Fabcoin crypto crash course
Elliptic curve preliminaries

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FA Enterprise System, Inc.

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1 Groups

Review of modular arithmetic ($\mathbb{Z}/n\mathbb{Z}$)

Definition (Modular arithmetic notation)

Let $n \geq 0$. If a, b have same remainder when divided by n, we say that:

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Example

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Finding the number between 0 and $n-1$ as described above is called "reducing a number modulo n".

Lemma

Let $a_1 \equiv a_2 \mod n$ *and* $b_1 \equiv b_2 \mod n$.

- $a_1 \pm b_1 \equiv a_2 \pm b_2 \mod n$ *(Mod. arithm. respects addition).*
- $a_1 \cdot b_1 \equiv a_2 \cdot b_2 \mod n$ *(Mod. arithm. respects multiplication).*

Proof. [Mult. respected].

Since $a_1 \equiv a_2 \mod n \Rightarrow a_1 = n \cdot p + a_2$ for some p.
Since $b_1 \equiv b_2 \mod n \Rightarrow b_1 = n \cdot q + b_2$ for some q.

$$\begin{aligned} a_1 \cdot b_1 &= (n \cdot p + a_2) \cdot (n \cdot q + b_2) \\ &= n^2(p+q) + n(b_2 + a_2) + b_2 + a_2 \\ &\equiv b_2 + a_2 \mod n \end{aligned}$$

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Example

Reduce $2030 \cdot 201800003 \mod 2018$.

$$\begin{aligned} 2030 &= 2018 + 12 \equiv 12 && \mod 2018 \\ 201800003 &= 20180000 + 3 = 2018 \cdot 10^4 + 3 \equiv 3 && \mod 2018 \\ 2030 \cdot 201800003 &\equiv 12 \cdot 3 = 36 && \mod 2018 \end{aligned}$$

Definition (Group, mathematics)

A group $\mathcal{G}$ is a set equipped with operation $\cdot$ with $a \cdot b \in \mathcal{G}$ so that:

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for every $a, b, c \in \mathcal{G}$. (Associativity)
- There exists $e \in \mathcal{G}$ with $e \cdot a = a \cdot e = a$ for every $a \in \mathcal{G}$. (Identity)
- For every a exists $b \in \mathcal{G}$ s.t. $a \cdot b = e$. Write $b = a^{-1}$. (Inverse)

Definition (Abelian (commutative) group)

The group is called abelian (commutative) if in addition:

- $a \cdot b = b \cdot a$ for all $a, b \in \mathcal{G}$.

- $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- Exists e s.t. $e \cdot a = a \cdot e = a$.
- Given a exists b s.t. $a \cdot b = e$.
- $a \cdot b = b \cdot a$.

Example

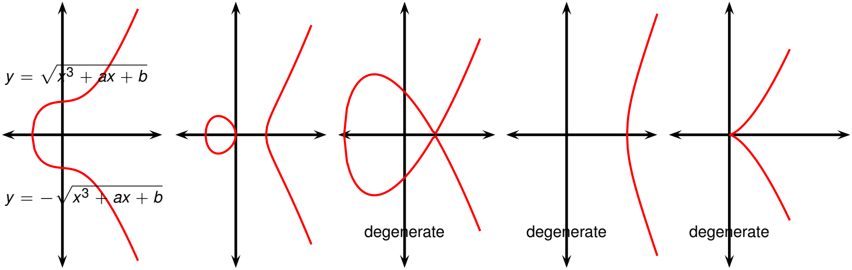
Take $G = \mathbb{Z}$, define $a \cdot b = a + b$.

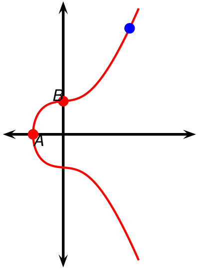
- $(a + b) + c = a + (b + c)$.
- Set $e = 0$. Then $0 + a = a + 0 = a$.
- For every a, take $b = -a$. Then $a + b = a + (-a) = 0 = e$.
- $a + b = b + a$ for all a, b.

Definition

The set of points $\{(x,y)\}$ for which
$$y^2 = x^3 + ax + b$$
is called an elliptic curve (possibly degenerate).

