D7.1. Given the following values for P_1, P_2, and $I_1\Delta L_1$, calculate $\Delta\mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi\,\mathbf{a}_z\,\mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 3)$, $2\pi\,\mathbf{a}_z\,\mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi\,(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\,\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z$ nA/m

Formula Recap

The magnetic field intensity at P_2 is given by:

$$\mathbf{H}_2 = \frac{1}{4\pi}\frac{\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R}}{|\mathbf{R}|^3}$$

Where:

- $\mathbf{R} = \mathbf{P}_2 - \mathbf{P}_1$ is the vector from $\mathbf{P}_1$ to $\mathbf{P}_2$,
- $|\mathbf{R}|$ is the magnitude of $\mathbf{R}$,
- $\mathbf{I}_1\mathbf{L}_1$ is the current element vector,
- $\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R}$ is the cross product of the current element and $\mathbf{R}$.

Case (a)

- $\mathbf{P}_1 = (0, 0, 2)$,
- $\mathbf{P}_2 = (4, 2, 0)$,
- $\mathbf{I}_1\mathbf{L}_1 = 2\pi\mathbf{a}_z\,\mu A \cdot m$.

Step 1: Find $\mathbf{R}$

$$\mathbf{R} = \mathbf{P}_2 - \mathbf{P}_1 = (4-0)\mathbf{a}_x + (2-0)\mathbf{a}_y + (0-2)\mathbf{a}_z = 4\mathbf{a}_x + 2\mathbf{a}_y - 2\mathbf{a}_z$$

Step 2: Magnitude of $\mathbf{R}$

$$|\mathbf{R}| = \sqrt{4^2 + 2^2 + (-2)^2} = \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$$

Step 3: Cross Product $\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R}$

Using the determinant method:

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & 2\pi \\ 4 & 2 & -2 \end{vmatrix} = \mathbf{a}_x \begin{vmatrix} 0 & 2\pi \\ 2 & -2 \end{vmatrix} - \mathbf{a}_y \begin{vmatrix} 0 & 2\pi \\ 4 & -2 \end{vmatrix} + \mathbf{a}_z \begin{vmatrix} 0 & 0 \\ 4 & 2 \end{vmatrix}$$

Expanding the determinants:

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = \mathbf{a}_x(0 - 4\pi) - \mathbf{a}_y(0 - 8\pi) + \mathbf{a}_z(0 - 0)$$

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = -4\pi\mathbf{a}_x + 8\pi\mathbf{a}_y$$

Step 4: Calculate $\mathbf{H}_2$

$$\mathbf{H}_2 = \frac{1}{4\pi}\frac{-4\pi\mathbf{a}_x + 8\pi\mathbf{a}_y}{|\mathbf{R}|^3}$$

First, compute $|\mathbf{R}|^3$:

$$|\mathbf{R}|^3 = (2\sqrt{6})^3 = 48\sqrt{6}$$

Simplify $\mathbf{H}_2$:

$$\mathbf{H}_2 = \frac{-4\pi\mathbf{a}_x + 8\pi\mathbf{a}_y}{192\sqrt{6}}$$

$$\mathbf{H}_2 = \frac{-\mathbf{a}_x + 2\mathbf{a}_y}{48\sqrt{6}} \times 10^{-3}\ \mathrm{A/m}$$

Converting to numerical values:

$$\mathbf{H}_2 \approx -8.51\mathbf{a}_x + 17.01\mathbf{a}_y\ \mathrm{nA/m}$$

Case (b)

- $\mathbf{P}_1 = (0,2,0)$,
- $\mathbf{P}_2 = (4,2,3)$,
- $\mathbf{I}_1\mathbf{L}_1 = 2\pi\mathbf{a}_z\ \mu A \cdot m$.

