NA14

Sunday, January 4, 2026 8:07 PM

$y = 4x^2 + 5t$

$\frac{dy}{dx} = 8x \qquad \frac{dy}{dt} = 5$

Partial differential equations

- Describes how a quantity depends on more than one independent variable

Types

1. Elliptic
2. Parabolic
3. Hyperbolic

Elliptic PDE

- describe system already in equilibrium
- Don't change with time

Poisson equation:

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = f(x,y). \rightarrow \text{Source function}$$

Special case

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0. \rightarrow \text{no internal source}$$

- $u(x,y) \rightarrow$ unknown function

Parabolic PDE

- Describe time-dependent processes

classical diffusion
Heat equation

$$\frac{\partial u}{\partial t}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0.$$

- t is time
- α is constant

Hyperbolic PDE

- Describe wave like motion
- Time dependent

Equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial^2 u}{\partial t^2}(x,t) = 0, \quad \text{for } 0 < x < l \quad \text{and} \quad 0 < t,$$

- Second time derivative causes oscillations

Elliptic PDE

d.) Determine the steady-state heat distribution in a thin square metal plate with dimensions 0.5 m by 0.5 m using $n = m = 4$. Two adjacent boundaries are held at 0°C, and the heat on the other boundaries increases linearly from 0°C at one corner to 100°C where the sides meet.

We need,

- Steady-state temperature on the metal

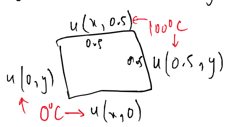
So • Temperature does not change with time

- Thus, using Laplace equation

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0.$$

$0 < x < 0.5 \qquad 0 < y < 0.5$

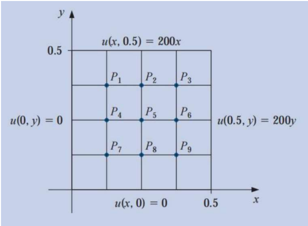


- Two edges are at 0°C
- other two edges increased linearly from 0°C to 100°C

$u(0,y) = 0 \qquad u(x, 0.5) = 200x$

$u(x, 0) = 0$ $\quad u(0.5, y) = 200y$

$200 \times 0.5 = 100°C$



(x-axis partitions)

$n = 4$

$m = 4$ (y-axis partitions)

$h = \frac{0.5}{4} = 0.125 \quad k = \frac{0.5}{4} = 0.125$

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right]w_{ij} - (w_{i+1,j} + w_{i-1,j}) - \left(\frac{h}{k}\right)^2 (w_{i,j+1} + w_{i,j-1}) = -h^2 f(x_i, y_j), \quad (12.4)$$

$$2\left[\left(\frac{0.125}{0.125}\right)^2 + 1\right] w_{ij} - w_{i+1,j} - w_{i-1,j} - \underbrace{\left(\frac{0.125}{0.125}\right)^2}_{1} (w_{i,j+1} + w_{i,j-1}) = -h^2 f(x_i, y_j)$$

$$4w_{ij} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = \underbrace{-(0.125)^2 \times 0}_{0}$$

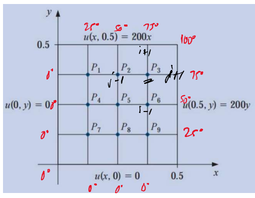
$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j-1} - w_{i,j+1} = 0, \quad \rightarrow \text{Simplified centered difference}$$

↳ Difference equation

Mesh Points :- $P_1, P_2 \ldots\ldots P_9$

Expressing this in terms of the relabeled interior grid points $w_i = u(P_i)$ implies that the equations at the points P_i are:

$$\begin{aligned}
P_1: &\quad 4w_1 - w_2 - w_4 = w_{0,3} + w_{1,4},\\
P_2: &\quad 4w_2 - w_3 - w_1 - w_5 = w_{2,4},\\
P_3: &\quad 4w_3 - w_2 - w_6 = w_{4,3} + w_{3,4},\\
P_4: &\quad 4w_4 - w_5 - w_1 - w_7 = w_{0,2},\\
P_5: &\quad 4w_5 - w_6 - w_4 - w_2 - w_8 = 0,\\
P_6: &\quad 4w_6 - w_5 - w_3 - w_9 = w_{4,2},\\
P_7: &\quad 4w_7 - w_8 - w_4 = w_{0,1} + w_{1,0},\\
P_8: &\quad 4w_8 - w_9 - w_7 - w_5 = w_{2,0},\\
P_9: &\quad 4w_9 - w_8 - w_6 = w_{3,0} + w_{4,1},
\end{aligned}$$



