Diskretne strukture UNI
Vaje 12

1. Dane so permutacije

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 3 & 7 & 8 & 6 & 9 & 1 & 2 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 3 & 5 & 9 & 6 & 7 & 8 & 1 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 6 & 9 & 1 & 5 & 2 & 4 & 7 \end{pmatrix}.$$

   (a) Poišči ciklično strukturo, red in parnost permutacije γ.

   (b) Poišči vse možne ciklične strukture permutacije π, ki zadošča enačbi

$$\alpha * \beta * \pi^2 * \beta^{-1} = \gamma.$$

   (c) Za vsako možno ciklično strukturo poišči eno rešitev enačbe.

a) $\gamma = (1\ 3\ 6\ 5)(2\ 8\ 4\ 9\ 7)$

ciklična struktura: $4+5$

red: $\text{lcm}(4,5) = 20$

parnost: $3+4 = 7$ transpozicij $\Rightarrow$ liha

$\alpha = (1\ 5\ 8)(2\ 4\ 7\ 9)(3)(6)$

$\beta = (1\ 2\ 4\ 5\ 9)(3)(6)(7)(8)$

b) $\alpha * \beta * \pi^2 * \beta^{-1} = \gamma \quad / * \beta$

$\alpha * \beta * \pi^2 * \underbrace{\beta^{-1} * \beta}_{id} = \gamma * \beta$

$\alpha^{-1} * \backslash \quad \alpha * \beta * \pi^2 = \gamma * \beta$

$\underbrace{\alpha^{-1} * \alpha}_{id} * \beta * \pi^2 = \alpha^{-1} * \gamma * \beta$

$\beta^{-1} * \backslash \quad \beta * \pi^2 = \alpha^{-1} * \gamma * \beta$

$\underbrace{\beta^{-1} * \beta}_{id} * \pi^2 = \beta^{-1} * \alpha^{-1} * \gamma * \beta$

$\pi^2 = \beta^{-1} * \alpha^{-1} * \gamma * \beta$ ★

★ $\beta^{-1} * \alpha^{-1} * \gamma * \beta =$

$= (9\ 5\ 4\ 2\ 1) * (9\ 7\ 4\ 2)(8\ 5\ 1) * (1\ 3\ 6\ 5)(2\ 8\ 4\ 9\ 7) * (1\ 2\ 4\ 5\ 9) =$

$= (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$

$$\underbrace{\boxed{\pi^2 = (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)}}_{\text{cikl. str. } 3+3+3}$$

Kandidati za ciklične strukture za π: $3+3+3$, $3+6$, 9

$(3+3+3)^2 = 3^2 + 3^2 + 3^2 = 3+3+3$ ✓

$(3+6)^2 = 3^2 + 6^2 = 3 + 3+3$ ✓

$(9^2) = 9$ //

$\gcd(6,2) = 2$
$\Rightarrow 6^2$ razpade na 2 3-cikla

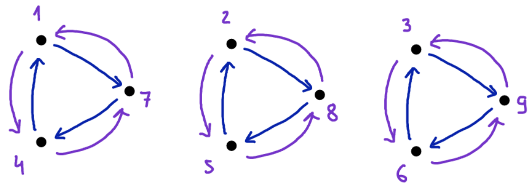
$\gcd(9,2) = 1$

$\boxed{(n\text{-cikel})^k \text{ razpade na } \gcd(k,n) \text{ ciklov dolžine } \frac{n}{\gcd(k,n)}}$

$\Rightarrow$ Možni ciklični strukturi za π sta $3+3+3$ in $3+6$.

