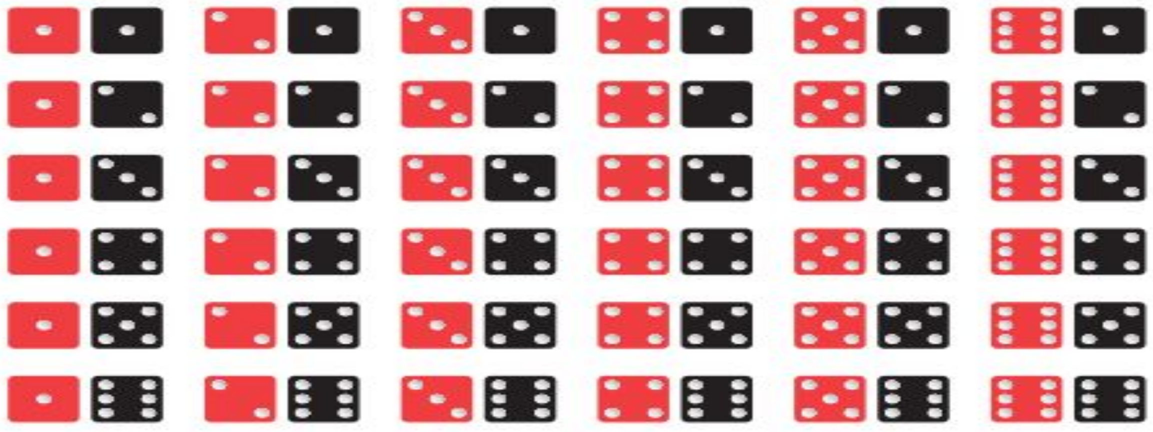

Introduction to Probability

Instructor

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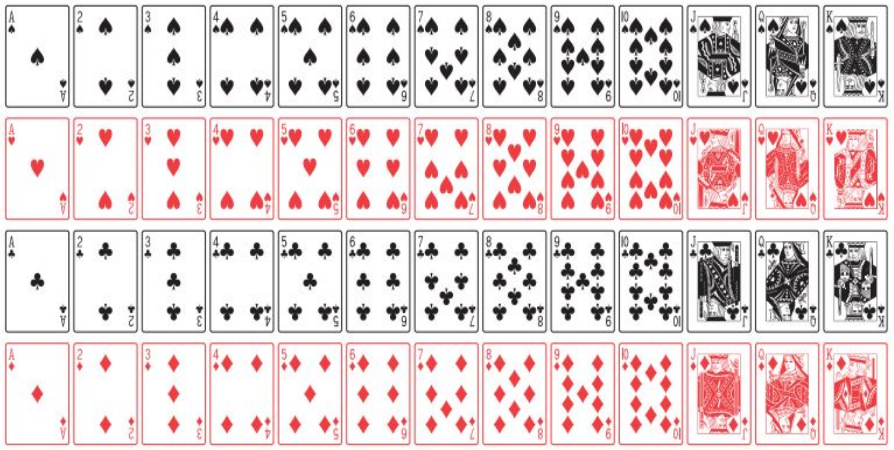


Content

- Sample Space and Event
- Tree diagram
- Set theory
- Venn diagram
- Counting techniques
- Additive and multiplicative rules for probability
- Conditional probability
- Bayes' Theorem

Sample Space

- What is an experiment?
- Any process or activity that generates a set of data is called experiment. For example:
  1. Tossing a coin
  2. Rolling dice
  3. Playing cards
  4. Opinion of voters
  5. Launching of missiles

Sample Space

- The set of all possible outcomes of a statistical experiment is called the **sample space** (S). For example:

Tree Diagram

- Tossing coin: 2 times, 3 times, 4 times
- Tossing die & coin together:
- Suppose that three items are selected at random from a manufacturing process. Each item is inspected and classified defective, *D*, or non-defective, *N*. List the elements of the sample space.

Events

- An **event** is a subset of a sample space. For example:

- The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

-

Intersection of Events

- The **intersection** of two events *A* and *B*, denoted by the symbol $A \cap B$, is the event containing all elements that are common to *A* and *B*.
  - Let *E* be the event that a person selected at random in a classroom is majoring in engineering, and let *F* be the event that the person is female. Then $E \cap F$ is the event of all female engineering students in the classroom.

  - Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$; then it follows that $V \cap C = \varphi$. That is, *V* and *C* have no elements in common and, therefore, cannot both simultaneously occur.

Mutually Exclusive events

- Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \varphi$, that is, if A and B have no elements in common.

