National University of Computer and Emerging Sciences
Lahore Campus

Complex Variables and Transforms (MT2003)

Final Exam

Date: December 19 2024

Course Instructor(s)
Mr. Tasaduque Hussain Shah

Solution

Total Time (Hrs): 3
Total Marks: 70
Total Questions: 6

Roll No ______ Section ______ Student Name & Signature ______

Instructions: Attempt all questions and write answers in the provided space. Rough work may be done on separate sheets but will not be graded.

[CLO-1] Q1: Answer the following questions briefly and precisely in the corresponding column. [20]

Questions	Short Answers	
What do you mean by half-range expansion?	Half interval in Fourier series	2
Which functions have a valid Fourier Series?	Periodic Functions	2
What is the advantage of Fourier Transform over Laplace Transform?	Fourier Transforms are valid for Negative domain ~~also~~ as well.	2
How can we determine whether to use a Fourier series or Fourier integral?	If the period is finite, we have to find Fourier series, otherwise Fourier integral.	2
Show that for N=8, $\omega = (1-i)/\sqrt{2}$	$\omega = e^{\frac{-2\pi i}{N}} = e^{\frac{-\pi}{4}i} = \cos(\frac{\pi}{4}) + i\sin(\frac{-\pi}{4}) = \cos(\frac{\pi}{4}) - i\sin(\frac{\pi}{4}) = \frac{1-i}{\sqrt{2}}$	3
Can the residue at a singularity be zero? Give reason.	Yes, Because Residue is just a limit at a particular value.	3
Determine $Arg(z)$ for $z = \pi - \pi i$	$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\frac{-\pi}{\pi}) = \tan^{-1}(-1) = -\frac{\pi}{4}$	2
What is meant by a simple closed path?	A closed path that don't cross itself.	2
Determine $\lvert e^z \rvert$ for $z = 3 - i$	$\lvert e^z \rvert = e^x = e^3$	2

[CLO-4] Q2 (b): Find the Fourier series of the function obtained by passing the voltage $v(t) = \cos(100\pi t)$ through a half-wave rectifier that clips the negative half-waves. [10]

$$f(t) = \begin{cases} \cos(100\pi t) & ; \ 0 < t < 0.01 \\ 0 & ; \ \text{otherwise} \end{cases}$$

2

$$a_0 = \frac{1}{L}\int_0^L f(t)\,dt = \frac{1}{0.01}\int_0^{0.01} \cos(100\pi t)\,dt = 100\left[\frac{\sin(100\pi t)}{100\pi t}\right]_0^{0.01}$$

$= 0$

2

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{2}{0.01}\int_0^{0.01} \cos(100\pi t)\cdot\cos\left(\frac{n\pi t}{0.01}\right)dt$$

$$= 200\int_0^{0.01} \underbrace{\cos(100\pi t)\cdot\cos(100n\pi t)}_{\text{product into sum}}\,dt$$

$$= 200\int_0^{0.01} \frac{1}{2}\left[\cos(100\pi t - 100n\pi t) + \cos(100\pi t + 100n\pi t)\right]dt$$

$= 0$ 3

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{2}{0.01}\int_0^{0.01} \underbrace{\cos(100\pi t)\sin(100n\pi t)}_{\text{product into sum}}\,dt$$

$$b_n = 200\int_0^{0.01} \frac{1}{2}\left[\sin(100\pi t + 100n\pi t) - \sin(100\pi t - 100n\pi t)\right]dt$$

$=$ 3

[10]

[CLO-2] Q3: Determine "a" such that the given function is harmonic and also find harmonic conjugate. 0134

[5]

$$u = e^{ax}\cos 5y$$

$u = e^{ax}\cos 5y$

$u_x = a e^{ax}\cos 5y$

$u_{xx} = a^2 e^{ax}\cos 5y$

$u_y = -5e^{ax}\sin 5y$

$u_{yy} = -25e^{ax}\cos 5y$

$u_{xx} + u_{yy} = 0$

$\Rightarrow a^2 e^{ax}\cos 5y - 25e^{ax}\cos 5y = 0$

$\Rightarrow a^2 - 25 = 0$

$\Rightarrow a^2 = 25$

$\Rightarrow a = \pm 5$

$\longrightarrow$ 2

For $a = 5$:

$u = e^{5x}\cos 5y$

$\Rightarrow u_x = 5e^{5x}\cos 5y$

$\Rightarrow v_y = 5e^{5x}\cos 5y$

$\Rightarrow \dfrac{\partial v}{\partial y} = 5e^{5x}\cos 5y$

$\Rightarrow \int \partial v = \int 5e^{5x}\cos 5y\, \partial y$

$\Rightarrow v = \dfrac{5e^{5x}\sin 5y}{5} + g(x)$

$\Rightarrow v = e^{5x}\sin 5y + g(x)$

$v = e^{5x}\sin 5y + g(x)$

$\Rightarrow v_x = 5e^{5x}\sin 5y + g'(x)$

$\Rightarrow -u_y = 5e^{5x}\sin 5y + g'(x)$

$5e^{5x}\sin 5y = 5e^{5x}\sin 5y + g'(x)$

$\Rightarrow g'(x) = 0$

$\Rightarrow g(x) = c$

Therfore: $v = e^{5x}\sin 5y + c$

Similarly for $a = -5$:

$v = -e^{-5x}\sin 5y + c$

Hence $v = \pm e^{\pm 5x}\sin 5y + c$ $\longrightarrow$ 3

01

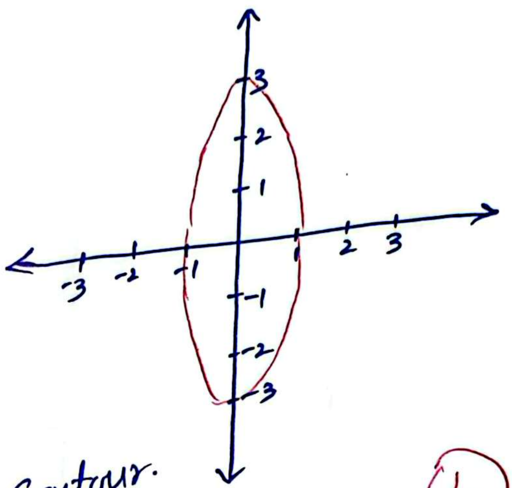
[CLO-3] Q4: Evaluate the following line integrals. **[5+10]**

(a) $\int_C \sinh(\pi z)\, dz$, C from i along the y-axis to 0.