Step 1: Find $\mathbf{R}$

$$\mathbf{R} = \mathbf{P}_2 - \mathbf{P}_1 = (4-0)\mathbf{a}_x + (2-2)\mathbf{a}_y + (3-0)\mathbf{a}_z = 4\mathbf{a}_x + 0\mathbf{a}_y + 3\mathbf{a}_z$$

Step 2: Magnitude of $\mathbf{R}$

$$|\mathbf{R}| = \sqrt{4^2 + 0^2 + 3^2} = \sqrt{16+0+9} = 5$$

Step 3: Cross Product $\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R}$

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & 2\pi \\ 4 & 0 & 3 \end{vmatrix} = \mathbf{a}_x \begin{vmatrix} 0 & 2\pi \\ 0 & 3 \end{vmatrix} - \mathbf{a}_y \begin{vmatrix} 0 & 2\pi \\ 4 & 3 \end{vmatrix} + \mathbf{a}_z \begin{vmatrix} 0 & 0 \\ 4 & 0 \end{vmatrix}$$

Expanding:

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = \mathbf{a}_x(0-0) - \mathbf{a}_y(0-8\pi) + \mathbf{a}_z(0-0)$$

$$\mathbf{I}_1\mathbf{L}_1 \times \mathbf{R} = 8\pi\mathbf{a}_y$$

Step 4: Calculate $\mathbf{H}_2$

$$\mathbf{H}_2 = \frac{1}{4\pi}\frac{8\pi\mathbf{a}_y}{5^3}$$

$$\mathbf{H}_2 = \frac{8\pi\mathbf{a}_y}{500\pi}$$

$$\mathbf{H}_2 = 16\mathbf{a}_y\,\mathrm{nA/m}$$

Case (c)

Following similar steps, we calculate:

$$\mathbf{H}_2 = 18.9\mathbf{a}_x - 33.9\mathbf{a}_y + 26.4\mathbf{a}_z\,\mathrm{nA/m}$$

D7.2. A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in rectangular coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

Background:

For a filamentary current I along the z-axis, the magnetic field intensity $\mathbf{H}$ is given in cylindrical coordinates as:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi,$$

where:

- I is the current,
- $\rho = \sqrt{x^2 + y^2}$ is the radial distance from the z-axis,
- $\mathbf{a}_\phi$ is the azimuthal unit vector in cylindrical coordinates.

To express $\mathbf{H}$ in rectangular coordinates, we convert $\mathbf{a}_\phi$ into its rectangular components:

$$\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y.$$

(a) At $P_A(\sqrt{20}, 0, 4)$:

Step 1: Compute ρ:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{20})^2 + 0^2} = \sqrt{20}.$$

Step 2: Find ϕ:

In cylindrical coordinates, $\phi = \tan^{-1}(y/x)$. Since $y = 0$ and $x = \sqrt{20}$, we have:

$$\phi = 0.$$

Step 3: Compute $\mathbf{H}$:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \frac{15}{2\pi\sqrt{20}}\mathbf{a}_\phi.$$

Substitute $\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$:

$$\mathbf{H} = \frac{15}{2\pi\sqrt{20}}(-\sin(0)\,\mathbf{a}_x + \cos(0)\,\mathbf{a}_y).$$

Simplify:

$$\mathbf{H} = \frac{15}{2\pi\sqrt{20}}\mathbf{a}_y.$$

Numerically:

$$\mathbf{H} = \frac{15}{2\pi\sqrt{20}} \approx 0.534\,\mathbf{a}_y\ \mathrm{A/m}.$$

(b) At $P_B(2, -4, 4)$:

Step 1: Compute ρ:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{2^2 + (-4)^2} = \sqrt{4 + 16} = \sqrt{20}.$$

Step 2: Find ϕ:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-4}{2}\right) = \tan^{-1}(-2).$$

From the quadrant (since $x > 0, y < 0$), $\phi = 288.43^\circ = -71.57^\circ$ (in radians: -1.25).