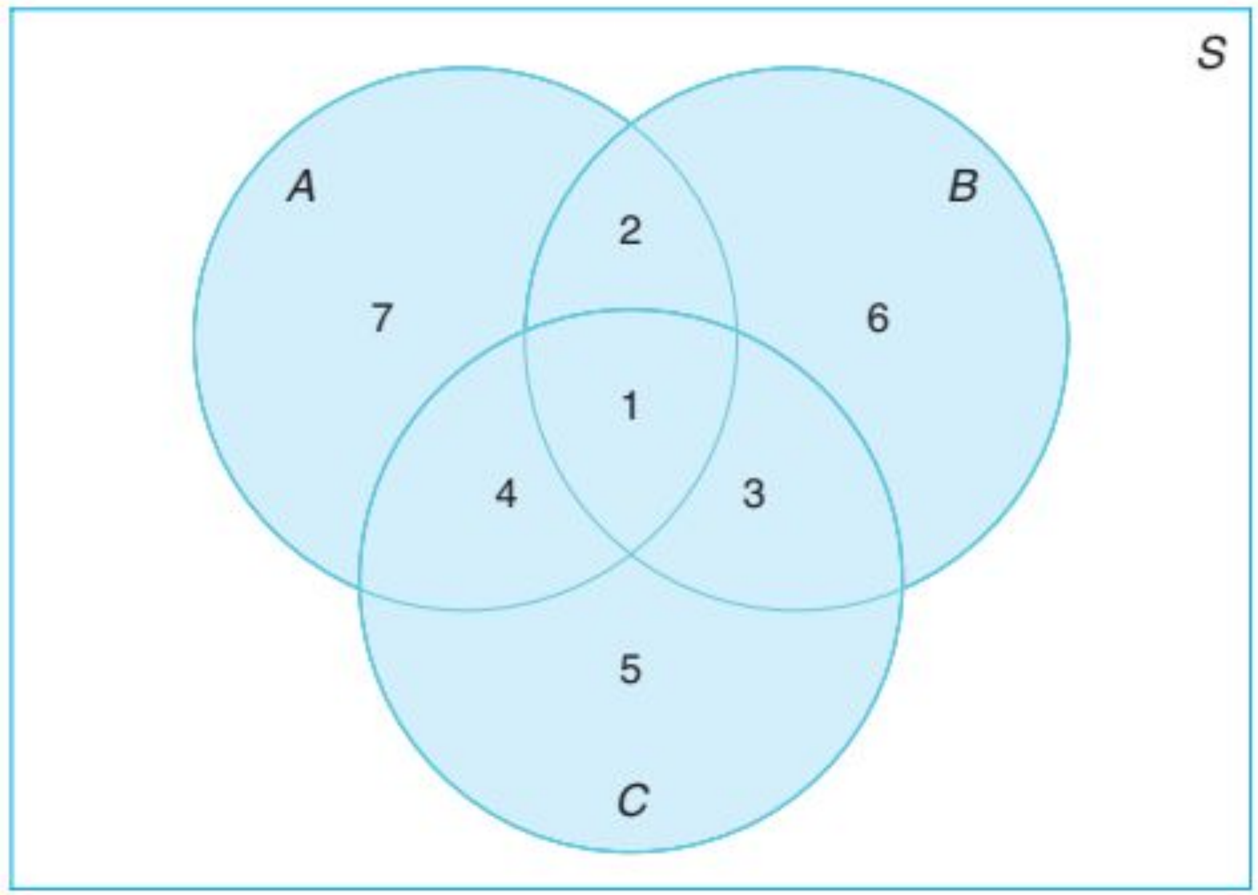
Union of events

- The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

  - Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$; then $A \cup B$ = ?

  - Let P be the event that an employee selected at random from an oil drilling company smokes cigarettes. Let Q be the event that the employee selected drinks alcoholic beverages. Then the event $P \cup Q$ $=?$