(b) $\oint_C \frac{ze^{\pi z}}{z^4 - 16}\, dz$, C is the ellipse $9x^2 + y^2 = 9$ (counterclockwise).

(a) $\int_C \sinh(\pi z)dz = \left.\frac{\cosh \pi z}{\pi}\right|_i^0 = \frac{1}{\pi}[1 - \cos\pi] = \frac{2}{\pi}$ → 5

(b) $\oint_C \frac{ze^{\pi z}}{z^4 - 16} dz$



$\oint_C \frac{ze^{\pi z}}{(z^2-4)(z^2+4)}$

$\Rightarrow$ $z = \pm 2$, $z = \pm 2i$

$z = \pm 2$ are outside of our contour. → 4

so $z_0 = \pm 2i$ and $n = 1$

$\underset{z=2i}{\text{Res}} f(z) = \lim_{z \to 2i} \left[\frac{(z-2i)\, ze^{\pi(2i)}}{(z-2i)(z+2i)(z^2-4)}\right] = \frac{-1}{16}$ → 4

Similarly $\underset{z=-2i}{\text{Res}} f(z) = \frac{-1}{16}$

so $\oint_C \frac{ze^{\pi z}}{z^4-16} dz = \left(\frac{-1}{16} - \frac{1}{16}\right) 2\pi i = -\frac{\pi}{4} i$ → 2

[5+10] 0

[CLO-4] Q5 (a): Show that the given integral represents the indicated function. **[5]** 0134

$$\int_0^\infty \frac{\cos(xw)}{1+w^2}\,dw = \frac{\pi}{2}e^{-x} \quad \text{if} \quad x > 0$$

Fourier integral $= \int_0^\infty [A(w)\cos(wx) + B(w)\sin wx]\,dw$

$$B(w) = 0$$

$$A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos(wx)\,dx = \frac{2}{\pi}\int_0^\infty \frac{\pi}{2} e^{-x}\cos wx\,dx = \int_0^\infty e^{-x}\cos wx\,dx$$

$$= \frac{1}{1+w^2} \qquad \left[\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}\right.$$

Hence Fourier integral $= \int_0^\infty \frac{\cos(wx)}{1+w^2}\,dw$ (proved) 5

[CLO-4] Q5 (b): Find the Fourier Transform of the following function. **[10]**

$$f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\,dx$$

$$F(w) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} |x|\, e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} |x|\, e^{-iwx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{1} |x|\, e^{-iwx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{0} (-x)\, e^{-iwx}\,dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{1} x\, e^{-iwx}\,dx \quad 4$$

$$= \frac{-1}{\sqrt{2\pi}}\left[\frac{-xe^{-iwx}}{iw} + \frac{e^{-iwx}}{iw(-iw)}\Big|_{-1}^{0}\right] + \frac{1}{\sqrt{2\pi}}\left[\frac{-xe^{-iwx}}{iw} + \frac{e^{-iwx}}{iw(-iw)}\Big|_{0}^{1}\right]$$

$$F(w) = \frac{-1}{\sqrt{2\pi}}\left[\frac{-1}{(iw)^2} - \frac{e^{iw}}{iw} + \frac{e^{iw}}{(iw)^2}\right] + \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(iw)^2} - \frac{e^{-iw}}{(iw)^2} - \frac{e^{-iw}}{iw}\right.$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{(iw)^2} + \frac{1}{iw}\left(e^{iw} - e^{-iw}\right) + \frac{1}{(iw)^2}\left(-e^{iw} - e^{-iw}\right)\right. \quad 4$$

$$= \frac{1}{\sqrt{2\pi}}\left[-\frac{2}{w^2} + \frac{2\sin w}{w} + \frac{2\cos w}{w^2}\right]$$

$$\Rightarrow F(w) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos w + w\sin w - 1}{w^2}\right] \quad 2$$

[CLO-3] Q6: Evaluate the following improper integral. [5] 0134

$$\int_0^\infty \sqrt{x}e^{-3\sqrt{x}}\,dx$$

Let $\boxed{y = 3\sqrt{x}} \Rightarrow \boxed{\sqrt{x} = \frac{y}{3}}$ → 1

$$\frac{dy}{dx} = \frac{3}{2\sqrt{x}}$$

$$\Rightarrow dy = \frac{3dx}{2\times\frac{y}{3}}$$

$$\Rightarrow dy = \frac{9}{2y}dx$$

$$\Rightarrow \boxed{\frac{2}{9}y\,dy = dx}$$ → 2

$$\because \int_0^\infty \sqrt{x}\,e^{-3\sqrt{x}}\,dx = \int_0^\infty \frac{y}{3}e^{-y}\frac{2}{9}y\,dy$$

$$= \frac{2}{27}\int_0^\infty y^2 e^{-y}\,dy$$

$$= \frac{2}{27}\int_0^\infty y^{3-1}e^{-y}\,dy$$

$$= \frac{2}{27}\times\Gamma(3)$$

$$= \frac{2}{27}(3-1)! = \frac{4}{27}$$ → 2