Diskretne strukture UNI

Vaje 6

1. Dane so množice $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ in $C = \{0, 1, 4, 5\}$. Določi naslednje množice:

   (a) $C + (A \cup C)$,

   (b) $(B \setminus A) \cap C$,

   (c) $\mathcal{P}(A \cap B) \setminus B$,

   (d) $\mathcal{P}(A \cap C) + \mathcal{P}(B \cap C)$.

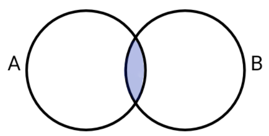
(a) $C + (A \cup C) = \{0,1,4,5\} + \left(\{1,2,3\} \cup \{0,1,4,5\}\right) = \{0,1,4,5\} + \{0,1,2,3,4,5\} = \underline{\underline{\{2,3\}}}$

(b) $(B \setminus A) \cap C = \left(\{2,3,4\} \setminus \{1,2,3\}\right) \cap \{0,1,4,5\} = \{4\} \cap \{0,1,4,5\} = \underline{\underline{\{4\}}}$

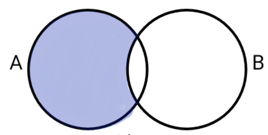
(c) $\mathcal{P}(A \cap B) \setminus B = \mathcal{P}(\{2,3\}) \setminus \{2,3,4\} = \{\emptyset, \{2\}, \{3\}, \{2,3\}\} \setminus \{2,3,4\} = \underline{\underline{\{\emptyset, \{2\}, \{3\}, \{2,3\}\}}}$

(d) $\mathcal{P}(A \cap C) + \mathcal{P}(B \cap C) = \mathcal{P}(\{1,2,3\} \cap \{0,1,4,5\}) + \mathcal{P}(\{2,3,4\} \cap \{0,1,4,5\}) =$

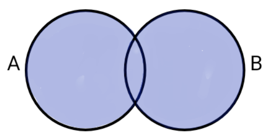
$= \mathcal{P}(\{1\}) + \mathcal{P}(\{4\}) = \{\emptyset, \{1\}\} + \{\emptyset, \{4\}\} = \underline{\underline{\{\{1\}, \{4\}\}}}$



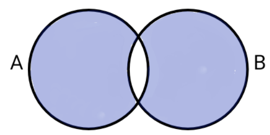
$A \cap B$



$A \setminus B$



$A \cup B$



$A + B$

2. Na ravni elementov pokaži, da velja

   (a) $A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$,

   (b) $A \subseteq B \Leftrightarrow A \cap B = A$.

(a) $A \subseteq B \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

Naj bo $A \subseteq B$ in $M \in \mathcal{P}(A)$. Potem je $M \subseteq A$. Za poljuben $x \in M$ velja, da je $x \in A$. Ker je $A \subseteq B$, je tudi $x \in B$. Torej je $M \subseteq B$ in zato je $M \in \mathcal{P}(B)$. Velja torej $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. ∎

(b) $A \subseteq B \Leftrightarrow A \cap B = A$

($\Rightarrow$) Naj bo $A \subseteq B$.

- $A \cap B \subseteq A$

  Naj bo $x \in A \cap B$. Potem je $x \in A$ in $x \in B$. Torej je $x \in A$ in zato $A \cap B \subseteq A$.

- $A \subseteq A \cap B$

  Naj bo $x \in A$ poljuben. Ker je $A \subseteq B$, je $x \in B$. Torej je $x \in A \cap B$ in zato $A \subseteq A \cap B$.

Ker je $A \cap B \subseteq A$ in $A \subseteq A \cap B$, je $A = A \cap B$.

($\Leftarrow$) Naj bo $A \cap B = A$.

