT(0,0));
    v.push_back(PT(5,0));
    v.push_back(PT(5,5));
    v.push_back(PT(0,5));

    // expected: 1 1 1 0 0
    cerr << PointInPolygon(v, PT(2,2)) << " "
        << PointInPolygon(v, PT(2,0)) << " "
        << PointInPolygon(v, PT(0,2)) << " "
        << PointInPolygon(v, PT(5,2)) << " "
        << PointInPolygon(v, PT(2,5)) << endl;

    // expected: 0 1 1 1 1
    cerr << PointOnPolygon(v, PT(2,2)) << " "
        << PointOnPolygon(v, PT(2,0)) << " "
        << PointOnPolygon(v, PT(0,2)) << " "
        << PointOnPolygon(v, PT(5,2)) << " "
        << PointOnPolygon(v, PT(2,5)) << endl;

    // expected: (1,6)
    // (5,4) (4,5)
    // blank line
    // (4,5) (5,4)
    // blank line
    // (4,5) (5,4)
    vector<PT> u = CircleLineIntersection(PT(0,6), PT(2,6), PT(1,1), 5);
    for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
    u = CircleLineIntersection(PT(0,9), PT(9,0), PT(1,1), 5);
    for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
    u = CircleCircleIntersection(PT(1,1), PT(10,10), 5, 5);
    for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
}

```

```

u = CircleCircleIntersection(PT(1,1), PT(8,8), 5, 5);
for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 10, sqrt(2.0)/2.0);
for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;
u = CircleCircleIntersection(PT(1,1), PT(4.5,4.5), 5, sqrt(2.0)/2.0);
for (int i = 0; i < u.size(); i++) cerr << u[i] << " "; cerr << endl;

// area should be 5.0
// centroid should be (1.1666666, 1.1666666)
PT pa[] = { PT(0,0), PT(5,0), PT(1,1), PT(0,5) };
vector<PT> p(pa, pa+4);
PT c = ComputeCentroid(p);
cerr << "Area: " << ComputeArea(p) << endl;
cerr << "Centroid: " << c << endl;

return 0;
}

```

4 Graphs/Trees

4.1 Graph structure example for our DFS and BFS algorithms

```

graph = {'A': set(['B', 'C']),
         'B': set(['A', 'D', 'E']),
         'C': set(['A', 'E']),
         'D': set(['B']),
         'E': set(['B', 'F']),
         'F': set(['C', 'E'])}

```

4.2 Breadth-First Search

```

def bfs(graph, start):
    visited, queue = set(), [start]
    while queue:
        vertex = queue.pop(0)
        if vertex not in visited:
            visited.add(vertex)
            queue.extend(graph[vertex] - visited)
    return visited

bfs(graph, 'A') # {'B', 'C', 'A', 'E', 'D', 'E'}

```

4.3 Breadth-First Search Paths

```

def bfs_paths(graph, start, goal):
    queue = [(start, [start])]
    while queue:
        (vertex, path) = queue.pop(0)
        for next in graph[vertex] - set(path):
            if next == goal:
                yield path + [next]
            else:
                queue.append((next, path + [next]))

list(bfs_paths(graph, 'A', 'F')) # [['A', 'C', 'F'], ['A', 'B', 'E', 'F']]

```

4.4 Breadth-First Search Shortest Path

```

def shortest_path(graph, start, goal):
    try:
        return next(bfs_paths(graph, start, goal))
    except StopIteration:
        return None

shortest_path(graph, 'A', 'F') # ['A', 'C', 'F']

```

4.5 Depth-First Search

```

def dfs(graph, start):
    visited, stack = set(), [start]
    while stack:
        vertex = stack.pop()
        if vertex not in visited:
            visited.add(vertex)
            stack.extend(graph[vertex] - visited)
    return visited

dfs(graph, 'A') # {'E', 'D', 'F', 'A', 'C', 'B'}

```

4.6 Depth-First Search Paths

```

#Returns all paths from start to goal
def dfs_paths(graph, start, goal):
    stack = [(start, [start])]
    while stack:
        (vertex, path) = stack.pop()
        for next in graph[vertex] - set(path):
            if next == goal:
                yield path + [next]
            else:
                stack.append((next, path + [next]))

list(dfs_paths(graph, 'A', 'F')) # [['A', 'C', 'F'], ['A', 'B', 'E', 'F']]

```

4.7 Dijkstra's Algorithm

```

from collections import defaultdict
from heapq import *

def dijkstra(edges, f, t):
    g = defaultdict(list)
    for l,r,c in edges:
        g[l].append((c,r))

    q, seen = [(0,f,())], set()
    while q:
        (cost,v1,path) = heappop(q)
        if v1 not in seen:
            seen.add(v1)
            path = (v1, path)
            if v1 == t: return (cost, path)
            for c, v2 in g.get(v1, ()):
                if v2 not in seen:
                    heappush(q, (cost+c, v2, path))

    return float("inf")

#Code example
edges = [("A", "B", 7), ("A", "D", 5), ("B", "C", 8),
         ("B", "D", 9), ("B", "E", 7), ("C", "E", 5)]
print "A -> E:"
print dijkstra(edges, "A", "E") # (14, ('E', ('B', ('A', ())))

```

4.8 Kruskal's Algorithm (including Merge-Find set)

```

parent = dict()
rank = dict()

def make_set(vertex):
    parent[vertex] = vertex
    rank[vertex] = 0

def find(vertex):
    if parent[vertex] != vertex:
        parent[vertex] = find(parent[vertex])
    return parent[vertex]

def union(vertex1, vertex2):
    root1 = find(vertex1)
    root2 = find(vertex2)
    if root1 != root2:
        if rank[root1] > rank[root2]:
            parent[root2] = root1
        else:
            parent[root1] = root2
    if rank[root1] == rank[root2]: rank[root2] += 1

```

```
def kruskal(graph):
    for vertex in graph['vertices']:
        make_set(vertex)
    minimum_spanning_tree = set()
    edges = list(graph['edges'])
    edges.sort()
    #print edges
    for edge in edges:
        weight, vertice1, vertice2 = edge
        if find(vertice1) != find(vertice2):
            union(vertice1, vertice2)
            minimum_spanning_tree.add(edge)

    return sorted(minimum_spanning_tree)
```

4.9 Bellman-Ford Algorithm

```
# Step 1: For each node prepare the destination and predecessor
def initialize(graph, source):
    d = {} # Stands for destination
    p = {} # Stands for predecessor
    for node in graph:
        d[node] = float('Inf') # We start admitting that the rest of nodes are very very far
        p[node] = None
    d[source] = 0 # For the source we know how to reach
    return d, p

def relax(node, neighbour, graph, d, p):
    # If the distance between the node and the neighbour is lower than the one I have now
    if d[neighbour] > d[node] + graph[node][neighbour]:
        # Record this lower distance
        d[neighbour] = d[node] + graph[node][neighbour]
        p[neighbour] = node

def bellman_ford(graph, source):
    d, p = initialize(graph, source)
    for i in range(len(graph)-1): #Run this until it converges
        for u in graph:
            for v in graph[u]: #For each neighbour of u
                relax(u, v, graph, d, p) #Lets relax it

    # Step 3: check for negative-weight cycles
    for u in graph:
        for v in graph[u]:
            assert d[v] <= d[u] + graph[u][v]

    return d, p

def test():
    graph = {
        'a': {'b': -1, 'c': 4},
        'b': {'c': 3, 'd': 2, 'e': 2},
        'c': {},
        'd': {'b': 1, 'c': 5},
        'e': {'d': -3}
    }
    d, p = bellman_ford(graph, 'a')
    # d = {'a':0, 'b':-1, 'c':2, 'd':-2, 'e':1},
    # p = {'a':None, 'b':'a', 'c':'b', 'd':'e', 'e':'b'}
```

4.10 Floyd-Warshall Algorithm

```
# Number of vertices in the graph
V = 4

# Define infinity as the large enough value. This value will be
# used for vertices not connected to each other
INF = 99999

# Solves all pair shortest path via Floyd Warshall Algorithm
def floydWarshall(graph):
    """ dist[i][j] will be the output matrix that will finally
    have the shortest distances between every pair of vertices """
    """ Initializing the solution matrix same as input graph matrix
    OR we can say that the initial values of shortest distances
    are based on shortest paths considering no
    intermediate vertices """
    dist = map(lambda i : map(lambda j : j, i), graph)

    """ Add all vertices one by one to the set of intermediate
    vertices.
```

```
---> Before start of a iteration, we have shortest distances
between all pairs of vertices such that the shortest
distances consider only the vertices in set
{0, 1, 2, .. k-1} as intermediate vertices.
----> After the end of a iteration, vertex no. k is
added to the set of intermediate vertices and the
set becomes {0, 1, 2, .. k}
"""
for k in range(V):

    # pick all vertices as source one by one
    for i in range(V):

        # Pick all vertices as destination for the
        # above picked source
        for j in range(V):

            # If vertex k is on the shortest path from
            # i to j, then update the value of dist[i][j]
            dist[i][j] = min(dist[i][j],
                             dist[i][k] + dist[k][j])

    printSolution(dist)
```

```
"""
      10
(0)----->(3)
 |           /\
5 |           |
 |/\         | 1
 |/\----->(1)
      3
graph = [[0,5,INF,10],
         [INF,0,3,INF],
         [INF, INF, 0, 1],
         [INF, INF, INF, 0]]

floydWarshall(graph) # [[0,5,8,9],[INF,0,3,4],[INF,INF,0,1],[INF,INF,INF,0]]
"""
```

4.11 Max Flow (Ford-Fulkerson Algorithm)