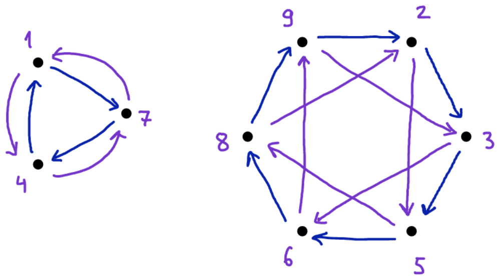
c) $\boxed{\pi^2 = (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)}$

π : $3+3+3$ $\qquad$ π^2 : $3+3+3$



$\pi = (1\ 7\ 4)(2\ 8\ 5)(3\ 9\ 6)$

π : $3+6$ $\qquad$ π^2 : $3+3+3$



$\pi = (1\ 7\ 4)(2\ 3\ 5\ 6\ 8\ 9)$

2. Dane so permutacije

$$\alpha = (1\,3\,5\,7\,9\,11),\ \ \beta = (2\,4\,6\,8\,10)\ \ \text{in}\ \ \gamma = (1\,9\,5)(2\,10\,8\,6\,4)(3\,11\,7).$$

(a) Preveri, da α in β komutirata.

(b) Preveri, da $\pi = \alpha^2 * \beta^2$ reši enačbo $\pi^2 = \gamma$.

(c) Poišči še eno rešitev $\pi^2 = \gamma$, ki ni enake parnosti kot $\alpha^2 * \beta^2$.

a) $\alpha * \beta = \beta * \alpha$?

$\alpha * \beta = (1\ 3\ 5\ 7\ 9\ 11) * (2\ 4\ 6\ 8\ 10) = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 6\ 8\ 10)$

disjunktni cikli komutirajo

$\beta * \alpha = (2\ 4\ 6\ 8\ 10) * (1\ 3\ 5\ 7\ 9\ 11) = (2\ 4\ 6\ 8\ 10)(1\ 3\ 5\ 7\ 9\ 11) = (1\ 3\ 5\ 7\ 9\ 11)(2\ 4\ 6\ 8\ 10)$

$\Rightarrow \alpha * \beta = \beta * \alpha$

ker $\alpha * \beta = \beta * \alpha$ in zato $\alpha^k * \beta^k = \beta^k * \alpha^k$ za vse k

b) $\pi^2 = (\alpha^2 * \beta^2)^2 = \alpha^2 * \beta^2 * \alpha^2 * \beta^2 = \alpha^2 * \alpha^2 * \beta^2 * \beta^2 = \alpha^4 * \beta^4 =$

$= (1\ 3\ 5\ 7\ 9\ 11)^4 * (2\ 4\ 6\ 8\ 10)^4 = (1\ 9\ 5)(3\ 11\ 7)(2\ 10\ 8\ 6\ 4) = \gamma$

$\Rightarrow (\alpha^2 * \beta^2)^2 = \gamma \Rightarrow \alpha^2 * \beta^2$ je rešitev enačbe $\pi^2 = \gamma$

c) parnost $\alpha^2 * \beta^2$?

Če ima α zapis z a transpozicijami, ima α^2 zapis z $2a$ transpozicijami, torej je α^2 soda. Podobno je β^2 soda (z $2b$ transpozicijami) in zato je $\alpha^2 * \beta^2$ soda (z $2a + 2b$ transpozicijami).

$\Rightarrow$ Iščemo še liho rešitev enačbe $\pi^2 = \gamma$.

$3+3+5$

Kandidati za cikl. strukturo za π: $3+3+5$, $6+5$

$2+2+4 = 8$ transpozicij $\to$ soda

$5+4 = 9$ transpozicij $\to$ liha ✓

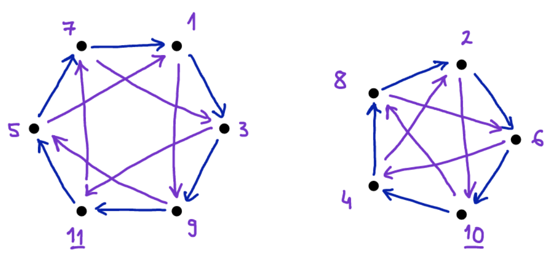
$(6+5)^2 = 6^2 + 5^2 = 3+3+5$ ✓

$\gcd(6,2) = 2$

$\gcd(5,2) = 1$

$\pi^2 = (1\ 9\ 5)(3\ 11\ 7)(2\ 10\ 8\ 6\ 4)$

π : $6+5$ $\qquad \pi^2$: $3+3+5$



$\underline{\pi = (1\ 3\ 9\ 11\ 5\ 7)(2\ 6\ 10\ 4\ 8)}$

3. Dane so permutacije

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 6 & 5 & 1 & 7 & 2 & 9 & 10 & 8 \end{pmatrix} \qquad \beta = (1\ 2)(1\ 6)(1\ 7)(1\ 3)(4\ 5)(4\ 10)(4\ 8)$$

$$\gamma = (1\ 4\ 9\ 3\ 6\ 7\ 2\ 8)$$

(a) Zapiši ciklične strukture permutacij α, β in γ ter določi njihove parnosti.

