Solution:

1(a) Domain : (-inf ,-2)U(2,inf)

Range : [0,inf)

1(b) f(x)=f(-x) Its even function

1(c)

$$m_{\text{sec}} = \frac{f(2) - f(1)}{2 - 1} = \frac{1/4 - 1}{1} = -\frac{3}{4}$$

1(d)

$$\Delta x = \frac{5}{n},\ x_k^* = \frac{5}{n}(k-1);$$

$$f(x_k^*)\Delta x = (5 - x_k^*)\Delta x = \left[5 - \frac{5}{n}(k-1)\right]\frac{5}{n} = \frac{25}{n} - \frac{25}{n^2}(k-1),$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{25}{n}\sum_{k=1}^{n} 1 - \frac{25}{n^2}\sum_{k=1}^{n}(k-1) = 25 - \frac{25}{2}\frac{n-1}{n},$$

$$A = \lim_{n \to +\infty}\left[25 - \frac{25}{2}\left(1 - \frac{1}{n}\right)\right] = 25 - \frac{25}{2} = \frac{25}{2}.$$

Question 2 Estimated Time: 30 minutes Marks (5+7+3)

a) Write Increasing and decreasing interval of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

b) Discuss Concavity , Inflection point and extreme values for the curve $f(x) = x^4 - 4x^3$

c) Find the absolute maxima and minima values of $f(x) = x^3 - 3x^2 + 1$, $[\frac{-1}{2}, 4]$

Solution a)

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x - 2)(x + 1)$$

divide the real line into intervals whose endpoints are the critical numbers -1, 0, and 2

Interval	$12x$	$x - 2$	$x + 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	$+$	increasing on $(2, \infty)$

Solution:

Q2(b) If $f(x) = x^4 - 4x^3$, then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 3$.

$$f''(0) = 0 \qquad f''(3) = 36 > 0$$

Since $f'(3) = 0$ and $f''(3) > 0$, $f(3) = -27$ is a local minimum.

First Derivative Test tells us that f does not have a local maximum or minimum at 0.

Since $f''(x) = 0$ when $x = 0$ or 2,

Interval	$f''(x) = 12x(x - 2)$	Concavity
$(-\infty, 0)$	$+$	upward
$(0, 2)$	$-$	downward
$(2, \infty)$	$+$	upward

$(2, -16)(0, 0)$ is an inflection point

Solution:

Q2(c)

Since f is continuous on $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method:

$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x = 3x(x - 2)$$

The values of f at these critical numbers are

$$f(0) = 1 \qquad f(2) = -3$$

The values of f at the endpoints of the interval are

$$f\left(-\tfrac{1}{2}\right) = \tfrac{1}{8} \qquad f(4) = 17$$

we see that the absolute maximum value is $f(4) = 17$

and the absolute minimum value is $f(2) = -3$.

Question 3 **Estimated Time: 25 minutes** **Marks (15)**

a) Find area of region enclosed by the curves $x^2 = y$ and $y = 2x - x^2$

b) Find the volume of the solid using washer method when the region enclosed by given curves $y = x^2$, $y = x$ is revolved about the x-axis.

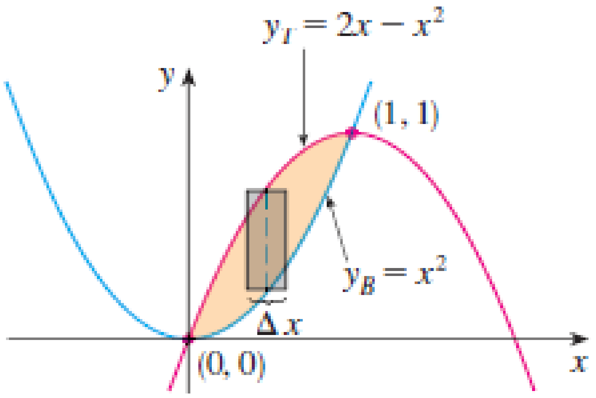
c) Determine the arc length of parabola $y^2 = x$ $from$ $(0,0)$ to $(1,1)$

Solution: a)

$x^2 = 2x - x^2$, or $2x^2 - 2x = 0$. Thus $2x(x - 1) = 0$,

The points of intersection are $(0, 0)$ and $(1, 1)$.

and the region lies between $x = 0$ and $x = 1$. So the total area is



$$A = \int_0^1 (2x - 2x^2)\,dx = 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

Solution: 3(b)

The curves $y = x$ and $y = x^2$ intersect at the points (0, 0) and (1, 1).