```
from collections import defaultdict

#This class represents a directed graph using adjacency matrix representation
class Graph:

    def __init__(self,graph):
        self.graph = graph # residual graph
        self.ROW = len(graph)
        #self.COL = len(gr[0])

    '''Returns true if there is a path from source 's' to sink 't' in
    residual graph. Also fills parent[] to store the path '''
    def BFS(self,s, t, parent):

        # Mark all the vertices as not visited
        visited=[False]*(self.ROW)

        # Create a queue for BFS
        queue=[]

        # Mark the source node as visited and enqueue it
        queue.append(s)
        visited[s] = True

        # Standard BFS Loop
        while queue:

            #Dequeue a vertex from queue and print it
            u = queue.pop(0)

            # Get all adjacent vertices of the dequeued vertex u
            # If a adjacent has not been visited, then mark it
            # visited and enqueue it
            for ind, val in enumerate(self.graph[u]):
                if visited[ind] == False and val > 0 :
                    queue.append(ind)
                    visited[ind] = True
                    parent[ind] = u

            # If we reached sink in BFS starting from source, then return
            # true, else false
            return True if visited[t] else False
```

```

# Returns the maximum flow from s to t in the given graph
def FordFulkerson(self, source, sink):

    # This array is filled by BFS and to store path
    parent = [-1]*(self.ROW)

    max_flow = 0 # There is no flow initially

    # Augment the flow while there is path from source to sink
    while self.BFS(source, sink, parent) :

        # Find minimum residual capacity of the edges along the
        # path filled by BFS. Or we can say find the maximum flow
        # through the path found.
        path_flow = float("Inf")
        s = sink
        while(s != source):
            path_flow = min(path_flow, self.graph[parent[s]][s])
            s = parent[s]

        # Add path flow to overall flow
        max_flow += path_flow

        # update residual capacities of the edges and reverse edges
        # along the path
        v = sink
        while(v != source):
            u = parent[v]
            self.graph[u][v] -= path_flow
            self.graph[v][u] += path_flow
            v = parent[v]

    return max_flow

# Create a graph given in the above diagram
graph = [[0, 16, 13, 0, 0, 0],
         [0, 0, 10, 12, 0, 0],
         [0, 4, 0, 0, 14, 0],
         [0, 0, 9, 0, 0, 20],
         [0, 0, 0, 7, 0, 4],
         [0, 0, 0, 0, 0, 0]]

g = Graph(graph)

source = 0; sink = 5

print ("The maximum possible flow is %d " % g.FordFulkerson(source, sink))

```

5 Mathematics

5.1 Gauss-Jordan Elimination (Matrix inversion and linear system solving)

```

def gauss_jordan(m, eps = 1.0/(10**10)):
    """Puts given matrix (2D array) into the Reduced Row Echelon Form.
    Returns True if successful, False if 'm' is singular.
    NOTE: make sure all the matrix items support fractions! Int matrix will NOT work!
    Written by Jarno Elonen in April 2005, released into Public Domain"""
    (h, w) = (len(m), len(m[0]))
    for y in range(0, h):
        maxrow = y
        for y2 in range(y+1, h): # Find max pivot
            if abs(m[y2][y]) > abs(m[maxrow][y]):
                maxrow = y2
        (m[y], m[maxrow]) = (m[maxrow], m[y])
        if abs(m[y][y]) <= eps: # Singular?
            return False
        for y2 in range(y+1, h): # Eliminate column y
            c = m[y2][y] / m[y][y]
            for x in range(y, w):
                m[y2][x] -= m[y][x] * c
        for y in range(h-1, 0-1, -1): # Backsubstitute
            c = m[y][y]
            for y2 in range(0, y):
                for x in range(w-1, y-1, -1):
                    m[y2][x] -= m[y][x] * m[y2][y] / c
            m[y][y] /= c
        for x in range(h, w): # Normalize row y
            m[y][x] /= c
    return True

```

```

def solve(M, b):
    """
    solves M*x = b
    return vector x so that M*x = b
    :param M: a matrix in the form of a list of list
    :param b: a vector in the form of a simple list of scalars
    """
    m2 = [row[:] + [right] for row, right in zip(M, b)]
    return [row[-1] for row in m2] if gauss_jordan(m2) else None

def inv(M):
    """
    return the inv of the matrix M
    """
    # clone the matrix and append the identity matrix
    # [int(i==j) for j in range_M] is nothing but the i(th row of the identity matrix
    m2 = [row[:] + [int(i==j) for j in range(len(M))] for i, row in enumerate(M)]
    # extract the appended matrix (kind of m2[m:,...])
    return [row[len(M[0]):] for row in m2] if gauss_jordan(m2) else None

def zeros(s, zero=0):
    """
    return a matrix of size 'size'
    :param size: a tuple containing dimensions of the matrix
    :param zero: the value to use to fill the matrix (by default it's zero)
    """
    return [zeros(s[1:]) for i in range(s[0])] if not len(s) else zero

```

5.2 Miller-Rabin Primality Test

```

def miller_rabin(n, k):
    # The optimal number of rounds (k) for this test is 40
    # for justification

    if n == 2:
        return True
    if n % 2 == 0:
        return False
    r, s = 0, n - 1
    while s % 2 == 0:
        r += 1
        s //= 2
    for _ in xrange(k):
        a = random.randrange(2, n - 1)
        x = pow(a, s, n)
        if x == 1 or x == n - 1:
            continue
        for _ in xrange(r - 1):
            x = pow(x, 2, n)
            if x == n - 1:
                break
        else:
            return False
    return True

```

5.3 Segment Tree

```

#encoding:utf-8
class SegmentTree(object):
    def __init__(self, start, end):
        self.start = start
        self.end = end
        self.max_value = {}
        self.sum_value = {}
        self.len_value = {}
        self._init(start, end)

    def add(self, start, end, weight=1):
        start = max(start, self.start)
        end = min(end, self.end)
        self._add(start, end, weight, self.start, self.end)
        return True

    def query_max(self, start, end):
        return self._query_max(start, end, self.start, self.end)

    def query_sum(self, start, end):
        return self._query_sum(start, end, self.start, self.end)

    def query_len(self, start, end):
        return self._query_len(start, end, self.start, self.end)

```



```

#####
def _init(self, start, end):
    self.max_value[start, end] = 0
    self.sum_value[start, end] = 0
    self.len_value[start, end] = 0
    if start < end:
        mid = start + int((end - start) / 2)
        self._init(start, mid)
        self._init(mid+1, end)

def _add(self, start, end, weight, in_start, in_end):
    key = (in_start, in_end)
    if in_start == in_end:
        self.max_value[key] += weight
        self.sum_value[key] += weight
        self.len_value[key] = 1 if self.sum_value[key] > 0 else 0
        return
    mid = in_start + int((in_end - in_start) / 2)
    if mid >= end:
        self._add(start, end, weight, in_start, mid)
    elif mid+1 <= start:
        self._add(start, end, weight, mid+1, in_end)
    else:
        self._add(start, mid, weight, in_start, mid)
        self._add(mid+1, end, weight, mid+1, in_end)
    self.max_value[key] = max(self.max_value[(in_start, mid)], self.max_value[(mid+1, in_end)])
    self.sum_value[key] = self.sum_value[(in_start, mid)] + self.sum_value[(mid+1, in_end)]
    self.len_value[key] = self.len_value[(in_start, mid)] + self.len_value[(mid+1, in_end)]

def _query_max(self, start, end, in_start, in_end):
    if start == in_start and end == in_end:
        ans = self.max_value[(start, end)]
    else:
        mid = in_start + int((in_end - in_start) / 2)
        if mid >= end:
            ans = self._query_max(start, end, in_start, mid)
        elif mid+1 <= start:
            ans = self._query_max(start, end, mid+1, in_end)
        else:
            ans = max(self._query_max(start, mid, in_start, mid),
                      self._query_max(mid+1, end, mid+1, in_end))
    #print start, end, in_start, in_end, ans
    return ans

def _query_sum(self, start, end, in_start, in_end):
    if start == in_start and end == in_end:
        ans = self.sum_value[(start, end)]
    else:
        mid = in_start + int((in_end - in_start) / 2)
        if mid >= end:
            ans = self._query_sum(start, end, in_start, mid)
        elif mid+1 <= start:
            ans = self._query_sum(start, end, mid+1, in_end)
        else:
            ans = self._query_sum(start, mid, in_start, mid) + self._query_sum(mid+1, end, mid+1, in_end)
    return ans

def _query_len(self, start, end, in_start, in_end):
    if start == in_start and end == in_end:
        ans = self.len_value[(start, end)]
    else:
        mid = in_start + int((in_end - in_start) / 2)
        if mid >= end:
            ans = self._query_len(start, end, in_start, mid)
        elif mid+1 <= start:
            ans = self._query_len(start, end, mid+1, in_end)
        else:
            ans = self._query_len(start, mid, in_start, mid) + self._query_len(mid+1, end, mid+1, in_end)
    #print start, end, in_start, in_end, ans
    return ans

```

5.4 Prime Number Sieve (generator)

