Lecture 02: Powers and Roots of Complex Numbers

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1 Powers of Complex Numbers

Given a complex number $z = r(\cos\theta + i\sin\theta)$ in polar form, the n-th power of z is calculated using De Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$$

where n is a positive integer.

Example:

Let's find $(1+i)^3$.

First, express $1+i$ in polar form:

$$z = 1+i \quad \text{where} \quad r = \sqrt{1^2+1^2} = \sqrt{2}, \quad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

So, $z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

Now, using De Moivre's Theorem for $n = 3$:

$$z^3 = \left(\sqrt{2}\right)^3\left[\cos\left(3\times\frac{\pi}{4}\right) + i\sin\left(3\times\frac{\pi}{4}\right)\right]$$

$$z^3 = 2\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}, \quad \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$z^3 = 2\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$$

Now, simplifying further:

$$z^3 = 2\sqrt{2}\times -\frac{1}{\sqrt{2}} + 2\sqrt{2}\times\frac{1}{\sqrt{2}}i$$

$$z^3 = -2+2i$$

2 Roots of Complex Numbers

To find the n-th root of a complex number $z = r(\cos\theta + i\sin\theta)$, we use the formula:

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right], \quad k = 0, 1, 2, \ldots, n-1$$

This expression gives all the n different roots of the complex number.

Problem Set 13.2

Question 22

Find and graph all roots of $\sqrt[3]{3+4i}$ in the complex plane. We are tasked with finding the cube roots of the complex number $z = 3 + 4i$ and graphing them in the complex plane.

Step 1: Convert z to Polar Form

First, we find the modulus r and the argument θ of the complex number $z = 3 + 4i$.

$$r = |z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.93 \text{ radians} \approx 53^\circ$$

Thus, the polar form of z is:

$$z = 5\left(\cos(0.93) + i\sin(0.93)\right)$$

Step 2: Finding the Cube Roots

The general formula for the n-th roots of a complex number $z = r(\cos\theta + i\sin\theta)$ is:

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

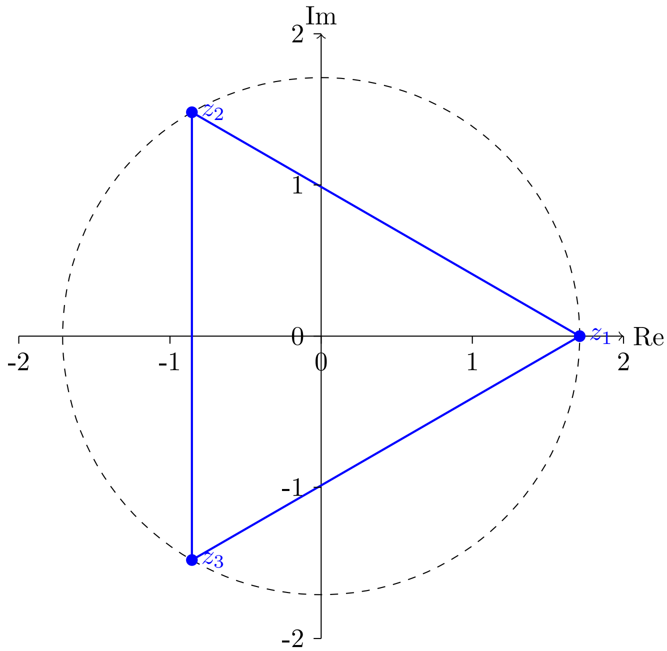
For $n = 3$ and $r = 5$, we find the cube roots as follows:

$$\text{Root 1: } z_1 = \sqrt[3]{5}\left[\cos(17.8^\circ) + i\sin(17.8^\circ)\right] \quad \text{(at angle } 17.8^\circ\text{)}$$

$$\text{Root 2: } z_2 = \sqrt[3]{5}\left[\cos(137.8^\circ) + i\sin(137.8^\circ)\right] \quad \text{(at angle } 137.8^\circ\text{)}$$

$$\text{Root 3: } z_3 = \sqrt[3]{5}\left[\cos(257.8^\circ) + i\sin(257.8^\circ)\right] \quad \text{(at angle } 257.8^\circ\text{)}$$

Step 3: Graphing the Roots in the Complex Plane



Question 26

Find and graph all roots of $\sqrt[8]{1}$ in the complex plane. The eighth roots of 1 are given by the formula:

$$z_k = \cos\left(\frac{2\pi k}{8}\right) + i\sin\left(\frac{2\pi k}{8}\right), \quad k = 0, 1, 2, \ldots, 7$$

These roots are:

$$
\begin{aligned}
z_0 &= 1 && \text{(at angle } 0^\circ\text{)}\\
z_1 &= \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) && \text{(at angle } 45^\circ\text{)}\\
z_2 &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) && \text{(at angle } 90^\circ\text{)}\\
z_3 &= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) && \text{(at angle } 135^\circ\text{)}\\
z_4 &= \cos\left(\pi\right) + i\sin\left(\pi\right) && \text{(at angle } 180^\circ\text{)}\\
z_5 &= \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) && \text{(at angle } 225^\circ\text{)}\\
z_6 &= \cos\left(\frac{3\pi/2}{}\right.) + i\sin\left(\frac{3\pi/2}{}\right.) && \text{(at angle } 270^\circ\text{)}\\
z_7 &= \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right) && \text{(at angle } 315^\circ\text{)}
\end{aligned}
$$

Graphing the Roots