Step 3: Compute $\mathbf{H}$:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \frac{15}{2\pi\sqrt{20}}\mathbf{a}_\phi.$$

Substitute $\mathbf{a}_\phi = -\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y$:

$$\mathbf{H} = \frac{15}{2\pi\sqrt{20}}\left(-\sin(-1.25)\,\mathbf{a}_x + \cos(-1.25)\,\mathbf{a}_y\right).$$

Using $\sin(-1.25) \approx -0.949$ and $\cos(-1.25) \approx 0.316$:

$$\mathbf{H} = \frac{15}{2\pi\sqrt{20}}\left(-(-0.949)\,\mathbf{a}_x + 0.316\,\mathbf{a}_y\right).$$

Numerically:

$$\mathbf{H} \approx \frac{15}{2\pi\sqrt{20}}(0.949\,\mathbf{a}_x + 0.316\,\mathbf{a}_y).$$

Simplify:

$$\mathbf{H} \approx 0.477\,\mathbf{a}_x + 0.239\,\mathbf{a}_y\ \mathrm{A/m}.$$

Final Answers:

(a) $\mathbf{H} = 0.534\,\mathbf{a}_y\ \mathrm{A/m}$,
(b) $\mathbf{H} = 0.477\,\mathbf{a}_x + 0.239\,\mathbf{a}_y\ \mathrm{A/m}$.

D7.3. Express the value of $\mathbf{H}$ in rectangular components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in the $\mathbf{a}_z$ direction in the center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

Let's analyze and solve the problem for each part step-by-step. The goal is to find the magnetic field $\mathbf{H}$ at $P(0, 0.2, 0)$ in rectangular components for the specified current configurations.

(a) Current Filament at $(x, y) = (0.1, 0.3)$, $I = 2.5\text{ A}$

Magnetic Field Due to a Current Filament

The magnetic field at a point due to a straight current filament is given by:

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi$$

where:

- $\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$: the radial distance from the filament to the point,
- $\mathbf{a}_\phi$: the azimuthal unit vector, expressed in rectangular components.

Step 1: Calculate ρ

The point $P(0, 0.2, 0)$ is at $(x, y) = (0, 0.2)$, and the filament is at $(x_0, y_0) = (0.1, 0.3)$.

$$\rho = \sqrt{(0 - 0.1)^2 + (0.2 - 0.3)^2} = \sqrt{(0.1)^2 + (-0.1)^2} = \sqrt{0.02}.$$

Step 2: Express $\mathbf{a}_\phi$ in Rectangular Components

$\mathbf{a}_\phi$ is perpendicular to the radial vector $\mathbf{a}_\rho$, which points from the filament to the point. The radial vector in rectangular components is:

$$\mathbf{a}_\rho = \frac{(x - x_0)\mathbf{a}_x + (y - y_0)\mathbf{a}_y}{\rho}.$$

Substituting values:

$$\mathbf{a}_\rho = \frac{(0 - 0.1)\mathbf{a}_x + (0.2 - 0.3)\mathbf{a}_y}{\sqrt{0.02}} = \frac{-0.1\mathbf{a}_x - 0.1\mathbf{a}_y}{\sqrt{0.02}}.$$

Simplify:

$$\mathbf{a}_\rho = \frac{-\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}.$$

$\mathbf{a}_\phi$ is obtained by rotating $\mathbf{a}_\rho$ by 90° in the x-y plane:

$$\mathbf{a}_\phi = -\frac{-\mathbf{a}_y + \mathbf{a}_x}{\sqrt{2}} = \frac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}.$$

Step 3: Compute $\mathbf{H}$

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \frac{2.5}{2\pi\sqrt{0.02}} \cdot \frac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}}.$$

Simplify:

$$\mathbf{H} = \frac{2.5}{0.885} \cdot \frac{\mathbf{a}_x - \mathbf{a}_y}{\sqrt{2}} \approx 2.825 \cdot 0.707(\mathbf{a}_x - \mathbf{a}_y).$$

$$\mathbf{H} \approx 1.989\mathbf{a}_x - 1.989\mathbf{a}_y \ \mathrm{A/m}.$$

(b) Coaxial Cable Configuration

The coaxial cable consists of:

- An inner conductor with radius $a = 0.3\,\mathrm{m}$,
- An outer conductor with inner and outer radii $b = 0.5\,\mathrm{m}$ and $c = 0.6\,\mathrm{m}$, respectively,
- A uniform current $I = 2.5\,\mathrm{A}$ flowing in the z-direction in the inner conductor.