```

from itertools import count

def postponed_sieve():
    yield 2; yield 3; yield 5; yield 7;
    sieve = ()
    ps = postponed_sieve()
    p = next(ps) and next(ps)
    q = p*p
    for c in count(9,2):
        if c in sieve:
            # c's a multiple of some base prime

```

```

s = sieve.pop(c) # i.e. a composite ; or
elif c < q:
    yield c # a prime
    continue
else: # (c==q): # or the next base prime's square:
    s=count(q+2*p,2+p) # (9+6, by 6 : 15,21,27,33,...)
    p=next(ps) # (5)
    q=p*p # (25)
    for m in s: # the next multiple
        if m not in sieve: # no duplicates
            break
    sieve[m] = s # original test entry: ideone.com/WFv4f

```

5.5 GCD and Euler's Totient Function

```

# Function to return gcd of a and b
def gcd(a, b):
    if a == 0:
        return b
    return gcd(b%a, a)

# A simple method to evaluate Euler Totient Function
def phi(n):
    result = 1
    for i in range(2, n):
        if gcd(i, n) == 1:
            result = result + 1
    return result

```

6 Strings

6.1 Knuth-Morris-Pratt Algorithm (fast pattern matching)

```

def KnuthMorrisPratt(text, pattern):

    """Yields all starting positions of copies of the pattern in the text.
    Calling conventions are similar to string.find, but its arguments can be
    lists or iterators, not just strings, it returns all matches, not just
    the first one, and it does not need the whole text in memory at once.
    Whenever it yields, it will have read the text exactly up to and including
    the match that caused the yield."""

    # allow indexing into pattern and protect against change during yield
    pattern = list(pattern)

    # build table of shift amounts
    shifts = [1] * (len(pattern) + 1)
    shift = 1
    for pos in range(len(pattern)):
        while shift <= pos and pattern[pos] != pattern[pos-shift]:
            shift += shifts[pos-shift]
        shifts[pos+1] = shift

    # do the actual search
    startPos = 0
    matchLen = 0
    for c in text:
        while matchLen == len(pattern) or \
              matchLen >= 0 and pattern[matchLen] != c:
            startPos += shifts[matchLen]
            matchLen -= shifts[matchLen]
            matchLen += 1
        if matchLen == len(pattern):
            yield startPos

```

6.2 Rabin-Karp Algorithm (multiple pattern matching)

```

# d is the number of characters in input alphabet
d = 256

# pat -> pattern
# txt -> text
# q -> A prime number

def search(pat, txt, q):

```

```
M = len(pat)
N = len(txt)
i = 0
j = 0
p = 0 # hash value for pattern
t = 0 # hash value for txt
h = 1

# The value of h would be "pow(d, M-1)%q"
for i in xrange(M-1):
    h = (h*d)%q

# Calculate the hash value of pattern and first window
# of text
for i in xrange(M):
    p = (d*p + ord(pat[i]))%q
    t = (d*t + ord(txt[i]))%q

# Slide the pattern over text one by one
for i in xrange(N-M+1):
    # Check the hash values of current window of text and
    # pattern if the hash values match then only check
    # for characters one by one
    if p==t:
        # Check for characters one by one
        for j in xrange(M):
            if txt[i+j] != pat[j]:
                break

        j+=1
        # if p == t and pat[0..M-1] = txt[i, i+1, ...i+M-1]
        if j==M:
            print "Pattern found at index " + str(i)

# Calculate hash value for next window of text: Remove
# leading digit, add trailing digit
if i < N-M:
    t = (d*(t-ord(txt[i])*h) + ord(txt[i+M]))%q

    # We might get negative values of t, converting it to
    # positive
    if t < 0:
        t = t+q

# Driver program to test the above function
txt = "GEEKS FOR GEEKS"
pat = "GEEK"
q = 101 # A prime number
search(pat,txt,q)
```

7 Techniques

7.1 Various algorithm techniques

```
Recursion
Divide and conquer
    Finding interesting points in N log N
Greedy algorithm
    Scheduling
    Max contiguous subvector sum
    Invariants
    Huffman encoding
Graph theory
    Dynamic graphs (extra book-keeping)
    Breadth first search
    Depth first search
    * Normal trees / DFS trees
    Dijkstra's algorithm
    MST: Prim's algorithm
    Bellman-Ford
    Konig's theorem and vertex cover
    Min-cost max flow
    Lovasz toggle
    Matrix tree theorem
    Maximal matching, general graphs
    Hopcroft-Karp
    Hall's marriage theorem
    Graphical sequences
    Floyd-Warshall
    Eulercykler
    Flow networks
    * Augmenting paths
    * Edmonds-Karp
    Bipartite matching
    Min. path cover
```