Naj bo $x \in A$ poljuben. Ker je $A = A \cap B$, je $x \in A \cap B$. Torej je $x \in A$ in $x \in B$. Zato je $A \subseteq B$. ∎

3. Ali velja

(a) $(A+B)\setminus A = B\setminus A$,
(b) $(A+B)+(A+C) = A+(B+C)$,
(c) $(A\setminus B)+(C\setminus B) = (A+C)\setminus B$,
(d) $(A+C)\setminus(A+B) = (A\cap B)+C$,
(e) $(A+C)\setminus(A+B) = (A\cap B)+C$ pod pogojem $C\subseteq A\cap B$,
(f) $(A+C)\setminus(A+B) \subseteq (A\cap B)+C$,
(g) $(A+B)\setminus C \;\subseteq\; (B\setminus(A+C))\cup(A\setminus(B\cup C))$,
(h) $(A+B)\setminus C \;=\; (B\setminus(A+C))\cup(A\setminus(B\cup C))$, če sta A in B disjunktni,
(i) $(A+B)\setminus C = (A\cup C)+(A\cup B)$,
(j) $(A\cap C)+(B\cap C) = C\setminus(A\cap B)$,
(k) $(A\cap C)+(B\cap C) \subseteq C\setminus(A\cap B)$,
(l) $(A\cap C)+(B\cap C) = C\setminus(A\cap B)$, če je $C\subseteq A\cup B$,
(m) $(A\setminus C)+B = (A+B)\setminus C$,
(n) $(A\setminus C)+B \subseteq (A+B)\setminus C$,
(o) $(A\setminus C)+B = (A+B)\setminus C$, če je $C\subseteq A\setminus B$?

(a) $(A+B)\setminus A = B\setminus A$

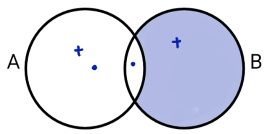
$A+B = A\cap B^c \cup A^c\cap B$

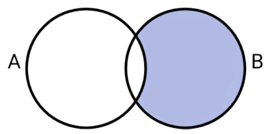
$A\setminus B = A\cap B^c$

$A\cap A = A$

$A\cap A^c = \emptyset$

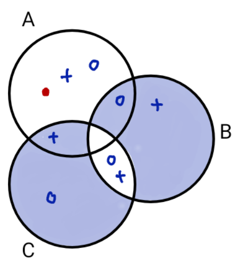
$A\cup\emptyset = A$

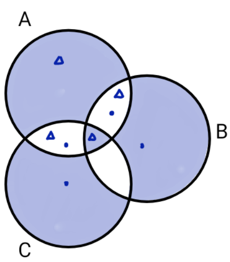




$L = (A\cap B^c \cup A^c\cap B)\cap A^c = \underbrace{A\cap B^c\cap A^c}_{\emptyset} \cup A^c\cap B\cap A^c = A^c\cap B = B\setminus A = D$

(b) $(A+B)+(A+C) = A+(B+C)$





$A = \{1\}$

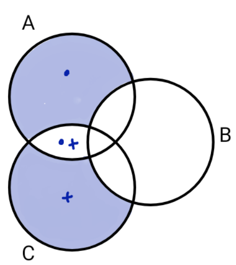
$B = C = \emptyset$

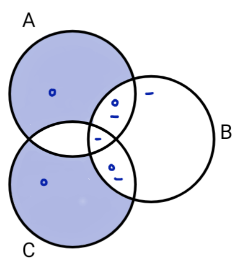
$L = (\{1\}+\emptyset)+(\{1\}+\emptyset) = \{1\}+\{1\} = \emptyset$

$D = \{1\}+(\emptyset+\emptyset) = \{1\}+\emptyset = \{1\}$

$L \neq D$

(c) $(A\setminus B)+(C\setminus B) = (A+C)\setminus B$





$L = A\cap B^c + C\cap B^c = (A\cap B^c)\cap(C\cap B^c)^c \cup (A\cap B^c)^c\cap(C\cap B^c) =$