Venn Diagram



$A \cap B =$ 1, 2

$B \cap C =$ 1, 3

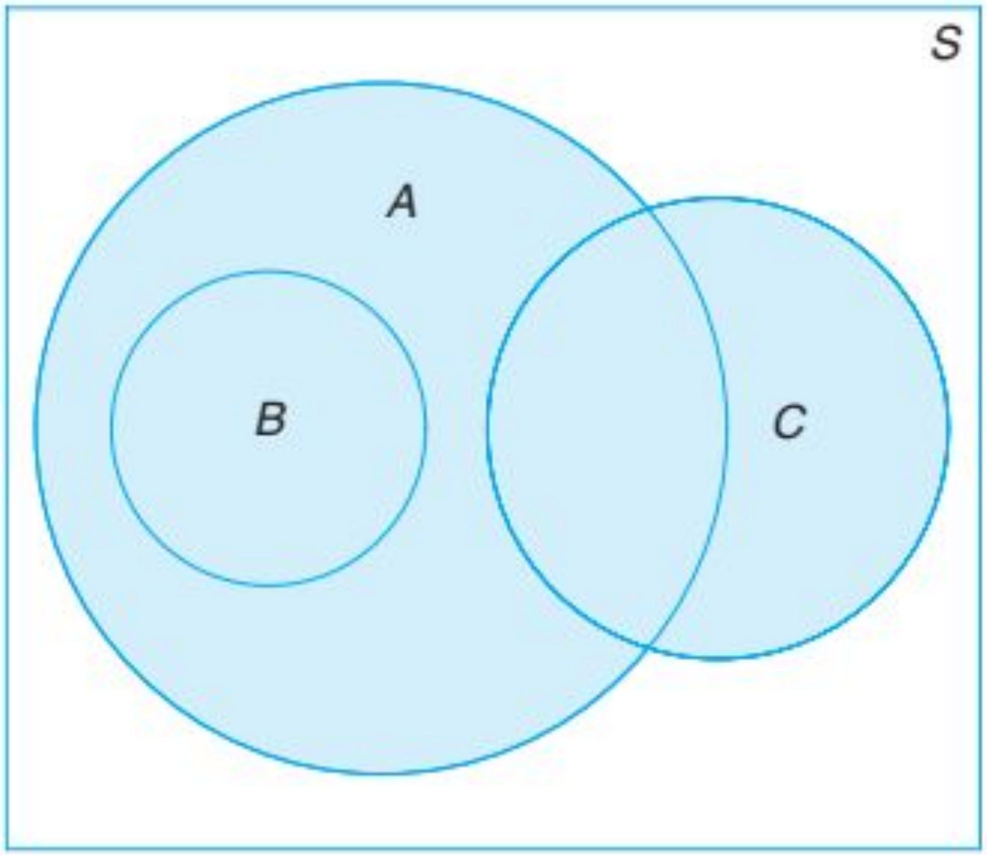
$A \cup C =$ 1,2,3,4,5,7

$B' \cap A =$ 4, 7

$A \cap B \cap C =$ 1

$(A \cup B) \cap C' =$7,2,6,

Venn Diagram



- $B \cap C =$ Null Set
- $A \cup B =$ A
- $A \cap C =$ At least one element

Venn Diagram

- Several results that follow from the foregoing definitions, which may easily be verified by means of Venn diagrams, are as follows:

1. $A \cap \phi = \phi$.
2. $A \cup \phi = A$.
3. $A \cap A' = \phi$.
4. $A \cup A' = S$.
5. $S' = \phi$.
6. $\phi' = S$.
7. $(A')' = A$.
8. $(A \cap B)' = A' \cup B'$.
9. $(A \cup B)' = A' \cap B'$.

2.1 List the elements of each of the following sample spaces:

(a) the set of integers between 1 and 50 divisible by 8;

(b) the set $S = \{x \mid x^2 + 4x - 5 = 0\}$;

(c) the set of outcomes when a coin is tossed until a tail or three heads appear;

(d) the set $S = \{x \mid x \text{ is a continent}\}$;

(e) the set $S = \{x \mid 2x - 4 \geq 0 \text{ and } x < 1\}$.

2.7 Four students are selected at random from a chemistry class and classified as male or female. List the elements of the sample space S_1, using the letter M for male and F for female. Define a second sample space S_2 where the elements represent the number of females selected.

Multiplication Or Fundamental Rule of counting

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdots n_k$ ways.

i. How many sample points are there in the sample space when a pair of dice is thrown once?

ii. How many there digit numbers can be formed from the digits 2, 4, 6, and 8 if: (i) repetitions are not allowed (ii) repetitions allowed

Set of Example (iii - viii)

(iii) A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?

(iv) If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

(v) Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

(vi) How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

(vii) How many new arrangements can be made from the letters of the word **FAVOUR** so that vowel occupy even place.

Permutation

- A **permutation** is an arrangement of all or part of a set of objects.
- The number of permutations of n objects is n!.
- Suppose you have to arrange 3 books: **Statistics**, **Maths**, **Physics** on a shelf. How many arrangements are possible?

Permutations of “n” objects taken “r” at a time

$$ {}_nP_r = \frac{n!}{(n-r)!}. $$

- In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Example # 10:

- A president and a treasurer are to be chosen from a student club consisting of 50 people. How many different choices of officers are possible if

(a) there are no restrictions;

(b) *A* will serve only if he is president;

(c) *B* and *C* will serve together or not at all;

(d) *D* and *E* will not serve together?

Circular Permutations

- The number of permutations of n objects arrange in a circle is (n –1)!.

2.43 In how many ways can 5 different trees be planted in a circle?

Permutations of n objects when they are not all different.

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, $\ldots$, n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

- Find the number of permutations of 9995
- In how many ways can the letters of the word STATISTICS be arranged?
- In how many ways can 2 red, 3 blue, and 4 green chips be arranged in a row, if the chips of same color are not distinguishable from each other?