(b) Poišči vse možne ciklične strukture za permutacijo π, ki reši enačbo

$$\alpha * \beta * \pi^4 * \beta^{-1} = \gamma$$

(c) Poišči vsaj eno rešitev zgornje enačbe, ki ima najvišji možni red.

a) $\alpha = (1\ 4\ 5)(2\ 3\ 6\ 7)(8\ 9\ 10) \rightsquigarrow \underline{\underline{3+3+4}} \rightsquigarrow 2+2+3 = 7$ transpozicij $\rightsquigarrow$ <u>liha</u>

$\beta = (1\ 2\ 6\ 7\ 3)(4\ 5\ 10\ 8)(9) \rightsquigarrow \underline{\underline{5+4+1}} \rightsquigarrow 4+3 = 7$ transpozicij $\rightsquigarrow$ <u>liha</u>

$\gamma = (1\ 4\ 9\ 3\ 6\ 7\ 2\ 8)(5)(10) \rightsquigarrow \underline{\underline{8+1+1}} \rightsquigarrow 7$ transpozicij $\rightsquigarrow$ <u>liha</u>

b) $\alpha * \beta * \pi^4 * \beta^{-1} = \gamma \quad / * \beta$

$\alpha^{-1} * \backslash \quad \alpha * \beta * \pi^4 = \gamma * \beta$

$\beta^{-1} * \backslash \quad \beta * \pi^4 = \alpha^{-1} * \gamma * \beta$

$\pi^4 = \beta^{-1} * \alpha^{-1} * \gamma * \beta^{*}$

$* \ \beta^{-1} * \alpha^{-1} * \gamma * \beta =$

$= (8\ 10\ 5\ 4)(3\ 7\ 6\ 2\ 1) * (10\ 9\ 8)(7\ 6\ 3\ 2)(5\ 4\ 1) * (1\ 4\ 9\ 3\ 6\ 7\ 2\ 8) * (1\ 2\ 6\ 7\ 3)(4\ 5\ 10\ 8) =$

$= (1\ 4\ 8)(2\ 10\ 9)(3)(5)(6)(7)$

$$\boxed{\pi^4 = (1\ 4\ 8)(2\ 10\ 9)(3)(5)(6)(7)}$$

$3+3+1+1+1+1$

$3+3,\ 6 \qquad\qquad 1+1+1+1,\ 2+1+1,\ 2+2,\ 3+1,\ 4$

$(3+3)^4 = 3^4+3^4 = 3^1+3^1 = 3+3$ ✓

gcd(6,4) = 2 $\quad 6^4 = 3+3$ ✓

$(1+1+1+1)^4 = 1+1+1+1$ ✓

$(2+1+1)^4 = 2^4+1^4+1^4 = 1+1+1+1$ ✓

$(2+2)^4 = 2^4+2^4 = 1+1+1+1$ ✓

$(3+1)^4 = 3^4+1^4 = 3+1$ //

$4^4 = 1+1+1+1$ ✓

2 možnosti

4 možnosti

8 možnih cikl. struktur za π

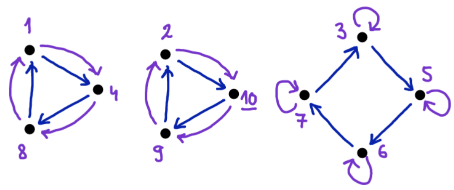
c)

ciklična struktura	red
3+3+1+1+1+1	lcm(3,3,1,1,1,1) = 3
3+3+2+1+1	lcm(3,2,1) = 6
3+3+2+2	lcm(3,2) = 6
3+3+4	lcm(3,4) = 12
6+1+1+1+1	lcm(6,1) = 6
6+2+1+1	lcm(6,2,1) = 6
6+2+2	lcm(6,2) = 6
6+4	lcm(6,4) = 12