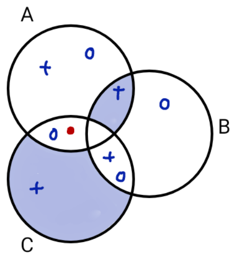
$= A\cap B^c\cap(C^c\cup B) \cup (A^c\cup B)\cap(C\cap B^c) =$

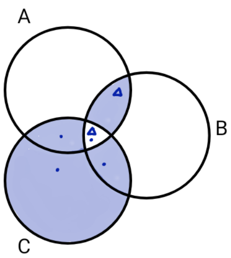
$= A\cap B^c\cap C^c \cup \underbrace{A\cap B^c\cap B}_{\emptyset} \cup A^c\cap C\cap B^c \cup \underbrace{B\cap C\cap B^c}_{\emptyset} =$

$= A\cap B^c\cap C^c \cup A^c\cap B^c\cap C = (\underbrace{A\cap C^c \cup A^c\cap C}_{A+C})\cap B^c =$

$= (A+C)\cap B^c = (A+C)\setminus B = D$

(d) $(A+C)\setminus(A+B) = (A\cap B)+C$





$A=\{1\}$

$B=\emptyset$

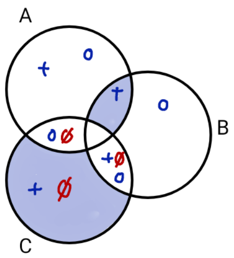
$C=\{1\}$

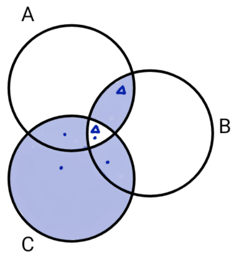
$L=(\{1\}+\{1\})\setminus(\{1\}+\emptyset)=\emptyset\setminus\{1\}=\emptyset$

$D=(\{1\}\cap\emptyset)+\{1\}=\emptyset+\{1\}=\{1\}$

$L\neq D$

(e) $(A+C)\setminus(A+B) = (A\cap B)+C$ pod pogojem $C\subseteq A\cap B$





$L=(A\cap C^c \cup A^c\cap C)\cap(A\cap B^c\cup A^c\cap B)^c=$

$=(A\cap C^c \cup A^c\cap C)\cap((A\cap B^c)^c\cap(A^c\cap B)^c)=$

$=(A\cap C^c \cup A^c\cap C)\cap((A^c\cup B)\cap(A\cup B^c))=$

$=(A\cap C^c \cup A^c\cap C)\cap(\underbrace{A^c\cap A}_{\emptyset}\cup A^c\cap B^c\cup B\cap A\cup \underbrace{B\cap B^c}_{\emptyset})=$

$=(A\cap C^c \cup A^c\cap C)\cap(A^c\cap B^c\cup A\cap B)=$

$=\underbrace{A\cap C^c\cap A^c\cap B^c}_{\emptyset}\cup A\cap C^c\cap \cancel{A}\cap B\cup A^c\cap C\cap\cancel{A^c}\cap B^c\cup\underbrace{A^c\cap C\cap A\cap B}_{\emptyset}=\underline{\underline{(A\cap B\cap C^c)\cup(A^c\cap B^c\cap C)}}$

$D=A\cap B\cap C^c\cup(A\cap B)^c\cap C=(A\cap B\cap C^c)\cup(A^c\cup B^c)\cap C=(A\cap B\cap C^c)\cup(\overset{B,B^c}{A^c\cap C})\cup(\overset{A,A^c}{B^c\cap C})=$

$=(A\cap B\cap C^c)\cup(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)\cup(A^c\cap B^c\cap C)=$

$=\underline{\underline{(A\cap B\cap C^c)\cup(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)}}$

Če je $C\subseteq A\cap B$, je $C\cap(A\cap B)^c=\emptyset$, torej je $C\cap(A^c\cup B^c)=(C\cap A^c)\cup(C\cap B^c)=$

$=(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)\cup(A^c\cap B^c\cap C)=$