```
Topological sorting
Strongly connected components
2-SAT
Cutvertices, cutedges och biconnected components
Edge coloring
* Trees
Vertex coloring
* Bipartite graphs (=> trees)
* 3^n (special case of set cover)
Diameter and centroid
K'th shortest path
Shortest cycle
Dynamic programming
    Knapsack
    Coin change
    Longest common subsequence
    Longest increasing subsequence
    Number of paths in a dag
    Shortest path in a dag
    Dynprog over intervals
    Dynprog over subsets
    Dynprog over probabilities
    Dynprog over trees
    3^n set cover
    Divide and conquer
    Knuth optimization
    Convex hull optimizations
    RMQ (sparse table a.k.a 2^k-jumps)
    Bitonic cycle
    Log partitioning (loop over most restricted)
Combinatorics
    Computation of binomial coefficients
    Pigeon-hole principle
    Inclusion/exclusion
    Catalan number
    Pick's theorem
Number theory
    Integer parts
    Divisibility
    Euklidian algorithm
    Modular arithmetic
    * Modular multiplication
    * Modular inverses
    * Modular exponentiation by squaring
    Chinese remainder theorem
    Fermat's small theorem
    Euler's theorem
    Phi function
    Frobenius number
    Quadratic reciprocity
    Pollard-Rho
    Miller-Rabin
    Hensel lifting
    Vieta root jumping
Game theory
    Combinatorial games
    Game trees
    Mini-max
    Nim
    Games on graphs
    Games on graphs with loops
    Grundy numbers
    Bipartite games without repetition
    General games without repetition
    Alpha-beta pruning
Probability theory
Optimization
    Binary search
    Ternary search
    Unimodality and convex functions
    Binary search on derivative
Numerical methods
    Numeric integration
    Newton's method
    Root-finding with binary/ternary search
    Golden section search
Matrices
    Gaussian elimination
    Exponentiation by squaring
Sorting
    Radix sort
Geometry
    Coordinates and vectors
    * Cross product
    * Scalar product
    Convex hull
    Polygon cut
    Closest pair
    Coordinate-compression
    Quadrees
    KD-trees
    All segment-segment intersection
```

Sweeping
 Discretization (convert to events and sweep)
 Angle sweeping
 Line sweeping
 Discrete second derivatives
Strings
 Longest common substring
 Palindrome subsequences
 Knuth-Morris-Pratt
 Tries
 Rolling polynomial hashes
 Suffix array
 Suffix tree
 Aho-Corasick
 Manacher's algorithm
 Letter position lists
Combinatorial search
 Meet in the middle

 Brute-force with pruning
 Best-first (A*)
 Bidirectional search
 Iterative deepening DFS / A*
Data structures
 LCA (2^k-jumps in trees in general)
 Pull/push-technique on trees
 Heavy-light decomposition
 Centroid decomposition
 Lazy propagation
 Self-balancing trees
 Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)
 Monotone queues / monotone stacks / sliding queues
 Sliding queue using 2 stacks
 Persistent segment tree

$f(n) = O(g(n))$	iff \exists positive c, n_0 such that $0 \leq f(n) \leq cg(n) \forall n \geq n_0$.	$\sum_{i=1}^n i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$
$f(n) = \Omega(g(n))$	iff \exists positive c, n_0 such that $f(n) \geq cg(n) \geq 0 \forall n \geq n_0$.	In general:
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.	$\sum_{i=1}^n i^m = \frac{1}{m+1} \left[(n+1)^{m+1} - 1 - \sum_{i=1}^n ((i+1)^{m+1} - i^{m+1} - (m+1)i^m) \right]$
$f(n) = o(g(n))$	iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$.	$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$
$\lim_{n \rightarrow \infty} a_n = a$	iff $\forall \epsilon > 0, \exists n_0$ such that $ a_n - a < \epsilon, \forall n \geq n_0$.	Geometric series:
$\sup S$	least $b \in \mathbb{R}$ such that $b \geq s, \forall s \in S$.	$\sum_{i=0}^n c^i = \frac{c^{n+1} - 1}{c - 1}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} c^i = \frac{1}{1 - c}, \quad \sum_{i=1}^{\infty} c^i = \frac{c}{1 - c}, \quad c < 1,$
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \leq s, \forall s \in S$.	$\sum_{i=0}^n i c^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \neq 1, \quad \sum_{i=0}^{\infty} i c^i = \frac{c}{(1-c)^2}, \quad c < 1.$
$\liminf_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \inf \{a_i \mid i \geq n, i \in \mathbb{N}\}.$	Harmonic series:
$\limsup_{n \rightarrow \infty} a_n$	$\lim_{n \rightarrow \infty} \sup \{a_i \mid i \geq n, i \in \mathbb{N}\}.$	$H_n = \sum_{i=1}^n \frac{1}{i}, \quad \sum_{i=1}^n i H_i = \frac{n(n+1)}{2} H_n - \frac{n(n-1)}{4}.$
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.	$\sum_{i=1}^n H_i = (n+1)H_n - n, \quad \sum_{i=1}^n \binom{i}{m} H_i = \binom{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right).$
$[n]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.	1. $\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad 2. \sum_{k=0}^n \binom{n}{k} = 2^n, \quad 3. \binom{n}{k} = \binom{n}{n-k},$
$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.	4. $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}, \quad 5. \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$
$\langle \begin{matrix} n \\ k \end{matrix} \rangle$	1st order Eulerian numbers: Permutations $\pi_1 \pi_2 \dots \pi_n$ on $\{1, 2, \dots, n\}$ with k ascents.	6. $\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}, \quad 7. \sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n},$
$\langle \langle \begin{matrix} n \\ k \end{matrix} \rangle \rangle$	2nd order Eulerian numbers.	8. $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}, \quad 9. \sum_{k=0}^n \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n},$
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.	10. $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}, \quad 11. \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1,$
14. $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!,$	15. $\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)!H_{n-1},$	16. $\begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad 17. \begin{bmatrix} n \\ k \end{bmatrix} \geq \left\{ \begin{matrix} n \\ k \end{matrix} \right\},$
18. $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix},$	19. $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \begin{bmatrix} n \\ n-1 \end{bmatrix} = \binom{n}{2},$	20. $\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!, \quad 21. C_n = \frac{1}{n+1} \binom{2n}{n},$
22. $\langle \begin{matrix} n \\ 0 \end{matrix} \rangle = \langle \begin{matrix} n \\ n-1 \end{matrix} \rangle = 1,$	23. $\langle \begin{matrix} n \\ k \end{matrix} \rangle = \langle \begin{matrix} n \\ n-1-k \end{matrix} \rangle,$	24. $\langle \begin{matrix} n \\ k \end{matrix} \rangle = (k+1) \langle \begin{matrix} n-1 \\ k \end{matrix} \rangle + (n-k) \langle \begin{matrix} n-1 \\ k-1 \end{matrix} \rangle,$
25. $\langle \begin{matrix} 0 \\ k \end{matrix} \rangle = \begin{cases} 1 & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$	26. $\langle \begin{matrix} n \\ 1 \end{matrix} \rangle = 2^n - n - 1,$	27. $\langle \begin{matrix} n \\ 2 \end{matrix} \rangle = 3^n - (n+1)2^n + \binom{n+1}{2},$
28. $x^n = \sum_{k=0}^n \langle \begin{matrix} n \\ k \end{matrix} \rangle \binom{x+k}{n},$	29. $\langle \begin{matrix} n \\ m \end{matrix} \rangle = \sum_{k=0}^m \binom{n+1}{k} (m+1-k)^n (-1)^k,$	30. $m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_{k=0}^n \langle \begin{matrix} n \\ k \end{matrix} \rangle \binom{k}{n-m},$
31. $\langle \begin{matrix} n \\ m \end{matrix} \rangle = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \binom{n-k}{m} (-1)^{n-k-m} k!,$	32. $\langle \langle \begin{matrix} n \\ 0 \end{matrix} \rangle \rangle = 1,$	33. $\langle \langle \begin{matrix} n \\ n \end{matrix} \rangle \rangle = 0 \quad \text{for } n \neq 0,$
34. $\langle \langle \begin{matrix} n \\ k \end{matrix} \rangle \rangle = (k+1) \langle \langle \begin{matrix} n-1 \\ k \end{matrix} \rangle \rangle + (2n-1-k) \langle \langle \begin{matrix} n-1 \\ k-1 \end{matrix} \rangle \rangle,$	35. $\sum_{k=0}^n \langle \langle \begin{matrix} n \\ k \end{matrix} \rangle \rangle = \frac{(2n)n}{2^n},$	36. $\left\{ \begin{matrix} x \\ x-n \end{matrix} \right\} = \sum_{k=0}^n \langle \langle \begin{matrix} n \\ k \end{matrix} \rangle \rangle \binom{x+n-1-k}{2n},$
37. $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} = \sum_{k=0}^n \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (m+1)^{n-k},$		

