# Diskretne strukture UNI
## Vaje 3

1. (a) Pokaži, da tromestni veznik $A(p,q,r) \equiv r \Rightarrow (\neg p \wedge \neg q)$ predstavlja poln nabor veznikov.

   (b) Zaporedje izrazov $A_n$ je definirano rekurzivno z

$$\begin{aligned} A_0 &= \neg p \\ A_n &= A(p, A_{n-1}, 1). \end{aligned}$$

   Izračunaj $A_{2019}$.

a) $A(0,0,0) = 0 \Rightarrow (\neg 0 \wedge \neg 0) \sim 1$

$A(1,1,1) = 1 \Rightarrow (\neg 1 \wedge \neg 1) \sim 1 \Rightarrow \neg 1 \sim 1 \Rightarrow 0 \sim 0$

Ne ohranja konstant. Morda je poln.

$\mathcal{P} = \{\neg, \Rightarrow\}$ $\quad \mathcal{N} = \{A\}$

Veznike iz $\mathcal{P}$ izrazimo z vezniki iz $\mathcal{N}$:

$A(p,p,p) = p \Rightarrow \neg p \wedge \neg p \sim p \Rightarrow \neg p \sim \neg p \vee \neg p \sim \neg p$

- $\underline{\underline{\neg p \sim A(p,p,p)}}$
- $\underline{\underline{p \Rightarrow q}} \sim p \Rightarrow q \wedge q \sim p \Rightarrow \neg(\neg q) \wedge \neg(\neg q) \sim A(\neg q, \neg q, p) \sim \underline{\underline{A(A(q,q,q), A(q,q,q), p)}}$

$\Rightarrow \{A\}$ je poln nabor.

b) $A_0 = \neg p$

$A(p,q,r) \equiv r \Rightarrow (\neg p \wedge \neg q)$

$A_n = A(p, A_{n-1}, 1).$

$A_1 = A(p, A_0, 1) = 1 \Rightarrow (\neg p \wedge \neg(\neg p)) \sim 1 \Rightarrow (\neg p \wedge p) \sim 1 \Rightarrow 0 \sim 0$

$A_2 = A(p, A_1, 1) = 1 \Rightarrow (\neg p \wedge \neg 0) \sim 1 \Rightarrow \neg p \sim \neg 1 \vee \neg p \sim 0 \vee \neg p \sim \neg p$

$A_3 = A(p, \underbrace{A_2}_{= A_0}, 1) = 0$

$A_4 = A(p, \underbrace{A_3}_{= A_1}, 1) = \neg p$

$\vdots$

$A_{2k} = \neg p \qquad A_{2k+1} = 0$

$\underline{\underline{A_{2019} = 0}}$

2. Naj bo $A$ veznik $A(p,q,r) \equiv (p \vee q) \Rightarrow r$.

   (a) Kateri izmed naborov $\{A\}$, $\{A,1\}$, $\{A,0\}$, $\{A,\neg\}$ so polni?

   (b) Zaporedje izrazov $A_n$ je definirano rekurzivno z

$$\begin{aligned} A_0 &= \neg p \\ A_1 &= \neg q \\ A_n &= A(p,q,A_{n-1} \wedge A_{n-2}) \end{aligned}$$

   Izračunaj $A_{2019}$.

a) $A(0,0,0) = 0 \vee 0 \Rightarrow 0 \sim 0 \Rightarrow 0 \sim 1$

$A(1,1,1) = 1 \vee 1 \Rightarrow 1 \sim 1 \Rightarrow 1 \sim 1$

$\Rightarrow$ $A$ ohranja enice $\Rightarrow$ $\{A\}$ in $\{A,1\}$ nista polna, ker ohranjata enice.

$\mathcal{N} = \{A, 0\}$, $\mathcal{P} = \{\neg, \Rightarrow\}$

$\mathcal{P}$ izrazimo z $\mathcal{N}$:

- $\underline{\underline{\neg p}} \sim p \Rightarrow 0 \sim p \vee p \Rightarrow 0 \sim \underline{\underline{A(p,p,0)}}$
- $\underline{\underline{p \Rightarrow q}} \sim p \vee p \Rightarrow q \sim \underline{\underline{A(p,p,q)}}$

$\Rightarrow$ $\{A,0\}$ je poln.

