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Homework Assignment 3
Image Compression with SVD

Abstract

In this homework, we review basic concepts related to singular value decomposition(SVD) and then focus on image compression as an application of SVD.

1 Theorem (SVD)

Let $A \epsilon \mathbb{R}^{m \times n}$, Then there are orthognal matrices $U \epsilon \mathbb{R}^{m \times m} and\ V \epsilon \mathbb{R}^{n \times n}$ such that:

$$\text{A = U } \Sigma V^T, \text{ with } \Sigma = diag\{\sigma_1, \sigma_2, ..., \sigma_p\} \epsilon \mathbb{R}^{m \times n}$$

and $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_p$ where p = min(m,n) is the maximum possible rank of the matrix A.

The columns of U are the left singular vectors u_i of A, while the columns of V are the right singular vectors v_i of A. Σ is a diagonal matrix with real positive entries called singular values of the matrix.

- $Av_i = \sigma_i u_i$ for i = 1,...,p
- $A^T u_i = \sigma_i v_i$ for i = 1,...,p

2 Basic Properties

- The SVD is a rank revealig matrix factorization, infact r = rank(A) is the number of non zero singular values of A.
- The first r vectors of U form an orthonormal basis for the range (column space) of A:

$$\text{range(A) = span}\{u_i, ..., u_r\}$$

- The last n-r+1 vectors of V form an orthonormal basis for the Kernel (Null space) of A:

$$\text{Kernel(A) = span}\{v_{r+1}, ..., u_n\}$$

3 Low rank approximation

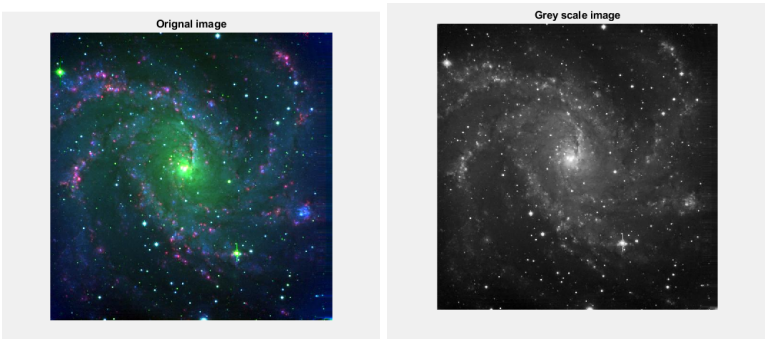
Given a high dimensional data represented as a matrix, we would like to approximate it with a much lower dimensional object. Any matrix with rank k , can be expressed using k singular values and the corresponding k singular vectors. Thus in order to reduce the dimensionality of the data, we approximate the matrix A with a law rank matrix $\tilde{A}_k$.
A possible application of this kind of approximation is image compression where we approximate the matrix corresponding to the image with a lower rank matrix.This allows us to define our approximate matrix $\tilde{A}_k$ as follows:

$$\tilde{A}_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

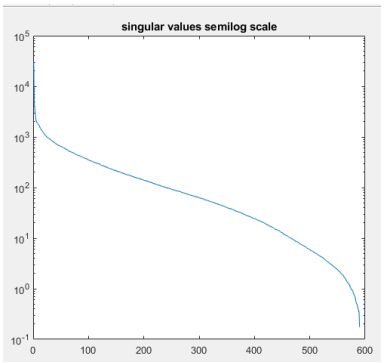
4 Image Compression

Now we perform image compression in the context of SVD. The image used during this homework can be downloaded from http://www.not.iac.es/general/photos/astronomical/extragalactic/ (credit: Nordic Optical Telescope, Søren Larsen). The idea is to represent the image as a matrix and then use the singullar value decomposition to find the closest matrix with lower rank compared to the orignal matrix.

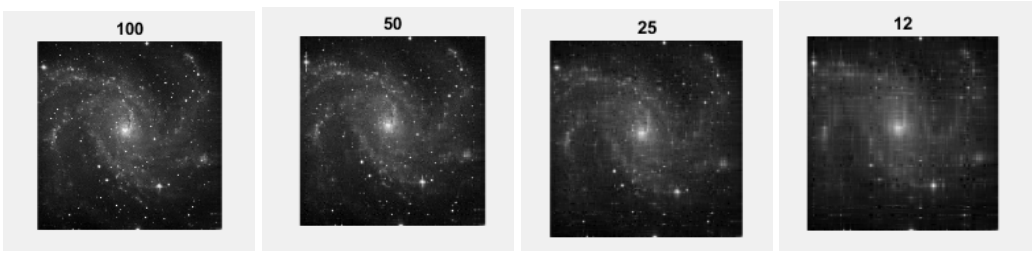
The analysis is performed using Matlab. The image can be read using the function imread which returns associated matrix as an output. We can display the image using the function imshow. This image can be transformed into grey scale using the function rgb2grey. Both the images are shown as follows:



The next step is to perform singular value decomposition. Matlab provides an inbuilt svd function that reutrns [U S V] as an output. Where the matrix S is σ, containing all the singular values. We can analyze these singular values by plotting them on a semilog scale:



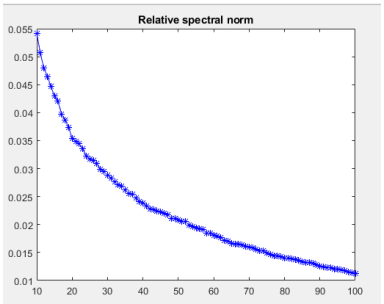
We can clearly notice that the singular values drop off very rapidly. With this in mind, we can compute low rank approximation with rank = 100,50,25,12. The corresponding low rank images are as shown below:



To evaluate these approximations analytically , We can compute the relative spectral norms of the approximate matrices with respect to the orignal matrix. The relative spectral norm can be computed as:

$$\frac{||A - \tilde{A}_k||_2}{||A||_2} = \frac{\sigma_{k+1}}{\sigma_1}$$

The choice of the rank k (i.e the number of singular values to keep) can be determined by plotting the relative spectral norm of the matrices.In this way we can choose k depending upon a certain amount of threshold criteria to be satisfied. Realative spectral norms in this case are shon as follows:



Notice that computation of matrices U and V are not necessary for this computation since only the singular values are involved. Hence computation can be easier and faster inorder to first choose the parameter k and then continue with SVD.