Lecture 03/04: Functions, Derivatives, Analytic Functions, and Cauchy-Riemann Equations, Laplace's Equation

Tasaduq Hussain

August 26, 2024

Function

A **function** between two sets A and B is a rule that assigns to each element in set A exactly one element in set B. This is formally written as:

$$f : A \to B$$

Where:

- A is the **domain** of the function, which is the set of all possible inputs.
- B is the **codomain** of the function.
- The **range** of the function is the set of all actual outputs. The range is a subset of the codomain B.

Real-Valued Function

If both A and B are from the set of real numbers $\mathbb{R}$, the function is called a **real-valued function** and is typically denoted by:

$$y = f(x)$$

Example: Real-Valued Function

Let $A = \mathbb{R}$ and $B = \mathbb{R}$.

Define the function $f : \mathbb{R} \to \mathbb{R}$ by:

$$y = f(x) = x^2$$

- Domain of $f = \mathbb{R}$
- Codomain of $f = \mathbb{R}$
- Range of $f = [0, \infty)$

Complex-Valued Function

If both A and B are from the set of complex numbers $\mathbb{C}$, the function is called a **complex-valued function** and is typically denoted by:

$$w = f(z)$$

Example: Complex-Valued Function

Let $A = \mathbb{C}$ and $B = \mathbb{C}$ the set of all complex numbers.
Define the function $f : \mathbb{C} \to \mathbb{C}$ by:

$$w = f(z) = z^2 + 1$$

- Domain of $f = \mathbb{C}$
- Codomain of $f = \mathbb{C}$
- The range of f will include all complex numbers that can be obtained by squaring any complex number z and adding 1.

Note

Note that a complex-valued function $f(z)$ can also be expressed in terms of its real and imaginary components. If z is a complex number represented as $z = x + iy$,the function can be written as:

$$w = f(z) = u(x, y) + iv(x, y)$$

Where:

- $u(x, y)$ is the real part of the function $f(z)$.
- $v(x, y)$ is the imaginary part of the function $f(z)$.

Example

Express the complex function $w = f(z) = z^2 + 1$ in terms of its real and imaginary parts.
For this we substitute $z = x + iy$

$$w = (x + iy)^2 + 1$$

$$w = \left(x^2 + 2ixy - y^2\right) + 1$$

$$w = \left(x^2 - y^2 + 1\right) + i(2xy)$$

Therefore,

$$\text{Real part: } u(x, y) = x^2 - y^2 + 1$$
$$\text{Imaginary part: } v(x, y) = 2xy$$

Remark

The standard rules of differentiation (such as the sum, product, quotient, power and chain rules) that apply to real-valued functions are also valid for complex-valued functions, provided that the complex function is differentiable.

Problem Set 13.3

Question 11

$$f(z) = \frac{1}{1-z}$$

Substitute $z = 1 - i$:

$$f(1-i) = \frac{1}{1-(1-i)} = \frac{1}{i} = -i$$

Real part:

$$\mathrm{Re}(f(1-i)) = 0$$

Imaginary part:

$$\mathrm{Im}(f(1-i)) = -1$$

Question 20

Find the value of the derivative of $\frac{1.5z+2i}{3iz-4}$ at any z. Explain the result.

Solution

The function given is

$$f(z) = \frac{1.5z + 2i}{3iz - 4}$$

To find the derivative, we apply the quotient rule:

$$f'(z) = \frac{(3iz-4)\left(\frac{d}{dz}(1.5z+2i)\right) - (1.5z+2i)\left(\frac{d}{dz}(3iz-4)\right)}{(3iz-4)^2}$$

