ODE → Ordinary differential equation
↳ Unknown function of $y(x)$, with derivatives $y', y'', \dots$

$$y' = 3x - y$$

IVP → Initial value Problem
↳ You know initial slope & initial value

$$y' = f(x, y) \qquad y(x_0) = y_0$$

BVP → Boundary value Problem
↳ Start and end values known

Two Point BVP
↳ BVP values given with second order differential equation

$$y'' = f(x, y, y')$$

$$y(a) = \alpha \qquad y(b) = \beta$$

- we don't know $y'(a)$ → initial slope
- So we must guess it

Linear
Shooting Method

Think of throwing a ball

→ You know $y(a)$ ← initial, $y(b)$ ← target → You don't know angle to throw (slope)

$$y'' = p(x)y' + q(x)y + r(x) \quad \text{(Linear Shooting)}$$

Steps

① Convert BVP into 2 IVPs

IVP 1

$$y_1'' = p(x)y_1' + q(x)y_1 + r(x)$$

$$y_1(a) = \alpha \qquad y_1'(a) = 0$$

IVP 2

$$y_2'' = p(x)y_2' + q(x)y_2$$

$$y_2(a) = 0 \qquad y_2'(a) = 1$$

$N = 5$

$1 \leq x \leq 2$

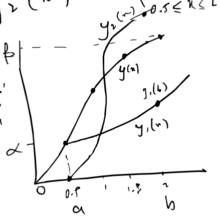
$\underbrace{1, 1.2, 1.4, 1.6, 1.8, 2}$

$y_1(x), y_2(x)$ to be found on each

② Solve IVPs then plug into formula

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)}\, y_2(x)$$

	y_0	y_0'
IVP_1	α	0
IVP_2	0	1



Q.) $y'' = y + x \qquad 0 \leq x \leq 1$

$$y(0) = 1 \qquad y(1) = 2$$

$N = 2$

$h = \frac{1}{2} = 0.5$

- $y'' = p(x)y' + q(x)y + r(x)$

  $p(x) = 0 \qquad q(x) = 1 \qquad r(x) = x$

- Construct 2 IVPs

IVP 1

$y_1'' = y_1 + x$

$y_1(0) = 1 \longrightarrow u_1$ initial value

$y_1'(0) = 0 \longrightarrow u_2$ initial value

IVP 2

$y_2'' = y_2$

$y_2(0) = 0 \longrightarrow v_1$ initial value

$y_2'(0) = 1 \longrightarrow v_2$ initial value

- Convert 2^{nd} order IVP into system of 1^{st} order IVP

IVP 1 System

$u_1 = y_1 \qquad u_2 = y_1'$

$u_1' = u_2$

$u_2' = y_1''$

$u_2' = y_1 + x$

$u_2' = u_1 + x$

$u_1(0) = 1$

$u_2(0) = 0$

IVP 2 System

$v_1 = y_2 \qquad v_2 = y_2'$

$v_1' = v_2$

$v_2' = y_2''$

$v_2' = y_2$

$v_2' = v_1$

$v_1(0) = 0$

$v_2(0) = 1$

- Solve the IVPs using Euler / Runge Kutta method

IVP 1

$u_{1,i+1} = u_{1,i} + h\, u_{2,i}$

$u_{2,i+1} = u_{2,i} + h(u_{1,i} + x_i)$

IVP 2

$v_{1,i+1} = v_{1,i} + h\, v_{2,i}$

$v_{2,i+1} = v_{2,i} + h\, v_{1,i}$

At $x_0 = 0$,

$u_1(0) = 1 \qquad u_2(0) = 0$

At $x_1 = 0.5$

$u_1(0.5) = 1 + 0.5(0) = 1$

$u_2(0.5) = 0 + 0.5(1+0) = 0.5$

At $x_2 = 1$

$u_1(1) = 1 + 0.5(0.5) = 1.25$

$u_2(1) = 0.5 + 0.5(1 + 0.5) = 1.25$

At $x_0 = 0$

$V_1(0) = 0 \qquad V_2(0) = 1$

At $x_1 = 0.5$

$V_1(0.5) = 0 + (0.5)\,1 = 0.5$

$V_2(0.5) = 1 + (0.5)\,0 = 1$

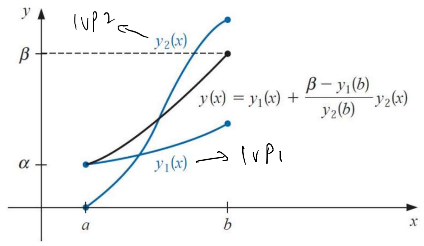
At $x_2 = 1$

$V_1(1) = 0.5 + (0.5)\,1 = 1$

$V_2(1) = 1 + (0.5)\,0.5 = 1.25$

$y_1(1) = 1.25 \qquad y_2(1) = 1$

* This graph is not for this question (only for understanding)



- Apply the shooting formula

$a \le x \le b$

$0 \le x \le 1$

$y(0) = 1 \to \alpha \qquad y(1) = 2 \to \beta$

$$y(x) = y_1(x) + \frac{\beta - y_1(1)}{y_2(1)} y_2(x)$$

✓ $y(0) = 1$

✓ $y(0.5) = 1 + \frac{2 - 1.25}{1}(0.5) = 1.375$

$y_1(0.5) = 1$ ↳ $u_1(0.5)$

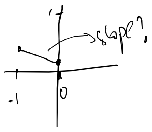
$y_2(0.5) = 0.5$ ↳ $v_1(0.5)$

✓ $y(1) = 2$

Non-linear Shooting Method

Q.1) $y'' = 2y^3 \qquad -1 \le x \le 0$

$h = 0.25$



Q.1 $y'' = 2y^3$ $\quad -1 \le x \le 0$

$y(-1) = \frac{1}{2}$ $\quad y(0) = \frac{1}{3}$ $\quad h = 0.25$

Step 1 Convert BVP → IVP

$y'' = 2y^3$ $\qquad F(t) = y(0,t) - \beta = 0$ → Goal according to slide eq 11.8

