Matrix multiplication

resultant

$\boxed{3 \times \boxed{3 : 3} \times 1}$

r c r c

Interpolation

- A method to find a function that exactly passes through given data points.

Data points : $(x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots$

x_1, x_2, x_3 $f(x) \rightarrow y$

$f(x_1), f(x_2), f(x_3)$

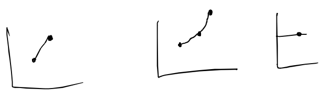
y_1, y_2, y_3

Why Polynomials?

- Approximate continuous functions well
- Easy to differentiate / integrate

Applications

- Image & signal Processing
- Weather / financial forecasting
- Engineering and Spacial Modeling



Lagrange Interpolating Polynomial

Gives polynomial that passes exactly through n data points, forming polynomial of degree n-1.

$$\text{Polynomial} = a_0 x^0 + a_1 x^1 + a_2 x^2 \ldots$$

Q.) Given n data points

1. Find $L_0, \ldots L_{n-1}$
2. Calculate $f(x_0), \ldots f(x_{n-1})$
3. Find Polynomial of degree n-1 : $P(x) = L_0(x) f(x_0) + \ldots L_{n-1} f(x_{n-1})$

Given 2 data points → linear interpolation

$$L_0 = \frac{x - x_1}{x_0 - x_1} \quad f(x_0)$$

$$L_1 = \frac{x - x_0}{x_1 - x_0} \quad f(x_1)$$

Given 3 data points → Quadratic interpolation

$$L_0 = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \quad f(x_0)$$

$$L_1 = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \quad f(x_1)$$

$$L_2 = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \quad f(x_2)$$

Limitation

- Adding/Removing points requires recomputing entire polynomial

Divided Difference

↳ Alternate to lagrange

- Express how function changes b/w different data points

1. Form the divided difference table

Form for Polynomial

x	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
		$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
x_2	$f[x_2]$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{}$		$f[x_1, x_2, x_3, x_4] = \frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{}$

① Form the divided difference table
② Form the equation for Polynomial

x_1	$f[x_1]$		$f[x_0,x_1,x_2] = \dfrac{\quad}{x_2 - x_0}$
		$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1}$	
x_2	$f[x_2]$		$f[x_1,x_2,x_3] = \dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$
		$f[x_2,x_3] = \dfrac{f[x_3]-f[x_2]}{x_3-x_2}$	
x_3	$f[x_3]$		$f[x_2,x_3,x_4] = \dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$
		$f[x_3,x_4] = \dfrac{f[x_4]-f[x_3]}{x_4-x_3}$	
x_4	$f[x_4]$		$f[x_3,x_4,x_5] = \dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$
		$f[x_4,x_5] = \dfrac{f[x_5]-f[x_4]}{x_5-x_4}$	
x_5	$f[x_5]$		

$$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$$

$$f[x_1,x_2,x_3,x_4] = \frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$$

$$f[x_2,x_3,x_4,x_5] = \frac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$$

Q.) (1, 2) (2, 3) (4, 6)

	x	$f(x)$ ↗ b_0	first divided Difference	↗ b_2
0	1	2		
			$\dfrac{f(x_1)-f(x_0)}{x_1-x_0} = \dfrac{3-2}{2-1} = 1$ ↗ b_1	$\dfrac{3/2 - 1}{4-1} = 0.1667$
1	2	3		
			$\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = \dfrac{6-3}{4-2} = \dfrac{3}{2} = 1.5$	
2	4	6		

- 3 data points ⟶ 2nd Divided difference (Answer in 2 degree)

$$P(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= 2 + 1(x-1) + 2(x-1)(x-2)$$

- 5 data points ⟶ 4th divided difference
- 4 data points ⟶ 3rd divided difference (Answer in 3 degree)

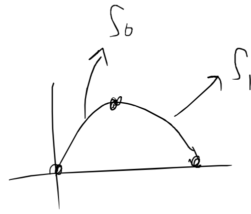
$$\hookrightarrow P(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Cubic Spline Interpolation

- Instead of using one big polynomial, we use piecewise cubic polynomial b/w each pair of data points.

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

① Write Spline equations



$x_0 = 0 \qquad x_1 = 1$

x	y	
0	0	$S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3$
1	1	$S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$
2		

1 1 $S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0 \dots$

2 0 $S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$

(2) Apply passes though points

At $x=0, y=0$

$$S_0(0) = a_0 + b_0(0) + c_0(0) + d_0(0) = 0$$

$$a_0 = 0$$

At $x=1, y=1$

$$S_0(1) = a_0 + b_0(1) + c_0(1) + d_0(1) = 1$$

$$b_0 + c_0 + d_0 = 1$$

$$S_1(1) = a_1 + b_1(1-1) + c_1(1-1)^2 + d_1(1-1)^3 = 1$$

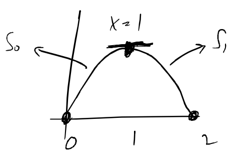
$$a_1 = 1$$

At $x=2, y=0$

$$S_1(2) = a_1 + b_1(2-1) + c_1(2-1)^2 + d_1(2-1)^3 = 0$$

$$b_1 + c_1 + d_1 = -1$$

(3) Continuity of 1st Derivative at $x=1$



$$S_0(x)' = b_0 + 2c_0 x + 3d_0 x^2$$

$$S_1(x)' = b_1 + 2c_1(x-1) + 3d_1(x-1)^2$$

$x=1$

$$S_1(1)' = S_0(1)'$$

$$b_1 = b_0 + 2c_0 + 3d_0$$

(4) Continuity of 2nd derivative $x=1$

$$S_0(x)'' = 2c_0 + 6d_0 x$$

$$S_1(x)'' = 2c_1 + 6d_1(x-1)$$

$x=1$

$$S_0(1)'' = S_1(1)''$$

$$\cancel{2}c_0 + \overset{3}{\cancel{6}}d_0 = \cancel{2}c_1 + \overset{3}{\cancel{6}}d_1(1-1)$$

$C_1 = C_0 + 3d_0$

⑤ Boundary Condition (Natural)

$S_0(0)'' = 2c_0 + 6d_0(0) = 0$

$= 2c_0 = 0$

$c_0 = 0$

$S_1(2)'' = 2c_1 + 6d_1(2-1) = 0$

$= 2c_1 + 6d_1 = 0$

⑥ find remaining constant terms with substitutions

$c_1 = 3d_0$

$b_0 + d_0 = 1$

$b_1 + c_1 + d_1 = -1$

$b_1 + 3d_0 - d_0 = -1$

$b_1 + 2d_0 = -1$

$b_0 + 3d_0 + 2d_0 = -1$

$b_0 + 5d_0 = -1$

$b_1 = 1.5 + 3(-0.5)$

$b_1 = 0$

$2(3d_0) + 6d_1 = 0$

$\cancel{6}d_0 = -\cancel{6}d_1$

$d_0 = -d_1$

$b_1 = b_0 + 3d_0$

$c_1 = 3(-0.5)$

$c_1 = -1.5$

$d_1 = 0.5$

$\cancel{b_1} + d_0 = 1$

$-(\cancel{b_0} + 5d_0 = -1)$

$-4d_0 = 2$

$d_0 = -0.5$

$b_0 = 1 - d_0$

$b_0 = 1.5$

$[0,1] \rightarrow S_0(x) = 0 + 1.5(x-0) + 0(x-0) - 0.5(x-0)^3$

$[1,2] \rightarrow S_1(x) = 1 + 0(x-1) - 1.5(x-1)^2 + 0.5(x-1)^3$