- Precise definition of all curves that are "elliptic": outside our scope.
- Precise definition of "degenerate": outside our scope.
- We do not fix the number types of x, y: possibly $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}_p, \dots$.
- Over $\mathbb{R}$, graph of elliptic has five types:

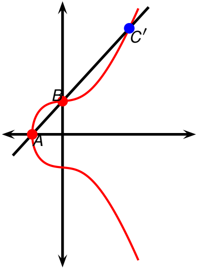




Definition (Elliptic curve group law)

Let $A = (x_A, y_A)$, $B = (x_B, y_B)$ - points on non-degenerate elliptic curve:

$$y^2 = x^3 + ax + b.$$



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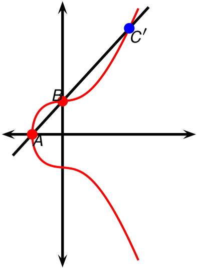

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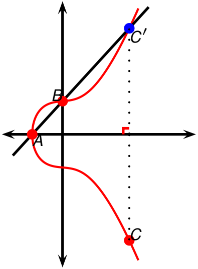

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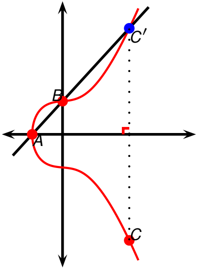

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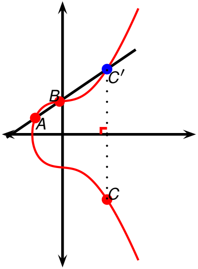

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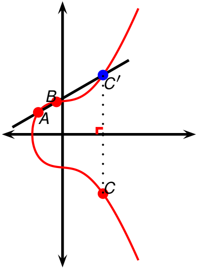

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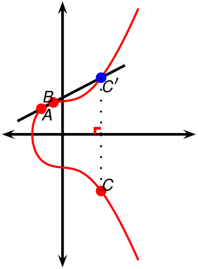

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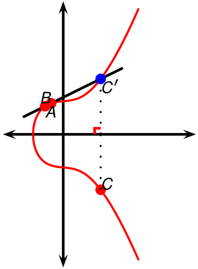

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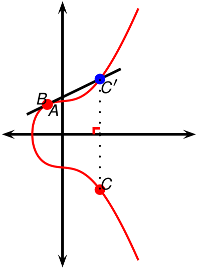

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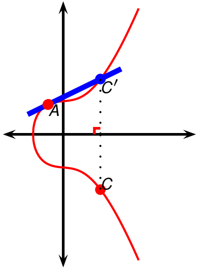

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- If line through A, B non-vertical, define $A \cdot B = C$.
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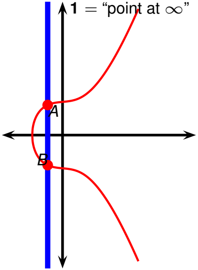

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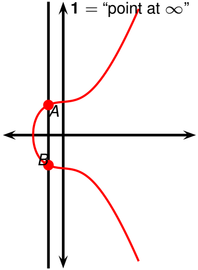

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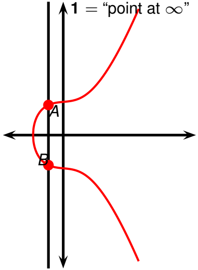

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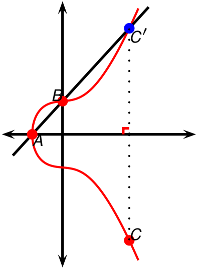

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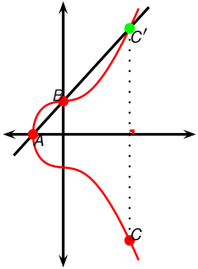
WARNING. Many authors use $+$ in place of $\cdot$ and $\mathbf{0}$ in place of $\mathbf{1}$.



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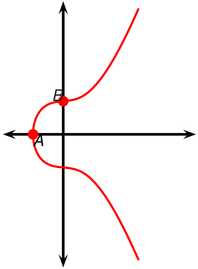

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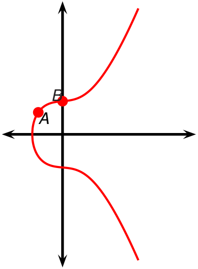
- $\cdot$ turns the points on the curve into a group.



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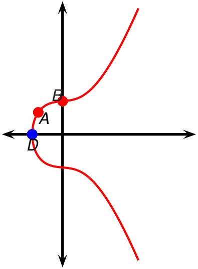

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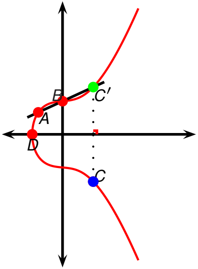

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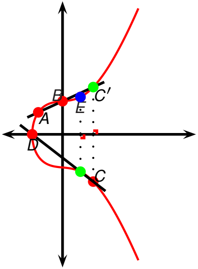