- In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Combinations

- Selection of “r” objects from “n” different objects and when the order is not important.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

- In how many ways a committee of 3 students can be selected from 4 students.
- From a group of 10 boys and 6 girls a committee of 3 boys and 2 girls are to be selected. In how many ways can this done?

Exercises

2.22 In a medical study, patients are classified in 8 ways according to whether they have blood type AB^+, AB^-, A^+, A^-, B^+, B^-, O^+, or O^-, and also according to whether their blood pressure is low, normal, or high. Find the number of ways in which a patient can be classified.

2.33 If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,

(a) in how many different ways can a student check off one answer to each question?

(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

2.37 In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

O

2.45 How many distinct permutations can be made from the letters of the word $INFINITY$?

2.47 How many ways are there to select 3 candidates from 8 equally qualified recent graduates for openings in an accounting firm?

2.48 How many ways are there that no two students will have the same birth date in a class of size 60?

Probability

- Probability is a measure of the chance that an uncertain event will occur.

Subjective:

- Personal experiences

Objective:

- Classical approach
- Relative frequency approach
- Axiomatic approach

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \leq P(A) \leq 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if A_1, A_2, A_3, $\ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

Examples (1 – 3)

1. A coin is tossed twice. What is the probability that at least 1 head occurs?
2. A die is tossed once. What is the probability of getting:

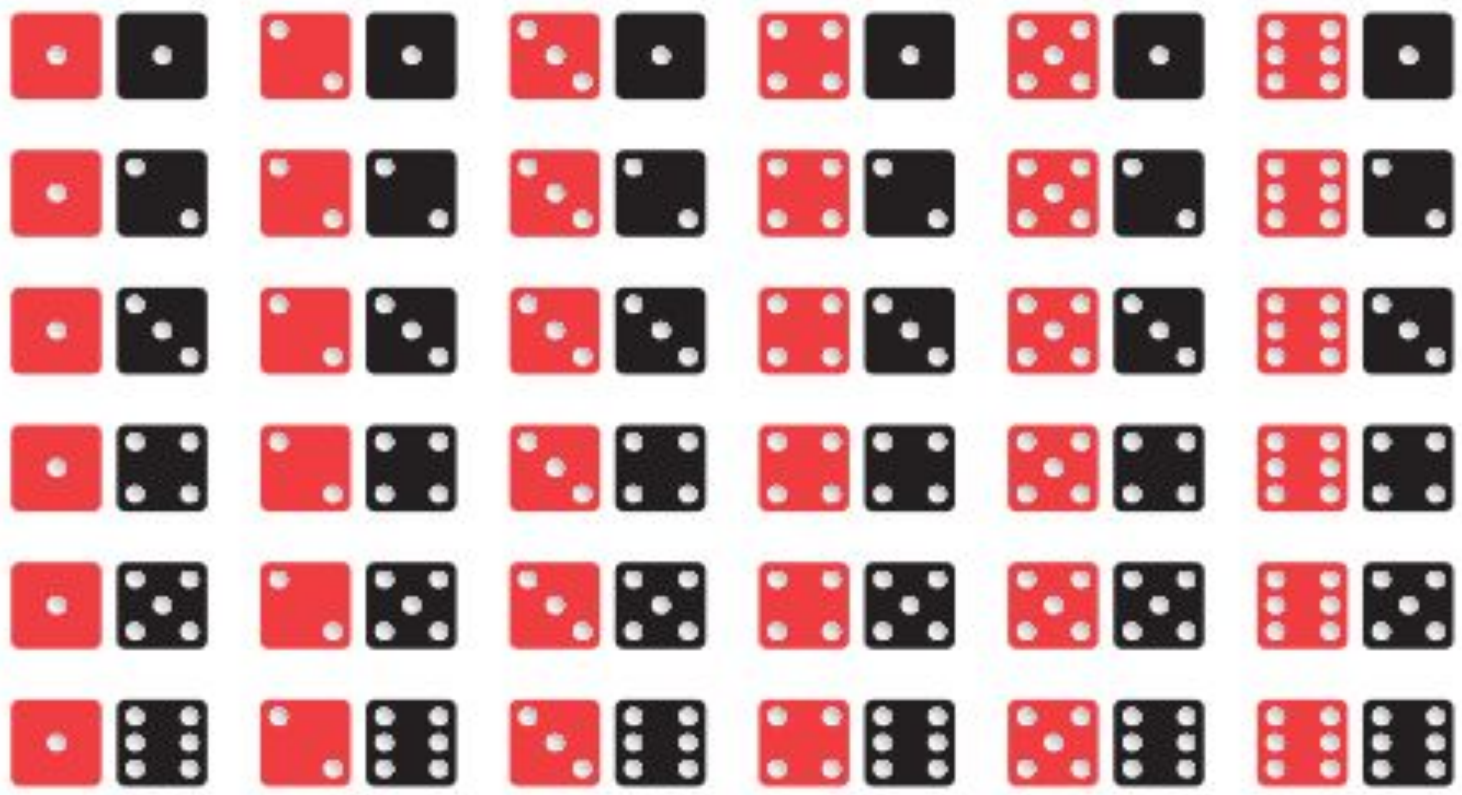
(a) an even number **(b)** a number less than 3 **(c)** a 4 or higher number **(d)** a 7 **(e)** A number from 1 to 6

