Q#1 Sol:

A quadric surface is the graph of a second degree equation in three variables x, y and z. The most general such equation is

$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0.$$

Now, to sketch the given surface $4x^2-y^2+2z^2+4=0$

Dividing by -4, we first put the equation in standard form

$$-x^2+\frac{y^2}{4}+\left(-\frac{z^2}{2}\right)=1$$

it represents a hyperboloid of two sheets where axis of the hyperboloid is the y-axis. The traces in the xy- and yz- planes are the hyperbolas

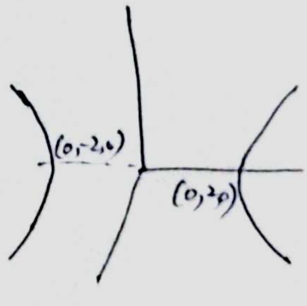
$$-x^2+\frac{y^2}{4}=1\ ,\ z=0 \quad \text{and} \quad \frac{y^2}{4}-\frac{z^2}{2}=1\ ,\ x=0.$$

The surface has no trace in the xz-plane, but traces in the vertical planes $y=k$ for $|k|>2$ are the ellipses

$$\frac{x^2}{1}+\frac{z^2}{2}=\frac{k^2}{4}-1\ ,\ y=k$$

which can be written as

$$\frac{x^2}{\frac{k^2}{4}-1}+\frac{z^2}{2\left(\frac{k^2}{4}-1\right)}=1\ ,\ y=k.$$



Q#2(a) Sol:

Let $F=x^3+y^3+z^3+6xyz-1$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x^2+2yz}{z^2+2xy} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{y^2+2xz}{z^2+2xy}$$

(b) Sol: we first locate the critical point's.

$f_x=4x^3-4y \quad , \quad f_y=4y^3-4x$

$=0 \Rightarrow y=x^3 \qquad\qquad =0 \qquad\qquad \Rightarrow x=0,1,-1.$

So, c.p. are $(0,0), (1,1), (-1,-1)$

$\Rightarrow f_{xx}=12x^2 \quad , \quad f_{xy}=-4 \quad , \quad f_{yy}=12y^2$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - 16$$

Since $D(0,0) = -16 < 0$, so, $(0,0)$ is a saddle point

i.e. f has no local maximum or minimum at $(0,0)$.

Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0 \Rightarrow f(1,1) = -1$ is a local minimum.

Similarly, $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$,

So, $f(-1,-1) = -1$ is also a local minimum.

Q#2.(c). Sol:

Let $f(x,y) = 2x^2 + y^2$

then $f_x = 4x$ and $f_y = 2y$

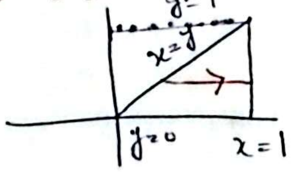
$f_x(1,1) = 4$, $f_y(1,1) = 2$

Then the eq of tangent plane at $(1,1,3)$ is

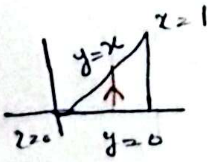
$$z - 3 = 4(x-1) + 2(y-1)$$

$$z = 4x + 2y - 3$$

Q#3.(a). Sol:



$$\int_{y=0}^{1}\left[\int_{x=y}^{1} \sin x^2 \, dx\right] dy$$



$$= \int_{x=0}^{1}\left[\int_{y=0}^{x} \sin x^2 \, dy\right] dx$$

$$= \int_0^1 \left[y \sin(x^2)\right]\Big|_{y=0}^{x} dx$$

$$= -\frac{1}{2}\cos x^2 \Big|_0^1 = \frac{1}{2}(1 - \cos 1)$$

Q#3(b). Sol: $r(\phi,\theta) = \sin\phi\cos\theta\, i + \sin\phi\sin\theta\, \hat{j} + \cos\phi\, \hat{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

$$F(r(\phi,\theta)) = \cos\phi\, i + \sin\phi\sin\theta\, j + \sin\phi\cos\theta\, \hat{k}$$

$$r_\phi \times r_\theta = \sin^2\phi\cos\theta\, i + \sin^2\phi\sin\theta\, j + \sin\phi\cos\phi\, \hat{k}$$

$$F(r(\phi,\theta)) \cdot (r_\phi \times r_\theta) = \cos\phi\sin^2\phi\cos\theta + \sin^2\phi\cos\phi\cos\theta$$

$$\iint_S F \cdot dS = \iint_D F \cdot (r_\phi \times r_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi\, d\theta$$

$$= \frac{4\pi}{3}$$

Q#3. (c). Sol:

$$\nabla \cdot F = \operatorname{div} F = ye^z + 2xyz^3$$

The divergence theorem states that

$$\iint_S F \cdot dS = \iiint_E \operatorname{div} F\, dV$$

$$= \int_0^3\int_0^2\int_0^1 (2xyz^3)\, dz\, dy\, dx$$

$$= \frac{9}{2}$$

Q#3. (d). Sol: The degree to which a curve deviates from a straight line, or a curved surface deviates from being a plane.