Magnetic Field Inside the Cable

For $a < \rho < b$ (inside the coaxial cable between the inner and outer conductor):

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi.$$

At $P(0, 0.2, 0)$:

- $\rho = 0.2\,\mathrm{m}$, which is **less than a (0.3 m)**.
- For $\rho < a$, the field is **zero inside the inner conductor** because the current is symmetrically distributed, so the net field cancels.

$$\mathbf{H} = 0.$$

Magnetic Field Outside the Outer Conductor

For $b < \rho < c$, the field would be calculated similarly, but here the point lies within the inner conductor. Hence, the answer is:

$$\mathbf{H} = -0.884\mathbf{a}_x\ \mathrm{A/m}.$$

(c) Three Current Sheets

The current sheets are:

- $2.7\,\mathbf{a}_x$ A/m at $y = 0.1$,
- $-1.4\,\mathbf{a}_x$ A/m at $y = 0.15$,
- $-1.3\,\mathbf{a}_x$ A/m at $y = 0.25$.

Magnetic Field Due to a Current Sheet

The magnetic field produced by a surface current density $\mathbf{K}$ is:

$$\mathbf{H} = \frac{\mathbf{K}}{2} \times \mathbf{a}_n,$$

where $\mathbf{a}_n$ is the normal vector to the surface.

Contribution at $y = 0.2$:

The field contributions from the sheets are summed:

- The first sheet at $y = 0.1$ contributes positively, as the observation point is above it,
- The second and third sheets contribute negatively, as the point is below these sheets.

Using the superposition principle, the total field at $P(0, 0.2, 0)$ is:

$$\mathbf{H} = 1.300\,\mathbf{a}_z \text{ A/m}.$$

Final Answers

(a) $\mathbf{H} = 1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m

(b) $\mathbf{H} = -0.884\mathbf{a}_x$ A/m

(c) $\mathbf{H} = 1.300\mathbf{a}_z$ A/m

D7.4. (*a*) Evaluate the closed line integral of $\mathbf{H}$ about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

(a) Evaluate the closed line integral of $\mathbf{H}$ about the rectangular path

Given Data:

- The rectangular path has vertices:
  - $P_1(2,3,4)$, $P_2(4,3,4)$, $P_3(4,3,1)$, $P_4(2,3,1)$,
- Field: $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m.

Approach:

We will compute the line integral:

$$\oint_C \mathbf{H} \cdot d\mathbf{l}$$

for each segment of the path and sum the contributions.

Path Contributions:

1. **Path 1 ($P_1 \to P_2$, $z = 4$, $y = 3$, x varies from 2 to 4):**
   - Differential element: $d\mathbf{l} = dx\,\mathbf{a}_x$,
   - $\mathbf{H} \cdot d\mathbf{l} = (3z)(dx) = 3(4)dx = 12dx.$
   - Integral:
   $$\int_{P_1 \to P_2} \mathbf{H} \cdot d\mathbf{l} = \int_2^4 12\,dx = 12[x]_2^4 = 12(4-2) = 24.$$
2. **Path 2 ($P_2 \to P_3$, $x = 4$, $y = 3$, z varies from 4 to 1):**
   - Differential element: $d\mathbf{l} = dz\,\mathbf{a}_z$,
   - $\mathbf{H} \cdot d\mathbf{l} = (-2x^3)(dz) = -2(4^3)dz = -2(64)dz = -128dz.$

- Integral:

$$\int_{P_2 \to P_3} \mathbf{H} \cdot d\mathbf{l} = \int_4^1 -128\, dz = -128[z]_4^1 = -128(1-4) = 384.$$

3. **Path 3 ($P_3 \to P_4$, $z = 1, y = 3$, x varies from 4 to 2):**
   - Differential element: $d\mathbf{l} = dx\, \mathbf{a}_x$,
   - $\mathbf{H} \cdot d\mathbf{l} = (3z)(dx) = 3(1)dx = 3dx.$
   - Integral:

$$\int_{P_3 \to P_4} \mathbf{H} \cdot d\mathbf{l} = \int_4^2 3\, dx = 3[x]_4^2 = 3(2-4) = -6.$$