$=(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)=\emptyset.$

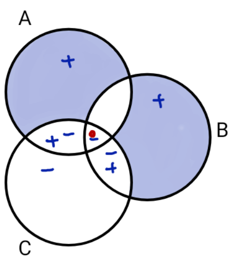
Sledi, da je $A^c\cap B\cap C=\emptyset$, $A^c\cap B^c\cap C=\emptyset$, $A\cap B^c\cap C=\emptyset$ in zato $L=(A\cap B\cap C^c)\cup\underbrace{(A^c\cap B^c\cap C)}_{0}=A\cap B\cap C^c$ ter

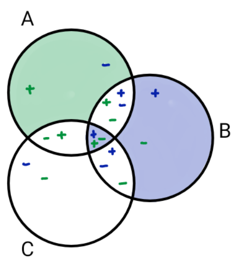
$D=(A\cap B\cap C^c)\cup(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)=A\cap B\cap C^c=L.$

(f) $(A+C)\setminus(A+B)\subseteq(A\cap B)+C$

$L=(A\cap B\cap C^c)\cup(A^c\cap B^c\cap C)\subseteq(A\cap B\cap C^c)\cup(A^c\cap B\cap C)\cup(A^c\cap B^c\cap C)\cup(A\cap B^c\cap C)=D\ \Rightarrow\ L\subseteq D$

(g) $(A+B)\setminus C\ \subseteq\ (B\setminus(A+C))\cup(A\setminus(B\cup C))$





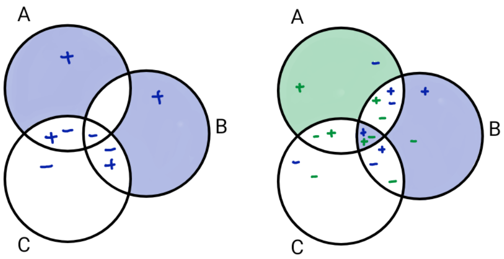
$A=B=C=\{1\}$

$L=(\{1\}+\{1\})\setminus\{1\}=\emptyset\setminus\{1\}=\emptyset$

$D=\{1\}\setminus(\{1\}+\{1\})\cup\{1\}\setminus(\{1\}\cap\{1\})=\{1\}\setminus\emptyset\cup\{1\}\setminus\{1\}=\{1\}\cup\emptyset=\{1\}$

$L\neq D$

(h) $(A + B) \setminus C = (B \setminus (A + C)) \cup (A \setminus (B \cup C))$, če sta A in B disjunktni



$L = (A \cap B^c \cup A^c \cap B) \cap C^c = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c)$

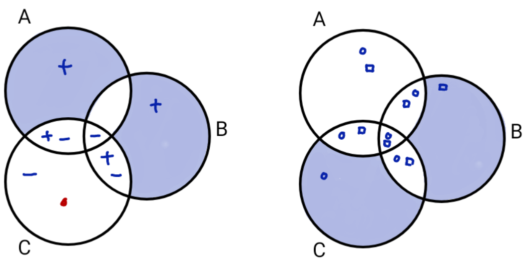
$D = B \cap (A \cap C^c \cup A^c \cap C)^c \cup A \cap (B \cup C)^c =$

$= B \cap ((A^c \overset{+}{\cup} C) \overset{\bullet}{\cap} (A \overset{+}{\cup} C^c)) \cup A \cap (B^c \cap C^c) =$

$= B \overset{\bullet}{\cap} (\underbrace{A^c \cap A}_{\emptyset} \cup A^c \cap C \overset{+}{\cup} C \cap A \cup \underbrace{C \cap C^c}_{\emptyset}) \cup (A \cap B^c \cap C^c) =$

$= (A^c \cap B \cap C) \cup \underbrace{(\underbrace{A \cap B}_{\emptyset} \cap C)}_{\emptyset} \cup (A \cap B^c \cap C^c) \underset{A \cap B = \emptyset}{=} (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) = L$

