Lecture 01: Complex Numbers and Their Geometric Representation, Polar Form of Complex Numbers

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1 Definition

A complex number z is typically expressed in the form:

$$z = x + iy$$

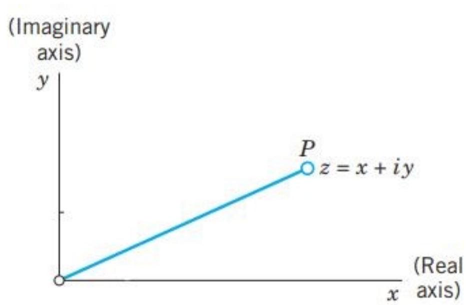
where x and y are real numbers, and i is the imaginary unit, defined by $i = \sqrt{-1}$.

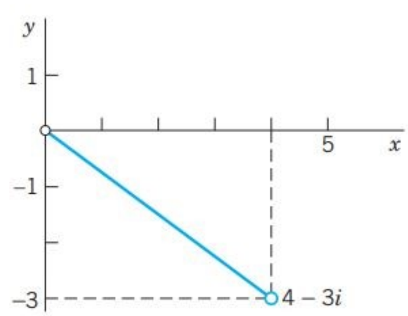
2 Geometric Representation of Complex Numbers

Complex numbers can be represented geometrically in the complex plane. In this representation:

- The real part x corresponds to the horizontal axis (real axis).
- The imaginary part y corresponds to the vertical axis (imaginary axis).

Thus, a complex number $z = x + iy$ can be represented as a point (x, y) or as a vector from the origin to the point (x, y) in the complex plane.





The modulus $|z|$ of the complex number $z = x + iy$ is the distance from the origin to the point (x, y) and is given by:

$$|z| = \sqrt{x^2 + y^2}$$

The argument θ of z is the angle made by the vector with the positive real axis, and it is given by:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Here, as in calculus, all angles are measured in radians and positive in the counterclockwise sense.

3 Polar Form of Complex Numbers

We employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We see that then $z = x + iy$ takes the so-called *polar form*

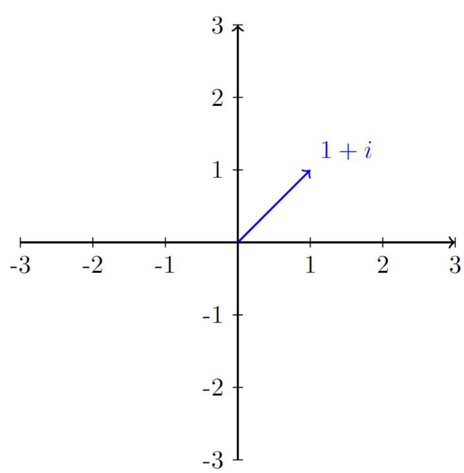
$$z = r(\cos\theta + i\sin\theta).$$

where $r = |z| = \sqrt{x^2 + y^2}$ and θ is the argument of z. This form is also known as the trigonometric form of a complex number.

Caution:

It is possible that you get an angle different from the quadrant where the complex number lies. To correct this, you need to add or subtract π radians to get the right angle (argument).

Polar Form of $1 + i$



$$r = |z| = \sqrt{x^2 + y^2}$$

For $z = 1 + i$, we have $x = 1$ and $y = 1$:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Next, we calculate the argument θ:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

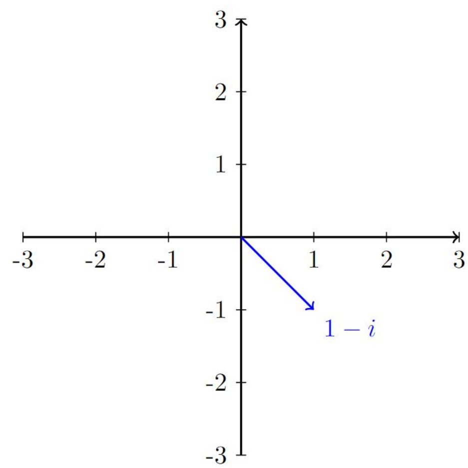
Substituting $x = 1$ and $y = 1$:

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Therefore, the polar form of $z = 1 + i$ is:

$$z = r\left(\cos\theta + i\sin\theta\right) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Polar Form of $1 - i$



$$r = |z| = \sqrt{x^2 + y^2}$$

For $z = 1 - i$, we have $x = 1$ and $y = -1$:

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{1 + 1} = \sqrt{2}$$

Next, we calculate the argument θ:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Substituting $x = 1$ and $y = -1$:

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

Since the complex number $z = 1 - i$ lies in the fourth quadrant, we add π to the argument to place θ in the correct quadrant:

$$\theta = -\frac{\pi}{4} + \pi = \frac{3\pi}{4} \quad and \quad again \quad \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$$

Therefore, the polar form of $z = 1 - i$ is:

$$z = r\left(\cos\theta + i\sin\theta\right) = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

Problem Set 13

Question 8

Represent in polar form and graph in the complex plane.

$$\frac{-4+19i}{2+5i}$$

Solution

To simplify the given complex expression, we first multiply the numerator and denominator by the conjugate of the denominator:

$$\frac{-4+19i}{2+5i} \times \frac{2-5i}{2-5i} = \frac{(-4+19i)(2-5i)}{(2+5i)(2-5i)}$$

Next, we expand and simplify the numerator:

$$(-4+19i)(2-5i) = -8 + 20i + 38i - 95i^2$$

Since $i^2 = -1$, the expression simplifies to:

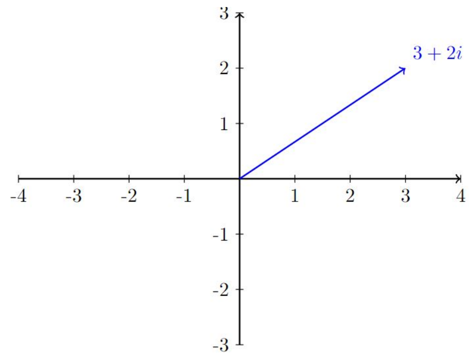
$$-8 + 58i + 95 = 87 + 58i$$

The denominator simplifies as follows:

$$(2+5i)(2-5i) = 4 - 25i^2 = 4 + 25 = 29$$

Thus, the complex number simplifies to:

$$\frac{87+58i}{29} = 3 + 2i$$



$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

For $3 + 2i$, $x = 3$ and $y = 2$:

$$r = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$$

The argument θ is:

$$\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.588 \text{ radians} \approx 34^\circ$$

Thus, the polar form of $3 + 2i$ is:

$$z = \sqrt{13}\,(\cos(0.588 \text{ radians}) + i \sin(0.588 \text{ radians}))$$