Notes for Modelling Qballs in 2+1 Dimensions

Muchen He

April 3, 2022

1 ODE for Qballs

Q - balls are a class of spherically symmetric solutions in 3 + 1 dimensions arising in a family of field theories.

Here, we examine the modeling of a 2 + 1 dimension case of these such solutions. In previous results, we already know that the Q ball is governed by the following equation:

$$\ddot{\phi} - \nabla^2\phi + \phi - A|\phi|\,\phi + B\left|\phi^2\right|\phi = 0 \tag{1}$$

where $\phi = \phi(r,t)$, a radially symmetric solution that is varying with respect to time. Previously, work has been done to seek solutions in form of

$$\phi(r,t) = \sigma(r)e^{i\omega t} \tag{2}$$

and generate numerical approximations accordingly.

The solutions to the QBall equations can be generated by the shooting method. The shooting method requires us to vary ω of the radial equation, and at each ω, we solve equation (4) to obtain the radial profile. When a radial profile has a good enough property (monotonically decays to 0 as we go out) we fit the tail of the Q balls exponentially.

2 Phaseshifted Qballs

We notice that we can add a phase shift factor, which motivates the following proposal:

$$\phi(r,t) = \sigma(r)e^{i(\omega t + k)} \tag{3}$$

where k is the phase shift. Then if we substitute this equation into equation (1), we still obtain the same equation governing $\sigma(r)$:

$$\nabla^2\sigma(r) + (\omega^2 - 1)\sigma(r) + A\sigma(r)^2 + B\sigma(r)^3 = 0 \tag{4}$$

This means that there are no change to the ordinary differential equations governing the Q balls with the additional phase change factor. Furthermore, this means the same method can be applied to solve the radial equation, but we do have a different time dependent term.

3 Anti-Qball

Now we want to consider an anti Qball, which takes the form

$$\phi(r,t) = f(r)e^{-i(\omega t)+ik} \tag{5}$$

This equation is motivated by the fact that the charge of a Qball is given by:

$$Q = \omega \int_{-\infty}^{\infty} \sigma^2(r)dr \tag{6}$$

Now, we substitute the new Qball equation into:

$$\ddot{\phi} - \nabla^2\phi + \phi - A|\phi|\,\phi + B\Big|\phi^2\Big|\,\phi = 0 \tag{7}$$

And we obtain

$$\nabla^2\sigma(r) + (\omega^2 - 1)\sigma(r) + A\sigma(r)^2 + B\sigma(r)^3 = 0 \tag{8}$$

which is the same governing differential equation as before. Thus we can simply implement the modelling of Anti-Qballs by having the frequency ω being a negative value.

4 Boost

But normally, we are interested at Q balls moving at relativistic speeds, in which we need to consider the Lorentz Transformations. These are normally called boost.

We use

$$\phi'(t',x',y') = \sigma(x',y')e^{i(\omega t+k)} = \phi'_1 + i\phi'_2 \tag{9}$$

to denote the Q ball field in the rest frame. Notice then,

$$\phi'_1(t',x',y') = \sigma(x',y')\cos(\omega t' + k) \tag{10}$$

and

$$\phi'_2(t',x',y') = \sigma(x',y')\sin(\omega t' + k) \tag{11}$$

and the corresponding time derivatives π'_1, π'_2 of ϕ'_1 and ϕ'_2 correspondingly are

$$\pi'_1(t',x',y') = \frac{\partial\phi'_1}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial\phi'_1}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial\phi'_1}{\partial y'}\frac{\partial y'}{\partial t} \tag{12}$$

$$\pi'_2(t',x',y') = \frac{\partial\phi'_2}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial\phi'_2}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial\phi'_2}{\partial y'}\frac{\partial y'}{\partial t} \tag{13}$$

Now we apply the lorentz transformations $x = \gamma(x' - vt')$, $y' = y$, and $t = \gamma(t' - vx')$

$$\begin{aligned}
\phi_1(t,x,y) =& \sigma(\gamma(x+vt),y)\cos(w\gamma(t+vx)+k) && (14)\\
\phi_2(t,x,y) =& \sigma(\gamma(x+vt),y)\sin(w\gamma(t+vx)+k) && (15)\\
& && (16)
\end{aligned}$$

and for π_1' and π_2', we have

$$\begin{aligned}
\pi_1(t,x,y) =& \gamma\frac{\partial\phi_1'}{\partial t'} + \gamma v\frac{\partial\phi_2'}{\partial x'} && (17)\\
=& -\omega\phi_2 + \gamma v\frac{\partial\sigma(\gamma(x+vt),y)}{\partial x}\cos(\omega\gamma(t+vx)+k) && (18)
\end{aligned}$$

$$\begin{aligned}
\pi_2(t,x,y) =& \gamma\frac{\partial\phi_1'}{\partial t'} + \gamma v\frac{\partial\phi_2'}{\partial x'} && (19)\\
=& \omega\phi_1 + \gamma v\frac{\partial\sigma(\gamma(x+vt),y)}{\partial x}\sin(\omega\gamma(t+vx)+k) && (20)
\end{aligned}$$

At t = 0 , we then have all together:

$$\begin{aligned}
\phi_1(t,x,y) =& \sigma(\gamma x,y)\cos(\omega\gamma vx+k) && (21)\\
\phi_2(t,x,y) =& \sigma(\gamma x,y)\sin(\omega\gamma vx+k) && (22)\\
\pi_1(t,x,y) =& -\omega\phi_2 + \gamma v\frac{\partial\sigma(\gamma x,y)}{\partial x}\cos(\omega\gamma vx+k) && (23)\\
\pi_2(t,x,y) =& \omega\phi_1 + \gamma v\frac{\partial\sigma(\gamma x,y)}{\partial x}\sin(\omega\gamma vx+k) && (24)
\end{aligned}$$

And this can be calculated numerically by the program. Notice the partial derivatives can be done using the np.grad function.

5 Endnote

Here are all the major steps of modelling a Qball. Now you can go and generate your own Qballs! For our results, you can refer to the slides that should be uploaded along my notes.