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15} \quad = 0.419$$

Solution : c)

Since $x = y^2$, we have $dx/dy = 2y$,

$$L = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

We make the trigonometric substitution $y = \frac{1}{2}\tan\theta$, which gives $dy = \frac{1}{2}\sec^2\theta\, d\theta$ and $\sqrt{1 + 4y^2} = \sqrt{1 + \tan^2\theta} = \sec\theta$. When $y = 0$, $\tan\theta = 0$, so $\theta = 0$; when $y = 1$, $\tan\theta = 2$, so $\theta = \tan^{-1} 2 = \alpha$, say. Thus

$$L = \int_0^\alpha \sec\theta \cdot \tfrac{1}{2}\sec^2\theta\, d\theta = \tfrac{1}{2}\int_0^\alpha \sec^3\theta\, d\theta$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^\alpha$$

$$= \tfrac{1}{4}\left(\sec\alpha\tan\alpha + \ln|\sec\alpha + \tan\alpha|\right)$$

$$L = \frac{\sqrt{5}}{2} + \frac{\ln\left(\sqrt{5} + 2\right)}{4} = 1.479$$

Q4 Solution: (a)

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

Just Put n=6 in reduction formula , no need to proof

Solution: b)

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du$$

$$\int \frac{dx}{1+\sin x + \cos x} = \int \frac{\dfrac{2\,du}{1+u^2}}{1+\left(\dfrac{2u}{1+u^2}\right)+\left(\dfrac{1-u^2}{1+u^2}\right)}$$

$$= \int \frac{2\,du}{(1+u^2)+2u+(1-u^2)}$$

$$= \int \frac{du}{1+u} = \ln|1+u| + C = \ln|1+\tan(x/2)| + C$$

Q5 Solution: a)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+1/x^2}}(-1/x^2) - 3\cosh(\cos 3x)\sin 3x.$$

Solution: b (II)

$$\int_0^{+\infty} x\, e^{-x^2} dx = \lim_{\ell\to+\infty} -\frac{1}{2}e^{-x^2}\Big]_0^{\ell} = \lim_{\ell\to+\infty} \frac{1}{2}\left(-e^{-\ell^2}+1\right) = 1/2.$$

Solution: b (i)

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+} \int_t^5 \frac{dx}{\sqrt{x-2}} \\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5 \\ &= \lim_{t\to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\,\right) \\ &= 2\sqrt{3} \quad = 3.464\end{aligned}$$

Q6 Solution: a)

$$\begin{array}{ccc}
\text{diff.} & & \text{antidiff.} \\
\hline\hline
x^3 & & \sqrt{2x+1} \\
 & \searrow + & \\
3x^2 & & \frac{1}{3}(2x+1)^{3/2} \\
 & \searrow - & \\
6x & & \frac{1}{15}(2x+1)^{5/2} \\
 & \searrow + & \\
6 & & \frac{1}{105}(2x+1)^{7/2} \\
 & \searrow - & \\
0 & & \frac{1}{945}(2x+1)^{9/2}
\end{array}$$

$$\int x^3\sqrt{2x+1}\,dx = \frac{1}{3}x^3(2x+1)^{3/2} - \frac{1}{5}x^2(2x+1)^{5/2} + \frac{2}{35}x(2x+1)^{7/2} - \frac{2}{315}(2x+1)^{9/2} + C.$$

Solution : b)

Let $x = 3\sin\theta$, where $-\pi/2 \leqslant \theta \leqslant \pi/2$. Then $dx = 3\cos\theta\, d\theta$ and

$$\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3|\cos\theta| = 3\cos\theta$$

$$\begin{aligned}
\int \frac{\sqrt{9 - x^2}}{x^2}\, dx &= \int \frac{3\cos\theta}{9\sin^2\theta}\, 3\cos\theta\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot^2\theta\, d\theta \\
&= \int (\csc^2\theta - 1)\, d\theta \\
&= -\cot\theta - \theta + C
\end{aligned}$$

$$\cot\theta = \frac{\sqrt{9 - x^2}}{x}$$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx = -\frac{\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$$