$\Rightarrow$ največji možni red je 12

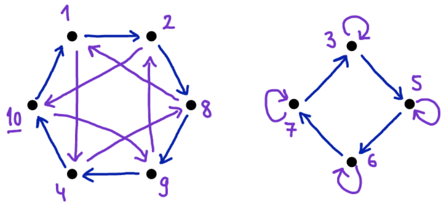
$$\boxed{\pi^4 = (1\ 4\ 8)(2\ 10\ 9)(3)(5)(6)(7)}$$

π: 3+3+4 $\quad \pi^4$: 3+3+1+1+1+1



$\underline{\underline{\pi = (1\ 4\ 8)(2\ 10\ 9)(3\ 5\ 6\ 7)}}$

π: 6+4 $\quad \pi^4$: 3+3+1+1+1+1



$\underline{\underline{\pi = (1\ 2\ 8\ 9\ 4\ 10)(3\ 5\ 6\ 7)}}$

4. (a) Zakaj lahko v skupini dveh ali več ljudi vedno najdemo 2, ki imata enako število prijateljev?

ljudje = vozlišča grafa

prijateljstvo = povezava grafa

stopnja vozlišča = število prijateljev

⇝ V grafu z n vozlišči $(n \geq 2)$ vedno obstajata dve vozlišči, ki imata isto stopnjo.

DOKAZ. S protislovjem. Denimo, da vsako od n vozlišč drugo stopnjo. Možne stopnje so $0, 1, 2, \dots, n-1$. Ker je n možnih stopenj in n vozlišč z različnimi stopnjami, morajo biti vse možne stopnje uporabljene. To pa pomeni, da imamo vozlišče stopnje 0, ki ni povezano z nobenim, in vozlišče stopnje $n-1$, ki je povezano z vsemi. To je protislovje (0 in $n-1$ se ne morta pojaviti v istem grafu). ■

DOKAZ. Imamo n možnih stopenj, $0, 1, \dots, n-1$, ampak 0 in $n-1$ se ne morta pojaviti v istem grafu. Torej imamo n vozlišč z največ $n-1$ različnimi stopnjami $(0, \dots, n-2$ ali $1, \dots, n-1)$. Po Dirichletovem principu obstajata vsaj dve vozlišči, ki imata isto stopnjo. ■

pigeonhole principle (princip golobnjaka)

Pigeonhole principle

From Wikipedia, the free encyclopedia

In mathematics, the **pigeonhole principle** states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item.[1] For example, if you have three gloves, then you must have at least two right-hand gloves, or at least two left-hand gloves, because you have three objects, but only two categories of handedness to put them into. This seemingly obvious statement, a type of counting argument, can be used to demonstrate possibly unexpected results. For example, if you know that the population of London is greater than the maximum number of hairs that can be present on a human's head, then the pigeonhole principle requires that there must be at least two people in London who have the same number of hairs on their heads.

Pigeons in holes. Here there are n = 10 pigeons in m = 9 holes. Since 10 is greater than 9, the pigeonhole principle says that at least one hole has more than one pigeon. (The top left hole has 2 pigeons.)

Although the pigeonhole principle appears as early as 1624 in a book attributed to Jean Leurechon,[2] it is commonly called **Dirichlet's box principle** or **Dirichlet's drawer principle** after an 1834 treatment of the principle by Peter Gustav Lejeune Dirichlet under the name *Schubfachprinzip* ("drawer principle" or "shelf principle").[3]

(b) Na zabavi se zbere 13 ljudi. Vsak je prinesel 3 darila, ki bi jih rad izmenjal z drugimi tremi udeleženci zabave. Ali jim lahko uspe?

ljudje = vozlišča, dva izmenjata darilo = povezava, število izmenjanih daril = stopnja točke

⇝ Ali obstaja kubičen graf na 13 vozliščih?

vsa vozlišča stopnje 3

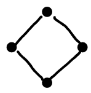
NE. Tak graf bi imel $\frac{13 \cdot 3}{2} = 19.5$ povezav, to pa ni celo število. ■

5. (a) Poišči vse (paroma neizomorfne) grafe na 4 točkah s 4 povezavami.