$\mathcal{N} = \{A, \neg\}$ $\mathcal{P} = \{\neg, \Rightarrow\}$

$\mathcal{P}$ izrazimo z $\mathcal{N}$:

- $\underline{\underline{\neg p}} \sim \underline{\underline{\neg p}}$
- $\underline{\underline{p \Rightarrow q}} \sim p \vee p \Rightarrow q \sim \underline{\underline{A(p,p,q)}}$

$\Rightarrow$ $\{A,\neg\}$ je poln.

$A(p,q,r) \equiv (p \vee q) \Rightarrow r$.

$$\begin{aligned} A_0 &= \neg p \\ A_1 &= \neg q \\ A_n &= A(p,q,A_{n-1} \wedge A_{n-2}) \end{aligned}$$

b) $A_0 = \neg p$

$A_1 = \neg q$

$A_2 = A(p,q, A_1 \wedge A_0) = A(p,q,\neg q \wedge \neg p) = p \vee q \Rightarrow \neg p \wedge \neg q \sim p \vee q \Rightarrow \neg(p \vee q) \sim \neg(p \vee q) \vee \neg(p \vee q) \sim$

$\sim \neg(p \vee q) \sim \neg p \wedge \neg q$

$A_3 = A(p,q, A_2 \wedge A_1) = A(p,q,\neg p \wedge \neg q \wedge \neg q) = A(p,q,\neg p \wedge \neg q) = \neg p \wedge \neg q$

$A_4 = A(p,q, A_3 \wedge A_2) = A(p,q,\neg p \wedge \neg q \wedge \neg p \wedge \neg q) = A(p,q,\neg p \wedge \neg q) = \neg p \wedge \neg q$

$\vdots$

$\underline{\underline{A_{2019} = \neg p \wedge \neg q}}$

3. Veznik $A$ je definiran s predpisom $A(p,q,r) \equiv (p \wedge q) \vee (\neg p \wedge \neg r)$.

(a) Samo z veznikom $A$ zapiši izraze $1$, $p \wedge q$ in $p \Rightarrow q$.

(b) Kateri izmed naborov $\{A\}$, $\{A,1\}$, $\{A,0\}$, $\{A,\Rightarrow\}$, $\{A,\veebar\}$ so polni?

(c) Zaporedje izrazov $I_n$ je definirano rekurzivno s predpisi

$$\begin{aligned} I_0 &= \neg p \\ I_1 &= p \\ I_n &= A(I_{n-1}, I_{n-2}, I_{n-2}) \end{aligned}$$

Izračunaj $I_{2019}$

a) $\underline{\underline{A(p,p,p)}} = (p \wedge p) \vee (\neg p \wedge \neg p) \sim p \vee \neg p \sim \underline{\underline{1}}$

$$A(p,q,r) \equiv \underbrace{(p \wedge q)}_{p \wedge q} \vee \underbrace{(\neg p \wedge \neg r)}_{0 \sim \neg p \wedge 0 \sim \neg p \wedge \neg 1}.$$

$A(p,q,1) \sim p \wedge q \vee \neg p \wedge \neg 1 \sim p \wedge q \vee 0 \sim \underline{\underline{p \wedge q \sim A(p,q,A(p,p,p))}}$

$A(p,q,r) \equiv (p \wedge q) \vee (\neg p \wedge \neg r) \sim \neg(p \vee r) \vee p \wedge q \sim \underbrace{\overset{p}{p} \vee \overset{p}{r}}_{p} \Rightarrow \underbrace{p \wedge q}_{q}$

$A(p,q,p) \sim p \vee p \Rightarrow p \wedge q \sim p \Rightarrow p \wedge q \sim \neg p \vee (p \wedge q) \sim \underbrace{(\neg p \vee p)}_{1} \wedge (\neg p \vee q) \sim \neg p \vee q \sim p \Rightarrow q$

$\underline{\underline{p \Rightarrow q \sim A(p,q,p)}}^{\star}$

b) $A(1,1,1) \sim 1 \Rightarrow \{A\}, \{A,1\}, \{A,\Rightarrow\}$ niso polni, ker (vsi vezniki) ohranjajo enice.

$\mathcal{N} = \{A,0\}$ $\quad \mathcal{P} = \{\neg, \Rightarrow\}$

$\mathcal{P}$ izrazimo z $\mathcal{N}$:

- $\underline{\underline{p \Rightarrow q \sim A(p,q,p)}}^{\star}$
- $\underline{\underline{\neg p}} \sim p \Rightarrow 0 \sim \underline{\underline{A(p,0,p)}}$ $\qquad \{A,0\}$ je poln nabor.