38. $\begin{bmatrix} n+1 \\ m+1 \end{bmatrix} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} = \sum_{k=0}^n \begin{bmatrix} k \\ m \end{bmatrix} n^{n-k} = n! \sum_{k=0}^n \frac{1}{k!} \begin{bmatrix} k \\ m \end{bmatrix},$	39. $\begin{bmatrix} x \\ x-n \end{bmatrix} = \sum_{k=0}^n \left\langle \begin{bmatrix} n \\ k \end{bmatrix} \right\rangle \begin{bmatrix} x+k \\ 2n \end{bmatrix},$	Every tree with n vertices has $n-1$ edges. Kraft inequality: If the depths of the leaves of a binary tree are d_1, \dots, d_n : $\sum_{i=1}^n 2^{-d_i} \leq 1,$ and equality holds only if every internal node has 2 sons.
40. $\left\{ \begin{matrix} n \\ m \end{matrix} \right\} = \sum_k \binom{n}{k} \left\{ \begin{matrix} k+1 \\ m+1 \end{matrix} \right\} (-1)^{n-k},$	41. $\begin{bmatrix} n \\ m \end{bmatrix} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$	
42. $\left\{ \begin{matrix} m+n+1 \\ m \end{matrix} \right\} = \sum_{k=0}^m k \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\},$	43. $\begin{bmatrix} m+n+1 \\ m \end{bmatrix} = \sum_{k=0}^m k(n+k) \begin{bmatrix} n+k \\ k \end{bmatrix},$	
44. $\binom{n}{m} = \sum_k \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \begin{bmatrix} k \\ m \end{bmatrix} (-1)^{m-k},$	45. $(n-m)! \binom{n}{m} = \sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \left\{ \begin{matrix} k \\ m \end{matrix} \right\} (-1)^{m-k}, \text{ for } n \geq m,$	
46. $\left\{ \begin{matrix} n \\ n-m \end{matrix} \right\} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \begin{bmatrix} m+k \\ k \end{bmatrix},$	47. $\begin{bmatrix} n \\ n-m \end{bmatrix} = \sum_k \binom{m-n}{m+k} \binom{m+n}{n+k} \left\{ \begin{matrix} m+k \\ k \end{matrix} \right\},$	
48. $\left\{ \begin{matrix} n \\ \ell+m \end{matrix} \right\} \binom{\ell+m}{\ell} = \sum_k \left\{ \begin{matrix} k \\ \ell \end{matrix} \right\} \left\{ \begin{matrix} n-k \\ m \end{matrix} \right\} \binom{n}{k},$	49. $\begin{bmatrix} n \\ \ell+m \end{bmatrix} \binom{\ell+m}{\ell} = \sum_k \begin{bmatrix} k \\ \ell \end{bmatrix} \begin{bmatrix} n-k \\ m \end{bmatrix} \binom{n}{k}.$	

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \geq 1, b > 1$$

If $\exists \epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a} \log_2 n).$$

If $\exists \epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \leq cf(n)$ for large n , then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two.

Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \frac{1}{2} + u_i, \quad u_1 = \frac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$.

Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side “telescope”

$$1(T(n) - 3T(n/2)) = n$$

$$3(T(n/2) - 3T(n/4)) = n/2$$

$$\vdots \quad \vdots \quad \vdots$$

$$3^{\log_2 n - 1} (T(2) - 3T(1)) = 2$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$n \sum_{i=0}^{m-1} c^i = n \left(\frac{c^m - 1}{c - 1} \right)$$

$$= 2n(c^{\log_2 n} - 1)$$

$$= 2n(c^{(k-1)\log_2 n} - 1)$$

$$= 2n^k - 2n,$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^i T_j.$$

Subtracting we find

$$T_{i+1} - T_i = 1 + \sum_{j=0}^i T_j - 1 - \sum_{j=0}^{i-1} T_j = T_i.$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i .
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i .

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i \geq 0} g_{i+1} x^i = \sum_{i \geq 0} 2g_i x^i + \sum_{i \geq 0} x^i.$$

We choose $G(x) = \sum_{i \geq 0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i \geq 0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

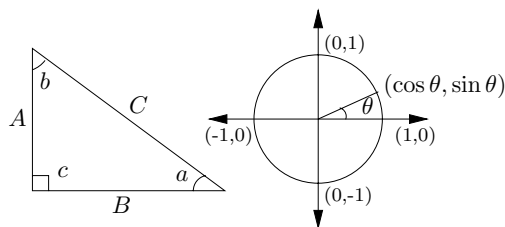
$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x \left(2 \sum_{i \geq 0} 2^i x^i - \sum_{i \geq 0} x^i \right) \\ &= \sum_{i \geq 0} (2^{i+1} - 1) x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

$n \sim 0.11109,$			$\psi - 2 \sim 1.01009,$	$\psi - 2 \sim 0.01009$
i	2^i	p_i	General	Probability
1	2	2	Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):	Continuous distributions: If
2	4	3	$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30},$	$\Pr[a < X < b] = \int_a^b p(x) dx,$
3	8	5	$B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}.$	then p is the probability density function of X . If
4	16	7	Change of base, quadratic formula:	$\Pr[X < a] = P(a),$
5	32	11	$\log_b x = \frac{\log_a x}{\log_a b}, \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$	then P is the distribution function of X . If P and p both exist then
6	64	13	Euler's number e :	$P(a) = \int_{-\infty}^a p(x) dx.$
7	128	17	$e = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$	Expectation: If X is discrete
8	256	19	$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$	$E[g(X)] = \sum_x g(x) \Pr[X = x].$
9	512	23	$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$	If X continuous then
10	1,024	29	$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$	$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x) dx = \int_{-\infty}^{\infty} g(x) dP(x).$
11	2,048	31	Harmonic numbers:	Variance, standard deviation:
12	4,096	37	$1, \frac{3}{2}, \frac{11}{6}, \frac{25}{12}, \frac{137}{60}, \frac{49}{20}, \frac{363}{140}, \frac{761}{280}, \frac{7129}{2520}, \dots$	$\text{VAR}[X] = E[X^2] - E[X]^2,$
13	8,192	41	$\ln n < H_n < \ln n + 1,$	$\sigma = \sqrt{\text{VAR}[X]}.$
14	16,384	43	$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$	For events A and B :
15	32,768	47	Factorial, Stirling's approximation:	$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$
16	65,536	53	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \dots$	$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$
17	131,072	59	$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$	iff A and B are independent.
18	262,144	61	Ackermann's function and inverse:	$\Pr[A B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$
19	524,288	67	$a(i, j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$	For random variables X and Y :
20	1,048,576	71	$\alpha(i) = \min\{j \mid a(j, j) \geq i\}.$	$E[X \cdot Y] = E[X] \cdot E[Y],$
21	2,097,152	73	Binomial distribution:	if X and Y are independent.
22	4,194,304	79	$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \quad q = 1 - p,$	$E[X + Y] = E[X] + E[Y],$
23	8,388,608	83	$E[X] = \sum_{k=1}^n k \binom{n}{k} p^k q^{n-k} = np.$	$E[cX] = cE[X].$
24	16,777,216	89	Poisson distribution:	Bayes' theorem:
25	33,554,432	97	$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad E[X] = \lambda.$	$\Pr[A_i B] = \frac{\Pr[B A_i] \Pr[A_i]}{\sum_{j=1}^n \Pr[A_j] \Pr[B A_j]}.$
26	67,108,864	101	Normal (Gaussian) distribution:	Inclusion-exclusion:
27	134,217,728	103	$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad E[X] = \mu.$	$\Pr\left[\bigvee_{i=1}^n X_i\right] = \sum_{i=1}^n \Pr[X_i] +$
28	268,435,456	107	The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is	$\sum_{k=2}^n (-1)^{k+1} \sum_{i_1 < \dots < i_k} \Pr\left[\bigwedge_{j=1}^k X_{i_j}\right].$
29	536,870,912	109	$nH_n.$	Moment inequalities:
30	1,073,741,824	113		$\Pr[X \geq \lambda E[X]] \leq \frac{1}{\lambda},$
31	2,147,483,648	127		$\Pr[X - E[X] \geq \lambda \cdot \sigma] \leq \frac{1}{\lambda^2}.$
32	4,294,967,296	131		Geometric distribution:
Pascal's Triangle				$\Pr[X = k] = pq^{k-1}, \quad q = 1 - p,$
1				$E[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$
1 1				
1 2 1				
1 3 3 1				
1 4 6 4 1				
1 5 10 10 5 1				
1 6 15 20 15 6 1				
1 7 21 35 35 21 7 1				
1 8 28 56 70 56 28 8 1				
1 9 36 84 126 126 84 36 9 1				
1 10 45 120 210 252 210 120 45 10 1				



Pythagorean theorem:

$$C^2 = A^2 + B^2.$$

Definitions:

$$\begin{aligned} \sin a &= A/C, & \cos a &= B/C, \\ \csc a &= C/A, & \sec a &= C/B, \\ \tan a &= \frac{\sin a}{\cos a} = \frac{A}{B}, & \cot a &= \frac{\cos a}{\sin a} = \frac{B}{A}. \end{aligned}$$

Area, radius of inscribed circle:

$$\frac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:

$$\begin{aligned} \sin x &= \frac{1}{\csc x}, & \cos x &= \frac{1}{\sec x}, \\ \tan x &= \frac{1}{\cot x}, & \sin^2 x + \cos^2 x &= 1, \\ 1 + \tan^2 x &= \sec^2 x, & 1 + \cot^2 x &= \csc^2 x, \\ \sin x &= \cos\left(\frac{\pi}{2} - x\right), & \sin x &= \sin(\pi - x), \\ \cos x &= -\cos(\pi - x), & \tan x &= \cot\left(\frac{\pi}{2} - x\right), \\ \cot x &= -\cot(\pi - x), & \csc x &= \cot\frac{x}{2} - \cot x, \\ \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y, \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y, \\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \\ \cot(x \pm y) &= \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y}, \\ \sin 2x &= 2 \sin x \cos x, & \sin 2x &= \frac{2 \tan x}{1 + \tan^2 x}, \\ \cos 2x &= \cos^2 x - \sin^2 x, & \cos 2x &= 2 \cos^2 x - 1, \\ \cos 2x &= 1 - 2 \sin^2 x, & \cos 2x &= \frac{1 - \tan^2 x}{1 + \tan^2 x}, \\ \tan 2x &= \frac{2 \tan x}{1 - \tan^2 x}, & \cot 2x &= \frac{\cot^2 x - 1}{2 \cot x}, \\ \sin(x + y) \sin(x - y) &= \sin^2 x - \sin^2 y, \\ \cos(x + y) \cos(x - y) &= \cos^2 x - \sin^2 y. \end{aligned}$$