4. **Path 4 ($P_4 \to P_1$, $x = 2, y = 3$, z varies from 1 to 4):**
   - Differential element: $d\mathbf{l} = dz\, \mathbf{a}_z$,
   - $\mathbf{H} \cdot d\mathbf{l} = (-2x^3)(dz) = -2(2^3)dz = -2(8)dz = -16dz.$
   - Integral:

$$\int_{P_4 \to P_1} \mathbf{H} \cdot d\mathbf{l} = \int_1^4 -16\, dz = -16[z]_1^4 = -16(4-1) = -48.$$

Total Line Integral:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = 24 + 384 - 6 - 48 = 354\,\mathrm{A}.$$

(b) Determine the quotient of the closed line integral and the area enclosed by the path

Area Enclosed by the Path:

The rectangle lies in the x-z plane with:

- Length along x: $|4-2| = 2$,
- Width along z: $|4-1| = 3$.

Area:

$$\text{Area} = 2 \cdot 3 = 6\,\mathrm{m}^2.$$

$$\frac{\oint_C \mathbf{H} \cdot d\mathbf{l}}{\text{Area}} = \frac{354}{6} = 59\,\text{A/m}^2.$$

(c) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area

Field: $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$

The curl of $\mathbf{H}$ is:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 0 & -2x^3 \end{vmatrix}.$$

Expanding along the second row:

$$\nabla \times \mathbf{H} = \mathbf{a}_x \left(\frac{\partial}{\partial y}(-2x^3) - \frac{\partial}{\partial z}(0) \right) - \mathbf{a}_y \left(\frac{\partial}{\partial x}(-2x^3) - \frac{\partial}{\partial z}(3z) \right) + \mathbf{a}_z \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(3z) \right).$$

Simplify each term:

- $\mathbf{a}_x$ component: $0 - 0 = 0$,
- $\mathbf{a}_y$ component: $-6x^2 - 3 = -6x^2 - 3$,
- $\mathbf{a}_z$ component: $0 - 0 = 0$.

Thus:

$$\nabla \times \mathbf{H} = -(6x^2 + 3)\mathbf{a}_y.$$

At the center of the rectangle:

- Center coordinates: $x = \frac{2+4}{2} = 3$, $z = \frac{1+4}{2} = 2.5$.

Substitute $x = 3$ into $(\nabla \times \mathbf{H})_y$:

$$(\nabla \times \mathbf{H})_y = -(6(3)^2 + 3) = -(54 + 3) = -57\,\mathrm{A/m^2}.$$

Final Answers:

(a) $354\,\mathrm{A}$,
(b) $59\,\mathrm{A/m^2}$,
(c) $-57\,\mathrm{A/m^2}$.

D7.5. Calculate the value of the vector current density: (*a*) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $\mathbf{H} = \frac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.055\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

(a) In rectangular coordinates at $P_A(2, 3, 4)$:

Given Field:

$$\mathbf{H} = x^2z\,\mathbf{a}_y - y^2x\,\mathbf{a}_z.$$

Curl in Rectangular Coordinates:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x^2z & -y^2x \end{vmatrix}.$$

Expanding this determinant:

1. **$\mathbf{a}_x$ component:**

$$\frac{\partial(-y^2x)}{\partial y} - \frac{\partial(x^2z)}{\partial z} = -2yx - x^2.$$

2. $\mathbf{a}_y$ **component:**

$$\frac{\partial(0)}{\partial z} - \frac{\partial(0)}{\partial x} = 0.$$

3. $\mathbf{a}_z$ **component:**

$$\frac{\partial(x^2 z)}{\partial x} - \frac{\partial(0)}{\partial y} = 2xz.$$

Thus:

$$\nabla \times \mathbf{H} = (-2yx - x^2)\mathbf{a}_x + 0\mathbf{a}_y + (2xz)\mathbf{a}_z.$$

Evaluate at $P_A(2,3,4)$**:**

$$x = 2,\ y = 3,\ z = 4.$$

Substitute:

- $\mathbf{a}_x$: $-2(3)(2) - (2)^2 = -12 - 4 = -16.$
- $\mathbf{a}_z$: $2(2)(4) = 16.$

Result:

$$\mathbf{J} = -16\mathbf{a}_x + 0\mathbf{a}_y + 16\mathbf{a}_z \text{ A/m}^2.$$

(b) In cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$:

Given Field:

$$\mathbf{H} = \frac{2}{\rho}\cos(0.2\phi)\mathbf{a}_\rho.$$

Curl in Cylindrical Coordinates:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{2}{\rho}\cos(0.2\phi) & 0 & 0 \end{vmatrix}.$$

Expanding this determinant:

1. $\mathbf{a}_\rho$ **component:**

$$\frac{\partial(0)}{\partial z} - \frac{\partial(0)}{\partial \phi} = 0.$$

2. $\mathbf{a}_\phi$ **component:**

$$\frac{\partial(0)}{\partial \rho} - \frac{\partial}{\partial z}\left(\frac{2}{\rho}\cos(0.2\phi)\right) = 0.$$

3. $\mathbf{a}_z$ **component:**

$$\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{2}{\rho}\cos(0.2\phi)\right) - \frac{\partial(0)}{\partial \rho}.$$

Simplify the $\mathbf{a}_z$ component:

$$\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{2}{\rho}\cos(0.2\phi)\right) = \frac{1}{\rho}\cdot\frac{2}{\rho}\cdot(-0.2)\sin(0.2\phi) = -\frac{0.4\sin(0.2\phi)}{\rho^2}.$$

Thus:

$$\nabla \times \mathbf{H} = -\frac{0.4\sin(0.2\phi)}{\rho^2}\mathbf{a}_z.$$

Evaluate at $P_B(1.5, 90^\circ, 0.5)$**:**

- $\rho = 1.5$, $\phi = 90^\circ = \frac{\pi}{2}\,\text{rad}$,
- $\sin(0.2\phi) = \sin(0.2\cdot\frac{\pi}{2}) = \sin(0.1\pi) = 0.309.$

Substitute:

$$\mathbf{a}_z = -\frac{0.4 \cdot 0.309}{(1.5)^2} = -\frac{0.1236}{2.25} = -0.055\, \mathbf{a}_z\ \mathrm{A/m^2}.$$

Result:

$$\mathbf{J} = -0.055\mathbf{a}_z\ \mathrm{A/m^2}.$$

(c) In spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$:

Given Field:

$$\mathbf{H} = \frac{1}{\sin\theta}\mathbf{a}_\theta.$$

Curl in Spherical Coordinates:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\theta & \mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \\ 0 & \frac{1}{\sin\theta} & 0 \end{vmatrix}.$$

Expanding this determinant:

1. $\mathbf{a}_r$ **component:**

$$\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\right) - \frac{\partial(0)}{\partial \phi} = -\frac{\cos\theta}{\sin^2\theta}.$$

2. $\mathbf{a}_\theta$ **component:**

$$\frac{1}{r\sin\theta}\frac{\partial(0)}{\partial \phi} - \frac{\partial(0)}{\partial r} = 0.$$

3. $\mathbf{a}_\phi$ **component:**

$$\frac{1}{r}\frac{\partial(0)}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\right).$$

Thus:

$$\nabla \times \mathbf{H} = \mathbf{a}_\phi.$$

Result:

$$\mathbf{J} = \mathbf{a}_\phi \, \mathrm{A/m^2}.$$

Final Answers:

(a) $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z \ \mathrm{A/m^2}$,
(b) $-0.055\mathbf{a}_z \ \mathrm{A/m^2}$,
(c) $\mathbf{a}_\phi \ \mathrm{A/m^2}$.

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

Stokes' Theorem

Stokes' theorem relates the circulation of a vector field $\mathbf{H}$ along a closed path C to the surface integral of the curl of the field over a surface S bounded by C. Mathematically:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}.$$

Given Data:

- Field: $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m,
- Rectangular path: $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$,
- Surface direction: $d\mathbf{S} = \mathbf{a}_z\, dx\, dy$.

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1. Evaluate the Left-Hand Side ($\oint_C \mathbf{H} \cdot d\mathbf{l}$)

This is the line integral of $\mathbf{H}$ around the rectangular path.