Now, calculate the derivatives of the numerator and denominator:

$$\frac{d}{dz}(1.5z + 2i) = 1.5$$

$$\frac{d}{dz}(3iz - 4) = 3i$$

Substitute these into the quotient rule formula:

$$f'(z) = \frac{(3iz-4)(1.5) - (1.5z+2i)(3i)}{(3iz-4)^2}$$

Simplify the numerator:

$$f'(z) = \frac{4.5iz - 6 - (4.5iz + 6)}{(3iz - 4)^2}$$

$$f'(z) = \frac{4.5iz - 6 - 4.5iz - 6}{(3iz - 4)^2}$$

$$f'(z) = \frac{-12}{(3iz - 4)^2}$$

Thus, the derivative is:

$$f'(z) = \frac{-12}{(3iz - 4)^2}$$

Analyticity

A function $f(z)$ is said to be analytic in a domain D if $f(z)$ is defined and differentiable at all points of D. A more modern term for analytic in D is holomorphic in D.

Cauchy-Riemann Equations

The **Cauchy–Riemann equations** provide a criterion for the analyticity of a complex function $f(z) = u(x, y) + iv(x, y)$. A function f is analytic in a domain D if the first partial derivatives of u and v satisfy:

$$u_x = v_y, \quad u_y = -v_x.$$

In polar coordinates, where $z = r(\cos\theta + i\sin\theta)$ and $f(z) = u(r, \theta) + iv(r, \theta)$, the equations become:

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta.$$

Problem Set 13.4

Question 2

Is the function

$$f(z) = iz\overline{z}$$

analytic?
Solution:

$$z = x + iy, \quad \overline{z} = x - iy$$

$$f(z) = iz\overline{z} = i(x+iy)(x-iy) = i(x^2+y^2)$$

$$u(x,y) = 0, \quad v(x,y) = x^2 + y^2$$

$$u_x = 0, \quad u_y = 0$$

$$v_x = 2x, \quad v_y = 2y$$

$$u_x \neq v_y, \quad u_y \neq -v_x$$

$$f(z) \text{ is not analytic.}$$

Question 3

Is the function $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$ analytic?
Solution: To determine whether the function $f(z) = e^{-2x}(\cos 2y - i \sin 2y)$ is analytic, we express it as $f(z) = u(x,y) + iv(x,y)$, where $u(x,y) = e^{-2x}\cos 2y$ and $v(x,y) = -e^{-2x}\sin 2y$.

The partial derivatives are:

$$\frac{\partial u}{\partial x} = -2e^{-2x}\cos 2y, \quad \frac{\partial u}{\partial y} = -2e^{-2x}\sin 2y$$

$$\frac{\partial v}{\partial x} = 2e^{-2x}\sin 2y, \quad \frac{\partial v}{\partial y} = -2e^{-2x}\cos 2y$$

Since the Cauchy-Riemann equations are satisfied, $f(z)$ is analytic.

Question 5

Is the function

$$f(z) = \mathrm{Re}(z^2) - i\mathrm{Im}(z^2)$$

analytic?
Solution:

$$z = x + iy$$

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$

$$\mathrm{Re}(z^2) = x^2 - y^2, \quad \mathrm{Im}(z^2) = 2xy$$

$$f(z) = \mathrm{Re}(z^2) - i\mathrm{Im}(z^2) = (x^2 - y^2) - i(2xy)$$

$$u(x, y) = x^2 - y^2, \quad v(x, y) = -2xy$$

$$u_x = 2x, \quad u_y = -2y$$

$$v_x = -2y, \quad v_y = -2x$$

$$u_x \neq v_y, \quad u_y \neq -v_x$$

$$f(z) \text{ is not analytic.}$$

Question 8

Is the function

$$f(z) = \operatorname{Arg}(2\pi z)$$

analytic?

Solution:

Given:

$$f(z) = \operatorname{Arg}(2\pi z)$$

Let $z = x + iy$, where x and y are real numbers.