3. A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

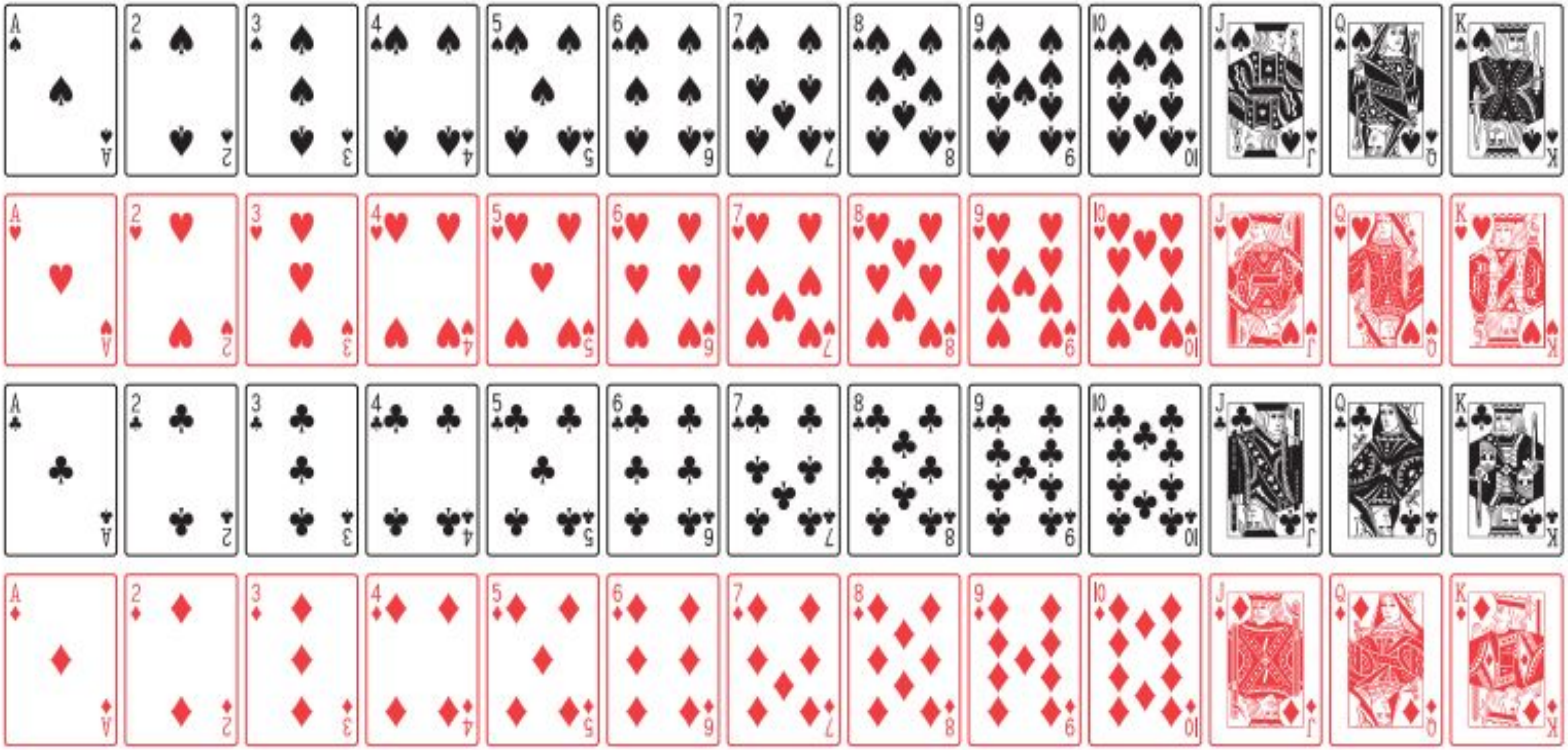
Example 4

- Two balanced dice are rolled once. What is the probability of getting

(a) A sum of 11 (b) same number on both dice (c) a sum of 13



A deck of playing Cards



Example 5 – 6

5. A card is drawn at random from the well shuffled pack of 52 playing cards. Find the probability that the card:

(a) is a Jack **(b)** is not a Jack

6. In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

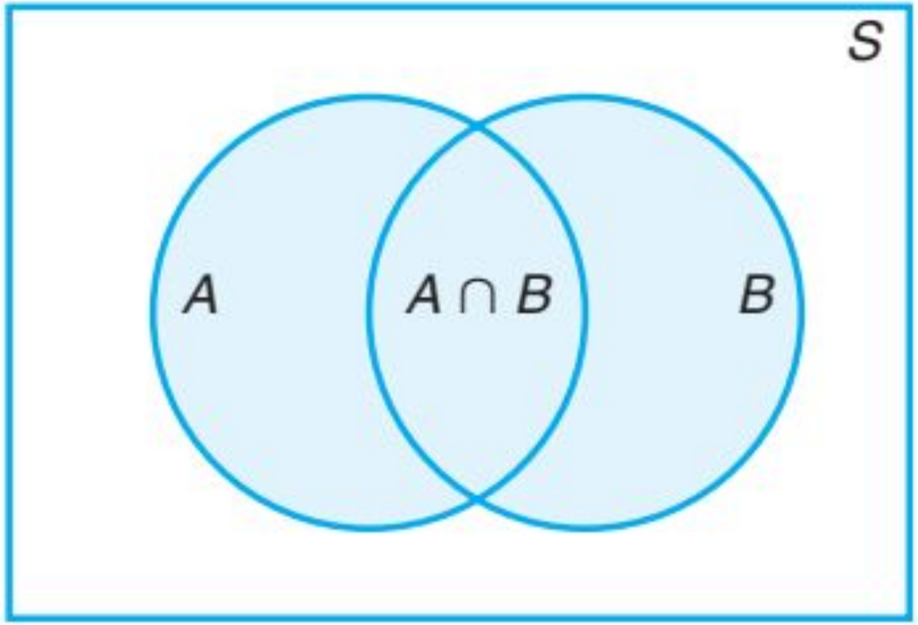
Example 7

- A statistics class for engineers consists of 25 industrial, 10 mechanical, 10 electrical, and 8 civil engineering students. If a person is randomly selected by the instructor to answer a question, find the probability that the student chosen is (a) an industrial engineering major and (b) a civil engineering or an electrical engineering major.

Additive Rule: Not - Mutually Exclusive Events

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$



For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Additive Rule: Mutually Exclusive Events

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Example # 08 – 10

8. John is going to graduate from an industrial engineering department in a university by the end of the semester. After being interviewed at two companies he likes, he assesses that his probability of getting an offer from company *A* is 0.8, and his probability of getting an offer from company *B* is 0.6. If he believes that the probability that he will get offers from both companies is 0.5, what is the probability that he will get at least one offer from these two companies?

9. What is the probability of getting a total of 7 or 11 when a pair of fair dice is tossed?

10. If the probabilities are, respectively, 0.09, 0.15, 0.21, and 0.23 that a person purchasing a new automobile will choose the color green, white, red, or blue, what is the probability that a given buyer will purchase a new automobile that comes in one of those colors?

Examples (11 – 12)

11. If the probabilities that an automobile mechanic will service 3, 4, 5, 6, 7, or 8 or more cars on any given workday are, respectively, 0.12, 0.19, 0.28, 0.24, 0.10, and 0.07, what is the probability that he will service at least 5 cars on his next day at work?