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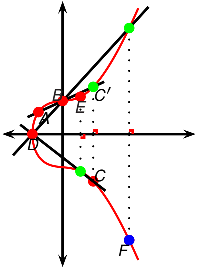

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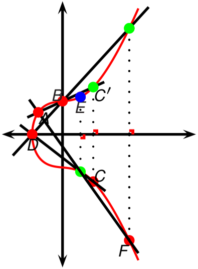

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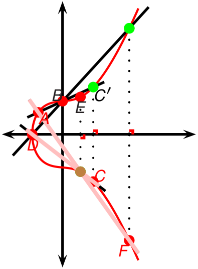

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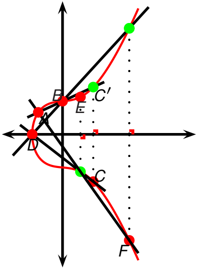

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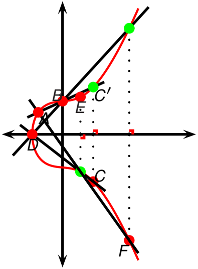

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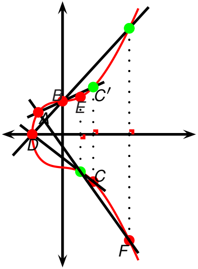

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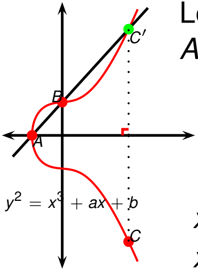
- An answer to why this all works is beyond current scope.

Niels Henrik Abel (1802-1829), pioneer of modern algebra and elliptic functions. Abelian groups are named after him.

Karl Weierstrass (1815-1897), pioneer of elliptic functions. The definition of elliptic curve given in our text is sometimes called "Weierstrass normal form".



Let $A = (x_A, y_A)$, $B = (x_B, y_B)$. Let s be the slope of line AB. Let x_C, y_C be the unknown coordinates of C.

$$s = \text{line slope}$$
$$\frac{-y - y_A}{x - x_A} = s$$
$$-y = s(x - x_A) + y_A \qquad (*)$$
$$x^3 + ax + b = (-y)^2 \qquad (**)$$
$$x^3 + ax + b = (s(x - x_A) + y_A)^2$$

$$\begin{aligned}
x^3 + ax + b - \left(s^2(x - x_A)^2 + 2sy_A(x - x_A) + y_A^2\right) &= 0\\
x^3 - s^2x^2 + 2s^2xx_A + s^2x_A^2 - 2sy_Ax + 2sy_Ax_A - y_A^2 + ax + b &= 0\\
x^3 - x^2s^2 + x(2s^2x_A - 2sy_A + a) + s^2x_A^2 + 2sy_Ax_A - y_A^2 + b &= 0
\end{aligned}$$

Setting $x = x_A$ solves $(*), (**)$. Similarly setting $x = x_B$ solves $(*), (**)$. Since $(**)$ is cubic $\Rightarrow$ its unknown 3^{rd} root is x_C. By Vieta's formulas,

$$\begin{aligned}
x_A + x_B + x_C &= s^2\\
x_C &= s^2 - x_A - x_B\\
y_C &= -s(x_C - x_A) - y_A
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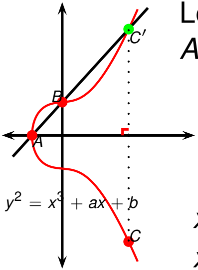

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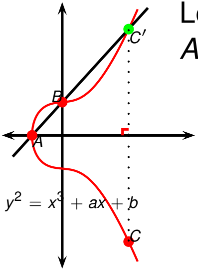

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(Elliptic curve group law, algebraic definition)

Let $(x_A, y_A), (x_B, y_B)$ *be two points on the elliptic curve.*

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- Full definition of field: outside of present scope.

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Let $y^2 = x^3 - x + 1$. Show $g = (3, 5)$ is a point on the curve. Compute $g^2 = g \cdot g$ and $g^3 = g \cdot g \cdot g$.

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- $x_2 = ?$

Product: $s = \begin{cases} \dfrac{y_B - y_A}{x_B - x_A} & \text{if } x_A \neq x_B \\ \dfrac{3x_A^2 + a}{2y_A} & \text{if } A = B \end{cases}$, $\begin{array}{rcl} y_C & = & -s(x_C - x_A) - y_A \\ x_C & = & s^2 - x_A - x_B \end{array}$

Example

Let $y^2 = x^3 - x + 1$. Show $g = (3,5)$ is a point on the curve. Compute $g^2 = g \cdot g$ and $g^3 = g \cdot g \cdot g$.

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