$\mathcal{N} = \{A,\veebar\}$ $\quad \mathcal{P} = \{\neg, \Rightarrow\}$

$\mathcal{P}$ izrazimo z $\mathcal{N}$:

- $\underline{\underline{p \Rightarrow q \sim A(p,q,p)}}^{\star}$
- $\underline{\underline{\neg p \sim p \veebar p}}$ $\qquad \{A,\veebar\}$ je poln nabor.

c) $I_0 = \neg p$

$I_1 = p$

$I_n = A(I_{n-1}, I_{n-2}, I_{n-2}) \qquad A(p,q,r) \equiv (p \wedge q) \vee (\neg p \wedge \neg r)$

$I_2 = A(I_1, I_0, I_0) = A(p, \neg p, \neg p) = (p \wedge \neg p) \vee (\neg p \wedge \neg\neg p) \sim p \wedge \neg p \sim 0$

$I_3 = A(I_2, I_1, I_1) = A(0, p, p) = (0 \wedge \neg p) \vee (\neg 0 \wedge \neg p) \sim 0 \vee \neg p \sim \neg p$

$I_4 = A(I_3, I_2, I_2) = A(\neg p, 0, 0) = (\neg p \wedge 0) \vee (\neg\neg p \wedge \neg 0) \sim 0 \vee p \sim p$

$I_5 = I_2 = 0$

$I_6 = I_3 = \neg p$

$\vdots$

$I_{3k} = \neg p \longleftarrow$ 2019 : 3 = 673 , 0 ost.

$I_{3k+1} = p \qquad I_{2019} = \neg p$

$I_{3k+2} = 0$

4. Kateri od naslednjih sklepov so pravilni?

(a) $p \vee q, \neg q \wedge r \Rightarrow \neg p \models q \vee r$,

(b) $p \Rightarrow q, r \Rightarrow s, p \vee r \models q \wedge s$,

(c) $p \wedge r, q \wedge p \Rightarrow \neg r \models \neg q$,

(d) $p \Rightarrow q, p \vee s, q \Rightarrow r, s \Rightarrow t, \neg r \models t$,

(e) $p \Rightarrow q, p \wedge s, q \wedge r \Rightarrow t, s \Rightarrow r \models t$,

(f) $p \Leftrightarrow q, \neg p, \neg(q \Rightarrow r) \vee t, s \vee t \Rightarrow r \models r \wedge \neg p$,

(a) $\overbrace{p \vee q}^{1 \checkmark}, \overbrace{\neg q \wedge r \Rightarrow \neg p}^{1 \checkmark} \models \overbrace{q \vee r}^{0 \checkmark}$,

$q \sim 0$

$r \sim 0$

$1 \wedge 0 \Rightarrow \neg p$

$1 \checkmark$

$p \vee 0$

$p \sim 1$

Pri $\overbrace{p \sim 1,\ q \sim r \sim 0}^{\text{protiprimer}}$ sta predpostavki pravilni, zaključek pa napačen. Sklep je napačen.

(b) $\overbrace{p \Rightarrow q}^{1 \checkmark}, \overbrace{r \Rightarrow s}^{1 \checkmark}, \overbrace{p \vee r}^{1 \checkmark} \models \overbrace{q \wedge s}^{0 \checkmark}$,

$q \sim 0$

$p \Rightarrow 0$

$p \sim 0$

$0 \vee r$

$r \sim 1$

$1 \Rightarrow s$

$s \sim 1$

Pri $p \sim q \sim 0$, $r \sim s \sim 1$ so predpostavke pravilne, zaključek pa napačen. Sklep je napačen.

