Lecture 2

Friday, September 26, 2025 5:51 PM

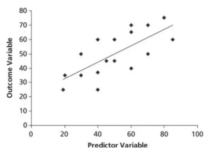
Approximation Theory → Branch of mathematics focused on replacing complicated functions or datasets with simple ones.

Purpose → Make Analysis and computation easy

Types of Problems

1. Function Approximation
   ↳ To find simpler version of a function

2. Data Approximation
   ↳ Fit a function to experimental data



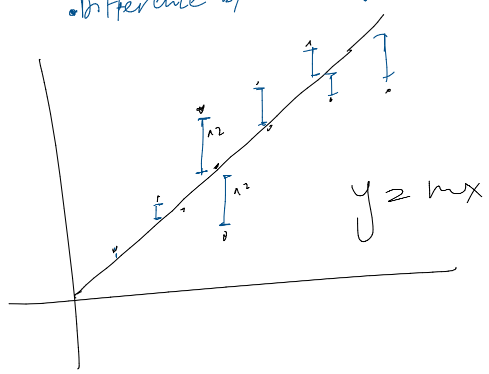
Applications

- Science and Engineering → Simplify computations
- Statistic and Machine Learning → fit models to data, capture trends

Approximation Methods

① Least Squares Approximation → Also called regression

Technique for fitting a function by minimizing sum of squared errors between observed and predicted values.



- Difference b/w these values is called error
- Sum of all errors (squared) gives sum of squared error

$y = mx + c$ → $\boxed{y = a_1 x + a_0}$

① Compute Σx, Σy, Σx^2, Σxy

② $$a_0 = \frac{\Sigma x^2 \Sigma y - \Sigma x \Sigma xy}{n \Sigma x^2 - (\Sigma x)^2}$$

no. of data points ← n

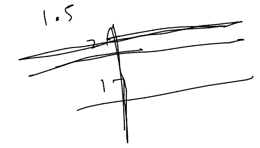
$$a_1 = \frac{n \Sigma xy - \Sigma x \Sigma y}{n \Sigma x^2 - (\Sigma x)^2}$$

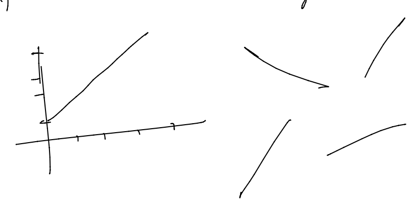
$f(x) = x^3 - 4x + 5$ → degree 3

$f(x) = 2x + 1$ → degree 1

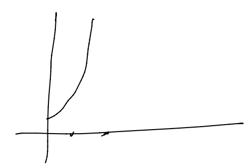
$f(x) = 2$ → degree 0

$f(x) = 1.5$





$f(x) = x^2 + 2x + 1$ → degree 2



1. Write a linear equation that "best fits" the data in the table shown below:

X	1	2	3	4	5	6	7
Y	1.5	3.8	6.7	9.0	11.2	13.6	16

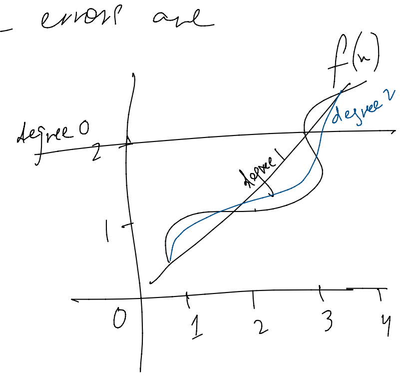
② Orthogonal Polynomial Approximation

Approximates functions by polynomials where errors are minimized in squared sense.

$y = a_0$ → degree 0

$y = a_1 x + a_0$ → degree 1

$y = a_2 x^2 + a_1 x + a_0$ → degree 2



Q.) degree? interval? $f(x)$?

Q.) $f(x) = x^2$ on interval $[0,1]$ degree $= 0$ $\longrightarrow P_0(x) = a_0$

$$\int_0^1 [x^2 - P_0(x)] \cdot 1 \, dx$$

Q.) Find the least squares approximating polynomial of degree 2 for the function $\boxed{f(x) = \sin \pi x}$ on the interval [0, 1].

