Li's criterion

For the Riemann $\xi(s)$ function, i.e.,

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

where $\lambda_n$is defined to be:

$$\lambda_n = \frac{1}{(n-1)!}\,\frac{d^n}{ds^n}\left[s^{n-1}\log\xi(s)\right]\Big|_{s=1}$$

then the Riemann hypothesis is equivelant to $> \lambda_n > 0$ for all positive integers n.

Alternative definition of Riemann zeta function:

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

. This is defined for $\Re(s) > 0$.

Consider

$$\alpha(s) = \int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx$$

We have the relation

$$\zeta(s)\Gamma(s) = \int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx.$$

We can then use this in the above and have

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx$$

The Taylor expansion of $x^s = e^{s\ln x}$ is given by

$$\begin{aligned}
x^s &= e^{s\ln x}\\
&= f(1) + \frac{1}{1!}f'(1)(s-1) + \frac{1}{2!}f''(1)(s-1)^2 + \frac{1}{3!}f'''(1)(s-1)^3 + \cdots\\
&= e^{\ln x} + \frac{1}{1!}\ln(x)e^{\ln x}(s-1) + \frac{1}{2!}\ln(x)^2e^{\ln x}(s-1)^2 + \frac{1}{3!}\ln(x)^3e^{\ln x}(s-1)^3 + \cdots\\
&= x^s + \frac{1}{1!}\ln(x)x^s(s-1) + \frac{1}{2!}\ln(x)^2x^s(s-1)^2 + \frac{1}{3!}\ln(x)^3x^s(s-1)^3 + \cdots
\end{aligned}$$

Let's look at the denominator:

$$\int_0^\infty \frac{x^{s-1}}{(e^x-1)}dx = \int_0^\infty \frac{x^{s-1}}{e^x\left(1-e^{-x}\right)}dx$$
$$= \int_0^\infty x^{s-1}e^{-x}(1+e^{-x}+e^{-2x}+\cdots)dx$$
$$= \int_0^\infty x^{s-1}(e^{-x}+e^{-2x}+e^{-3x}+\cdots)dx$$
$$= \int_0^\infty x^{s-1}e^{-x}dx + \int_0^\infty x^{s-1}e^{-2x}dx + \int_0^\infty x^{s-1}e^{-3x}dx + \cdots$$
$$\int_0^\infty x^{s-1}e^{-x}dx$$
$$+\frac{1}{2^s}\int_0^\infty x^{s-1}e^{-x}dx$$
$$+\frac{1}{3^s}\int_0^\infty x^{s-1}e^{-x}dx + \cdots$$
$$= \Gamma(s)\zeta(s)$$
$$= \lim_{m\to\infty}\int_0^m \frac{x^{s-1}e^{-x}}{\left(1-(1-\frac{x}{m})^m\right)}dx$$
$$= \lim_{m\to\infty}\int_0^m \frac{x^{s-1}}{}\left[1+(1+\frac{x}{m})^m\right]dx$$

$$f(x) = e^x - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This satisfies the equation, $\frac{df}{dx} - 1 = f(x)$ or $\frac{df}{dx} = f(x)+1$. We know that $f(x) = -1$ is a solution. Hence the general solution is $f(x) = Ce^x - 1$. The Runge Kutta solution is had by approximation. Initial condition is $f(0) = 0$. Divide the unit interval into n-intervals. Then

$$f(1/n) \approx 1 + f(0)(1/n)$$

The Riemann ζ function (product formulation). Let $p_i = \{2, 3, 5, \ldots\}$be the distinct primes in order, then

$$\zeta(s) = \prod\left(1-p_i^{-s}\right)^{-1}$$

So

$$\begin{aligned}
\ln \zeta(s) &= -\sum \ln\left(1 - p_i^{-s}\right) \\
&= -\sum \ln\left(1 - \frac{1}{p_i^s}\right) \\
&= -\sum \ln\left(\frac{p_i^s}{p_i^s} - \frac{1}{p_i^s}\right) \\
&= -\sum \ln\left(\frac{p_i^s - 1}{p_i^s}\right) \\
&= \sum \ln\left(\frac{p_i^s}{p_i^s - 1}\right) \\
&= \sum \ln p_i^s - \ln(p_i^s - 1)
\end{aligned}$$

Suppose we have $y = \ln f(x)$. Then $e^y = f(x)$. Differentiating,

$$e^y y' = f'(x)$$

$$e^y \left(y'\right)^2 + = f'(x)$$