(c) $\overbrace{p \wedge r}^{1 \checkmark}, \overbrace{q \wedge p \Rightarrow \neg r}^{1\ /\!/} \models \overbrace{\neg q}^{0 \checkmark}$,

$p \sim r \sim 1$

$q \sim 1$

$1 \wedge 1 \Rightarrow \neg 1$

$1 \Rightarrow 0\ /\!/$

Ne najdemo protiprimera. Sklep je pravilen. Dokaz:

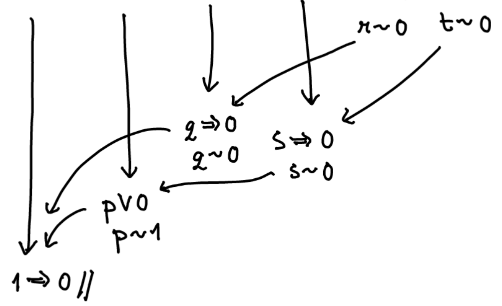
| | | |
|---|---|---|
| 1. $p \wedge r$ | predp. | $\models \neg q$ |
| 2. $q \wedge p \Rightarrow \neg r$ | predp. | |
| 3. $p$ | Po(1) | |
| 4. $r$ | Po(1) | |
| 5. $\neg(q \wedge p)$ | MT(2,4) | |
| 6. $\neg q \vee \neg p$ | $\sim$ 5. | |
| 7. $\neg q$ | DS(6,3) | |

$\boxed{\text{POENOSTAVITEV Po} \quad A \wedge B \models A}$

$\boxed{\text{MODUS TOLLENS} \quad A \Rightarrow B, \neg B \models \neg A}$

$\boxed{\text{DISJUNKTIVNI SILOGIZEM} \quad A \vee B, \neg A \models B}$

(d) $\overbrace{p \Rightarrow q}^{1 /\!/}, \overbrace{p \vee s}^{1 \checkmark}, \overbrace{q \Rightarrow r}^{1 \checkmark}, \overbrace{s \Rightarrow t}^{1 \checkmark}, \overbrace{\neg r}^{1 \checkmark} \models \overbrace{t}^{0 \checkmark},$




Dokaz:

$$\left.\begin{array}{l} 1.\ p \Rightarrow q \\ 2.\ p \vee s \\ 3.\ q \Rightarrow r \\ 4.\ s \Rightarrow t \\ 5.\ \neg r \end{array}\right\} \text{predp.} \qquad \models t$$

| | | |
|---|---|---|
| 6. | $\neg q$ | MT(3,5) |
| 7. | $\neg p$ | MT(1,6) |
| 8. | $s$ | DS(2,7) |
| 9. | $t$ | MP(4,8) |

MODUS PONENS

$A \Rightarrow B, A \models B$

(e) $p \Rightarrow q, p \wedge s, q \wedge r \Rightarrow t, s \Rightarrow r \models t,$

$$\left.\begin{array}{l} 1.\ p \Rightarrow q \\ 2.\ p \wedge s \\ 3.\ q \wedge r \Rightarrow t \\ 4.\ s \Rightarrow r \end{array}\right\} \text{predp.} \qquad \models t$$

| | | |
|---|---|---|
| 5. | $p$ | Po(2) |
| 6. | $s$ | Po(2) |
| 7. | $r$ | MP(4,6) |
| 8. | $q$ | MP(1,5) |
| 9. | $q \wedge r$ | Zd(7,8) |
| 10. | $t$ | MP(3,9) |

ZDRUŽITEV

$A, B \models A \wedge B$

★ $A \Leftrightarrow B \sim (A \Rightarrow B) \wedge (B \Rightarrow A)$

(f) $p \Leftrightarrow q, \neg p, \neg(q \Rightarrow r) \vee t, s \vee t \Rightarrow r \models r \wedge \neg p.$

$$\left.\begin{array}{l} 1.\ p \Leftrightarrow q^{\star} \\ 2.\ \neg p \\ 3.\ \neg(q \Rightarrow r) \vee t \\ 4.\ s \vee t \Rightarrow r \end{array}\right\} \text{predp.} \qquad \models r \wedge \neg p$$

| | | |
|---|---|---|
| 5. | $(p \Rightarrow q) \wedge (q \Rightarrow p)$ | $\sim$ 1. ★ |
| 6. | $p \Rightarrow q$ | Po(5) |
| 7. | $q \Rightarrow p$ | Po(5) |
| 8. | $\neg(\neg q \vee r) \vee t$ | $\sim$ 3. |
| 9. | $\neg q$ | MT(7,2) |
| 10. | $\neg q \vee r$ | Pr(9,r) |

PRIDRUŽITEV Pr

$A \models A \vee B$

11. $t$ DS(8, 10)

12. $s \vee t$ Pr(11, s)

13. $r$ MP(4, 12)

14. $r \wedge \neg p$ Zd(13, 2)

HIPOTETIČNI SILOGIZEM

$A \Rightarrow B,\ B \Rightarrow C \models A \Rightarrow C$