$$\boxed{P_2(x) = a_0 + a_1 x + a_2 x^2}$$

$$\int_0^1 [f(x) - P_2(x)] \cdot 1 \, dx = \int_0^1 [\sin \pi x - (a_0 + a_1 x + a_2 x^2)] \cdot 1$$

$$\int_0^1 [f(x) - P_2(x)] \cdot x \, dx = \int_0^1 [\sin \pi x - (a_0 + a_1 x + a_2 x^2)] \cdot x$$

$$\int_0^1 [f(x) - P_2(x)] \cdot x^2 \, dx = \int_0^1 [\sin \pi x - (a_0 + a_1 x + a_2 x^2)] \cdot x^2$$

$$\int_0^1 a_0 + \int_0^1 a_1 x + \int_0^1 a_2 x^2 = \int_0^1 \sin \pi x \, dx$$

$$\int_0^1 a_0 x + \int_0^1 a_1 x^2 + \int_0^1 a_2 x^3 = \int_0^1 x \sin \pi x \, dx$$

$$\int_0^1 a_0 x^2 + \int_0^1 a_1 x^3 + \int_0^1 a_2 x^4 = \int_0^1 x^2 \sin \pi x \, dx$$

$$a_0 \int_0^1 1 \, dx + a_1 \int_0^1 x \, dx + a_2 \int_0^1 x^2 \, dx = \int_0^1 \sin \pi x \, dx,$$

$$a_0 \int_0^1 x \, dx + a_1 \int_0^1 x^2 \, dx + a_2 \int_0^1 x^3 \, dx = \int_0^1 x \sin \pi x \, dx,$$

$$a_0 \int_0^1 x^2 \, dx + a_1 \int_0^1 x^3 \, dx + a_2 \int_0^1 x^4 \, dx = \int_0^1 x^2 \sin \pi x \, dx.$$

Q.) $f(x) = x^2$ $[0,2]$ degree $= 1$ $\longrightarrow P_1(x) = a_0 + a_1 x$

$$\int_0^2 [x^2 - P_1(x)] \cdot 1 \, dx \longrightarrow \int_0^2 x^2 - (a_0 + a_1 x) \, dx$$

$$\int_0^2 [x^2 - P_1(x)] \cdot x \, dx \longrightarrow \int_0^2 x^3 - x(a_0 + a_1 x)$$

$$\int_0^2 x^3 - \int_0^2 a_0 x - \int_0^2 a_1 x^2$$

$$\Big|_0^2 \frac{x^4}{4} - \Big|_0^2 \frac{a_0 x^2}{2} - \Big|_0^2 \frac{a_1 x^3}{3}$$

$$\frac{16}{4} - \frac{a_0 \cdot 4}{2} - \frac{a_1 \cdot 8}{3} = 0$$

$$4 - 2a_0 - \frac{\left(\frac{4}{3} - a_0\right) \cdot 8}{3} = 0$$

$$4 - 2a_0 - \frac{32}{9} - \frac{8a_0}{3} = 0$$

$$4 - \frac{32}{9} = \frac{2a_0}{1}\frac{\times 3}{\times 3} + \frac{8a_0}{3}$$

$$12 - \frac{32}{3} = 6a_0 + 8a_0$$

$$\frac{12 - \frac{32}{3}}{14} = a_0$$

$$a_0 = -0.476$$

$$\int_0^2 x^2 - \int_0^2 a_0 - \int_0^2 a_1 x$$

$$\Big|_0^2 \frac{x^3}{3} - \Big|_0^2 a_0 x - \Big|_0^2 \frac{a_1 x^2}{2}$$

$$\frac{8}{3} - a_0 \cdot 2 - a_1 \cdot 2 = 0$$

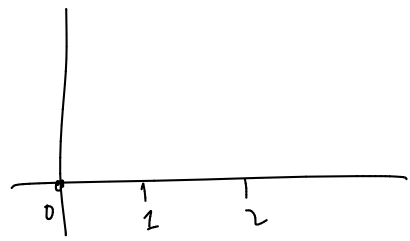
$$\frac{8}{3} - 2a_0 = 2a_1$$

$$\frac{\frac{4\cancel{8}}{3}}{\cancel{2}} - \frac{\cancel{2}a_0}{\cancel{2}} = a_1 \longrightarrow \frac{4}{3} + 0.476 = 1.809$$

$$a_1 = \frac{4}{3} - a_0$$

$$\frac{\frac{8}{3}}{2} \qquad \frac{8}{3} \div 2 \qquad \frac{8}{3} \times \frac{1}{2}$$

$$P_1(x) = 1.809x - 0.476$$



Polynomial Approximation

Advantages

- Works for any continuous function on closed interval
- Easy to evaluate, differentiate & integrate