Euler's equation:

$$e^{ix} = \cos x + i \sin x, \quad e^{i\pi} = -1.$$

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Multiplication:

$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^n a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.

$$\det A \cdot B = \det A \cdot \det B,$$

$$\det A = \sum_{\pi} \prod_{i=1}^n \text{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= ad - bc, \\ \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix} \\ &= aei + bfg + cdh \\ &\quad - ceg - fha - ibd. \end{aligned}$$

Permanents:

$$\text{perm } A = \sum_{\pi} \prod_{i=1}^n a_{i,\pi(i)}.$$

Hyperbolic Functions

Definitions:

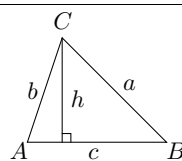
$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2}, & \cosh x &= \frac{e^x + e^{-x}}{2}, \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}}, & \text{csch } x &= \frac{1}{\sinh x}, \\ \text{sech } x &= \frac{1}{\cosh x}, & \coth x &= \frac{1}{\tanh x}. \end{aligned}$$

Identities:

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= 1, & \tanh^2 x + \text{sech}^2 x &= 1, \\ \coth^2 x - \text{csch}^2 x &= 1, & \sinh(-x) &= -\sinh x, \\ \cosh(-x) &= \cosh x, & \tanh(-x) &= -\tanh x, \\ \sinh(x + y) &= \sinh x \cosh y + \cosh x \sinh y, \\ \cosh(x + y) &= \cosh x \cosh y + \sinh x \sinh y, \\ \sinh 2x &= 2 \sinh x \cosh x, \\ \cosh 2x &= \cosh^2 x + \sinh^2 x, \\ \cosh x + \sinh x &= e^x, & \cosh x - \sinh x &= e^{-x}, \\ (\cosh x + \sinh x)^n &= \cosh nx + \sinh nx, & n \in \mathbb{Z}, \\ 2 \sinh^2 \frac{x}{2} &= \cosh x - 1, & 2 \cosh^2 \frac{x}{2} &= \cosh x + 1. \end{aligned}$$

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

... in mathematics
you don't under-
stand things, you
just get used to
them.
- J. von Neumann



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Area:

$$\begin{aligned} A &= \frac{1}{2}hc, \\ &= \frac{1}{2}ab \sin C, \\ &= \frac{c^2 \sin A \sin B}{2 \sin C}. \end{aligned}$$

Heron's formula:

$$\begin{aligned} A &= \sqrt{s \cdot s_a \cdot s_b \cdot s_c}, \\ s &= \frac{1}{2}(a + b + c), \\ s_a &= s - a, \\ s_b &= s - b, \\ s_c &= s - c. \end{aligned}$$

More identities:

$$\begin{aligned} \sin \frac{x}{2} &= \sqrt{\frac{1 - \cos x}{2}}, \\ \cos \frac{x}{2} &= \sqrt{\frac{1 + \cos x}{2}}, \\ \tan \frac{x}{2} &= \sqrt{\frac{1 - \cos x}{1 + \cos x}}, \\ &= \frac{1 - \cos x}{\sin x}, \\ &= \frac{\sin x}{1 + \cos x}, \\ \cot \frac{x}{2} &= \sqrt{\frac{1 + \cos x}{1 - \cos x}}, \\ &= \frac{1 + \cos x}{\sin x}, \\ &= \frac{\sin x}{1 - \cos x}, \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i}, \\ \cos x &= \frac{e^{ix} + e^{-ix}}{2}, \\ \tan x &= -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}, \\ &= -i \frac{e^{2ix} - 1}{e^{2ix} + 1}, \\ \sin x &= \frac{\sinh ix}{i}, \\ \cos x &= \cosh ix, \\ \tan x &= \frac{\tanh ix}{i}. \end{aligned}$$

The Chinese remainder theorem: There exists a number C such that:

$$C \equiv r_1 \pmod{m_1}$$

$$\vdots \quad \vdots \quad \vdots$$

$$C \equiv r_n \pmod{m_n}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x . If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^n p_i^{e_i-1} (p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \pmod{b}.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \pmod{p}.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^n \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \pmod{n}.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of } r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln \ln n - n + n \frac{\ln \ln n}{\ln n}$$

$$+ O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3}$$

$$+ O\left(\frac{n}{(\ln n)^4}\right).$$

Definitions:

Loop An edge connecting a vertex to itself.

Directed Simple Each edge has a direction. Graph with no loops or multi-edges.

Walk A sequence $v_0 e_1 v_1 \dots e_\ell v_\ell$.

Trail A walk with distinct edges.

Path A trail with distinct vertices.

Connected A graph where there exists a path between any two vertices.

Component A maximal connected subgraph.

Tree A connected acyclic graph.

Free tree A tree with no root.

DAG Directed acyclic graph.

Eulerian Graph with a trail visiting each edge exactly once.

Hamiltonian Graph with a cycle visiting each vertex exactly once.

Cut A set of edges whose removal increases the number of components.

Cut-set A minimal cut.

Cut edge A size 1 cut.

k-Connected A graph connected with the removal of any $k-1$ vertices.

k-Tough $\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G-S) \leq |S|$.

k-Regular A graph where all vertices have degree k .

k-Factor A k -regular spanning subgraph.

Matching A set of edges, no two of which are adjacent.

Clique A set of vertices, all of which are adjacent.

Ind. set A set of vertices, none of which are adjacent.

Vertex cover A set of vertices which cover all edges.

Planar graph A graph which can be embedded in the plane.

Plane graph An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5 .

Notation:

$E(G)$ Edge set

$V(G)$ Vertex set

$c(G)$ Number of components

$G[S]$ Induced subgraph

$\deg(v)$ Degree of v

$\Delta(G)$ Maximum degree

$\delta(G)$ Minimum degree

$\chi(G)$ Chromatic number

$\chi_E(G)$ Edge chromatic number

G^c Complement graph

K_n Complete graph

K_{n_1, n_2} Complete bipartite graph

$r(k, \ell)$ Ramsey number

Geometry

Projective coordinates: triples (x, y, z) , not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian Projective

$$(x, y) \quad (x, y, 1)$$

$$y = mx + b \quad (m, -1, b)$$

$$x = c \quad (1, 0, -c)$$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

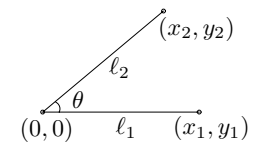
$$[|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p},$$

$$\lim_{p \rightarrow \infty} [|x_1 - x_0|^p + |y_1 - y_0|^p]^{1/p}.$$

Area of triangle $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) :

$$\frac{1}{2} \text{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos \theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{l_1 l_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1) :

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \quad V = \frac{4}{3} \pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.

– Issac Newton

Wallis' identity:

$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:

$$\frac{\pi}{4} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \cdots}}}}$$

Gregory's series:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots \right)$$

Euler's series:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x . We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D , divide N by D , obtaining

$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$

where the degree of N' is less than that of D . Second, factor $D(x)$. Use the following rules: For a non-repeated factor:

$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$

where

$$A = \left[\frac{N(x)}{D(x)} \right]_{x=a}.$$

For a repeated factor:

$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$

where

$$A_k = \frac{1}{k!} \left[\frac{d^k}{dx^k} \left(\frac{N(x)}{D(x)} \right) \right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Derivatives:

$$1. \frac{d(cu)}{dx} = c \frac{du}{dx}, \quad 2. \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \quad 3. \frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx},$$

$$4. \frac{d(u^n)}{dx} = nu^{n-1} \frac{du}{dx}, \quad 5. \frac{d(u/v)}{dx} = \frac{v \left(\frac{du}{dx} \right) - u \left(\frac{dv}{dx} \right)}{v^2}, \quad 6. \frac{d(e^{cu})}{dx} = ce^{cu} \frac{du}{dx},$$

$$7. \frac{d(c^u)}{dx} = (\ln c) c^u \frac{du}{dx}, \quad 8. \frac{d(\ln u)}{dx} = \frac{1}{u} \frac{du}{dx},$$

$$9. \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}, \quad 10. \frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx},$$

$$11. \frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}, \quad 12. \frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx},$$

$$13. \frac{d(\sec u)}{dx} = \tan u \sec u \frac{du}{dx}, \quad 14. \frac{d(\csc u)}{dx} = -\cot u \csc u \frac{du}{dx},$$

$$15. \frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}, \quad 16. \frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx},$$

$$17. \frac{d(\arctan u)}{dx} = \frac{1}{1+u^2} \frac{du}{dx}, \quad 18. \frac{d(\operatorname{arccot} u)}{dx} = \frac{-1}{1+u^2} \frac{du}{dx},$$