12. Suppose the manufacturer's specifications for the length of a certain type of computer cable are 2000 ± 10 millimeters. In this industry, it is known that small cable is just as likely to be defective (not meeting specifications) as large cable. That is, the probability of randomly producing a cable with length exceeding 2010 millimeters is equal to the probability of producing a cable with length smaller than 1990 millimeters. The probability that the production procedure meets specifications is known to be 0.99.

(a) What is the probability that a cable selected randomly is too large?

(b) What is the probability that a randomly selected cable is larger than 1990 millimeters?

Exercises

2.49 Find the errors in each of the following statements:

(a) The probabilities that an automobile salesperson will sell 0, 1, 2, or 3 cars on any given day in February are, respectively, 0.19, 0.38, 0.29, and 0.15.

(b) The probability that it will rain tomorrow is 0.40, and the probability that it will not rain tomorrow is 0.52.

(c) The probabilities that a printer will make 0, 1, 2, 3, or 4 or more mistakes in setting a document are, respectively, $0.19, 0.34, -0.25, 0.43$, and 0.29.

(d) On a single draw from a deck of playing cards, the probability of selecting a heart is $1/4$, the probability of selecting a black card is $1/2$, and the probability of selecting both a heart and a black card is $1/8$.

2.51 A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

2.53 The probability that an American industry will locate in Shanghai, China, is 0.7, the probability that it will locate in Beijing, China, is 0.4, and the probability that it will locate in either Shanghai or Beijing or both is 0.8. What is the probability that the industry will locate

(a) in both cities?

(b) in neither city?

2.56 An automobile manufacturer is concerned about a possible recall of its best-selling four-door sedan. If there were a recall, there is a probability of 0.25 of a defect in the brake system, 0.18 of a defect in the transmission, 0.17 of a defect in the fuel system, and 0.40 of a defect in some other area.