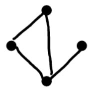
Drevo na n vozliščih ima $n-1$ povezav. ⇝ povezan graf brez ciklov	Gozd na n vozliščih ima $< n-1$ povezav. ⇝ unija dreves (= ne nujno povezan acikličen graf)

4 vozlišča, 4 povezave ⇒ vsaj en cikel

4-cikel



3-cikel + 1 dodatna povezava

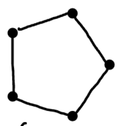


(b) Poišči vse (paroma neizomorfne) grafe na 5 točkah, ki so izomorfni svojemu komplementu.

5 vozlišč ⇒ max $\frac{5\cdot 4}{2} = \binom{5}{2} = 10$ povezav, od tega 5 v G in 5 v $\overline{G}$

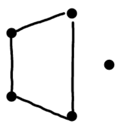
⇒ 5 vozlišč in 5 povezav ⇒ vsaj en cikel

max. cikel 5-cikel

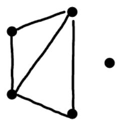


G_1 ✓

max. 4-cikel

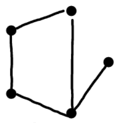


+ 1 povezava



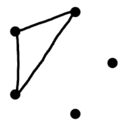
G_2 //

ali

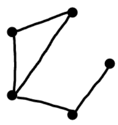


G_3 //

max. 3-cikel

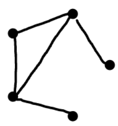


+ 2 povezavi



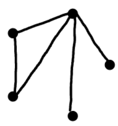
G_4 //

ali

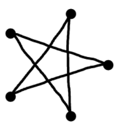


G_5 ✓

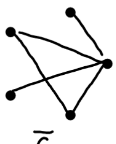
ali



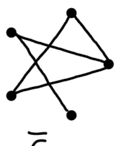
G_6 //



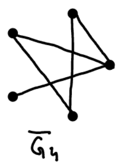
$\overline{G_1}$



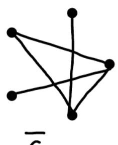
$\overline{G_2}$



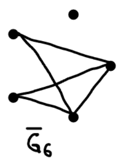
$\overline{G_3}$



$\overline{G_4}$

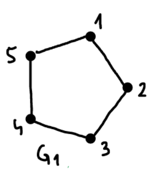


$\overline{G_5}$



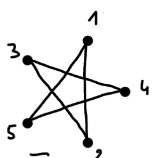
$\overline{G_6}$

- $G_1 = \overline{G_1}$ (G_1 in $\overline{G_1}$ sta oba izomorfna C_5)
- $\overline{G_2} = G_6 \neq G_2$ in $\overline{G_6} = G_2 \neq G_6$ ⇝ G_2 in G_6 nista izomorfna svojemu komplementu
- $\overline{G_3} = G_4 \neq G_3$ in $\overline{G_4} = G_3 \neq G_4$ ⇝ G_3 in G_4 nista izomorfna svojemu komplementu
- $G_5 = \overline{G_5}$ (3-cikel z dvema dodatnima povezavama iz sosednjih vozlišč)



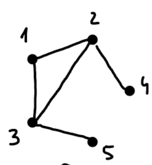
G_1

=



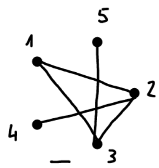
$\overline{G_1}$

$V = \{1,2,3,4,5\}$
$E = \{12,23,34,45,15\}$
$V(G_1) = V(\overline{G_1})$ in $E(G_1) = E(\overline{G_1})$
⇝ sta izomorfna



G_5

=



$\overline{G_5}$

$V = \{1,2,3,4,5\}$
$E = \{12,23,13,24,35\}$
$V(G_5) = V(\overline{G_5})$ in $E(G_5) = E(\overline{G_5})$
⇝ sta izomorfna