(i) $(A \overset{+}{+} B) \setminus \overset{-}{C} = (A \overset{\circ}{\cup} C) + (A \overset{\square}{\cup} B)$



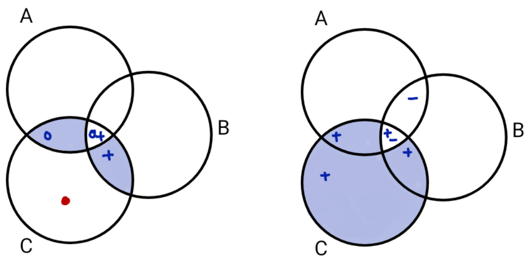
$A = B = \emptyset, \; C = \{1\}$

$L = (\emptyset + \emptyset) \setminus \{1\} = \emptyset \setminus \{1\} = \emptyset$

$D = (\emptyset \cup \{1\}) + (\emptyset \cup \emptyset) = \{1\} + \emptyset = \{1\}$

$L \neq D$

(j) $(A \overset{\circ}{\cap} C) + (B \overset{+}{\cap} C) = \overset{+}{C} \setminus (A \overset{-}{\cap} B)$



$A = B = \emptyset, \; C = \{1\}$

$L = (\emptyset \cap \{1\}) + (\emptyset \cap \{1\}) = \emptyset + \emptyset = \emptyset$

$D = \{1\} \setminus (\emptyset \cap \emptyset) = \{1\} \setminus \emptyset = \{1\}$

$L \neq D$

(k) $(A \cap C) + (B \cap C) \subseteq C \setminus (A \cap B)$

$L = \cdots = (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \subseteq (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C) = \cdots = D$

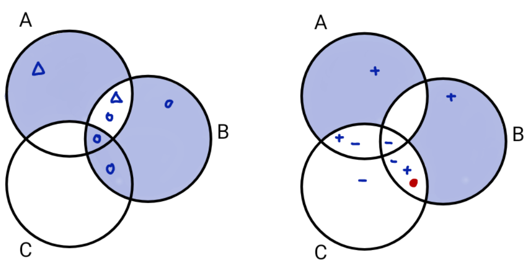
$L \subseteq D$

(l) $(A \cap C) + (B \cap C) = C \setminus (A \cap B)$, če je $C \subseteq A \cup B$

Če je $C \subseteq A \cup B$, potem je $C \cap (A \cup B)^c = \emptyset$, torej je $C \cap A^c \cap B^c = \emptyset$ in je $D = (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup \overbrace{(A^c \cap B^c \cap C)}^{\emptyset}$

$= (A \cap B^c \cap C) \cup (A^c \cap B \cap C) = L.$

(m) $(A \overset{\triangle}{\setminus} C) + \overset{\circ}{B} = (A \overset{+}{+} B) \setminus \overset{-}{C}$



$A = \emptyset, \; B = C = \{1\}$

$L = (\emptyset \setminus \{1\}) + \{1\} = \emptyset + \{1\} = \{1\}$

$D = (\emptyset + \{1\}) \setminus \{1\} = \{1\} \setminus \{1\} = \emptyset$

$L \neq D$

(n) $(A \setminus C) + B \subseteq (A + B) \setminus C$

$$L = A \cap C^c + B = (A \cap C^c) \cap B^c \cup (A \cap C^c)^c \cap B = (A \cap B^c \cap C^c) \cup (A^c \overset{+}{\cup} C) \overset{\cdot}{\cap} B =$$

$$= (A \cap B^c \cap C^c) \cup (A^c \overset{C,\,C^c}{\cap} B) \cup (\overset{A,\,A^c}{B} \cap C) = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B \cap C) \cup \cancel{(A^c \cap B \cap C)} =$$

$$= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C) \cup (A^c \cap B \cap C^c) \cup (A \cap B \cap C)$$

$$D = (A \cap B^c \cup A^c \cap B) \cap C^c = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \subseteq L$$

$\Rightarrow D \subseteq L$

Ni pa $L \subseteq D$: za $A = \emptyset$, $B = C = \{1\}$ je $L = \{1\}$ in $D = \emptyset$, torej $L \nsubseteq D$.