$$r(t) = a\cos t\, i + a\sin t\, j$$

$$r'(t) = -a\sin t\, i + a\cos t\, j \quad \text{and} \quad |r'(t)| = a$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = -\sin t\, i + \cos t\, j \Rightarrow T'(t) = -\cos t\, i - \sin t\, j$$

$$\Rightarrow |T'(t)| = 1 \text{, So, } K(t) = |T'(t)| / |r'(t)| = 1/a$$

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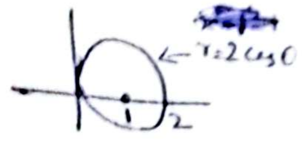
Q#3. (f). Sol:

After completing square, $x^2+y^2=2x$ becomes $(x-1)^2+y^2=1$

In polar coordinates, we have

$$x^2+y^2=r^2 \text{ and } x=r\cos\theta$$

$$\Rightarrow \quad r^2 = 2r\cos\theta \quad , -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \quad , 0 \le r \le 2\cos\theta$$



$$\Rightarrow V = \iint_D (x^2+y^2)\, dA = \int_{-\frac{\pi}{2}}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta = \frac{3\pi}{2}$$

Q#3. (h). Sol: As, r=1, so parametric eqs are

$$x=\cos\theta \quad , \quad y=y \quad . \quad z=\sin\theta$$

Given function becomes

$$x+y+z=\cos\theta+y+\sin\theta$$

$$\iint_S (x+y+z)\, dS = \int_{y=-2}^{0} \int_{\theta=\pi}^{2\pi} (\cos\theta+y+\sin\theta)\, d\theta\, dy$$

$$= 0+\pi(-2)-2(2) = -2\pi-4$$

Q#3. (e). Sol: As, $z=y^2-x^2$, so $r(t) = \cos t\, i + \sin t\, j + (\sin^2 t - \cos^2 t)k$

$$\int_0^{2\pi} F \cdot dr = \int_0^{2\pi} (-\sin^2 t - \cos^2 t + 4\sin t\cos^3 t)\, dt = -2\pi \quad (L.H.S.)$$

Now, to solve R.H.S. of stokes th. i.e. $\nabla \times F \cdot n$,

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = -2xj - 2k = -(2r\cos\theta)j - 2k$$

So,

$$\iint_S \nabla \times F \cdot n\, dS = \int_0^{2\pi} \int_0^1 \nabla \times F \cdot (r_r \times r_\theta)\, dr\, d\theta$$

After taking "Dot" product

$$= \int_0^{2\pi} \int_0^1 \left[4r^3(2\sin\theta\cos^3\theta + \sin^3\theta\cos\theta + \sin\theta\cos^3\theta) - 2r\right] dr\, d\theta$$

$$= -2\pi \quad \Rightarrow \quad \text{Hence, } L.H.S. = R.H.S.$$

Sol. 3 (g):

$$x = \cos u \cos v, \quad y = \cos u \sin v, \quad z = u, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2} \quad \text{and} \quad 0 \le v \le 2\pi,$$

where v represents the angle of rotation from the xz-plane about the z-axis. Substituting this parametrization into the expression for G gives

$$\sqrt{1 - x^2 - y^2} = \sqrt{1 - (\cos^2 u)(\cos^2 v + \sin^2 v)} = \sqrt{1 - \cos^2 u} = |\sin u|.$$

The surface area differential for the parametrization was found to be Section 16.5)

$$d\sigma = \cos u \sqrt{1 + \sin^2 u}\, du\, dv.$$

These calculations give the surface integral

$$\iint_S \sqrt{1 - x^2 - y^2}\, d\sigma = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= 2\int_0^{2\pi} \int_0^{\pi/2} \sin u \cos u \sqrt{1 + \sin^2 u}\, du\, dv$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{w}\, dw\, dv$$

$w = 1 + \sin^2 u,$
$dw = 2 \sin u \cos u\, du$
When $u = 0, w = 1.$
When $u = \pi/2, w = 2.$

$$= 2\pi \cdot \frac{2}{3} w^{3/2} \Big]_1^2 = \frac{4\pi}{3}\left(2\sqrt{2} - 1\right).$$