Solution: c)

Since $x^3 + 4x = x(x^2 + 4)$ can't be factored further, we write

$$\frac{2x^2 - x + 4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A \end{aligned}$$

$$A + B = 2 \qquad C = -1 \qquad 4A = 4$$

Thus $A = 1$, $B = 1$, and $C = -1$ and so

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx$$

Solution: c)

$$\int \frac{x-1}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$\int \frac{2x^2-x+4}{x(x^2+4)}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}(x/2) + K$$

Solution: d)

$$3 - 2x - x^2 = 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1)$$
$$= 4 - (x + 1)^2$$

we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{u - 1}{\sqrt{4 - u^2}}\,du$$

now substitute $u = 2\sin\theta$, giving $du = 2\cos\theta\,d\theta$ and $\sqrt{4 - u^2} = 2\cos\theta$,

$$\int \frac{x}{\sqrt{3 - 2x - x^2}}\,dx = \int \frac{2\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta$$
$$= \int (2\sin\theta - 1)\,d\theta$$
$$= -2\cos\theta - \theta + C$$
$$= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C$$
$$= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C$$

Q7 Solution: a)

The vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\mathbf{a} = \langle 2, -4, 4 \rangle \qquad \mathbf{b} = \langle 4, -1, -2 \rangle$$

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\,\mathbf{i} + 20\,\mathbf{j} + 14\,\mathbf{k}$$

point $P(1, 3, 2)$ and the normal vector $\mathbf{n}$, an equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

$$6x + 10y + 7z = 50$$

Solution: b)

vector to both planes must be orthogonal to both $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ (the direction of L_1) and $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$ (the direction of L_2). So a normal vector is

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -1 \\ 2 & 1 & 4 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}$$

If we put $s = 0$ in the equations of L_2, we get the point $(0, 3, -3)$ on L_2 and so an equation for P_2 is

$$13(x - 0) - 6(y - 3) - 5(z + 3) = 0 \qquad \text{or} \qquad 13x - 6y - 5z + 3 = 0$$

If we now set $t = 0$ in the equations for L_1, we get the point $(1, -2, 4)$ on P_1. So the distance between L_1 and L_2 is the same as the distance from $(1, -2, 4)$ to $13x - 6y - 5z + 3 = 0$. By Formula 9, this distance is

$$D = \frac{|13(1) - 6(-2) - 5(4) + 3|}{\sqrt{13^2 + (-6)^2 + (-5)^2}} = \frac{8}{\sqrt{230}} \approx 0.53$$

Solution: 8 a)

The normal vectors of these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \qquad \mathbf{n}_2 = \langle 1, -2, 3 \rangle$$

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1(1) + 1(-2) + 1(3)}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{42}}\right) \approx 72^\circ$$

Solution: 8b)

$$\overrightarrow{PQ} = (-2 - 1)\,\mathbf{i} + (5 - 4)\,\mathbf{j} + (-1 - 6)\,\mathbf{k} = -3\mathbf{i} + \mathbf{j} - 7\mathbf{k}$$

$$\overrightarrow{PR} = (1 - 1)\,\mathbf{i} + (-1 - 4)\,\mathbf{j} + (1 - 6)\,\mathbf{k} = -5\mathbf{j} - 5\mathbf{k}$$

We compute the cross product of these vectors:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix}$$

$$= (-5 - 35)\,\mathbf{i} - (15 - 0)\,\mathbf{j} + (15 - 0)\,\mathbf{k} = -40\,\mathbf{i} - 15\,\mathbf{j} + 15\,\mathbf{k}$$

So the vector $\langle -40, -15, 15 \rangle$ is perpendicular to the given plane.

we computed that $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -40, -15, 15 \rangle$.

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{(-40)^2 + (-15)^2 + 15^2} = 5\sqrt{82}$$

The area of th parallelogram with adjacent sides PQ and PR

The area A of the triangle PQR is half the area of this parallelogram, that is, $\frac{5}{2}\sqrt{82}$.