Line Integral in Complex Plane

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Complex Line Integral

Complex definite integrals are called (complex) **line integrals**. They are written

$$\int_C f(z)\,dz.$$

Here the integrand $f(z)$ is integrated over a given curve C. This curve C in the complex plane is called the **path of integration**. if C is a **closed path** (one whose terminal point z coincides with its initial point) , then the line integral will be written as:

$$\oint_C f(z)\,dz$$

Methods of Evaluation of Complex Line Integral

First Evaluation Method: Indefinite Integration and Substitution of Limits

This method is analogous to the evaluation of definite integrals in calculus, using the well-known formula:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F'(x) = f(x)$. It is simpler than other methods but is suitable only for analytic functions.

Second Evaluation Method: Use of a Representation of a Path

This method is not limited to analytic functions but applies to any continuous complex function. It involves the use of path parametrization. Let C be a piecewise smooth path represented by $z = z(t)$, where $a \le t \le b$. Let $f(z)$ be a continuous function on C. Then, the complex line integral is evaluated as:

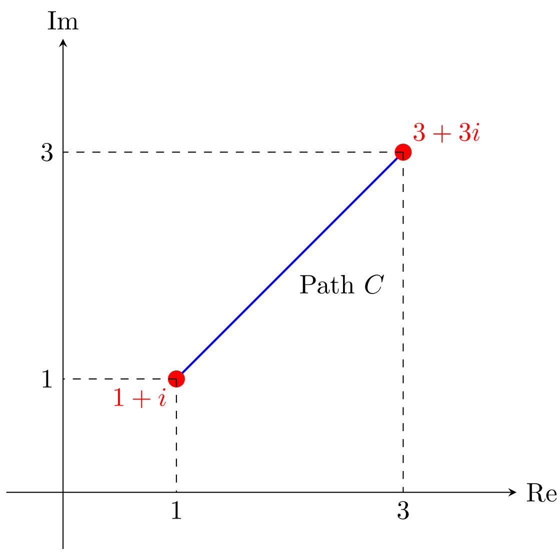
$$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right)$$

This approach is useful when the path C is represented in a parametric form, enabling the evaluation of integrals for more general cases.

Steps for Evaluation

1. Represent the path C in the form $z(t)$, where $a \le t \le b$.
2. Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$.
3. Substitute $z(t)$ for every z in $f(z)$.

Second Method: Use of a Representation of a Path



Step 1: Parametrize the Path

The shortest path from $1+i$ to $3+3i$ is a straight line, which can be represented by:

$$z(t) = (1+i) + t[(3+3i) - (1+i)] = (1+i) + t(2+2i), \quad 0 \leq t \leq 1$$

Simplifying,

$$z(t) = (1+2t) + i(1+2t), \quad 0 \leq t \leq 1$$

Step 2: Calculate the Derivative

The derivative of $z(t)$ with respect to t is:

$$\dot{z}(t) = \frac{dz}{dt} = 2+2i$$

Step 3: Substitute $z(t)$ in $f(z)$

The real part of $z(t)$, denoted as $f[z(t)]$, is:

$$f[z(t)] = \mathrm{Re}(z(t)) = 1+2t$$

Step 4: Evaluate the Line Integral

We now evaluate the line integral along C:

$$\int_C \mathrm{Re}\, z\, dz = \int_0^1 f[z(t)]\, \dot{z}(t)\, dt = \int_0^1 (1+2t)\cdot(2+2i)\, dt$$

Expanding the integrand:

$$(1 + 2t)(2 + 2i) = 2 + 2i + 4t + 4ti = (2 + 4t) + i(2 + 4t)$$

Now integrate each part:

$$\int_0^1 (2 + 4t)\,dt + i\int_0^1 (2 + 4t)\,dt$$
$$= \left[2t + 2t^2\right]_0^1 + i\left[2t + 2t^2\right]_0^1$$
$$= (2 + 2) + i(2 + 2) = 4 + 4i$$

Question 23

$$\int_C e^z\,dz$$

where C is the shortest path from πi to $2\pi i$.

Solution

The first method involves finding an antiderivative of the function $f(z) = e^z$. Since e^z is an analytic function everywhere, we can apply the first method.

$$\int_C e^z\,dz = e^{2\pi i} - e^{\pi i}$$

Now, evaluating the exponential values:

$$e^{\pi i} = -1, \quad e^{2\pi i} = 1$$

Therefore:

$$\int_C e^z\,dz = 1 - (-1) = 2$$

Problem 25

Evaluate the integral

$$\int_C z\exp(z^2)\,dz$$

where C is the path from 1 along the axes to i.

Solution

This function is analytic everywhere in the complex plane, and its integral can be found as follows:

$$\int_C z\exp(z^2)\,dz = \left[\frac{1}{2}\exp(z^2)\right]_1^i = \frac{1}{2e} - \frac{1}{2}e = \frac{1}{2}\left(e^{-1} - e^1\right) = -\sinh 1$$

Question 26

$$\int_C \left(z + z^{-1}\right)\,dz$$

C is the unit circle counterclockwise.

Solution

First Method: Direct Evaluation Using Antiderivative

$z^{-1} = \frac{1}{z}$ is not analytic at $z = 0$. So the first method does not apply.

Second Method: Parametrization of the Path

Since the first method cannot be used, we use the second method by parametrizing the unit circle.

Step 1: Parametrize the Path The unit circle can be parametrized as:

$$z(t) = e^{it}, \quad 0 \le t \le 2\pi$$

The differential dz is:

$$dz = ie^{it}\,dt$$

Step 2: Substitute and Integrate Substituting $z(t)$ and dz into the integral:

$$\int_C \left(z + z^{-1}\right)\,dz = \int_0^{2\pi} \left(e^{it} + e^{-it}\right) ie^{it}\,dt$$

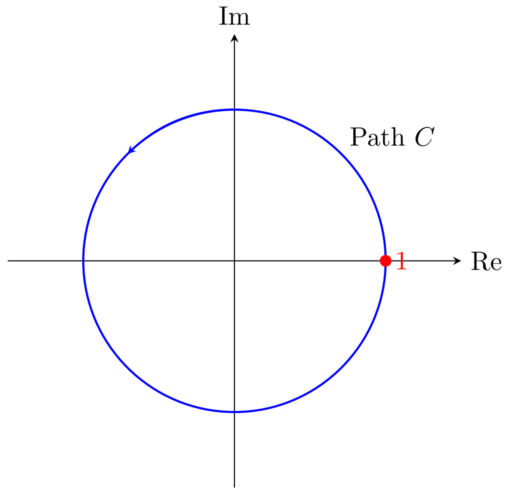
Expanding the integrand:

$$= i\int_0^{2\pi} \left(e^{2it} + 1\right)\,dt$$

Step 3: Evaluate the Integral

$$\int_C \left(z + z^{-1}\right)\,dz = i \cdot 0 + i \cdot 2\pi = 2\pi i$$

Plot of the Path



The path C is the unit circle centered at the origin, traversed counterclockwise.