$P_3 = 4w_3 - w_2 - w_6 = w_{3,4} + w_{4,3}$

where the right sides of the equations are obtained from the boundary conditions. In fact, the boundary conditions imply that

$$w_{1,0} = w_{2,0} = w_{3,0} = w_{0,1} = w_{0,2} = w_{0,3} = 0,$$

$$w_{1,4} = w_{4,1} = 25, \quad w_{2,4} = w_{4,2} = 50, \quad \text{and} \quad w_{3,4} = w_{4,3} = 75.$$

So the linear system associated with this problem has the form

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}.$$

The values of $w_1, w_2, \ldots, w_9$, found by applying the Gauss-Seidel method to this matrix, are given in Table 12.1.

Table 12.1

i	w_i
1	18.75
2	37.50
3	56.25
4	12.50
5	25.00
6	37.50
7	6.25
8	12.50
9	18.75

Parabolic PDE

- Diffusion processes → heat flow
- Time dependent → Temperature changes with time

The *parabolic* partial differential equation we consider is the heat, or diffusion, equation

$u(x,t) \rightarrow$ Temperature

$u(x,t)$ ↳ temperature of position x at that t

$$\frac{\partial u}{\partial t}(x, t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x, t), \quad 0 < x < l, \quad t > 0, \quad (12.6)$$

constant ↳ time

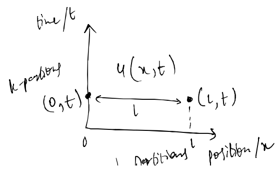
subject to the conditions

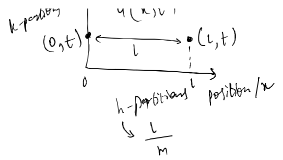
$$\underbrace{u(0, t) = u(l, t) = 0}_{\text{Boundary Conditions}}, \quad t > 0, \quad \text{and} \quad \underbrace{u(x, 0) = f(x)}_{\text{Initial condition}}, \quad 0 \le x \le l.$$

upper ↰

(x,1) (0,y) (1,y) (x,0)

↳ starting temperature





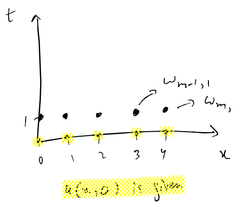
$$w_{i,j+1} = \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{ij} + \alpha^2 \frac{k}{h^2}(w_{i+1,j} + w_{i-1,j}),$$ → Heat equation (rearranged)

So we have

$$w_{0,0} = f(x_0), \quad w_{1,0} = f(x_1), \quad \ldots w_{m,0} = f(x_m).$$

Then we generate the next t-row by

$$\begin{aligned}
w_{0,1} &= u(0, t_1) = 0;\\
w_{1,1} &= \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{1,0} + \alpha^2 \frac{k}{h^2}(w_{2,0} + w_{0,0});\\
w_{2,1} &= \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{2,0} + \alpha^2 \frac{k}{h^2}(w_{3,0} + w_{1,0});\\
&\vdots\\
w_{m-1,1} &= \left(1 - \frac{2\alpha^2 k}{h^2}\right) w_{m-1,0} + \alpha^2 \frac{k}{h^2}(w_{m,0} + w_{m-2,0});\\
w_{m,1} &= u(m, t_1) = 0.
\end{aligned}$$



$m = 4$

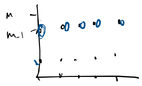
Now we can use the $w_{i,1}$ values to generate all the $w_{i,2}$ values and so on.

where $\lambda = \alpha^2(k/h^2)$.