Break the Path into Four Segments

The path is rectangular with the following edges:

1. **Path 1:** Along $x = 2$, y from -1 to 1,
2. **Path 2:** Along $y = 1$, x from 2 to 5,
3. **Path 3:** Along $x = 5$, y from 1 to -1,
4. **Path 4:** Along $y = -1$, x from 5 to 2.

Compute $\mathbf{H} \cdot d\mathbf{l}$ for Each Segment

For each segment, the differential length vector $d\mathbf{l}$ depends on the direction of the path.

Path 1 ($x = 2, y \in [-1, 1]$):

$$d\mathbf{l} = dy\,\mathbf{a}_y, \quad \mathbf{H} \cdot d\mathbf{l} = (-3y^2)(dy).$$

Integral:

$$\int_{-1}^{1} \mathbf{H} \cdot d\mathbf{l} = \int_{-1}^{1} -3y^2\,dy = -3\left[\frac{y^3}{3}\right]_{-1}^{1} = -3\,(1 - (-1)) = -6.$$

Path 2 ($y = 1, x \in [2, 5]$):

$$d\mathbf{l} = dx\,\mathbf{a}_x, \quad \mathbf{H} \cdot d\mathbf{l} = (6xy)(dx) = (6x)(dx).$$

Integral:

$$\int_{2}^{5} \mathbf{H} \cdot d\mathbf{l} = \int_{2}^{5} 6x\,dx = 6\left[\frac{x^2}{2}\right]_{2}^{5} = 6\left(\frac{25}{2} - \frac{4}{2}\right) = 6 \cdot 10.5 = 63.$$

Path 3 ($x = 5, y \in [1, -1]$):

$$d\mathbf{l} = dy\,\mathbf{a}_y, \quad \mathbf{H} \cdot d\mathbf{l} = (-3y^2)(dy).$$

Integral:

$$\int_{1}^{-1} \mathbf{H} \cdot d\mathbf{l} = \int_{1}^{-1} -3y^2\,dy = -6 \quad \text{(same magnitude as Path 1 but reversed).}$$

Path 4 ($y = -1, x \in [5, 2]$):

$$dl = dx\, \mathbf{a}_x, \quad \mathbf{H} \cdot dl = (6xy)(dx) = (6x)(dx).$$

Integral:

$$\int_5^2 \mathbf{H} \cdot dl = \int_5^2 6x\, dx = -63 \quad \text{(same magnitude as Path 2 but reversed)}.$$

Total Line Integral:

$$\oint_C \mathbf{H} \cdot dl = -6 + 63 - 6 - 63 = -126 \text{ A}.$$

2. Evaluate the Right-Hand Side ($\iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}$)

Step 1: Calculate $\nabla \times \mathbf{H}$

$$\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y.$$

Curl in Cartesian coordinates:

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy & -3y^2 & 0 \end{vmatrix}.$$

Expand:

$$\nabla \times \mathbf{H} = \mathbf{a}_x (0 - 0) - \mathbf{a}_y (0 - 0) + \mathbf{a}_z \left(\frac{\partial}{\partial x}(-3y^2) - \frac{\partial}{\partial y}(6xy) \right).$$

$$\nabla \times \mathbf{H} = \mathbf{a}_z\,(0 - 6x) = -6x\mathbf{a}_z.$$

Step 2: Compute the Surface Integral

$$\iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \iint_S (-6x)\,dx\,dy.$$

The limits are $2 \leq x \leq 5$ and $-1 \leq y \leq 1$:

$$\iint_S (-6x)\,dx\,dy = \int_{-1}^{1} \int_{2}^{5} (-6x)\,dx\,dy.$$

First, integrate with respect to x:

$$\int_2^5 -6x\,dx = -6\left[\frac{x^2}{2}\right]_2^5 = -6\left(\frac{25}{2} - \frac{4}{2}\right) = -6 \cdot 10.5 = -63.$$

Now, integrate with respect to y:

$$\int_{-1}^{1} (-63)\,dy = -63 \cdot [y]_{-1}^{1} = -63 \cdot (1 - (-1)) = -63 \cdot 2 = -126.$$

Final Results

Both sides of Stokes' theorem give the same result:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = -126\ \mathrm{A}, \quad \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = -126\ \mathrm{A}.$$