(a) What is the probability that the defect is the brakes or the fueling system if the probability of defects in both systems simultaneously is 0.15?

(b) What is the probability that there are no defects in either the brakes or the fueling system?

The Product Rule: Independent Events

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Some important results for Independent events

- (i) A’ and B are independent = $P(A' \cap B) = P(A').P(B)$
- (ii) A and B’ are independent = $P(A \cap B') = P(A).P(B')$
- (iii) A’ and B’ are independent = $P(A' \cap B') = P(A').P(B')$
- If A and B are independent then they are not mutually exclusive.
- If A, B, and C are independent, then $P(A \cap B \cap C)' = 1 - P(A).P(B).P(C)$
- If A, B, and C are independent, then $P(A \cup B \cup C) = 1 – P(A' \cap B' \cap C')$

Relationship among events

(not E): The event "E does not occur"

(A & B): The event "both A and B occur"

(A or B): The event "either A or B or both occur"

Examples # 13 – 14

- A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.
- A bag contains 5 red and 7 black balls. A ball is drawn at random from the bag, the color is noted and the ball is replaced. A second balls is then drawn. Find the probability that the first balls is red and the second is black.

Examples 15 – 17

(15). A die is rolled two times. Find the probability of obtaining a 5 on the first thrown and an even number on the second thrown.

(16). The probability that Ahsan will be alive in 30 years is 0.4 and the probability that Bilawal will be alive in 30 years is 0.8. What is the probability that: **(a)** both will be alive in 30 years

(b) both of them die **(c)** Ahsan will be alive and B dead.

(17). A town has two fire engines operating independently. The probability that a specific engine is available when needed is

0.96.

(a) What is the probability that neither is available when needed? **(b)** What is the probability that a fire engine is available when needed?

The Product Rule: Dependent Events

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

The conditional probability of B, given A, denoted by $P(B|A)$, is defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad \text{provided} \quad P(A) > 0.$$

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Example # 18

(18). Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

(19). Two cards are drawn in succession from a deck of 52 playing cards without replacement. What is the probability that both cards a are spades.

(20). A box contains 8 tickets bearing the numbers 1, 2, 3, 4, 5, 6, 8, 10. One ticket is drawn and kept aside. Then a second ticket is drawn. What is the probability that both the tickets show even numbers.

Example # 21

(21). In a certain college 25% of the students passed Mathematics, 15% of the students passed statistics and 10% of the students passed both mathematics and Statistics. A students is selected at random.

(a) if he passed statistics, what is the probability that he passed mathematics.

(b) if he passed mathematics, what is the probability that he passed statistics.

(22). Suppose a pair of dice is tossed once. If it is known that one die shows a 3. what is the probability that other die shows a 6.

Example # 22

2.76 In an experiment to study the relationship of hypertension and smoking habits, the following data are collected for 180 individuals:

	Nonsmokers	Moderate Smokers	Heavy Smokers
H	21	36	30
NH	48	26	19

where H and NH in the table stand for *Hypertension* and *Nonhypertension*, respectively. If one of these individuals is selected at random, find the probability that the person is

(a) experiencing hypertension, given that the person is a heavy smoker;

(b) a nonsmoker, given that the person is experiencing no hypertension.

Example # 23

(2.91) Find the probability of randomly selecting 4 good quarts of milk in succession from a cooler containing 20 quarts of which 5 have spoiled, by using

(a) the first formula of Theorem 2.12 on page 68.

(b) the formulas of Theorem 2.6 and Rule 2.3 on pages 50 and 54, respectively.