$$19. \frac{d(\operatorname{arcsec} u)}{dx} = \frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 20. \frac{d(\operatorname{arccsc} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx},$$

$$21. \frac{d(\sinh u)}{dx} = \cosh u \frac{du}{dx}, \quad 22. \frac{d(\cosh u)}{dx} = \sinh u \frac{du}{dx},$$

$$23. \frac{d(\tanh u)}{dx} = \operatorname{sech}^2 u \frac{du}{dx}, \quad 24. \frac{d(\coth u)}{dx} = -\operatorname{csch}^2 u \frac{du}{dx},$$

$$25. \frac{d(\operatorname{sech} u)}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}, \quad 26. \frac{d(\operatorname{csch} u)}{dx} = -\operatorname{csch} u \coth u \frac{du}{dx},$$

$$27. \frac{d(\operatorname{arcsinh} u)}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}, \quad 28. \frac{d(\operatorname{arccosh} u)}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx},$$

$$29. \frac{d(\operatorname{arctanh} u)}{dx} = \frac{1}{1-u^2} \frac{du}{dx}, \quad 30. \frac{d(\operatorname{arccoth} u)}{dx} = \frac{1}{u^2-1} \frac{du}{dx},$$

$$31. \frac{d(\operatorname{arcsech} u)}{dx} = \frac{-1}{u\sqrt{1-u^2}} \frac{du}{dx}, \quad 32. \frac{d(\operatorname{arccsch} u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}} \frac{du}{dx}.$$

Integrals:

$$1. \int cu \, dx = c \int u \, dx, \quad 2. \int (u+v) \, dx = \int u \, dx + \int v \, dx,$$

$$3. \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1, \quad 4. \int \frac{1}{x} \, dx = \ln x, \quad 5. \int e^x \, dx = e^x,$$

$$6. \int \frac{dx}{1+x^2} = \arctan x, \quad 7. \int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx,$$

$$8. \int \sin x \, dx = -\cos x, \quad 9. \int \cos x \, dx = \sin x,$$

$$10. \int \tan x \, dx = -\ln |\cos x|, \quad 11. \int \cot x \, dx = \ln |\cos x|,$$

$$12. \int \sec x \, dx = \ln |\sec x + \tan x|, \quad 13. \int \csc x \, dx = \ln |\csc x + \cot x|,$$

$$14. \int \arcsin \frac{x}{a} \, dx = \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0,$$

15. $\int \arccos \frac{x}{a} dx = \arccos \frac{x}{a} - \sqrt{a^2 - x^2}, \quad a > 0,$
16. $\int \arctan \frac{x}{a} dx = x \arctan \frac{x}{a} - \frac{a}{2} \ln(a^2 + x^2), \quad a > 0,$
17. $\int \sin^2(ax) dx = \frac{1}{2a}(ax - \sin(ax) \cos(ax)),$
18. $\int \cos^2(ax) dx = \frac{1}{2a}(ax + \sin(ax) \cos(ax)),$
19. $\int \sec^2 x dx = \tan x,$
20. $\int \csc^2 x dx = -\cot x,$
21. $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx,$
22. $\int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x dx,$
23. $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx, \quad n \neq 1,$
24. $\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx, \quad n \neq 1,$
25. $\int \sec^n x dx = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x dx, \quad n \neq 1,$
26. $\int \csc^n x dx = -\frac{\cot x \csc^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \csc^{n-2} x dx, \quad n \neq 1,$
27. $\int \sinh x dx = \cosh x,$
28. $\int \cosh x dx = \sinh x,$
29. $\int \tanh x dx = \ln |\cosh x|,$
30. $\int \coth x dx = \ln |\sinh x|,$
31. $\int \operatorname{sech} x dx = \arctan \sinh x,$
32. $\int \operatorname{csch} x dx = \ln \left| \tanh \frac{x}{2} \right|,$
33. $\int \sinh^2 x dx = \frac{1}{4} \sinh(2x) - \frac{1}{2} x,$
34. $\int \cosh^2 x dx = \frac{1}{4} \sinh(2x) + \frac{1}{2} x,$
35. $\int \operatorname{sech}^2 x dx = \tanh x,$
36. $\int \operatorname{arcsinh} \frac{x}{a} dx = x \operatorname{arcsinh} \frac{x}{a} - \sqrt{x^2 + a^2}, \quad a > 0,$
37. $\int \operatorname{arctanh} \frac{x}{a} dx = x \operatorname{arctanh} \frac{x}{a} + \frac{a}{2} \ln |a^2 - x^2|,$
38. $\int \operatorname{arccosh} \frac{x}{a} dx = \begin{cases} x \operatorname{arccosh} \frac{x}{a} - \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} > 0 \text{ and } a > 0, \\ x \operatorname{arccosh} \frac{x}{a} + \sqrt{x^2 + a^2}, & \text{if } \operatorname{arccosh} \frac{x}{a} < 0 \text{ and } a > 0, \end{cases}$
39. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left(x + \sqrt{a^2 + x^2} \right), \quad a > 0,$
40. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a}, \quad a > 0,$
41. $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
42. $\int (a^2 - x^2)^{3/2} dx = \frac{x}{8} (5a^2 - 2x^2) \sqrt{a^2 - x^2} + \frac{3a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
43. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a}, \quad a > 0,$
44. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|,$
45. $\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}},$
46. $\int \sqrt{a^2 \pm x^2} dx = \frac{x}{2} \sqrt{a^2 \pm x^2} \pm \frac{a^2}{2} \ln \left| x + \sqrt{a^2 \pm x^2} \right|,$
47. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right|, \quad a > 0,$
48. $\int \frac{dx}{ax^2 + bx} = \frac{1}{a} \ln \left| \frac{x}{a+bx} \right|,$
49. $\int x \sqrt{a+bx} dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$
50. $\int \frac{\sqrt{a+bx}}{x} dx = 2\sqrt{a+bx} + a \int \frac{1}{x\sqrt{a+bx}} dx,$
51. $\int \frac{x}{\sqrt{a+bx}} dx = \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{a+bx} - \sqrt{a}}{\sqrt{a+bx} + \sqrt{a}} \right|, \quad a > 0,$
52. $\int \frac{\sqrt{a^2 - x^2}}{x} dx = \sqrt{a^2 - x^2} - a \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
53. $\int x \sqrt{a^2 - x^2} dx = -\frac{1}{3} (a^2 - x^2)^{3/2},$
54. $\int x^2 \sqrt{a^2 - x^2} dx = \frac{x}{8} (2x^2 - a^2) \sqrt{a^2 - x^2} + \frac{a^4}{8} \arcsin \frac{x}{a}, \quad a > 0,$
55. $\int \frac{dx}{\sqrt{a^2 - x^2}} = -\frac{1}{a} \ln \left| \frac{a + \sqrt{a^2 - x^2}}{x} \right|,$
56. $\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2},$
57. $\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a}, \quad a > 0,$
58. $\int \frac{\sqrt{a^2 + x^2}}{x} dx = \sqrt{a^2 + x^2} - a \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|,$
59. $\int \frac{\sqrt{x^2 - a^2}}{x} dx = \sqrt{x^2 - a^2} - a \arccos \frac{a}{|x|}, \quad a > 0,$
60. $\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} (x^2 \pm a^2)^{3/2},$
61. $\int \frac{dx}{x \sqrt{x^2 + a^2}} = \frac{1}{a} \ln \left| \frac{x}{a + \sqrt{a^2 + x^2}} \right|,$

$$\begin{aligned}
62. \int \frac{dx}{x\sqrt{x^2 - a^2}} &= \frac{1}{a} \arccos \frac{a}{|x|}, \quad a > 0, & 63. \int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} &= \mp \frac{\sqrt{x^2 \pm a^2}}{a^2 x}, \\
64. \int \frac{x dx}{\sqrt{x^2 \pm a^2}} &= \sqrt{x^2 \pm a^2}, & 65. \int \frac{\sqrt{x^2 \pm a^2}}{x^4} dx &= \mp \frac{(x^2 + a^2)^{3/2}}{3a^2 x^3}, \\
66. \int \frac{dx}{ax^2 + bx + c} &= \begin{cases} \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right|, & \text{if } b^2 > 4ac, \\ \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}}, & \text{if } b^2 < 4ac, \end{cases} \\
67. \int \frac{dx}{\sqrt{ax^2 + bx + c}} &= \begin{cases} \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a}\sqrt{ax^2 + bx + c} \right|, & \text{if } a > 0, \\ \frac{1}{\sqrt{-a}} \arcsin \frac{-2ax - b}{\sqrt{b^2 - 4ac}}, & \text{if } a < 0, \end{cases} \\
68. \int \sqrt{ax^2 + bx + c} dx &= \frac{2ax + b}{4a} \sqrt{ax^2 + bx + c} + \frac{4ac - b^2}{8a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
69. \int \frac{x dx}{\sqrt{ax^2 + bx + c}} &= \frac{\sqrt{ax^2 + bx + c}}{a} - \frac{b}{2a} \int \frac{dx}{\sqrt{ax^2 + bx + c}}, \\
70. \int \frac{dx}{x\sqrt{ax^2 + bx + c}} &= \begin{cases} \frac{-1}{\sqrt{c}} \ln \left| \frac{2\sqrt{c}\sqrt{ax^2 + bx + c} + bx + 2c}{x} \right|, & \text{if } c > 0, \\ \frac{1}{\sqrt{-c}} \arcsin \frac{bx + 2c}{|x|\sqrt{b^2 - 4ac}}, & \text{if } c < 0, \end{cases} \\
71. \int x^3 \sqrt{x^2 + a^2} dx &= \left(\frac{1}{3}x^2 - \frac{2}{15}a^2\right)(x^2 + a^2)^{3/2}, \\
72. \int x^n \sin(ax) dx &= -\frac{1}{a}x^n \cos(ax) + \frac{n}{a} \int x^{n-1} \cos(ax) dx, \\
73. \int x^n \cos(ax) dx &= \frac{1}{a}x^n \sin(ax) - \frac{n}{a} \int x^{n-1} \sin(ax) dx, \\
74. \int x^n e^{ax} dx &= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \\
75. \int x^n \ln(ax) dx &= x^{n+1} \left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2} \right), \\
76. \int x^n (\ln ax)^m dx &= \frac{x^{n+1}}{n+1} (\ln ax)^m - \frac{m}{n+1} \int x^n (\ln ax)^{m-1} dx.
\end{aligned}$$

$$\begin{aligned}
x^1 &= x^1 & x^{\bar{1}} &= x^{\bar{1}} \\
x^2 &= x^2 + x^1 & x^{\bar{2}} &= x^{\bar{2}} - x^{\bar{1}} \\
x^3 &= x^3 + 3x^2 + x^1 & x^{\bar{3}} &= x^{\bar{3}} - 3x^{\bar{2}} + x^{\bar{1}} \\
x^4 &= x^4 + 6x^3 + 7x^2 + x^1 & x^{\bar{4}} &= x^{\bar{4}} - 6x^{\bar{3}} + 7x^{\bar{2}} - x^{\bar{1}} \\
x^5 &= x^5 + 15x^4 + 25x^3 + 10x^2 + x^1 & x^{\bar{5}} &= x^{\bar{5}} - 15x^{\bar{4}} + 25x^{\bar{3}} - 10x^{\bar{2}} + x^{\bar{1}} \\
x^{\bar{1}} &= x^1 & x^1 &= x^1 \\
