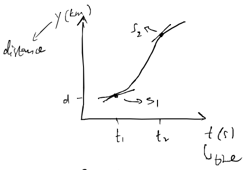


$a = \dfrac{s_2 - s_1}{t_2 - t_1}$

speed = $\dfrac{\text{distance}}{\text{time}}$ acceleration = $\dfrac{\text{speed}}{\text{time}}$

$y' = \dfrac{dy}{dt}$ $y'' = \dfrac{d^2y}{dt^2}$

ODE (ordinary differential equation)

- Relates a function $y(x)$ to its derivatives

General form: $F(x, y, y', \dots, y^{(n)}) = 0$

$y' = f(x, y)$

higher order derivative → $n \geq 2$ $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x)$

IVP (Initial value problems)

Given, $y(x_0) = y_0$ $y'(x_0) = y_1$

BVP (Boundary value problems)

Given, $y(a) = c_1$ $y(b) = c_2$

Lipschitz Condition

$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$

$|3 - 1| \leq 2|3 - 1|$ $|2 - 6| \leq 2|2 - 6|$

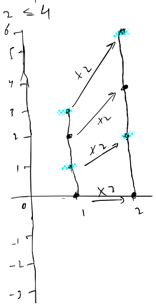
$2 \leq 4$ $4 \leq 8$

Example

Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Solution For each pair of points (t, y_1) and (t, y_2) in D we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\,||y_1| - |y_2|| \leq 2|y_1 - y_2|.$$



$f(t, y) = t|y|$

Domain,

$1 \leq t \leq 2$ $-3 \leq y \leq 4$

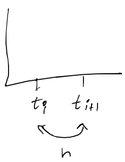
Result,

$|f(t, y_1) - f(t, y_2)| \leq 2|y_1 - y_2|$

Euler's Method → Used to approximate IVPs

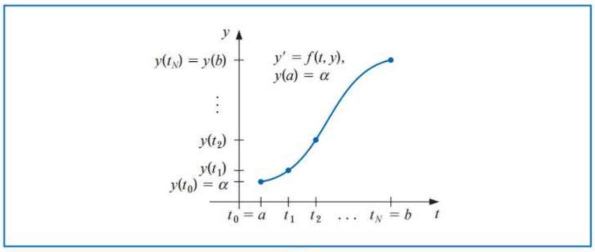
$\dfrac{dy}{dt} = f(t, y)$ $y(a) = \alpha$

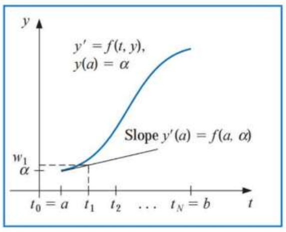
Interval $[a, b]$ divided into N equal steps

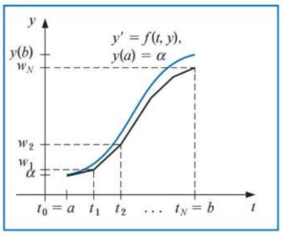


step size → $h = \dfrac{b - a}{N} = t_{i+1} - t$

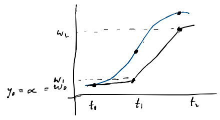
- Euler's Method derivation → based on Taylor's Theorem







$w_0 = y_0 = \alpha$



$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h\underbrace{f(t_i, w_i)}_{\text{slope at specific point}}, \quad \text{for each } i = 0, 1, \ldots, N-1$$

$\frac{dy}{dx} = x \qquad y(0) = 0 \qquad h = 0.5$

$\longrightarrow \qquad x_0 = 0 \qquad y_0 = 0$

$\frac{dy}{dx} = x = 0$

$y_1 = 0 + 0.5(0) = 0$

$\longrightarrow \qquad x_1 = 0.5$

$y_2 = 0 + 0.5(0.5) = 0.25$

To use an algorithm for Euler's method to approximate the solution to

$$\text{slope} \quad y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5,$$

at $t = 2$. Here we will simply illustrate the steps in the technique when we have $h = 0.5$.

$t = 0, 0.5, 1, 1.5, 2 \qquad 0 \le t \le 2$

$t_0 = 0 \qquad w_0 = 0.5$

$w_{i+1} = w_i + h\underbrace{f(t, y)}_{\text{find slope}} \rightarrow = y - t^2 + 1$

$= 0.5 - 0^2 + 1 = 1.5$

$w_1 = 0.5 + 0.5(1.5) = 1.25$

$f(0.5, 1.25) = 1.25 - 0.5^2 + 1 = 2$

$w_2 = 1.25 + 0.5(2) = 2.25$

$w_3 = 3.375 \qquad w_4 = 4.4375$

Runge-Kutta Method

- Unlike Euler method RK uses 4 slopes instead of 1

$k_1 \rightarrow$ slope at start } Average

$w_0 = \alpha,$

$k_1 = h f(t_i, w_i),$

- Unlike Euler method RK uses ...

$k_1 \rightarrow$ slope at start
$k_2 \rightarrow$ slope at middle (using k_1)
$k_3 \rightarrow$ slope at middle (using k_2)
$k_4 \rightarrow$ slope at end

} Average

$$w_0 = \alpha,$$
$$k_1 = hf(t_i, w_i),$$
$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$
$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$
$$k_4 = hf(t_{i+1}, w_i + k_3),$$

Runga-Kutta Formula : $w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$

Q.) To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

Sol.)

Step 1 Set $h = (b-a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5.