Thus,

$$2\pi z = 2\pi(x + iy) = 2\pi x + i2\pi y$$

The argument of a complex number $z = x + iy$ is given by:

$$\operatorname{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Therefore,

$$f(z) = \operatorname{Arg}(2\pi z) = \operatorname{Arg}(2\pi x + i2\pi y) = \tan^{-1}\left(\frac{2\pi y}{2\pi x}\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

Since the argument $\operatorname{Arg}(2\pi z)$ represents an angle, it is a real-valued function, and the imaginary part $v(x, y)$ is zero, and $u(x, y)$ is given by:

$$u(x, y) = \operatorname{Arg}(2\pi z) = \tan^{-1}\left(\frac{y}{x}\right)$$

and the imaginary part $v(x, y)$ is:

$$v(x, y) = 0$$

Next, let's compute the partial derivatives of $u(x, y)$ and $v(x, y)$ with respect to x and y.

$$u_x = \frac{\partial}{\partial x} \tan^{-1}\left(\frac{y}{x}\right)$$

Using the chain rule and the derivative of $\tan^{-1}(z)$, which is $\frac{1}{1+z^2}$, we get:

$$u_x = \frac{\partial}{\partial x} \tan^{-1}\left(\frac{y}{x}\right) = \frac{-y}{x^2+y^2}$$

Similarly, for u_y:

$$u_y = \frac{\partial}{\partial y} \tan^{-1}\left(\frac{y}{x}\right) = \frac{x}{x^2+y^2}$$

Since $v(x,y) = 0$, its partial derivatives are:

$$v_x = 0, \quad v_y = 0$$

For $f(z)$ to be analytic, the Cauchy-Riemann equations must be satisfied:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Substituting the computed derivatives:

$$u_x = \frac{-y}{x^2+y^2}, \quad v_y = 0$$

$$u_y = \frac{x}{x^2+y^2}, \quad v_x = 0$$

Clearly, $u_x \neq v_y$ and $u_y \neq -v_x$. Therefore, the Cauchy-Riemann equations are not satisfied.

Hence, the function $f(z) = \text{Arg}(2\pi z)$ is **not analytic**.

Laplace's Equation

Laplace's equation is a second-order partial differential equation of the form:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

∇^2 read as "nabla squared".

Harmonic Function

A function $u(x,y)$ is called a **harmonic function** if it satisfies Laplace's equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

- If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then both u and v satisfy Laplace's equation, and so both are harmonic functions.

Harmonic Conjugate

Given a harmonic function $u(x, y)$, a function $v(x, y)$ is called a **harmonic conjugate** of $u(x, y)$ if the complex function $f(z) = u(x, y)+iv(x, y)$ is analytic, where $z = x+iy$. The relationship between u and v is governed by the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

The function $v(x, y)$ is also harmonic, and together with $u(x, y)$, they form the real and imaginary parts of the analytic function $f(z)$.

Problem Set 13.4

Question 12

Is the function $u(x, y) = x^2 + y^2$ harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

Solution:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(2x) = 2$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(2y) = 2$$

Therefore,

$$u_{xx} + u_{yy} = 2 + 2 = 4$$

Since the Laplacian of u is not zero, $u(x, y)$ is not a harmonic function.

Question 14

Is the function $v(x, y) = xy$ harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

Solution: To check if $v(x, y)$ is harmonic, we compute its second partial derivatives:

$$v_{xx} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}(y) = 0$$

$$v_{yy} = \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(x) = 0$$

Therefore,

$$v_{xx} + v_{yy} = 0 + 0 = 0$$

Since the Laplacian $v_{xx} + v_{yy}$ is zero, $v(x, y)$ is a harmonic function.

To find the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$, we assume $u(x, y)$ satisfies the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Given $v(x, y) = xy$:

$$v_y = x \quad \text{and} \quad v_x = y$$

From $u_x = v_y$:

$$u_x = x$$

Integrating with respect to x, we get:

$$u(x, y) = \frac{x^2}{2} + g(y)$$

From $u_y = -v_x$:

$$\frac{\partial u}{\partial y} = -y$$

$$g'(y) = -y$$

Integrating with respect to y, we get:

$$g(y) = -\frac{y^2}{2} + C$$

Thus,

$$u(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + C$$

The corresponding analytic function is:

$$f(z) = u(x, y) + iv(x, y) = \frac{x^2}{2} - \frac{y^2}{2} + C + ixy$$

Question 16

Is the function $u(x, y) = \sin x \cosh y$ harmonic? If your answer is yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$.