Theorem 2.12

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

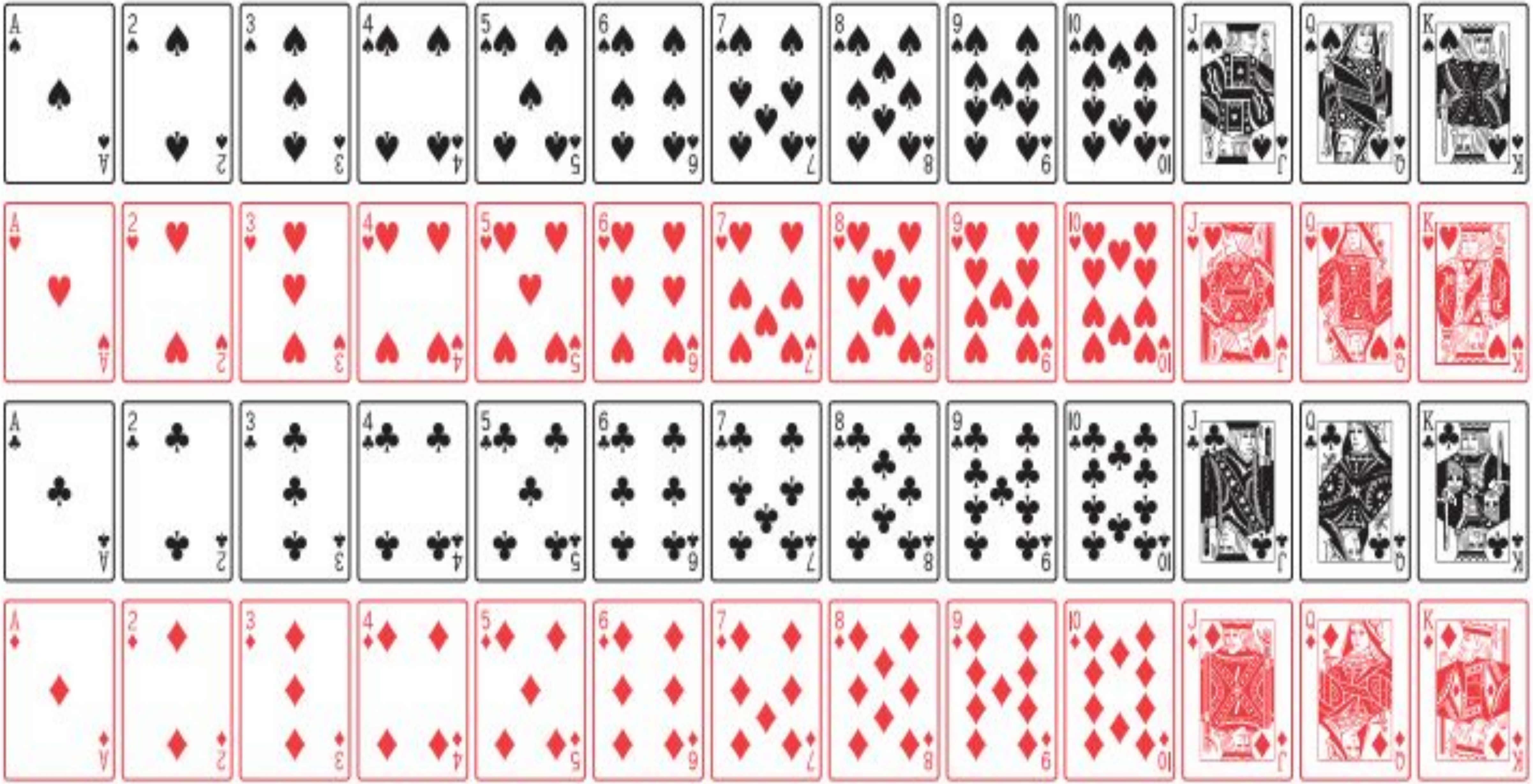
$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

Theorem 2.6

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Rule 2.3

$$P(A) = \frac{n}{N}.$$



Example # 23

- A card is drawn is random from a deck of ordinary playing cards. What is the probability that it is a diamond, a face card or a king?

Solution:

Let A = the card drawn is diamond

B = the card drawn is face card, &

C = the card drawn is a king

P (A U B U C) =

Example # 24

- A man tosses two fair dice. What is the conditional probability that the sum of two dice will be 7, given that

  (i) the sum is odd. (B)

  (ii) the sum is greater than 6. (C)

  (iii) the two dice had the same outcome. (D)

Solution:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

When order matter

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

when order does not matter

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	
(3,3)	(3,4)	(3,5)	(3,6)		
(4,4)	(4,5)	(4,6)			
(5,5)	(5,6)				
(6,6)					

Example # 25

- Two coins are tossed. What is the conditional probability that two heads results, given that there is at least one head?

Example # 25

- Two events A & B are such that P(A) = 1/4, P(A|B)=1/2, P(B|A) = 2/3

(i) Are A and B independent events?

(ii) Are A and B mutually exclusive events?

(iii) Find $P(A \cap B)$

Solution:

Example: Roommate Compatibility

Baber is off to college. There are 1000 additional new male students, and one of them will be randomly assigned to share Baber's dorm room. He is hoping that it won't be someone who likes to party or who snores. The table for the 1000 students is given below. What is the probability that Baber will be disappointed and get a roommate who either likes to party or snores or both?

	Snores	Doesn't Snore	Total
Likes to Party	150	100	250
Doesn't Like to Party	200	550	750
Total	350	650	1000

$$P(\mathrm{A}) = \frac{250}{1000} = .25$$

$$P(\mathrm{B}) = \frac{350}{1000} = .35$$

$$P(\mathrm{A\ and\ B}) = \frac{150}{1000} = .15$$

$$P(\mathrm{A\ or\ B}) = P(\mathrm{A}) + P(\mathrm{B}) - P(\mathrm{A\ and\ B}) = .25 + .35 - .15 = .45$$