Step 3 Set $K_1 = hf(t, w)$;
$K_2 = hf(t + h/2, w + K_1/2)$;
$K_3 = hf(t + h/2, w + K_2/2)$;
$K_4 = hf(t + h, w + K_3)$.

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 5 OUTPUT (t, w).

Step 6 STOP.

1. Calculate h
2. Start from initial value
3. Compute 4 slopes
4. Take average
5. Next t_i
   (Repeat for next step)
6. Stop at w_N

Q.)

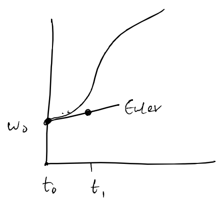
$y' = t + y \quad y(0) = 1 \quad N = 5$

$0 \le t \le 1$

$h = 0.2 \quad h = \frac{1-0}{5} = 0.2$

$t_0 = 0 \quad w_0 = 1$

$t_1 = 0.2 \quad w_1 = ?$



$k_1 = hf(t_0, w_0)$

$k_1 = 0.2(0+1) = 0.2$

$k_2 = hf(\underbrace{t_0 + \frac{h}{2}}_{t}, \underbrace{w_0 + \frac{k_1}{2}}_{y})$

$k_2 = 0.2\left(0.1 + 1 + \frac{0.2}{2}\right)$

$= 0.2(0.1 + 1.1) = 0.2(1.2) = 0.24$

$k_3 = hf\left(t_0 + \frac{h}{2}, w_0 + \frac{k_2}{2}\right)$

$k_3 = 0.2\left(0.1 + 1 + \frac{0.24}{2}\right)$

$k_3 = 0.2(1.22) = 0.244$

$k_4 = hf(\overbrace{t_0 + h}^{t_1}, w + k_3)$

$k_4 = 0.2(0.2, 1 + 0.244) = 0.2(0.2 + 1.244)$

$= 0.2(1.444)$

$k_4 = 0.2888$

$w_1 = w_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$w_1 = 1 + \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$

$w_1 = 1.2428$

Use the Runge-Kutta method of order four with $h = 0.2$, $N = 10$, and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

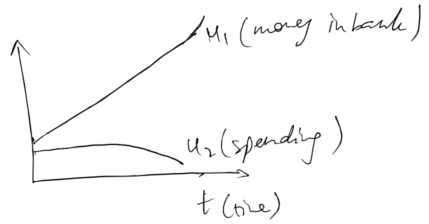
System of differential Equations

↳ Group of differential equations
↳ multiple quantities change together

$\frac{dy}{dt}$ = Rate of change of y with respect to t

$\frac{d u_1}{dt} = f(\overbrace{t, u_1, \ldots}^{\text{dependencies}}) \rightarrow$ Rate of change of u_1

$$\frac{du_1}{dt} = f_1(\overbrace{t, u_1, u_2}^{\text{dependencies}}) \rightarrow$$ Rate of change of u_1 with respect to t



d.) $I_1' = f_1(t, I_1, I_2) = -4I_1 + 3I_2 + 6, \quad I_1(0) = 0,$

$I_2' = f_2(t, I_1, I_2) = 0.6I_1' - 0.2I_2 = -2.4I_1 + 1.6I_2 + 3.6, \quad I_2(0) = 0.$

We will apply the Runge-Kutta method of order four to this system with $\underbrace{h = 0.1}_{\text{stepsize}}$. Since $w_{1,0} = I_1(0) = 0$ and $w_{2,0} = I_2(0) = 0$,

initial values

- Calculate slopes $k_{1,1}, k_{1,2}$
  $k_{2,1}, k_{2,2}$
  $k_{3,1}, k_{3,2}$
  $k_{4,1}, k_{4,2}$

- Combine slopes (avg)

$t_0 = 0 \xrightarrow{+h \rightarrow 0.1} t_1 = 0.1$

$\underset{w_{1,1}}{I_1(0.1)} = w_{1,0} + \frac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1})$

$\underset{w_{2,1}}{I_2(0.1)} = w_{2,0} + \frac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2})$

Higher order differential equation

$y' = f(t, y)$

$y''(t), y'''(t)$

$y(t)$ = displacement

$y'(t)$ = velocity

$y''(t)$ = acceleration

- How to deal with differential eq system with order greater than 1
- Euler & Runge Kutta method capable of solving 1st order IVPs $\rightarrow y'$

Q.1
Transform the the second-order initial-value problem

$$y'' - 2y' + 2y = e^{2t} \sin t, \quad \text{for } 0 \le t \le 1, \quad \text{with } y(0) = -0.4, \; y'(0) = -0.6$$

into a system of first order initial-value problems, and use the Runge-Kutta method with $h = 0.1$ to approximate the solution.

Steps

① Define new variables, $u_1(t) = y(t)$ $\quad u_2(t) = y'(t)$

$u_1 = y \quad u_2 = y'$

② Write system as 1st order IVP for new variables,

$u_1'(t) = y'(t)$

$u_1'(t) = u_2(t)$

$u_2'(t) = y''(t)$

$u_2'(t) = e^{2t} \sin t - 2y + 2y'$

$u_2'(t) = e^{2t} \sin t - 2u_1 + 2u_2$

$u_1'(t) =$

$u_2'(t) = e^{2t}\sin t - 2u_1 + 2u_2$

③ Now using IVPs $u_1(0) = -0.4$, $u_2(0) = -0.6$
& h & $u_1'(t)$, $u_2'(t)$ eqs, solve the system $\rightarrow$ Euler/Runge Kutta