x^{\bar{2}} &= x^2 + x^1 & x^2 &= x^2 - x^1 \\
x^{\bar{3}} &= x^3 + 3x^2 + 2x^1 & x^3 &= x^3 - 3x^2 + 2x^1 \\
x^{\bar{4}} &= x^4 + 6x^3 + 11x^2 + 6x^1 & x^4 &= x^4 - 6x^3 + 11x^2 - 6x^1 \\
x^{\bar{5}} &= x^5 + 10x^4 + 35x^3 + 50x^2 + 24x^1 & x^{\bar{5}} &= x^5 - 10x^4 + 35x^3 - 50x^2 + 24x^1
\end{aligned}$$

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$

$$\mathbb{E} f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x) \delta x = F(x) + C.$$

$$\sum_a^b f(x) \delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$$\Delta(cu) = c\Delta u, \quad \Delta(u+v) = \Delta u + \Delta v,$$

$$\Delta(uv) = u\Delta v + \mathbb{E} v \Delta u,$$

$$\Delta(x^n) = nx^{n-1},$$

$$\Delta(H_x) = x^{-1}, \quad \Delta(2^x) = 2^x,$$

$$\Delta(c^x) = (c-1)c^x, \quad \Delta\binom{x}{m} = \binom{x}{m-1}.$$

Sums:

$$\sum cu \delta x = c \sum u \delta x,$$

$$\sum (u+v) \delta x = \sum u \delta x + \sum v \delta x,$$

$$\sum u \Delta v \delta x = uv - \sum \mathbb{E} v \Delta u \delta x,$$

$$\sum x^n \delta x = \frac{x^{n+1}}{n+1}, \quad \sum x^{-1} \delta x = H_x,$$

$$\sum c^x \delta x = \frac{c^x}{c-1}, \quad \sum \binom{x}{m} \delta x = \binom{x}{m+1}.$$

Falling Factorial Powers:

$$x^{\underline{n}} = x(x-1) \cdots (x-n+1), \quad n > 0,$$

$$x^{\underline{0}} = 1,$$

$$x^{\underline{n}} = \frac{1}{(x+1) \cdots (x+|n|)}, \quad n < 0,$$

$$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$$

Rising Factorial Powers:

$$x^{\overline{n}} = x(x+1) \cdots (x+n-1), \quad n > 0,$$

$$x^{\overline{0}} = 1,$$

$$x^{\overline{n}} = \frac{1}{(x-1) \cdots (x-|n|)}, \quad n < 0,$$

$$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$$

Conversion:

$$x^{\underline{n}} = (-1)^n (-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$$

$$= 1/(x+1)^{-n},$$

$$x^{\overline{n}} = (-1)^n (-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$$

$$= 1/(x-1)^{-n},$$

$$x^n = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^{n-k} x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] x^k.$$

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \dots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{i=0}^{\infty} x^i, \\ \frac{1}{1-cx} &= 1 + cx + c^2x^2 + c^3x^3 + \dots = \sum_{i=0}^{\infty} c^i x^i, \\ \frac{1}{1-x^n} &= 1 + x^n + x^{2n} + x^{3n} + \dots = \sum_{i=0}^{\infty} x^{ni}, \\ \frac{x}{(1-x)^2} &= x + 2x^2 + 3x^3 + 4x^4 + \dots = \sum_{i=0}^{\infty} ix^i, \\ x^k \frac{d^n}{dx^n} \left(\frac{1}{1-x} \right) &= x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \dots = \sum_{i=0}^{\infty} i^n x^i, \\ e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!}, \\ \ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i}, \\ \ln \frac{1}{1-x} &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots = \sum_{i=1}^{\infty} \frac{x^i}{i}, \\ \sin x &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}, \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!}, \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)}, \\ (1+x)^n &= 1 + nx + \frac{n(n-1)}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{n}{i} x^i, \\ \frac{1}{(1-x)^{n+1}} &= 1 + (n+1)x + \binom{n+2}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i, \\ \frac{x}{e^x - 1} &= 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \dots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!}, \\ \frac{1}{2x}(1 - \sqrt{1-4x}) &= 1 + x + 2x^2 + 5x^3 + \dots = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i, \\ \frac{1}{\sqrt{1-4x}} &= 1 + x + 2x^2 + 6x^3 + \dots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i, \\ \frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n &= 1 + (2+n)x + \binom{4+n}{2}x^2 + \dots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i, \\ \frac{1}{1-x} \ln \frac{1}{1-x} &= x + \frac{3}{2}x^2 + \frac{11}{6}x^3 + \frac{25}{12}x^4 + \dots = \sum_{i=1}^{\infty} H_i x^i, \\ \frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2 &= \frac{1}{2}x^2 + \frac{3}{4}x^3 + \frac{11}{24}x^4 + \dots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i}, \\ \frac{x}{1-x-x^2} &= x + x^2 + 2x^3 + 3x^4 + \dots = \sum_{i=0}^{\infty} F_i x^i, \\ \frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} &= F_n x + F_{2n} x^2 + F_{3n} x^3 + \dots = \sum_{i=0}^{\infty} F_{ni} x^i. \end{aligned}$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x) dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

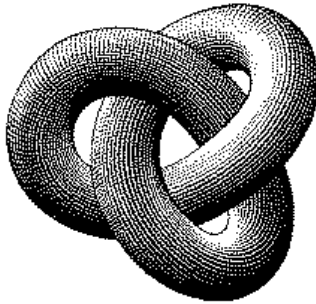
Summation: If $b_i = \sum_{j=0}^i a_j$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Expansions:					
$\frac{1}{(1-x)^{n+1}} \ln \frac{1}{1-x}$	$= \sum_{i=0}^{\infty} (H_{n+i} - H_n) \binom{n+i}{i} x^i,$		$\left(\frac{1}{x}\right)^{\overline{-n}}$	$= \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} x^i,$	
$x^{\overline{n}}$	$= \sum_{i=0}^{\infty} \left[\begin{matrix} n \\ i \end{matrix} \right] x^i,$		$(e^x - 1)^n$	$= \sum_{i=0}^{\infty} \left\{ \begin{matrix} i \\ n \end{matrix} \right\} \frac{n! x^i}{i!},$	
$\left(\ln \frac{1}{1-x}\right)^n$	$= \sum_{i=0}^{\infty} \left[\begin{matrix} i \\ n \end{matrix} \right] \frac{n! x^i}{i!},$		$x \cot x$	$= \sum_{i=0}^{\infty} \frac{(-4)^i B_{2i} x^{2i}}{(2i)!},$	
$\tan x$	$= \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^{2i} (2^{2i} - 1) B_{2i} x^{2i-1}}{(2i)!},$		$\zeta(x)$	$= \sum_{i=1}^{\infty} \frac{1}{i^x},$	
$\frac{1}{\zeta(x)}$	$= \sum_{i=1}^{\infty} \frac{\mu(i)}{i^x},$	$\frac{\zeta(x-1)}{\zeta(x)}$	$= \sum_{i=1}^{\infty} \frac{\phi(i)}{i^x},$		
$\zeta(x)$	$= \prod_p \frac{1}{1 - p^{-x}},$				
$\zeta^2(x)$	$= \sum_{i=1}^{\infty} \frac{d(i)}{x^i} \quad \text{where } d(n) = \sum_{d n} 1,$	Stieltjes Integration			
$\zeta(x)\zeta(x-1)$	$= \sum_{i=1}^{\infty} \frac{S(i)}{x^i} \quad \text{where } S(n) = \sum_{d n} d,$	If G is continuous in the interval $[a, b]$ and F is nondecreasing then			
$\zeta(2n)$	$= \frac{2^{2n-1} B_{2n} }{(2n)!} \pi^{2n}, \quad n \in \mathbb{N},$	$\int_a^b G(x) dF(x)$			
$\frac{x}{\sin x}$	$= \sum_{i=0}^{\infty} (-1)^{i-1} \frac{(4^i - 2) B_{2i} x^{2i}}{(2i)!},$	exists. If $a \leq b \leq c$ then			
$\left(\frac{1 - \sqrt{1-4x}}{2x}\right)^n$	$= \sum_{i=0}^{\infty} \frac{n(2i+n-1)!}{i!(n+i)!} x^i,$	$\int_a^c G(x) dF(x) = \int_a^b G(x) dF(x) + \int_b^c G(x) dF(x).$			
$e^x \sin x$	$= \sum_{i=1}^{\infty} \frac{2^{i/2} \sin \frac{i\pi}{4}}{i!} x^i,$	If the integrals involved exist			
$\sqrt{\frac{1 - \sqrt{1-x}}{x}}$	$= \sum_{i=0}^{\infty} \frac{(4i)!}{16^i \sqrt{2} (2i)! (2i+1)!} x^i,$	$\int_a^b (G(x) + H(x)) dF(x) = \int_a^b G(x) dF(x) + \int_a^b H(x) dF(x),$			
$\left(\frac{\arcsin x}{x}\right)^2$	$= \sum_{i=0}^{\infty} \frac{4^i i!^2}{(i+1)(2i+1)!} x^{2i}.$	$\int_a^b G(x) d(F(x) + H(x)) = \int_a^b G(x) dF(x) + \int_a^b G(x) dH(x),$			
		$\int_a^b c \cdot G(x) dF(x) = \int_a^b G(x) d(c \cdot F(x)) = c \int_a^b G(x) dF(x),$			
		$\int_a^b G(x) dF(x) = G(b)F(b) - G(a)F(a) - \int_a^b F(x) dG(x).$			
		If the integrals involved exist, and F possesses a derivative F' at every point in $[a, b]$ then			
		$\int_a^b G(x) dF(x) = \int_a^b G(x) F'(x) dx.$			
Cramer's Rule				Fibonacci Numbers	
If we have equations:		00 47 18 76 29 93 85 34 61 52 86 11 57 28 70 39 94 45 02 63 95 80 22 67 38 71 49 56 13 04 59 96 81 33 07 48 72 60 24 15 73 69 90 82 44 17 58 01 35 26 68 74 09 91 83 55 27 12 46 30 37 08 75 19 92 84 66 23 50 41 14 25 36 40 51 62 03 77 88 99 21 32 43 54 65 06 10 89 97 78 42 53 64 05 16 20 31 98 79 87		1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...	
$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1$				Definitions:	
$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2$				$F_i = F_{i-1} + F_{i-2}, \quad F_0 = F_1 = 1,$	
\vdots				$F_{-i} = (-1)^{i-1} F_i,$	
$a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n$				$F_i = \frac{1}{\sqrt{5}} \left(\phi^i - \hat{\phi}^i \right),$	
Let $A = (a_{i,j})$ and B be the column matrix (b_i) . Then there is a unique solution iff $\det A \neq 0$. Let A_i be A with column i replaced by B . Then				Cassini's identity: for $i > 0$:	
$x_i = \frac{\det A_i}{\det A}.$				$F_{i+1}F_{i-1} - F_i^2 = (-1)^i.$	
Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius. – William Blake (The Marriage of Heaven and Hell)		The Fibonacci number system: Every integer n has a unique representation $n = F_{k_1} + F_{k_2} + \cdots + F_{k_m},$ where $k_i \geq k_{i+1} + 2$ for all $i,$ $1 \leq i < m$ and $k_m \geq 2.$		Additive rule: $F_{n+k} = F_k F_{n+1} + F_{k-1} F_n,$ $F_{2n} = F_n F_{n+1} + F_{n-1} F_n.$ Calculation by matrices: $\begin{pmatrix} F_{n-2} & F_{n-1} \\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n.$	