Solution:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} (\cos x \cosh y) = -\sin x \cosh y$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} (\sinh y \sin x) = \cosh y \sin x$$

Therefore,

$$u_{xx} + u_{yy} = -\sin x \cosh y + \sin x \cosh y = 0$$

Since the Laplacian $u_{xx} + u_{yy}$ is zero, $u(x, y)$ is a harmonic function.

To find the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$, we assume $v(x, y)$ satisfies the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Given $u(x, y) = \sin x \cosh y$:

$$u_x = \cos x \cosh y \quad \text{and} \quad u_y = \sin x \sinh y$$

From $u_x = v_y$:

$$v_y = \cos x \cosh y$$

Integrating with respect to y, we get:

$$v(x, y) = \cos x \sinh y + h(x)$$

From $u_y = -v_x$:

$$\sin x \sinh y = -v_x = -\frac{\partial}{\partial x}\left(\cos x \sinh y + h(x)\right)$$
$$\sin x \sinh y = \sin x \sinh y - h'(x)$$

Therefore, $h'(x) = 0$, and $h(x)$ is a constant. We can take $h(x) = 0$ for simplicity.

Thus, $v(x, y) = \cos x \sinh y$.

The corresponding analytic function is:

$$f(z) = u(x, y) + iv(x, y) = \sin x \cosh y + i \cos x \sinh y$$

Question 21

Determine a and b so that the given function is harmonic and find a harmonic conjugate.

$$u(x, y) = e^{\pi x} \cos(ay)$$

Solution: First, we need to compute the second partial derivatives of $u(x, y)$:

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\pi e^{\pi x} \cos(ay)\right) = \pi^2 e^{\pi x} \cos(ay)$$

$$u_{yy} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-ae^{\pi x}\sin(ay)\right) = -a^2 e^{\pi x}\cos(ay)$$

For $u(x, y)$ to be harmonic, the Laplacian must be zero:

$$u_{xx} + u_{yy} = \pi^2 e^{\pi x}\cos(ay) - a^2 e^{\pi x}\cos(ay) = e^{\pi x}\cos(ay)(\pi^2 - a^2) = 0$$

This equation is satisfied when:

$$\pi^2 - a^2 = 0 \quad \Rightarrow \quad a^2 = \pi^2 \quad \Rightarrow \quad a = \pi$$

Thus, $a = \pi$.

To find the harmonic conjugate $v(x, y)$, we use the Cauchy-Riemann equations:

$$u_x = v_y \quad \text{and} \quad u_y = -v_x$$

Given $u(x, y) = e^{\pi x}\cos(\pi y)$:

$$u_x = \pi e^{\pi x}\cos(\pi y) \quad \text{and} \quad u_y = -\pi e^{\pi x}\sin(\pi y)$$

From $v_y = \pi e^{\pi x}\cos(\pi y)$, integrating with respect to y, we get:

$$v(x, y) = e^{\pi x}\sin(\pi y) + h(x)$$

From $u_y = -v_x$:

$$-\pi e^{\pi x}\sin(\pi y) = -h'(x)$$

Thus, $h'(x) = 0$, and $h(x)$ is a constant. We can take $h(x) = 0$ for simplicity. Therefore, the harmonic conjugate is:

$$v(x, y) = e^{\pi x}\sin(\pi y)$$

The corresponding analytic function is:

$$f(z) = u(x, y) + iv(x, y) = e^{\pi x}\cos(\pi y) + ie^{\pi x}\sin(\pi y) = e^{\pi z}$$