$y(-1) = \frac{1}{2}$ $\qquad \beta = \frac{1}{3}$

$y'(-1) = t$ → unknown slope

Step 2 Convert to 1st order system

$u_1 = y \quad u_2 = y'$

$u_1' = u_2 \quad u_2' = 2u_1^3$

$u_1(-1) = 0.5 \quad u_2(-1) = t$

Step 3 Choose 2 initial guesses

$t_0 = 0 \quad t_1 = -1$

→ How are these guesses decided?

- Simple, look at BVP overall slope

$$t = \frac{y(b) - y(a)}{b - a} = \frac{1/3 - 1/2}{0 - (-1)} = -0.1667$$

- One guess is overshoot → $t_0 = 0$ } -0.1667
- One is undershoot → $t_1 = -1$

Step 4 Solve IVP using RK4 (↳ slide says)

- Find $u_{1,i+1} = u_{1,i} + \frac{1}{6}\overbrace{(k_1 + 2k_2 + 2k_3 + k_4)}^{\text{for } u_1}$

$\quad\quad u_{2,i+1} = u_{2,i} + \frac{1}{6}\underbrace{(k_1 + 2k_2 + 2k_3 + k_4)}_{\text{for } u_2}$

initial values

$u_1(-1) = 0.5$

$u_2(-1) = 0$ ↳ from t_0

- Use u_1, u_2 at -1

↳ to find u_1, u_2 at -0.75

↳ to find u_1, u_2 at -0.5

↳ " " at -0.25

↳ " u_1 u_2 at 0

repeat

repeat with $t_1 = -1$

t_1, u_1, u_2 at 0

Step 5

$$y(0, t_0) = u_{1,final}$$

$$F(t_0) = u_{1,final} - \underbrace{\tfrac{1}{3}}_{\beta} \qquad F(t_1) = u_{1,final} - \underbrace{\tfrac{1}{3}}_{\beta}$$

Step 6

$$t_2 = t_1 - \frac{F(t_1)(t_1 - t_0)}{F(t_1) - F(t_0)}$$

Step 7

With, $u_1(-1) = \overbrace{0.5}^{\text{same } \alpha}$ & $u_2(-1) = t_2$

run IVP with RK or Euler 3rd n final time

- This will give actual solution table

Finite difference method for linear systems

Now write all the equations together.

i = 1	$-3y_1$	$+y_2$		$= -3$
i = 2	y_1	$-3y_2$	$+y_3$	$= -4$
i = 3		y_2	$-3y$	$= -3$

Here diagonal (d) is -3

So,

$d = (-3, -3, -3)$

$a = (1, 1, 0)$

$b = (0, 1, 1)$

$r = (-3, -4, -3)$

Tridiagonal

$Aw = b$

$3R_2 + R_1$

$$\begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 1 & 0 \\ 0 & -8 & 3 \\ 0 & 1 & -3 \end{bmatrix} 8R_3 + R_2 \Rightarrow \begin{bmatrix} -3 & 1 & 0 \\ 0 & -8 & 3 \\ 0 & 0 & -21 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ -3 \end{bmatrix}$$

$3 \times 3 \quad 3 \times 1$

$$-3w_1 + w_2 + 0 = -3$$
$$w_1 - 3w_2 + w_3 = -4$$
$$0 + w_2 - 3w_3 = -3$$

$$w_1 - 3w_2 + \frac{3 + w_2}{3} = -4$$
$$w_1 - 3w_2 + 1 + \frac{w_2}{3} = -4$$
$$-8w_2 = -5 \quad] \times 3$$

$$+21 w_3 = +3$$
$$w_3 = \frac{3}{21} = \frac{1}{7}$$

$$-8w_2 + 3w_3 = -4$$
$$-8w_2 \; \frac{3}{7} = -4$$
$$-8w_2 = -4 - \frac{3}{7}$$

$w_1 - 3w_2 + w_3 = -4$
$0 + w_2 - 3w_3 = -3$

$-3w_3 = -3 - w_2$
$w_3 = \frac{-3 - w_2}{-3} = \frac{3 + w_2}{3}$

$w_1 - 3w_2 + 1 + \frac{w_2}{3} = -4$
$[\, w_1 - \frac{8}{3} w_2 = -5 \,] \times 3$
$-3w_1 + w_2 = -7$

$0w_1 + 7w_2 = 18$
$w_2 = \frac{18}{7}$

$\checkmark w_1 \quad \checkmark w_3$

$w_3 = \frac{\ }{21} \quad \frac{\ }{7}$

$-8 w_2 = \frac{-4}{17} - \frac{3}{7}$

$w_2 = \frac{-31}{56}$

Use Algorithm 11.3 with $N = 9$ to approximate the solution to the linear boundary-value problem

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}, \quad \text{for } 1 \le x \le 2, \text{ with } y(1) = 1 \text{ and } y(2) = 2,$$