Disadvantages

- Can oscillate heavily

③ Rational Function Approximation

- Ratio of two polynomials $\rightarrow$ $R(x) = \dfrac{P(x)}{Q(x)}$ $\begin{matrix}\rightarrow \\ \rightarrow\end{matrix}$ Polynomials
- Gives superior approximation for same effort
- Padé approximation technique uses rational function approximation

- Can oscillate heavily
- Errors may be larger near edges

$2 = \frac{4 \div 2}{2 \div 2}$ $\quad 2 = \frac{2}{1}$

$2 = \frac{4 \times 2}{2 \times 2}$

$2 = \frac{8}{4}$

Q.) find Padé approximation for e^x $n=1$ $m=1$

$$r(x) = \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x \quad \div q_0}{q_0 + q_1 x \quad \div q_0}$$

$$\frac{p(x) \div p_0}{q(x) \div p_0}$$

$$\frac{1 + p_1 x}{q_0 + q_1 x}$$

$$= \frac{p_0/q_0 + p_1/q_0 \, x}{1 \, \cancel{q_0/q_0} + q_1/q_0 \, x}$$

$$= \frac{a + b x}{1 + c x} = \frac{p_0 + p_1 x}{1 + q_1 x}$$

$\to e^x$

$f(x) \to$ maclaurin series

$$\left(1 + x + \frac{x^2}{2} \ldots\right) = \frac{p_0 + p_1 x}{1 + q_1 x}$$

$$\left(1 + x + \frac{x^2}{2} \ldots\right)\left(1 + q_1 x\right) = p_0 + p_1 x$$

$$1 + q_1 x + x + q_1 x^2 + \frac{x^2}{2} + q_1 \frac{x^3}{2} \ldots = p_0 + p_1 x$$

$\longrightarrow$ Match coefficients now to find unknowns

Constant Term

$p_0 = 1$

x Term

$q_1 \cancel{x} + \cancel{x} = p_1 \cancel{x}$

x^2 Term

$q_1 \cancel{x^2} + \underline{\cancel{x^2}} = 0 \cancel{x^2}$

$p_0 = 1$

$q_1 x + x = p_1 x$

$p_1 = q_1 + 1$

$p_1 = -\frac{1}{2} + 1 = \frac{1}{2}$

$q_1 \cancel{x^2} + \frac{x^2}{2} = 0\cancel{x^2}$

$q_1 + \frac{1}{2} = 0$

$q_1 = -\frac{1}{2}$

$$r(x) = \frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x} \;\times 2 = \frac{2 + x}{2 - x}$$

Now verifying our approximation

From calculator	From our eq	
$e^0 = 1$	$e^0 = \frac{2+0}{2-0} = \frac{2}{2} = 1$	error $= 0$
$e^{0.5} = 1.648$	$e^{0.5} = \frac{2+0.5}{2-0.5} = 1.666$	error $= 0.018$
$e^1 = 2.718$	$e^1 = \frac{2+1}{2-1} = 3$	error $= 0.288$

(4) Trigonometric Polynomial Approximation

- Used for periodic functions $\rightarrow$ functions that repeat $\rightarrow$ eg wave, music, sounds, signals
- Based on sums of sines & cosines

→ for function $f(x)$ on interval $[-\pi,\pi]$

$|x| = \pm x$

$|-7| = |7|$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

Q.) find $T_2(x)$ for $f(x) = x$ on interval $[-\pi,\pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right] = 0$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos kx\,dx = vu - \int v\,du = x\frac{\sin kx}{k} - \int_{-\pi}^{\pi}\frac{\sin kx}{k}\cdot 1\,dx$$

$u = x \qquad dv = \cos(kx)$

$du = 1 \qquad v = \frac{\sin kx}{k}$

$$= x\frac{\sin kx}{k} + \left[\frac{\cos kx}{k^2}\right]_{-\pi}^{\pi}$$

$$= x\frac{\sin kx}{k} + \left(\frac{\cos k\pi}{k^2} - \frac{\cos k(-\pi)}{k^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{x\sin kx}{k}\right) + 0$$

$$\xrightarrow{\text{integrate}} \sin 2x$$

$$\frac{\int \sin 2x}{2} \quad \Big\downarrow \text{differential}$$

$$2\cos 2x$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin kx\,dx$$

Integration by parts

$\int x \ln x$

$\downarrow \quad \downarrow$

$dv \quad u$

$du = 1/x$

$v = \frac{x^2}{2}$

u ↑ Log
Inverse
Algebraic
Trigonometric
Exponent

$= uv - \int v\,du$

$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \frac{1}{\cancel{x}} \; dx$

$= \frac{x^2 \ln x}{2} - \frac{x^2}{4}$