(o) $(A \setminus C) + B = (A + B) \setminus C$, če je $C \subseteq A \setminus B$

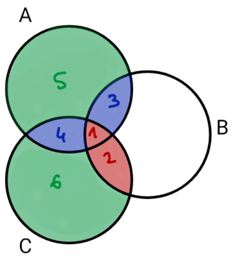
Če je $C \subseteq A \setminus B = A \cap B^c$, je $C \cap (A \cap B^c)^c = \emptyset$, torej je

$$C \cap (A^c \cup B) = (A^c \cap C) \cup (B \cap C) = (A^c \cap B \cap C) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C) \cup (A^c \cap B \cap C) = \emptyset,$$

zato je $A^c \cap B \cap C = \emptyset$, $A^c \cap B^c \cap C = \emptyset$, $A \cap B \cap C = \emptyset$ in $A^c \cap B \cap C = \emptyset$. Sledi:

$$L = (A \cap B^c \cap C^c) \cup \underbrace{(A^c \cap B \cap C)}_{\emptyset} \cup (A^c \cap B \cap C^c) \cup \underbrace{(A \cap B \cap C)}_{\emptyset} = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) = D.$$

4. Pokaži, da množice $\overset{A_1}{\underline{B \cap C}}$, $\overset{A_2}{\underline{(B + C) \cap A}}$ in $\overset{A_3}{\underline{(A + C) \setminus B}}$ predstavljajo razbitje za množico $\underbrace{A \cup C}_{M}$.



Preveriti moramo: $A_i \cap A_j = \emptyset$ za vse $i \neq j$, $i,j \in \{1,2,3\}$ ter $A_1 \cup A_2 \cup A_3 = M$.

Najprej poenostavimo:

$A_1 = B \cap C = \overset{1}{(A \cap B \cap C)} \cup \overset{2}{(A^c \cap B \cap C)}$,

$A_2 = (B \cap C^c \cup B^c \cap C) \cap A = \overset{3}{(A \cap B \cap C^c)} \cup \overset{4}{(A \cap B^c \cap C)}$,

$A_3 = (A \cap C^c \cup A^c \cap C) \cap B^c = \overset{5}{(A \cap B^c \cap C^c)} \cup \overset{6}{(A^c \cap B^c \cap C)}$.

$$A_1 \cap A_2 = (B \cap C) \cap \big((A \cap B \cap C^c) \cup (A \cap B^c \cap C)\big) = \underbrace{B \cap C \cap A \cap B \cap C^c}_{\emptyset} \cup \underbrace{B \cap C \cap A \cap B^c \cap C}_{\emptyset} = \emptyset \cup \emptyset = \emptyset$$

Podobno $A_1 \cap A_3 = \emptyset$ in $A_2 \cap A_3 = \emptyset$.

$$A_1 \cup A_2 \cup A_3 = (A \cap B \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A \cap B^c \cap C^c) \cup (A^c \cap B^c \cap C)$$

$$= A \cap C \cap \underbrace{(B \cup B^c)}_{\mathcal{U}} \cup A^c \cap C \cap \underbrace{(B \cup B^c)}_{\mathcal{U}} \cup A \cap C^c \cap \underbrace{(B \cup B^c)}_{\mathcal{U}} =$$

$$= A \cap C \cup A^c \cap C \cup A \cap C^c = \underbrace{(A \cup A^c)}_{\mathcal{U}} \cap C \cup A \cap C^c = C \cup (A \cap C^c) = (C \cup A) \cap \underbrace{(C \cup C^c)}_{\mathcal{U}} = A \cup C. \blacksquare$$