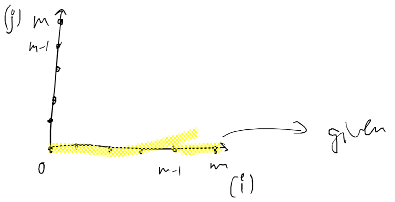
The explicit nature of the difference method implies that the $(m-1) \times (m-1)$ matrix associated with this system can be written in the tridiagonal form

$$A = \begin{bmatrix} (1-2\lambda) & \lambda & 0 & \cdots & 0 \\ \lambda & (1-2\lambda) & \lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \lambda \\ 0 & \cdots & 0 & \lambda & (1-2\lambda) \end{bmatrix}, \begin{bmatrix} w_{0,j-1} \\ w_{1,j-1} \\ w_{2,j-1} \\ w_{3,j-1} \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$A\, w^{(j-1)} = w^{j}$



Hyperbolic PDE



① $$w_{i,j+1} = 2(1-\lambda^2)w_{i,j} + \lambda^2(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}.$$

$\lambda = \alpha \dfrac{k}{h}$

②
$$\begin{bmatrix} w_{1,j+1} \\ w_{2,j+1} \\ \vdots \\ w_{m-1,j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix} \begin{bmatrix} w_{1,j} \\ w_{2,j} \\ \vdots \\ w_{m-1,j} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ w_{2,j-1} \\ \vdots \\ w_{m-1,j-1} \end{bmatrix}. \tag{12.22}$$

Use steps sizes **(a)** $h = 0.1$ and $k = 0.0005$ and **(b)** $h = 0.1$ and $k = 0.01$ to approximate the solution to the heat equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < 1, \quad 0 \le t,$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t,$$

and initial conditions

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1.$$

Compare the results at $t = 0.5$ to the exact solution

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x).$$

Solution **(a)** Forward-Difference method with $h = 0.1$, $k = 0.0005$ and $\lambda = (1)^2(0.0005/(0.1)^2) = 0.05$ gives the results in the third column of Table 12.3. As can be seem from the fourth column, these results are quite accurate.

(b) Forward-Difference method with $h = 0.1$, $k = 0.01$ and $\lambda = (1)^2(0.01/(0.1)^2) = 1$ gives the results in the fifth column of Table 12.3. As can be seem from the sixth column, these results are worthless. ■

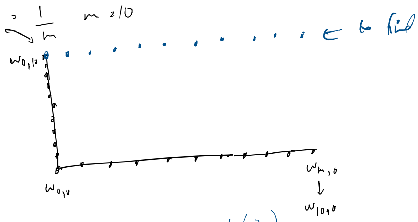
Table 12.3

x_i	$u(x_i, 0.5)$	$w_{i,1000}$ $k = 0.0005$	$\lvert u(x_i,0.5) - w_{i,1000}\rvert$	$w_{i,50}$ $k = 0.01$	$\lvert u(x_i,0.5) - w_{i,50}\rvert$
0.0	0	0		0	
0.1	0.00222241	0.00228652	6.411×10^{-5}	8.19876×10^{7}	8.199×10^{7}
0.2	0.00422728	0.00434922	1.219×10^{-4}	-1.55719×10^{8}	1.557×10^{8}
0.3	0.00581836	0.00598619	1.678×10^{-4}	2.13833×10^{8}	2.138×10^{8}
0.4	0.00683989	0.00703719	1.973×10^{-4}	-2.50642×10^{8}	2.506×10^{8}
0.5	0.00719188	0.00739934	2.075×10^{-4}	2.62685×10^{8}	2.627×10^{8}
0.6	0.00683989	0.00703719	1.973×10^{-4}	-2.49015×10^{8}	2.490×10^{8}
0.7	0.00581836	0.00598619	1.678×10^{-4}	2.11200×10^{8}	2.112×10^{8}
0.8	0.00422728	0.00434922	1.219×10^{-4}	-1.53086×10^{8}	1.531×10^{8}
0.9	0.00222241	0.00228652	6.511×10^{-5}	8.03604×10^{7}	8.036×10^{7}
1.0	0	0		0	

$\underbrace{0 < x < 1}$ → $l = 1$

$h = \dfrac{l}{m}$

$0.1 = \dfrac{1}{m}$ $\quad m = 10$



① find $\lambda \to \alpha^2(k/h^2)$

② make A matrix

③ find 9th row $w_{m-1,0.9}$
↳ second eqn

④ $A\, w_{m-1,0.9} = w_{m,1}$
$(9\times9)\times(9\times1) = (9\times1)$ ↳ $w_{i,1}$ (10th row)
$9\times1 = 9\times1$